

## PARTITIONS OF NATURAL NUMBERS AND THEIR WEIGHTED REPRESENTATION FUNCTIONS

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### Abstract

For any positive integers  $k_1, k_2$  and any set  $A \subseteq \mathbb{N}$ , let  $R_{k_1, k_2}(A, n)$  be the number of solutions of the equation  $n = k_1 a_1 + k_2 a_2$  with  $a_1, a_2 \in A$ . Let  $g$  be a fixed integer. We prove that if  $k_1$  and  $k_2$  are two integers with  $2 \leq k_1 < k_2$  and  $(k_1, k_2) = 1$ , then there does not exist any set  $A \subseteq \mathbb{N}$  such that  $R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = g$  for all sufficiently large integers  $n$ , and if  $1 = k_1 < k_2$ , then there exists a set  $A$  such that  $R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = 1$  for all positive integers  $n$ .

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### 1. Introduction

Let  $\mathbb{N}$  be the set of all nonnegative integers. For a set  $A \subseteq \mathbb{N}$ , let  $R_1(A, n)$ ,  $R_2(A, n)$  and  $R_3(A, n)$  denote the number of solutions of  $a_1 + a_2 = n, a_1, a_2 \in A$ ;  $a_1 + a_2 = n, a_1, a_2 \in A, a_1 < a_2$  and  $a_1 + a_2 = n, a_1, a_2 \in A, a_1 \leq a_2$ , respectively. For  $i = 1, 2, 3$ , Sárközy asked whether there exist two sets  $A$  and  $B$  with  $|(A \cup B) \setminus (A \cap B)| = +\infty$  such that  $R_i(A, n) = R_i(B, n)$  for all sufficiently large integers  $n$ . We call this problem the Sárközy problem. In 2002, Dombi [2] proved that the answer is negative for  $i = 1$  and positive for  $i = 2$ . For  $i = 3$ , Chen and Wang [1] proved that the answer is also positive. In 2004, Lev [3] provided a new proof by using generating functions. Later, Sándor [5] determined the partitions of  $\mathbb{N}$  into two sets with the same representation functions by using generating functions. In 2008, Tang [6] provided a simple proof by using the characteristic function.

In 2012, Yang and Chen [7] first considered the Sárközy problem with weighted representation functions. For any positive integers  $k_1, \dots, k_t$  and any set  $A \subseteq \mathbb{N}$ , let  $R_{k_1, \dots, k_t}(A, n)$  be the number of solutions of the equation  $n = k_1 a_1 + \dots + k_t a_t$  with  $a_1, \dots, a_t \in A$ . They posed the following question.

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**PROBLEM 1.1** [7, Problem 1]. Does there exist a set  $A \subseteq \mathbb{N}$  such that  $R_{k_1, \dots, k_t}(A, n) = R_{k_1, \dots, k_t}(\mathbb{N} \setminus A, n)$  for all  $n \geq n_0$ ?

They answered this question for  $t = 2$  and proved the following results.

**THEOREM 1.2** [7, Theorem 1]. *If  $k_1$  and  $k_2$  are two integers with  $k_2 > k_1 \geq 2$  and  $(k_1, k_2) = 1$ , then there does not exist any set  $A \subseteq \mathbb{N}$  such that  $R_{k_1, k_2}(A, n) = R_{k_1, k_2}(\mathbb{N} \setminus A, n)$  for all sufficiently large integers  $n$ .*

**THEOREM 1.3** [7, Theorem 2]. *If  $k$  is an integer with  $k > 1$ , then there exists a set  $A \subseteq \mathbb{N}$  such that*

$$R_{1,k}(A, n) = R_{1,k}(\mathbb{N} \setminus A, n) \quad (1.1)$$

for all integers  $n \geq 1$ .

Furthermore, if  $0 \in A$ , then (1.1) holds for all integers  $n \geq 1$  if and only if

$$A = \{0\} \cup \left( \bigcup_{i=0}^{\infty} [(k+1)k^{2i}, (k+1)k^{2i+1} - 1] \right),$$

where  $[x, y] = \{n : n \in \mathbb{Z}, x \leq n \leq y\}$ .

Later, Li and Ma [4] proved the same results by using generating functions.

Let  $g$  be a fixed integer. In this paper, we consider whether there exists a set  $A \subseteq \mathbb{N}$  such that  $R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = g$  for all  $n \geq n_0$ . First, we answer this problem in the negative if  $k_1$  and  $k_2$  are two integers with  $2 \leq k_1 < k_2$  and  $(k_1, k_2) = 1$ .

**THEOREM 1.4.** *Let  $g$  be a fixed integer. If  $k_1$  and  $k_2$  are two integers with  $2 \leq k_1 < k_2$  and  $(k_1, k_2) = 1$ , then there does not exist any set  $A \subseteq \mathbb{N}$  such that*

$$R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = g$$

for all sufficiently large integers  $n$ .

Similar to Theorem 1.3, we seek a set  $A \subseteq \mathbb{N}$  such that  $R_{1,k}(A, n) - R_{1,k}(\mathbb{N} \setminus A, n) = g$  for all integers  $n \geq 1$ . In fact, if  $|g| > 1$ , then such a set  $A$  does not exist by the simple observation that  $0 \leq R_{1,k}(A, n) \leq 1$  and  $0 \leq R_{1,k}(\mathbb{N} \setminus A, n) \leq 1$  for all positive integers  $n < k$ . So we only need to consider the case  $g = 1$ .

**THEOREM 1.5.** *If  $k$  is an integer with  $k > 1$ , then there exists a set  $A \subseteq \mathbb{N}$  such that*

$$R_{1,k}(A, n) - R_{1,k}(\mathbb{N} \setminus A, n) = 1 \quad (1.2)$$

for all integers  $n \geq 1$ .

Furthermore, (1.2) holds for all integers  $n \geq 1$  if and only if

$$A = \{0\} \cup \left( \bigcup_{i=0}^{\infty} [k^{2i}, k^{2i+1} - 1] \right).$$

### 2. Proofs

**LEMMA 2.1.** *Let  $k_1 < k_2$  be two positive integers,  $\{a(n)\}_{n=-\infty}^{+\infty}$  be a sequence of integers with  $a(n) = 0$  for  $n < 0$  and  $A \subseteq \mathbb{N}$ . Then the equality*

$$R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = a(n) \tag{2.1}$$

holds for all nonnegative integers  $n$  if and only if

$$\chi_A\left(\left\lfloor \frac{n}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{n}{k_2} \right\rfloor\right) = 1 + \sum_{j=0}^{k_1-1} (a(n-j) - a(n-k_2-j))$$

holds for all nonnegative integers  $n$ , where  $\chi_A(i)$  is the characteristic function of  $A$ , that is,  $\chi_A(i) = 1$  if  $i \in A$  and  $\chi_A(i) = 0$  if  $i \notin A$ .

**PROOF.** Let  $f(x)$  be the generating function associated with  $A$ , that is,

$$f(x) = \sum_{a \in A} x^a = \sum_{i=0}^{\infty} \chi_A(i)x^i.$$

Then,

$$\begin{aligned} & \sum_{n=0}^{\infty} (R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n))x^n \\ &= f(x^{k_1})f(x^{k_2}) - \left(\frac{1}{1-x^{k_1}} - f(x^{k_1})\right)\left(\frac{1}{1-x^{k_2}} - f(x^{k_2})\right) \\ &= \frac{f(x^{k_1})}{1-x^{k_2}} + \frac{f(x^{k_2})}{1-x^{k_1}} - \frac{1}{(1-x^{k_1})(1-x^{k_2})}. \end{aligned}$$

Let

$$p(x) = \sum_{n=0}^{\infty} a(n)x^n.$$

It follows that (2.1) holds for all nonnegative integers  $n$  if and only if

$$\frac{f(x^{k_1})}{1-x^{k_2}} + \frac{f(x^{k_2})}{1-x^{k_1}} - \frac{1}{(1-x^{k_1})(1-x^{k_2})} = p(x),$$

that is,

$$f(x^{k_1})\frac{1-x^{k_1}}{1-x} + f(x^{k_2})\frac{1-x^{k_2}}{1-x} = \frac{1}{1-x} + (1-x^{k_2})\frac{1-x^{k_1}}{1-x}p(x). \tag{2.2}$$

Note that

$$f(x^{k_1})\frac{1-x^{k_1}}{1-x} = (1+x+\dots+x^{k_1-1})\sum_{n=0}^{\infty} \chi_A(n)x^{k_1n} = \sum_{n=0}^{\infty} \chi_A\left(\left\lfloor \frac{n}{k_1} \right\rfloor\right)x^n,$$

$$f(x^{k_2}) \frac{1 - x^{k_2}}{1 - x} = (1 + x + \dots + x^{k_2-1}) \sum_{n=0}^{\infty} \chi_A(n) x^{k_2 n} = \sum_{n=0}^{\infty} \chi_A\left(\left\lfloor \frac{n}{k_2} \right\rfloor\right) x^n,$$

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$$

and

$$(1 - x^{k_2}) \frac{1 - x^{k_1}}{1 - x} p(x) = (1 - x^{k_2})(1 + x + \dots + x^{k_1-1}) \sum_{n=0}^{\infty} a(n) x^n$$

$$= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{k_1-1} (a(n - j) - a(n - k_2 - j)) \right) x^n.$$

It follows from (2.2) that for all nonnegative integers  $n$ ,

$$\chi_A\left(\left\lfloor \frac{n}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{n}{k_2} \right\rfloor\right) = 1 + \sum_{j=0}^{k_1-1} (a(n - j) - a(n - k_2 - j)).$$

This completes the proof of Lemma 2.1. □

**LEMMA 2.2.** *Let  $n_0$  be a positive integer and  $k_1 < k_2$  be two positive integers with  $(k_1, k_2) = 1$  and  $A \subseteq \mathbb{N}$  be a set with*

$$\chi_A\left(\left\lfloor \frac{i}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{i}{k_2} \right\rfloor\right) = 1 \quad \text{for all } i \geq k_1 + k_2 + n_0. \tag{2.3}$$

*If  $n \geq k_1 + k_2 + n_0$  and  $\chi_A(n) + \chi_A(n + 1) = 1$ , then  $k_2 \mid n + 1$ .*

**PROOF.** Since  $\chi_A(n) + \chi_A(n + 1) = 1$ , it follows that

$$\chi_A\left(\left\lfloor \frac{(n + 1)k_1 - 1}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{(n + 1)k_1}{k_1} \right\rfloor\right) = \chi_A(n) + \chi_A(n + 1) = 1. \tag{2.4}$$

By (2.3),

$$\chi_A\left(\left\lfloor \frac{(n + 1)k_1 - 1}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{(n + 1)k_1 - 1}{k_2} \right\rfloor\right) = 1$$

and

$$\chi_A\left(\left\lfloor \frac{(n + 1)k_1}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{(n + 1)k_1}{k_2} \right\rfloor\right) = 1.$$

It follows from (2.4) that

$$\chi_A\left(\left\lfloor \frac{(n + 1)k_1 - 1}{k_2} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{(n + 1)k_1}{k_2} \right\rfloor\right) = 1.$$

Let  $t$  and  $r$  be integers with

$$(n + 1)k_1 = tk_2 + r, \quad 0 \leq r \leq k_2 - 1.$$

If  $r \geq 1$ , then

$$1 = \chi_A\left(\left\lceil \frac{(n+1)k_1 - 1}{k_2} \right\rceil\right) + \chi_A\left(\left\lceil \frac{(n+1)k_1}{k_2} \right\rceil\right) = 2\chi_A(t),$$

which is a contradiction. Hence,  $r = 0$  and  $(n+1)k_1 = tk_2$ . Noting that  $(k_1, k_2) = 1$ , we have  $k_2 \mid n+1$ . This completes the proof of Lemma 2.2.  $\square$

**PROOF OF THEOREM 1.4.** Let  $g$  be an integer and let  $k_1, k_2$  be integers with  $2 \leq k_1 < k_2$  and  $(k_1, k_2) = 1$ . Suppose that

$$R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = g \tag{2.5}$$

for all integers  $n \geq n_0$ . Let  $\{a(n)\}_{n=-\infty}^{+\infty}$  be a sequence of integers with  $a(n) = 0$  for  $n < 0$  and  $a(n) = g$  for all integers  $n \geq n_0$ . It follows from Lemma 2.1 that for all integers  $i \geq k_1 + k_2 + n_0$ ,

$$\chi_A\left(\left\lceil \frac{i}{k_1} \right\rceil\right) + \chi_A\left(\left\lceil \frac{i}{k_2} \right\rceil\right) = 1. \tag{2.6}$$

If  $A$  is a finite set, then  $R_{k_1, k_2}(A, n) = 0$  for all sufficiently large integers  $n$ , and  $R_{k_1, k_2}(\mathbb{N} \setminus A, n)$  cannot be a fixed constant as  $n \rightarrow +\infty$ , which implies that (2.5) cannot hold. So  $A$  is an infinite set. Similarly,  $\mathbb{N} \setminus A$  is also an infinite set.

Since  $2 \leq k_1 < k_2$ , it follows that there exists an integer  $t > 1$  such that  $k_2 < k_1^t$ . Note that both  $A$  and  $\mathbb{N} \setminus A$  are infinite sets. So there exists an integer  $n = k_1^\alpha k_2^\beta h - 1 > (k_1 + k_2 + n_0)^{t+1}$  such that  $n \in A$  and  $n + 1 \notin A$ , where  $\alpha$  and  $\beta$  are nonnegative integers and  $h$  is a positive integer with  $(h, k_1 k_2) = 1$ . It follows from (2.6) and Lemma 2.2 that  $k_2 \mid n + 1$  and  $\beta \geq 1$ . Since

$$(k_1 + k_2 + n_0)^{t+1} < n < k_1^\alpha k_2^\beta h < k_1^{t(\alpha+\beta)} h,$$

it follows that  $k_1^{\alpha+\beta} > k_1 + k_2 + n_0$  or  $h > k_1 + k_2 + n_0$ . Hence, for any  $0 \leq i \leq \beta$ ,

$$k_1^{\alpha+i} k_2^{\beta-i} h \geq k_1^{\alpha+\beta} h > k_1 + k_2 + n_0. \tag{2.7}$$

By (2.6),

$$\chi_A\left(\left\lceil \frac{k_1^{\alpha+1} k_2^\beta h}{k_1} \right\rceil\right) + \chi_A\left(\left\lceil \frac{k_1^{\alpha+1} k_2^\beta h}{k_2} \right\rceil\right) = 1 \tag{2.8}$$

and

$$\chi_A\left(\left\lceil \frac{k_1^{\alpha+1} k_2^\beta h - k_1}{k_1} \right\rceil\right) + \chi_A\left(\left\lceil \frac{k_1^{\alpha+1} k_2^\beta h - k_1}{k_2} \right\rceil\right) = 1. \tag{2.9}$$

Since  $k_1^\alpha k_2^\beta h = n + 1 \notin A$  and  $k_1^\alpha k_2^\beta h - 1 = n \in A$ , it follows from (2.8) and (2.9) that

$$\chi_A(k_1^{\alpha+1} k_2^{\beta-1} h - 1) + \chi_A(k_1^{\alpha+1} k_2^{\beta-1} h) = 1.$$

By Lemma 2.2,  $k_2 \mid k_1^{\alpha+1} k_2^{\beta-1} h$  and so  $\beta \geq 2$ . Continuing this procedure yields

$$\chi_A(k_1^{\alpha+\beta} h - 1) + \chi_A(k_1^{\alpha+\beta} h) = 1.$$

By (2.7) and Lemma 2.2, we also have  $k_2 \mid k_1^{\alpha+\beta} h$ , which is impossible. Hence, there does not exist any set  $A \subseteq \mathbb{N}$  such that (2.5) holds for all sufficiently large integers  $n$ . This completes the proof of Theorem 1.4.  $\square$

**PROOF OF THEOREM 1.5.** Suppose that there is a set  $A$  such that

$$R_{1,k}(A, n) - R_{1,k}(\mathbb{N} \setminus A, n) = 1 \quad (2.10)$$

for all integers  $n \geq 1$ . Then  $0 \in A$  and (2.10) holds for all integers  $n \geq 0$ . Let  $\{a(n)\}_{n=-\infty}^{+\infty}$  be a sequence of integers with  $a(n) = 0$  for  $n < 0$  and  $a(n) = 1$  for  $n \geq 0$ . By Lemma 2.1,

$$R_{1,k}(A, n) - R_{1,k}(\mathbb{N} \setminus A, n) = a(n)$$

for all nonnegative integers  $n$  if and only if

$$\chi_A(n) + \chi_A\left(\left\lfloor \frac{n}{k} \right\rfloor\right) = 1 + a(n) - a(n-k)$$

for all nonnegative integers  $n$ , that is,

$$\begin{aligned} \chi_A(n) + \chi_A(0) &= 2 \quad \text{for } 0 \leq n \leq k-1, \\ \chi_A(n) + \chi_A\left(\left\lfloor \frac{n}{k} \right\rfloor\right) &= 1 \quad \text{for } n \geq k. \end{aligned}$$

Thus,

$$A = \{0\} \cup \left( \bigcup_{i=0}^{\infty} [k^{2i}, k^{2i+1} - 1] \right). \quad \square$$

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