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Existence of nodal solutions of nonlinear elliptic equations

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We establish that, for $n \geqslant 3$, the elliptic equation

$$-\Delta u = \lambda |x|^{\mu} |u|^{q-2} u + |x|^{\nu} |u|^{p-2} u$$

on a ball with zero Dirichlet data possesses a pair of nodal radial solutions for all $\lambda > 0$ provided that

$$\mu, \nu > -2, \quad \max\left\{2, \frac{n+2\mu+2}{n-2}\right\} < q < \frac{2(n+\mu)}{n-2} \quad \text{and} \quad p = \frac{2(n+\nu)}{n-2}.$$

When $q = 2$ and $n > 2\mu + 6$, the same result holds for $\lambda > 0$ small. Canonical transformations convert the equation into a quasi-linear elliptic equation and an equation with Hardy term. Then the results correspond to the results for the transformed equations. For example, the equation

$$-\Delta w - \frac{\chi}{|y|^2} w = \tilde{\lambda} |y|^{a} |w|^{q-2} w + |y|^{\nu} |w|^{p-2} w,$$

on a ball with zero Dirichlet data, possesses a pair of nodal radial solutions for all $\tilde{\lambda} > 0$ provided that $a, \nu > -2$ and

$$\max\left\{2, \frac{n+a-\sqrt{\bar{\chi}-\chi}}{\sqrt{\bar{\chi}}}\right\} < q < \frac{n+a}{\sqrt{\bar{\chi}}} \quad \text{with } \bar{\chi} = \left(\frac{n-2}{2}\right)^2.$$

When $q = 2$, $n > 2a + 6$ and $0 < \chi < \bar{\chi} - (a+2)^2$, the same result holds for $\tilde{\lambda} > 0$ small.

1. Introduction

In this paper, we study the existence of solutions for the following elliptic equation on the unit ball $B_1 = \{x : |x| < 1\}$, $n \geqslant 3$:

$$\left.\begin{aligned} -\Delta u = \lambda |x|^{\mu} |u|^{q-2} u + |x|^{\nu} |u|^{p-2} u \quad \text{in } B_1, \\ u = 0 \quad \text{on } \partial B_1, \end{aligned}\right\} \tag{1.1}$$

where $\mu, \nu > -2$, $2 \leqslant q < 2(n+\mu)/(n-2)$, $p = 2(n+\nu)/(n-2)$ and λ is a real parameter.

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When $\nu = 0$, $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Namely, for a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact. In 1983, Brezis and Nirenberg [3] added a lower-order perturbation as in (1.1) with $\mu = \nu = 0$ and verified a compactness of local type in a range of q to ensure the existence of positive solutions. Later, Cerami *et al*. [6] studied nodal solutions (changing-sign solutions) of (1.1) with $\mu = \nu = 0$ and $q = 2$. In [14], Tarantello obtained the existence of nodal solutions by the convergence of nodal solutions of subcritical problems with $2^* - \varepsilon$ as $\varepsilon \to 0$. There are many papers dealing with elliptic problems involving the critical Sobolev exponent. For known results and related topics including the non-existence of nodal solutions, we refer the reader to [1, 2, 5, 8–12] and the references therein.

In general, subcritical perturbations indicate that $\mu \geqslant \nu$ and $q < p$.

In order to state our result in the case of $q = 2$, we need the number

$$\lambda_{1,\alpha,\mu}(B_1) = \min_{0 \neq \varphi \in H_0^{1,2}(B_1)} \frac{\int_{B_1} |x|^\alpha |\nabla\varphi|^2}{\int_{B_1} |x|^\mu \varphi^2},$$

where $\alpha > 2 - n$. For $\alpha = 0$, we simply write $\lambda_{1,\mu}$ instead of $\lambda_{1,0,\mu}$.

The main result of this paper is the following assertion.

Theorem 1.1.

(i) *Let*

$$\max\left\{2, \frac{2(2+\mu)}{n-2}\right\} < q < \frac{2(n+\mu)}{n-2} \quad \textit{and} \quad \lambda > 0.$$

Then (1.1) has a positive radial solution. Moreover, if

$$\max\left\{2, \frac{n+2\mu+2}{n-2}\right\} < q < \frac{2(n+\mu)}{n-2},$$

(1.1) has a pair of nodal radial solutions.

(ii) *Let $q = 2$ and $0 < \lambda < \lambda_{1,\mu}$. If $n \geqslant \mu + 4$, then (1.1) has a positive radial solution. Moreover, if $n > 2\mu + 6$, (1.1) has a pair of nodal radial solutions.*

For the cases when $\mu = \nu = 0$ and $q < p$ in theorem 1.1, the existence of both positive solutions and nodal solutions was established in [3] and [6, 14], respectively. However, the important observation of theorem 1.1 is that the conditions on q are independent of p but depend only on μ. In other words, the class of subcritical exponents is subcritical for the class of critical exponents in the perturbational sense.

We now consider two equations which can be transformed to (1.1) by suitable transformations and state the corresponding results. The first is

$$\left.\begin{aligned} -\nabla\cdot(|y|^\alpha \nabla v) &= \tilde{\lambda}|y|^{\tilde{\mu}}|v|^{q-2}v + |y|^{\tilde{\nu}}|v|^{p-2}v \quad \text{in } B_1,\\ v &= 0 \quad \text{on } \partial B_1, \end{aligned}\right\} \tag{1.2}$$

where $\tilde{\mu}, \tilde{\nu} > \alpha - 2 > -n$ and $p = 2(n+\tilde{\nu})/(\alpha+n-2)$. Let

$$u(x) = \left(\frac{n-2}{\alpha+n-2}\right)^{2/(p-2)} v(y) \quad \text{and} \quad |x| = |y|^{(\alpha+n-2)/(n-2)}.$$

Then (1.2) is transformed into (1.1) with

$$\mu = \frac{(n-2)\tilde{\mu} - n\alpha}{\alpha + n - 2}, \qquad \nu = \frac{(n-2)\tilde{\nu} - n\alpha}{\alpha + n - 2}, \qquad \lambda = \tilde{\lambda}\left(\frac{n-2}{\alpha+n-2}\right)^{2(p-q)/(p-2)}.$$

From theorem 1.1, we reach the following conclusion.

THEOREM 1.2.

(i) *Let*

$$\max\left\{2, \frac{2(2+\tilde{\mu}-\alpha)}{\alpha+n-2}\right\} < q < \frac{2(n+\tilde{\mu})}{\alpha+n-2} \quad \textit{and} \quad \tilde{\lambda} > 0.$$

Then (1.2) has a positive radial solution. Moreover, if

$$\max\left\{2, \frac{n+2\tilde{\mu}+2-\alpha}{\alpha+n-2}\right\} < q < \frac{2(n+\tilde{\mu})}{\alpha+n-2},$$

(1.2) has a pair of nodal radial solutions.

(ii) *Let $q = 2$ and $0 < \tilde{\lambda} < \lambda_{1,\alpha,\tilde{\mu}}$. If $n \geqslant 4 + \tilde{\mu} - 2\alpha$, then (1.2) has a positive radial solution. Moreover, if $n > 2\tilde{\mu} + 6 - 3\alpha$, (1.2) has a pair of nodal radial solutions.*

For the existence of a positive solution of (1.2) with $q = 2$, Egnell [8] showed that when $n \geqslant 4+\tilde{\mu}-2\alpha$, (1.2) has a positive radial solution if and only if $0 < \lambda < \lambda_{1,\alpha,\tilde{\mu}}$. Obviously,

$$\lambda_{1,\alpha,\tilde{\mu}} = \left(\frac{\alpha+n-2}{n-2}\right)^2 \lambda_{1,\mu}.$$

The second equation is that with Hardy terms:

$$\left.\begin{aligned} -\Delta w - \frac{\chi}{|y|^2} w + \tilde{\lambda}|y|^a|w|^{q-2}w + |y|^{\nu}|w|^{p-2}w \quad \text{in } B_1, \\ w = 0 \quad \text{on } \partial B_1, \end{aligned}\right\} \tag{1.3}$$

where $0 < \chi < \bar{\chi} = \frac{1}{2}(n-2)^2$, $a > -2$, $2 \leqslant q < 2(n+a)/(n-2)$ and $p = 2(n+\nu)/(n-2)$.

Setting $v(|y|) = |y|^{\sigma} w(|y|)$ with $\sigma = \sqrt{\bar{\chi}} - \sqrt{\bar{\chi} - \chi}$, we see that v satisfies

$$-v''(|y|) - \frac{n-2\sigma-1}{|y|} v'(|y|) = \tilde{\lambda}|y|^{a-\sigma(q-2)}|v|^{q-2}v + |y|^{\nu-\sigma(p-2)}|v|^{p-2}v.$$

Letting $\alpha = -2\sigma = 2 - n + 2\sqrt{\bar{\chi} - \chi}$, we have

$$\begin{aligned} -\nabla \cdot (|y|^{\alpha}\nabla v) &= -|y|^{\alpha}v'' - (\alpha+n-1)|y|^{\alpha-1}v' \\ &= -|y|^{\alpha}v'' - (n-2\sigma-1)|y|^{\alpha-1}v' \\ &= \tilde{\lambda}|y|^{\tilde{\mu}}|v|^{q-2}v + |y|^{\tilde{\nu}}|v|^{p-2}v, \end{aligned}$$

with $\tilde{\mu} = a - \sigma(q-2) + \alpha$ and $\tilde{\nu} = \nu - \sigma(p-2) + \alpha$. Hence, (1.3) is also transformed into (1.1) with

$$\mu = \check{\mu} = -2 + \frac{(n-2)[2 + a - \sigma(q-2)]}{\alpha + n - 2} > -2, \tag{1.4}$$

$$\nu = \check{\nu} = -2 + \frac{(n-2)[2 + \nu - \sigma(p-2)]}{\alpha + n - 2} > -2, \tag{1.5}$$

and

$$\lambda = \tilde{\lambda}\left(\frac{n-2}{\alpha + n - 2}\right)^{2(p-q)/(p-2)}.$$

It is easy to see that

$$p = \frac{2(n+\nu)}{n-2}, \qquad 2 \leqslant q < \frac{2(n+a)}{n-2} < \frac{2(n+\check{\mu})}{n-2}$$

and

$$\lambda_1(\chi) = \min_{\phi \in H_0^1(B)} \frac{\int_B |\nabla\phi(x)|^2 - \chi \int_B \phi^2/|x|^2}{\int_B \phi^2(x)} = \lambda_{1,\alpha,\tilde{\mu}}.$$

Therefore, theorem 1.1 is translated into the following result.

Theorem 1.3. *Let* $a, \nu > -2$ *and* $0 < \chi < \bar{\chi}$.

(i) *Let*

$$\max\left\{2, \frac{n + a - 2\sqrt{\bar{\chi} - \chi}}{\sqrt{\bar{\chi}}}\right\} < q < \frac{n+a}{\sqrt{\bar{\chi}}}$$

and $\tilde{\lambda} > 0$. *Then (1.3) has a positive radial solution. Moreover, if*

$$\max\left\{2, \frac{n + a - \sqrt{\bar{\chi} - \chi}}{\sqrt{\bar{\chi}}}\right\} < q < \frac{n+a}{\sqrt{\bar{\chi}}},$$

(1.3) has a pair of nodal radial solutions.

(ii) *Let* $q = 2$, $n \geqslant a + 4$ *and* $0 < \chi \leqslant \bar{\chi} - \frac{1}{2}(a+2)^2$. *For* $0 < \tilde{\lambda} < \lambda_1(\chi)$, *(1.3) has a positive radial solution. Moreover, if* $n > 2a + 6$ *and* $0 < \chi < \bar{\chi} - (a+2)^2$, *(1.3) has a pair of nodal radial solutions.*

Note that

$$\bar{\chi} - \left(\frac{a+2}{2}\right)^2 = \frac{(n+a)(n-a-4)}{4},$$

$$\bar{\chi} - (a+2)^2 = \frac{(n+2a+2)(n-2a-6)}{4}.$$

Recently, (1.3) has been studied intensively. However, we understand (1.3) through (1.1) and (1.2) rather than by studying (1.3) directly. Every structure for (1.1) generates a structure for (1.3). In [10], Jannelli studied the existence of positive solutions of (1.3) with $a = \nu = 0$ and $q = 2$, and derived the first assertion of theorem 1.3(ii). On the other hand, Cao and Peng [5] considered the existence of nodal solutions and obtained the second assertion of theorem 1.3(ii). By straightforward computations, the second result of theorem 1.3(i) improves a recent result in [11] where $-2 < \nu \leqslant 0$, $a = 0$, and

$$\max\left\{2, \frac{n+\beta}{\sqrt{\bar\chi}+\beta}, \frac{n-\beta}{\sqrt{\bar\chi}}\right\} < q < \frac{n+2\beta}{\sqrt{\bar\chi}+\beta} \quad \text{with } \beta = \sqrt{\bar\chi - \chi}.$$

Under a weak perturbation in the following sense, (1.1) has no sign-changing radial solution. More precisely, if $\mu, \nu > -2$ and $2 \leqslant q \leqslant (p-1)\min\{1, (2+\mu)/(2+\nu)\}$, then there exists a constant $\lambda^* > 0$ such that, for $\lambda \in (0, \lambda^*)$, (1.1) has no radial solution which changes sign. See [2] for the proof. In the case when $q \geqslant 2$ and

$$\frac{n+2\nu+2}{n-2}\min\left\{1, \frac{2+\mu}{2+\nu}\right\} < q \leqslant \max\left\{2, \frac{n+2\mu+2}{n-2}\right\}$$

and except for the case when $q = 2$ and $n > 2\mu + 6$, whether or not (1.1) has a sign-changing radial solution is an open question. The non-existence is transformed into the following two non-existence results.

Theorem 1.4. *Let $\tilde\mu, \tilde\nu > \alpha - 2 > -n$. If $2 \leqslant q \leqslant (p-1)\min\{1, (2+\tilde\mu-\alpha)/(2+\tilde\nu-\alpha)\}$, then there exists a constant $\tilde\lambda^* > 0$ such that, for $\tilde\lambda \in (0, \tilde\lambda^*)$, (1.2) has no radial solution which changes sign.*

Theorem 1.5. *Let $0 < \chi < \bar\chi$ and $a, \nu > -2$. If $2 \leqslant q \leqslant (p-1)\min\{1, (2+a+2\sigma)/(2+\nu+\sigma)\}$ with $\sigma = \sqrt{\bar\chi} - \sqrt{\bar\chi - \chi}$, then there exists a constant $\tilde\lambda^* > 0$ such that, for $\tilde\lambda \in (0, \tilde\lambda^*)$, (1.3) has no radial solution which changes sign.*

Proof. If $2 \leqslant q \leqslant (p-1)\min\{1, (2+\check\mu)/(2+\check\nu)\}$, then there exists a constant $\lambda^* > 0$ such that, for $\lambda \in (0, \lambda^*)$, (1.3) has no sign-changing radial solution. The inequality

$$q \leqslant (p-1)\frac{2+\check\mu}{2+\check\nu} = (p-1)\frac{2+a-\sigma(q-2)}{2+\nu-\sigma(p-2)}$$

is equivalent to

$$q \leqslant (p-1)\frac{2+a+2\sigma}{2+\nu+\sigma}.$$

□

Corollary 1.6. *Let $0 < \chi < \bar\chi$ and $a + \sigma \geqslant \nu > -2$. If $2 \leqslant q \leqslant p-1$, then there exists a constant $\tilde\lambda^* > 0$ such that, for $\tilde\lambda \in (0, \tilde\lambda^*)$, (1.3) has no nodal radial solutions.*

Furthermore, if $\nu = 0$ and $a + \sigma \geqslant 0$, then the result holds for $n = 3, 4, 5, 6$ because $p - 1 = (n+2)/(n-2) \geqslant 2$.

This paper is organized as follows. Some preliminaries, including the existence of positive solutions, are reviewed in § 2. In § 3 we study the existence of nodal solutions. In § 4 the transformation to conclude theorem 1.3 is presented in detail. Finally, we make related remarks in § 5.

2. Preliminaries

In this section, we collect some known facts and consider the existence of positive solutions of (1.1). Let $0 \geqslant \nu > -2$ and let S_ν be the best constant for the Sobolev–Hardy embedding $\mathcal{D}^{1,2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n, |x|^\nu)$ with $p = 2(n+\nu)/(n-2)$, where the space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is the completion of $\mathbb{C}_0^\infty(\mathbb{R}^n)$ in the norm

$$\|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla u|^2 \,\mathrm{d}x.$$

In the radial context, we may regard S_ν for $\nu > -2$ as the best constant for the embedding $H_r(B_1) \to L^p(B_1, |x|^\nu)$, where $H_r(B_1)$ is the class of radial functions in $H_0^{1,2}(B_1)$. The constant

$$S_\nu = \inf_{\mathcal{D}^{1,2}(\mathbb{R}^n), u \not\equiv 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^2}{(\int_{\mathbb{R}^n} |x|^\nu |u|^p)^{2/p}}$$

is achieved by the function

$$\bar{u}_\varepsilon(x) = \frac{[(n+\nu)(n-2)\varepsilon]^{(n-2)/2(2+\nu)}}{(\varepsilon + |x|^{2+\nu})^{(n-2)/(2+\nu)}}$$

for each $\varepsilon > 0$. In fact, these functions are minimizers of S_ν in the set of radial functions in the case when $\nu > -2$. Moreover, $\bar{u}_\varepsilon$ are the only positive radial solutions of

$$-\Delta u = |x|^\nu u^{p-1} \quad \text{in } \mathbb{R}^n.$$

See [4, 7, 13] for details and more general results.

Lemma 2.1. *Let $0 \in \Omega$ be a bounded domain and let $1 \leqslant l \leqslant 2(n+\nu)/(n-2)$. Then $H_0^1(\Omega)$ is continuously embedded into $L^l(\Omega, |x|^\nu \,\mathrm{d}x)$ if $0 \geqslant \nu > -2$ and, moreover, the embedding is compact if $l < 2(n+\nu)/(n-2)$. On the radial space*

$$H_r(B_1) = \{u \in H_0^1(B_1) \mid u(x) = u(|x|)\}$$

in the case when $\Omega = B_1$, the two results hold even if the condition $0 \geqslant \nu > -2$ is replaced by $\nu > -2$.

In a variational approach to obtain a positive solution of (1.1), one looks for critical points in $H_0^1(B_1)$ or $H_r(B_1)$ of an energy functional

$$I_\lambda(u) = \int_{B_1} \left(\frac{|\nabla u|^2}{2} - \frac{\lambda}{q}|x|^\mu |u|^q - \frac{1}{p}|x|^\nu |u|^p \right).$$

Under the assumptions of lemma 2.1, I_λ is $\mathbb{C}^1$-functional on $H_0^1(B_1)$ or $H_r(B_1)0$, respectively. By the method initiated by Brezis and Nirenberg in [3], we first observe that I_λ satisfies $(\mathrm{PS})_\beta$ for each $\beta \in (-\infty, (2+\nu/(2(n+\nu))S_\nu^{(n+\nu)/(2+\nu)}))$.

Lemma 2.2. *Any sequence $\{u_m\}$ in $H_r(B_1)$ such that*

$$I_\lambda(u_m) \to \beta < \frac{2+\nu}{2(n+\nu)} S_\nu^{(n+\nu)/(2+\nu)}, \qquad I'_\lambda(u_m) \to 0,$$

as $m \to \infty$, is relatively compact.

We can prove the compactness of local type by the argument in [3]. To concentrate on the compactness conditions we consider the energy level

$$c_1 = \min_{u \in M_1} I_\lambda(u),$$

where $M_1 = \{u \in H_r(B_1) \setminus \{0\} \mid I'_\lambda(u)u = 0\}$. In order to establish the existence of positive solutions, we follow a similar argument to that in [3].

THEOREM 2.3.

(i) *Let* $\max\{2, 2(2+\mu)/(n-2)\} < q < 2(n+\mu)/(n-2)$. *Then (1.1) has a positive radial solution for all* $\lambda > 0$.

(ii) *Let* $q = 2$ *and* $0 < \lambda < \lambda_{1,\mu}$. *If* $n \geqslant \mu + 4$, *then (1.1) has a positive radial solution.*

Proof. Fix a radial function $\phi \in \mathbb{C}_0^\infty(B_1)$, $\phi \geqslant 0$ such that $\phi(x) \equiv 1$ for $|x| < R$ for some $0 < R < 1$. Set

$$u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^{2+\nu})^{(n-2)/(2+\nu)}}$$

for $\varepsilon > 0$ and $v_\varepsilon(x) = u_\varepsilon(x)/\|u_\varepsilon\|_{L^p(B_1,|x|^\nu)}$ so that $\|v_\varepsilon\|_{L^p(B_1,|x|^\nu)} = 1$. We claim that v_ε satisfies

$$\sup_{t \geqslant 0} I_\lambda(tv_\varepsilon) < \frac{2+\nu}{2(n+\nu)} S_\nu^{(n+\nu)/(2+\nu)}$$

for $\varepsilon > 0$ sufficiently small. The computations show that

$$\|u_\varepsilon\|_{L^p(B_1,|x|^\nu)} = \frac{C}{\varepsilon^{(n-2)/(2(2+\nu))}} + o(1) \tag{2.1}$$

for some $C = C(n, \nu)$ and

$$\|\nabla v_\varepsilon\|^2_{L^2(B_1)} = S_\nu + O(\varepsilon^{(n-2)/(2+\nu)}). \tag{2.2}$$

In fact, we have

$$\int_{B_1} |\nabla u_\varepsilon|^2 = \frac{C_1}{\varepsilon^{(n-2)/(2+\nu)}} + O(1), \tag{2.3}$$

$$\left(\int_{B_1} |x|^\nu u_\varepsilon^p\right)^{2/p} = \frac{C_2}{\varepsilon^{(n-2)/(2+\nu)}} + O(\varepsilon), \tag{2.4}$$

where C_1 and C_2 denote the positive constants which depend on n and ν.

Verification of (2.3): we have

$$\nabla u_\varepsilon(x) = \frac{\nabla\phi(x)}{(\varepsilon + |x|^{2+\nu})^{(n-2)/(2+\nu)}} - \frac{(n-2)|x|^\nu \phi(x) x}{(\varepsilon + |x|^{2+\nu})^{(n+\nu)/(2+\nu)}}.$$

Since $\phi \equiv 1$ near 0, it follows that

$$\begin{aligned}\int_{B_1} |\nabla u_\varepsilon|^2 &= (n-2)^2 \int_{B_1} \frac{|x|^{2(\nu+1)}}{(\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}} + O(1)\\ &= (n-2)^2 \int_{\mathbb{R}^n} \frac{|x|^{2(\nu+1)}}{(\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}} + O(1)\\ &= \frac{C_1}{\varepsilon^{(n-2)/(2+\nu)}} + O(1),\end{aligned}$$

where

$$C_1 = (n-2)^2 \int_{\mathbb{R}^n} \frac{|x|^{2(\nu+1)}}{(1 + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}}.$$

Verification of (2.4):

$$\begin{aligned}\int_{B_1} |x|^\nu u_\varepsilon^p &= \int_{B_1} \frac{|x|^\nu \phi^p(x)}{(\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}}\\ &= \int_{B_1} \frac{|x|^\nu (\phi^p(x) - 1)}{(\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}} + \int_{B_1} \frac{1}{|x|^\nu (\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}}\\ &= O(1) + \int_{\mathbb{R}^n} \frac{1}{|x|^\nu (\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}}\\ &= \frac{C_2'}{\varepsilon^{(n+\nu)/(2+\nu)}} + O(1),\end{aligned}$$

where

$$C_2' = \int_{\mathbb{R}^n} \frac{|x|^\nu}{(1 + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}}.$$

Thus, (2.4) follows with $C_2 = (C_2')^{2/p}$ and $C_1/C_2 = S_\nu$.

Setting $X_\varepsilon = \|\nabla v_\varepsilon\|^2_{L^2(B_1)}$, we have

$$I_\lambda(tv_\varepsilon) = \tfrac{1}{2} t^2 X_\varepsilon - \frac{1}{p} t^p - \frac{\lambda}{q} t^q \int_{B_1} |x|^\mu |v_\varepsilon|^q.$$

Since $I_\lambda(tv_\varepsilon) \leqslant \frac{1}{2} t^2 X_\varepsilon - t^p/p$, we have

$$\lim_{t\to\infty} I_\lambda(tv_\varepsilon) = -\infty$$

and thus $\sup_{t\geqslant 0} I_\lambda(tv_\varepsilon)$ is achieved at some $t_\varepsilon > 0$. Therefore, we have

$$t_\varepsilon X_\varepsilon - t_\varepsilon^{p-1} - \lambda t_\varepsilon^{q-1} \int_{B_1} |x|^\mu |v_\varepsilon|^q = 0 \tag{2.5}$$

and

$$t_\varepsilon \leqslant X_\varepsilon^{1/(p-2)}. \tag{2.6}$$

Obviously, $t_\varepsilon v_\varepsilon \in M_1$. Set

$$Y_\varepsilon = \sup_{t\geqslant 0} I_\lambda(tv_\varepsilon) = I_\lambda(t_\varepsilon v_\varepsilon).$$

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Since the function $t \mapsto (\frac{1}{2}t^2 X_\varepsilon - t^p/p)$ is increasing on the interval $[0, {X_\varepsilon}^{1/(p-2)}]$, by (2.6) we have

$$\begin{aligned} Y_\varepsilon &= \tfrac{1}{2}t_\varepsilon^2 X_\varepsilon - \frac{1}{p}t_\varepsilon^p - \frac{\lambda}{q}t_\varepsilon^q \int_{B_1} |x|^\mu |v_\varepsilon|^q \\ &\leqslant \frac{p-2}{2p} X_\varepsilon^{p/(p-2)} - \frac{\lambda}{q}t_\varepsilon^q \int_{B_1} |x|^\mu |v_\varepsilon|^q \\ &= \frac{2+\nu}{2(n+\nu)} X_\varepsilon^{(n+\nu)/(2+\nu)} - \frac{\lambda}{q}t_\varepsilon^q \int_{B_1} |x|^\mu |v_\varepsilon|^q. \end{aligned}$$

By (2.2), we obtain

$$Y_\varepsilon \leqslant \frac{2+\nu}{2(n+\nu)} S_\nu^{(n+\nu)/(2+\nu)} + O(\varepsilon^{(n-2)/(2+\nu)}) - \frac{\lambda}{q}t_\varepsilon^q \int_{B_1} |x|^\mu |v_\varepsilon|^q. \tag{2.7}$$

To get

$$Y_\varepsilon < \frac{2+\nu}{2(n+\nu)} S_\nu^{(n+\nu)/(2+\nu)}$$

for $\varepsilon > 0$ sufficiently small, it suffices to show that

$$\lim_{\varepsilon \to 0} \varepsilon^{-(n-2)/(2+\nu)} \int_{B_1} |x|^\mu |v_\varepsilon|^q = \infty.$$

In fact, if $q > \max\{2, 2(2+\mu)/(n-2)\}$, then

$$\begin{aligned} &\lim_{\varepsilon \to 0} \varepsilon^{-(n-2)/(2+\nu)} \int_{B_1} \frac{\varepsilon^{(n-2)q/(2(2+\nu))}|x|^\mu}{(\varepsilon + |x|^{2+\nu})^{(n-2)q/(2+\nu)}} \\ &\qquad = \lim_{\varepsilon \to 0} \omega_n \varepsilon^{(2(2+\mu)-(n-2)q)/(2(2+\nu))} \int_0^{\varepsilon^{-1/(2+\nu)}} \frac{s^{\mu+n-1}}{(1+s^{2+\nu})^{(n-2)q/(2+\nu)}} \, \mathrm{d}s \\ &\qquad = \infty. \end{aligned}$$

Therefore,

$$c_1 < \frac{2+\nu}{2(n+\nu)} S_\nu^{(n+\nu)/(2+\nu)}.$$

Combining a variational principle and lemma 2.2, we see that (1.1) has a positive solution attaining c_1 (see [3, 9]).

For $q = 2$, by (2.1) and

$$\begin{aligned} \int_{B_1} |x|^\mu u_\varepsilon^2 &= \int_{B_1} \frac{|x|^\mu}{(\varepsilon + |x|^{2+\nu})^{2(n-2)/(2+\nu)}} + O(1) \\ &= \begin{cases} K_1 \varepsilon^{(4+\mu-n)/(2+\nu)} + O(1), & n > \mu + 4, \\ K_1 |\log \varepsilon| + O(1), & n = \mu + 4, \\ O(1), & n < \mu + 4, \end{cases} \end{aligned}$$

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we have the following estimates:

$$\|v_\varepsilon\|^2_{L^2(B_1,|x|^\mu)} = \begin{cases} O(\varepsilon^{(2+\mu)/(2+\nu)}), & n > \mu+4, \\ O(\varepsilon^{(n-2)/(2+\nu)}|\log\varepsilon|), & n = \mu+4, \\ O(\varepsilon^{(n-2)/(2+\nu)}), & n < \mu+4. \end{cases}$$

When $q = 2$, we impose the extra condition $\lambda \in (0, \lambda_{1,\mu})$. Then, for any $u \in H^1_0(\Omega)\setminus\{0\}$, there exists $t_u > 0$ satisfying (2.5). Let $2(2+\mu)/(n-2) \leqslant 2$, i.e. $n \geqslant \mu+4$. From (2.7), we have

$$Y_\varepsilon \leqslant \frac{2+\nu}{2(n+\nu)} S_\nu^{(n+\nu)/(2+\nu)} + \begin{cases} O(\varepsilon^{(n-2)/(2+\nu)}) - O(\varepsilon^{(2+\mu)/(2+\nu)}), & n > \mu+4, \\ O(\varepsilon^{(2+\mu)/(2+\nu)}) - O(\varepsilon^{(2+\mu)/(2+\nu)}|\log\varepsilon|), & n = \mu+4, \\ O(\varepsilon^{(n-2)/(2+\nu)}), & n < \mu+4. \end{cases}$$

Therefore, (1.1) possesses a positive solution if $n \geqslant \mu+4$. □

Note that u_0 obtained in theorem 2.3 is a weak solution of (1.1). Moreover, it is well known that u_0 is bounded. See [2, proposition 2.2 and lemma 2.3] or [7].

Brezis and Nirenberg [3] established that (1.1) with $\mu = \nu = 0$, $q = 2$ and $n = 3$ has a positive solution if and only if $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$. For the case when $n < \mu+4$ and $2 \leqslant q \leqslant 2(2+\mu)/(n-2)$, we refer the reader to [12].

3. Nodal solution

In this section we consider the existence of nodal solutions. We denote the positive solution of (1.1) by u_0. Let $c_1 = I_\lambda(u_0) = I_\lambda(-u_0)$. Note that c_1 can be characterized by

$$c_1 = \min_{u\in H_r(B_1), u\not\equiv 0} \{I_\lambda(u) : u \geqslant 0,\ f_\lambda(u) = 1\},$$

where $f_\lambda(u)$ is the functional in $H_r(B_1)$ defined by $f_\lambda(u) = 0$ for $u = 0$ and, for $u \neq 0$,

$$f_\lambda(u) = \begin{cases} \dfrac{\int_{B_1}(|x|^\nu|u|^p + \lambda|x|^\mu|u|^q)}{\int_{B_1}|\nabla u|^2} & \text{if } 2 < q < \dfrac{2(n+\mu)}{n-2},\ \lambda > 0, \\ \dfrac{\int_{B_1}|x|^\nu|u|^p}{\int_{B_1}(|\nabla u|^2 - \lambda|x|^\mu|u|^2)} & \text{if } q = 2,\ \lambda_{1,\mu} > \lambda > 0. \end{cases}$$

Set

$$M_2 = \{u \in M_1 : f_\lambda(u^\pm) = 1\}.$$

Then, for any $u = u^+ - u^- \in H_r(B_1)$ with $u^\pm \not\equiv 0$, we may choose α and β in $\mathbb{R}$ such that

$$\tilde{u} = \alpha u^+ - \beta u^- \in M_2,$$

and thus $M_2 \neq \emptyset$. We give its proof for the convenience of the reader (see the proof of lemma 3.2 in [6]).

LEMMA 3.1. *Let u_1 and u_2 be two non-negative linearly independent functions in $H_r(B_1)$. Then there exist $r, s \in \mathbb{R}$ such that $ru_1 + su_2 \in M_2$.*

Proof. Let $v_s = (1-s)u_1 - su_2$ for $0 \leqslant s \leqslant 1$. Then it follows that

$$\lim_{\gamma\to\infty} \inf_{s\in[0,1]} f_\lambda(\gamma v_s) = \infty.$$

Hence, we may choose a $\gamma_0 > 0$ such that, for all $s \in [0,1]$, $f_\lambda(\gamma_0 v_s) \geqslant 2$. Define $K = (K_1, K_2) : [0,1] \times [0,1] \to \mathbb{R}^2$ by

$$K_1(s,t) = f_\lambda(\gamma_0 t v_s^-) - f_\lambda(\gamma_0 t v_s^+),$$
$$K_2(s,t) = f_\lambda(\gamma_0 t v_s^-) + f_\lambda(\gamma_0 t v_s^+) - 2.$$

Then for $(s,t) \in \partial([0,1]\times[0,1])$, we have $K(s,t)\cdot\nu(s,t) \geqslant 0$, where $\nu(s,t)$ denotes the unit outward normal. Hence, from Miranda's theorem, there exists $(s_0,t_0) \in (0,1)\times(0,1)$ such that $K(s_0,t_0) = 0$. Therefore, $\gamma_0 t_0 v_{s_0} \in M_2$. □

Let

$$N = \{u \in H_r(B_1) : |f_\lambda(u^\pm) - 1| < \tfrac{1}{2}\}.$$

Then we have

$$u \in N \Rightarrow \int_{B_1} |\nabla u^\pm|^2 > \delta > 0$$

since

$$\begin{aligned}\int_{B_1} |x|^\nu |u^\pm|^p &> \frac{1}{2}\int_{B_1} (|\nabla u^\pm|^2 - \lambda |x|^\mu |u^\pm|^q) \\ &\geqslant c\int_{B_1} |\nabla u^\pm|^2 \\ &\geqslant cS_\nu\biggl(\int_{B_1} |x|^\nu |u^\pm|^p\biggr)^{2/p}\end{aligned}$$

for a constant $c > 0$.

By the similar arguments as in [6, lemma 3.1], I_λ satisfies a Palais–Smale condition in

$$I_\lambda^{-1}\biggl(-\infty, c_1 + \frac{2+\nu}{2(n+\nu)} S_\nu^{(n+\nu)/(2+\nu)}\biggr) \cap N.$$

LEMMA 3.2. *If a sequence $\{u_m\} \subset N$ satisfies*

$$I_\lambda(u_m) \to \beta < c_1 + \frac{2+\nu}{2(n+\nu)} S_\nu^{(n+\nu)/(2+\nu)}, \qquad I'_\lambda(u_m) \to 0$$

as $m \to \infty$, then $\{u_m\}$ is relatively compact in $H_r(B_1)$.

See the proof of [6, theorem A] to discover a Palais–Smale sequence in N. We consider the energy level

$$c_2 = \min_{u\in M_2} I_\lambda(u).$$

In the case when $q > 2$, c_2 is less than the critical level in lemma 3.2 when $q > (n + 2\mu + 2)/(n-2)$.

THEOREM 3.3. *Let*

$$q^* = \frac{2(n+\mu)}{n-2} \quad \textit{and} \quad \lambda \in (0,\infty).$$

If $\max\{2, q^*-1\} < q < q^*$, *then (1.1) has a pair of nodal solutions attaining* c_2.

Proof. We claim that

$$c_2 < c_1 + \frac{2+\nu}{2(n+\nu)} S_\nu^{(n+\nu)/(2+\nu)}.$$

It suffices to show that

$$\sup_{\alpha,\beta\in\mathbb{R}} I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) < c_1 + \frac{2+\nu}{2(n+\nu)} S_\nu^{(n+\nu)/(2+\nu)}, \tag{3.1}$$

where $\psi_\varepsilon \equiv \phi\bar{u}_\varepsilon$ with a radial function $\phi \in \mathbb{C}_0^\infty(B_{1/2})$, $\phi \geqslant 0$, $\phi \equiv 1$ on $B_{1/4}$. By lemma 3.1, there exist some $\alpha, \beta \in \mathbb{R}$ such that $\alpha u_0 + \beta\psi_\varepsilon \in M_2$. Since $I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) \leqslant 0$ for large $\alpha^2+\beta^2$, we may assume that α and β are bounded. Since u_0 is a positive radial solution of (1.1) obtained in theorem 2.3, we get

$$\int_{B_1} \tfrac{1}{2}|\nabla(\alpha u_0 + \beta\psi_\varepsilon)|^2 = \frac{1}{2}\int_{B_1} |\nabla(\alpha u_0)|^2 + \frac{1}{2}\int_{B_1} |\nabla(\beta\psi_\varepsilon)|^2 + \alpha\beta\biggl[\lambda\int_{B_1} |x|^\mu |u_0|^{q-2}u_0\psi_\varepsilon + \int_{B_1} |x|^\nu |u_0|^{p-2}u_0\psi_\varepsilon\biggr].$$

Using the equality

$$\frac{1}{p}\int_{B_1} |u|^p = \int_{B_1}\biggl(\int_0^u |s|^{p-2}s\,\mathrm{d}s\biggr),$$

we obtain

$$\begin{aligned}
&\frac{1}{p}\biggl[\int_{B_1} |x|^\nu|\alpha u_0 + \beta\psi_\varepsilon|^p - \int_{B_1} |x|^\nu|\beta\psi_\varepsilon|^p - \int_{B_1} |x|^\nu|\alpha u_0|^p\biggr] \\
&\qquad = \int_0^1 \biggl[\int_{B_1} |x|^\nu(|\beta\psi_\varepsilon + s\alpha u_0|^{p-2}(\beta\psi_\varepsilon + s\alpha u_0) - |s\alpha u_0|^{p-2}(s\alpha u_0))\alpha u_0\biggr]\,\mathrm{d}s \\
&\qquad = (p-1)\int_0^1 \biggl[\int_{B_1} |x|^\nu|\beta\psi_\varepsilon\theta + s\alpha u_0|^{p-2}\alpha\beta u_0\psi_\varepsilon\biggr]\,\mathrm{d}s,
\end{aligned}$$

where $\theta = \theta(x)$ is a measurable function such that $0 < \theta < 1$. From the above equality, it follows that

$$\begin{aligned}
&\frac{1}{p}\biggl|\int_{B_1} |x|^\nu|\alpha u_0 + \beta\psi_\varepsilon|^p - \int_{B_1} |x|^\nu|\beta\psi_\varepsilon|^p - \int_{B_1} |x|^\nu|\alpha u_0|^p\biggr| \\
&\qquad \leqslant C\int_0^1 \biggl[\int_{B_1} |x|^\nu(|\beta\psi_\varepsilon|^{p-2} + |\alpha u_0|^{p-2})u_0\psi_\varepsilon\biggr]\,\mathrm{d}s.
\end{aligned}$$

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Hence,

$$I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) \leqslant c_1 + \frac{\beta^2}{2}\int_{B_1} |\nabla\psi_\varepsilon|^2 - \lambda\frac{|\beta|^q}{q}\int_{B_1} |x|^\mu|\psi_\varepsilon|^q - \frac{|\beta|^p}{p}\int_{B_1} |x|^\nu|\psi_\varepsilon|^p$$
$$+ \lambda\, C_1|u_0|_\infty^{q-1}\int_{B_1} |x|^\mu\psi_\varepsilon + C_2|u_0|_\infty^{p-1}\int_{B_1} |x|^\nu\psi_\varepsilon$$
$$+ \lambda\, C_3|u_0|_\infty\int_{B_1} |x|^\mu|\psi_\varepsilon|^{q-1} + C_4|u_0|_\infty\int_{B_1} |x|^\nu|\psi_\varepsilon|^{p-1}.$$

We claim that, as $\varepsilon \to 0$, we have

(a) $\displaystyle\int_{B_1} |\nabla\psi_\varepsilon|^2 = S_\nu^{(n+\nu)/(2+\nu)} + O(\varepsilon^{(n-2)/(2+\nu)})$,

(b) $\displaystyle\int_{B_1} |x|^\nu|\psi_\varepsilon|^p = S_\nu^{(n+\nu)/(2+\nu)} + O(\varepsilon^{(n+\nu)/(2+\nu)})$,

(c) $\displaystyle\int_{B_1} |x|^\mu|\psi_\varepsilon|^q = K_1\varepsilon^{(2(n+\mu)-(n-2)q)/(2(2+\nu))} + O(\varepsilon^{(n-2)q/2(2+\nu)}), \quad q > \frac{n+\mu}{n-2}$,

(d) $\displaystyle\int_{B_1} |x|^\nu\psi_\varepsilon \leqslant K_2\ \varepsilon^{(n-2)/2(2+\nu)}$,

(e) $\displaystyle\int_{B_1} |x|^\mu\psi_\varepsilon \leqslant K_3\ \varepsilon^{(n-2)/2(2+\nu)}$,

(f) $\displaystyle\int_{B_1} |x|^\nu|\psi_\varepsilon|^{p-1} \leqslant K_4\varepsilon^{(n-2)/2(2+\nu)}$,

(g) $\displaystyle\int_{B_1} |x|^\mu|\psi_\varepsilon|^{q-1}$
$$= \begin{cases} K_5\varepsilon^{(n-2)(q-1)/2(2+\nu)}, & q < \dfrac{n+\mu}{n-2} + 1, \\ K_5\varepsilon^{(n+\mu)/2(2+\nu)}|\log\varepsilon| + O(\varepsilon^{(n+\mu)/2(2+\nu)}), & q = \dfrac{n+\mu}{n-2} + 1, \\ K_5\varepsilon^{[2(n+\mu)-(n-2)(q-1)]/2(2+\nu)}, & q > \dfrac{n+\mu}{n-2} + 1. \end{cases}$$

Verification of (a): let

$$C_\nu = [(n+\nu)(n-2)]^{(n-2)/2(2+\nu)}$$

and

$$C_{\nu,\varepsilon} = [(n+\nu)(n-2)\varepsilon]^{(n-2)/2(2+\nu)}.$$

Since

$$\nabla\psi_\varepsilon(x) = C_{\nu,\varepsilon}\left[\frac{\nabla\phi(x)}{(\varepsilon+|x|^{2+\nu})^{(n-2)/(2+\nu)}} - \frac{(n-2)\phi(x)|x|^\nu x}{(\varepsilon+|x|^{2+\nu})^{(n+\nu)/(2+\nu)}}\right]$$

and $\phi(x) = 1$ near 0, it follows that

$$\begin{aligned}\int_{B_1} |\nabla\psi_\varepsilon|^2 &= C_{\nu,\varepsilon}^2 (n-2)^2 \int_{B_{1/4}} \frac{|x|^{2\nu+2}}{(\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}} + O(\varepsilon^{(n-2)/(2+\nu)}) \\ &= C_{\nu,\varepsilon}^2 (n-2)^2 \int_{\mathbb{R}^n} \frac{|x|^{2\nu+2}}{(\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}} + O(\varepsilon^{(n-2)/(2+\nu)}) \\ &= \int_{\mathbb{R}^n} \frac{C_\nu^2 (n-2)^2 |x|^{2\nu+2}}{(1 + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}} + O(\varepsilon^{(n-2)/(2+\nu)}) \\ &= \int_{\mathbb{R}^n} |\nabla \bar{u}_1|^2 \, \mathrm{d}x + O(\varepsilon^{(n-2)/(2+\nu)}) \\ &= S_\nu^{(n+\nu)/(2+\nu)} + O(\varepsilon^{(n-2)/(2+\nu)}).\end{aligned}$$

Verification of (b):

$$\begin{aligned}\int_{B_1} |x|^\nu |\psi_\varepsilon|^p &= C_{\nu,\varepsilon}^p \int_{B_1} \frac{|x|^\nu \phi^p(x)}{(\varepsilon + |x|^{2+\nu})^{(n-2)p/(2+\nu)}} \\ &= C_\nu^p \varepsilon^{(n+\nu)/(2+\nu)} \int_{B_{1/2}} \frac{|x|^\nu [\phi^p(x) - 1]}{(\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}} \\ &\quad + C_\nu^p \varepsilon^{(n+\nu)/(2+\nu)} \int_{B_{1/2}} \frac{|x|^\nu}{(\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}} \\ &= C_\nu^p \varepsilon^{(n+\nu)/(2+\nu)} \int_{B_{1/2}} \frac{|x|^\nu}{(\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}} + O(\varepsilon^{(n+\nu)/(2+\nu)}) \\ &= C_\nu^p \varepsilon^{(n+\nu)/(2+\nu)} \int_{\mathbb{R}^n} \frac{|x|^\nu}{(\varepsilon + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}} + O(\varepsilon^{(n+\nu)/(2+\nu)}) \\ &= \int_{\mathbb{R}^n} \frac{C_\nu^p |x|^\nu}{(1 + |x|^{2+\nu})^{2(n+\nu)/(2+\nu)}} + O(\varepsilon^{(n+\nu)/(2+\nu)}) \\ &= \int_{\mathbb{R}^n} |x|^\nu |\bar{u}_1|^p + O(\varepsilon^{(n+\nu)/(2+\nu)}) \\ &= S_\nu^{(n+\nu)/(2+\nu)} + O(\varepsilon^{(n+\nu)/(2+\nu)}).\end{aligned}$$

Verification of (c):

$$\begin{aligned}\int_{B_1} |x|^\mu |\psi_\varepsilon|^q &= C_{\nu,\varepsilon}^q \int_{B_1} \frac{|x|^\nu \phi^q(x)}{(\varepsilon + |x|^{2+\nu})^{(n-2)q/(2+\nu)}} \\ &= C_{\nu,\varepsilon}^q \int_{B_{1/2}} \frac{|x|^\mu [\phi^q(x) - 1]}{(\varepsilon + |x|^{2+\nu})^{(n-2)q/(2+\nu)}} \\ &\quad + C_{\nu,\varepsilon}^q \int_{B_{1/2}} \frac{|x|^\mu}{(\varepsilon + |x|^{2+\nu})^{(n-2)q/(2+\nu)}} \\ &= C_{\nu,\varepsilon}^q \int_{B_{1/2}} \frac{|x|^\mu}{(\varepsilon + |x|^{2+\nu})^{(n-2)q/(2+\nu)}} + O(\varepsilon^{(n-2)q/2(2+\nu)})\end{aligned}$$

$$
\begin{aligned}
&= C^q_{\nu,\varepsilon}\int_{\mathbb{R}^n}\frac{|x|^{\mu}}{(\varepsilon+|x|^{2+\nu})^{(n-2)q/(2+\nu)}} \\
&\quad + O(\varepsilon^{(n-2)q/2(2+\nu)}) \\
&= C^q_{\nu}\varepsilon^{[2(n+\mu)-(n-2)q]/2(2+\nu)}\int_{\mathbb{R}^n}\frac{|x|^{\mu}}{(1+|x|^{2+\nu})^{(n-2)q/(2+\nu)}} \\
&\quad + O(\varepsilon^{(n-2)q/2(2+\nu)}) \\
&= K_3\varepsilon^{[2(n+\mu)-(n-2)q]/2(2+\nu)} + O(\varepsilon^{(n-2)q/2(2+\nu)}), \qquad q > \frac{n+\mu}{n-2}.
\end{aligned}
$$

Verification of (d):

$$
\begin{aligned}
\int_{B_1}|x|^{\nu}|\psi_\varepsilon| &= C_{\nu,\varepsilon}\int_{B_1}\frac{|x|^{\nu}\phi(x)}{(\varepsilon+|x|^{2+\nu})^{(n-2)/(2+\nu)}} \\
&\leqslant C_{\nu}\varepsilon^{(n-2)/2(2+\nu)}\int_{B_{1/2}}\frac{|x|^{\nu}}{(\varepsilon+|x|^{2+\nu})^{(n-2)/(2+\nu)}} \\
&= K_2\varepsilon^{(n-2)/2(2+\nu)}.
\end{aligned}
$$

Verification of (e):

$$
\begin{aligned}
\int_{B_1}|x|^{\mu}|\psi_\varepsilon| &= C_{\nu,\varepsilon}\int_{B_1}\frac{|x|^{\mu}\phi(x)}{(\varepsilon+|x|^{2+\nu})^{(n-2)/(2+\nu)}} \\
&\leqslant C_{\nu}\varepsilon^{(n-2)/2(2+\nu)}\int_{B_{1/2}}\frac{|x|^{\mu}}{(\varepsilon+|x|^{2+\nu})^{(n-2)/(2+\nu)}} \\
&= K_2\varepsilon^{(n-2)/2(2+\nu)}.
\end{aligned}
$$

Verification of (f):

$$
\begin{aligned}
\int_{B_1}|x|^{\nu}|\psi_\varepsilon|^{p-1} &= C^{p-1}_{\nu,\varepsilon}\int_{B_1}\frac{|x|^{\nu}\phi^{p-1}(x)}{(\varepsilon+|x|^{2+\nu})^{(n-2)(p-1)/(2+\nu)}} \\
&\leqslant C^{p-1}_{\nu}\varepsilon^{(n+2+2\nu)/2(2+\nu)}\int_{B_{1/2}}\frac{|x|^{\nu}}{(\varepsilon+|x|^{2+\nu})^{(n+2+2\nu)/(2+\nu)}} \\
&= C^{p-1}_{\nu}\varepsilon^{(n-2)/2(2+\nu)}\int_{|x|\leqslant\varepsilon^{-1/(2+\nu)}/2}\frac{|x|^{\nu}}{(1+|x|^{2+\nu})^{(n+2+2\nu)/(2+\nu)}} \\
&\leqslant K_4\varepsilon^{(n-2)/2(2+\nu)}.
\end{aligned}
$$

Verification of (g): since

$$
\int_{B_1}|x|^{\mu}|\psi_\varepsilon|^{q-1} = \int_{B_1}|x|^{\mu}\phi^{q-1}|\bar{u}_\varepsilon|^{q-1}
$$

and

$$
\int_{B_{1/4}}|x|^{\mu}|\bar{u}_\varepsilon|^{q-1} \leqslant \int_{B_1}|x|^{\mu}|\psi_\varepsilon|^{q-1} \leqslant \int_{B_{1/2}}|x|^{\mu}|\bar{u}_\varepsilon|^{q-1},
$$

we have to calculate

$$\int_{B_R} |x|^{\mu} |\bar{u}_\varepsilon|^{q-1}, \quad 0 < R < 1.$$

(i) Let $q < (n+\mu)/(n-2) + 1$. Then,

$$\begin{aligned}\int_{B_R} |x|^{\mu} |\bar{u}_\varepsilon|^{q-1} &= C_{\nu,\varepsilon}^{q-1} \int_{B_R} \frac{|x|^{\mu}}{(\varepsilon + |x|^{2+\nu})^{(n-2)(q-1)/(2+\nu)}} \\ &= K_5 \varepsilon^{(n-2)(q-1)/2(2+\nu)}.\end{aligned}$$

(ii) Let $q = (n+\mu)/(n-2) + 1$. Then,

$$\begin{aligned}&\int_{B_R} |x|^{\mu} |\bar{u}_\varepsilon|^{q-1} \\ &\quad = C_{\nu,\varepsilon}^{q-1} \int_{B_R} \frac{|x|^{\mu}}{(\varepsilon + |x|^{2+\nu})^{(n-2)(q-1)/(2+\nu)}} \\ &\quad = C_{\nu}^{q-1} \varepsilon^{(n+\mu)/2(2+\nu)} \int_{|x| \leqslant R\varepsilon^{-1/(2+\nu)}} \frac{|x|^{\mu}}{(1 + |x|^{2+\nu})^{(n-2)(q-1)/(2+\nu)}} \\ &\quad = C_{\nu}^{q-1} \varepsilon^{(n+\mu)/2(2+\nu)} \left[C + \int_1^{R\varepsilon^{-1/(2+\nu)}} \frac{r^{n+\mu-1}}{(1 + r^{2+\nu})^{(n-2)(q-1)/(2+\nu)}} \,\mathrm{d}r \right] \\ &\quad = C_{\nu}^{q-1} \varepsilon^{(n+\mu)/2(2+\nu)} \left[O(1) + \frac{1}{2+\nu} |\log \varepsilon| \right] \\ &\quad = K_5 \varepsilon^{(n+\mu)/2(2+\nu)} |\log \varepsilon| + O(\varepsilon^{(n+\mu)/2(2+\nu)}).\end{aligned}$$

(iii) Let $q > (n+\mu)/(n-2) + 1$. Then,

$$\begin{aligned}&\int_{B_R} |x|^{\mu} |\bar{u}_\varepsilon|^{q-1} \\ &\quad = C_{\nu,\varepsilon}^{q-1} \int_{B_R} \frac{|x|^{\mu}}{(\varepsilon + |x|^{2+\nu})^{(n-2)(q-1)/(2+\nu)}} \\ &\quad = C_{\nu}^{q-1} \varepsilon^{(2(n+\mu)-(n-2)(q-1))/2(2+\nu)} \int_{|x| \leqslant R\varepsilon^{1/(2+\nu)}} \frac{|x|^{\mu}}{(1 + |x|^{2+\nu})^{(n-2)(q-1)/(2+\nu)}} \\ &\quad = C_{\nu}^{q-1} \varepsilon^{(2(n+\mu)-(n-2)(q-1))/2(2+\nu)} \int_0^{R\varepsilon^{-1/(2+\nu)}} \frac{r^{n+\mu-1}}{(1 + r^{2+\nu})^{(n-2)(q-1)/(2+\nu)}} \,\mathrm{d}r \\ &\quad = K_5 \varepsilon^{[2(n+\mu)-(n-2)(q-1)]/2(2+\nu)}.\end{aligned}$$

Note

$$\frac{\beta^2}{2} - \frac{|\beta|^p}{p} \leqslant \frac{1}{2} - \frac{1}{p} = \frac{2+\nu}{2(n+\nu)}.$$

From the above estimates, we have

$$\begin{aligned}&I_\lambda(\alpha u_0 + \beta \psi_\varepsilon) \\ &\quad \leqslant c_1 + \frac{2+\nu}{2(n+\nu)} S_{\nu}^{(n+\nu)/(2+\nu)} + O(\varepsilon^{(n-2)/(2+\nu)}) - O(\varepsilon^{(n+\nu)/(2+\nu)}) \\ &\qquad - \lambda\, K_6 \varepsilon^{[2(n+\mu)-(n-2)q]/2(2+\nu)} - O(\varepsilon^{(n-2)q/2(2+\nu)})\end{aligned}$$

$$+ K_7[\lambda|u_0|_\infty^{q-1} + |u_0|_\infty^{p-1} + |u_0|_\infty]\varepsilon^{(n-2)/2(2+\nu)}$$

$$+ \lambda|u_0|_\infty \begin{cases} K_8\varepsilon^{(n-2)(q-1)/2(2+\nu)}, & q < \dfrac{n+\mu}{n-2}+1, \\ K_8\varepsilon^{(n+\mu)/2(2+\nu)}|\log\varepsilon| + O(\varepsilon^{(n+\mu)/2(2+\nu)}), & q = \dfrac{n+\mu}{n-2}+1, \\ K_8\varepsilon^{[2(n+\mu)-(n-2)(q-1)]/2(2+\nu)}, & q > \dfrac{n+\mu}{n-2}+1. \end{cases}$$

To show (3.1), we consider the following three cases on q separately.

CASE 1. Let $q < (n+\mu)/(n-2)+1$. We consider the inequalities

$$\frac{(n-2)(q-1)}{2(2+\nu)} > \frac{n-2}{2(2+\nu)} > \frac{[2(n+\mu)-(n-2)q]}{2(2+\nu)}.$$

The second inequality shows that

$$\frac{n+2+2\mu}{n-2} < q.$$

Note that

$$n > \mu+4 \longleftrightarrow \frac{n+2+2\mu}{n-2} < \frac{n+\mu}{n-2}+1.$$

Hence, if

$$\frac{n+2+2\mu}{n-2} < q < \frac{n+\mu}{n-2}+1 \quad \text{and} \quad n > \mu+4,$$

then we have (3.1).

CASE 2. Let $q = (n+\mu)/(n-2)+1$. Then

$$\frac{[2(n+\mu)-(n-2)q]}{2(2+\nu)} = \frac{2+\mu}{n-2}.$$

Note that

$$\frac{n-2}{2(2+\nu)} > \frac{2+\mu}{2(2+\nu)} \longleftrightarrow n > \mu+4.$$

Hence, if $q = (n+\mu)/(n-2)+1$ and $n > \mu+4$, then we have (3.1).

CASE 3. Let $q > (n+\mu)/(n-2)+1$. Observe that

$$\frac{[2(n+\mu)-(n-2)q]}{2(2+\nu)} < \frac{n-2}{2(2+\nu)}$$

and, equivalently,

$$\frac{n+2+2\mu}{n-2} < q.$$

Then

$$q > \max\left\{\frac{n+2+2\mu}{n-2}, \frac{n+\mu}{n-2}+1\right\} = \begin{cases} \dfrac{n+2+2\mu}{n-2}, & n \geqslant \mu+4, \\ \dfrac{n+\mu}{n-2}+1, & n < \mu+4. \end{cases}$$

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Therefore, if $q > (n+\mu)/(n-2)+1$ and $n \geqslant \mu+4$, or if $q > (n+2+2\mu)/(n-2)$ and $n < \mu+4$, then we have (3.1).

Then from lemma 3.2, there exists $u \in M_2$ such that $I_\lambda(u) = c_2$, and $I'_\lambda(u_2) = 0$ (see [6] for details). □

In the case when $q = 2$, we add the condition $\lambda \in (0, \lambda_{1,\mu})$.

THEOREM 3.4. *Let $q = 2$ and $\lambda \in (0, \lambda_{1,\mu})$. If $n > 2\mu + 6$, then (1.1) has a pair of nodal solutions attaining c_2.*

Proof. Similarly, we get

$$\begin{aligned} I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) \leqslant c_1 + \frac{\beta^2}{2}\int_{B_1} |\nabla\psi_\varepsilon|^2 - \lambda\frac{\beta^2}{2}\int_{B_1}|x|^\mu|\psi_\varepsilon|^2 - \frac{|\beta|^p}{p}\int_{B_1}|x|^\nu|\psi_\varepsilon|^p \\ + C_1|u_0|_\infty^{p-1}\int_{B_1}|x|^\nu\psi_\varepsilon + \lambda\, C_2|u_0|_\infty\int_{B_1}|x|^\nu|\psi_\varepsilon|^{p-1}. \end{aligned}$$

Since

$$\int_{B_1}|x|^\mu|\psi_\varepsilon|^2 = \begin{cases} K_1\varepsilon + O(\varepsilon^{(n-2)/(2+\nu)}), & \text{if } n > 4+\mu, \\ K_1\varepsilon|\log\varepsilon| + O(\varepsilon^{(2+\mu)/(2+\nu)}), & \text{if } n = 4+\mu, \\ K_1\varepsilon^{(n-2)/(2+\nu)}, & \text{if } n < 4+\mu, \end{cases}$$

we have

$$\begin{aligned} I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) \leqslant c_1 &+ \frac{2+\nu}{2(n+\nu)}S_\nu^{(n+\nu)/(2+\nu)} + O(\varepsilon^{(n-2)/(2+\nu)}) - O(\varepsilon^{(n+\nu)/(2+\nu)}) \\ &+ [K_1|u_0|_\infty^{p-1} + K_2|u_0|_\infty]\varepsilon^{(n-2)/2(2+\nu)} \\ &- \lambda\begin{cases} K_4\varepsilon^{(2+\mu)/(2+\nu)} + O(\varepsilon^{(n-2)/(2+\nu)}), & n > 4+\mu, \\ K_4\varepsilon^{(2+\mu)/(2+\nu)}|\log\varepsilon| + O(\varepsilon^{(2+\mu)/(2+\nu)}), & n = 4+\mu, \\ K_4\varepsilon^{(n-2)/(2+\nu)}, & n < 4+\mu. \end{cases} \end{aligned}$$

By considering the inequality

$$\frac{n-2}{2(2+\nu)} > \frac{2+\mu}{2+\nu} \quad \text{for } n > 4+\mu,$$

we have (3.1). Then from lemma 3.2, there exists $u \in M_2$ such that $I_\lambda(u) = c_2$ and $I'_\lambda(u_2) = 0$. □

4. Proof of theorem 1.3

(i) Let $(n+2+2\check{\mu})/(n-2) < q$. By

$$\check{\mu} = \frac{(n-2)[a - \sigma(q-2) + \alpha] - n\alpha}{2\sqrt{\bar{\chi} - \chi}},$$

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we see that

$$\begin{aligned}4+2\breve{\mu} &= 4+\frac{(n-2)[a-\sigma(q-2)+\alpha]-n\alpha}{\sqrt{\bar{\chi}-\chi}}\\ &= \frac{4\sqrt{\bar{\chi}-\chi}+(n-2)a-(n-2)\sigma q-n\alpha}{\sqrt{\bar{\chi}-\chi}}\\ &= \frac{(n-2)(n+a)-2(n-2)\sqrt{\bar{\chi}-\chi}-(n-2)\sigma q}{\sqrt{\bar{\chi}-\chi}}\end{aligned}$$

and

$$\frac{4+2\breve{\mu}}{n-2} = \frac{n+a-2\sqrt{\bar{\chi}-\chi}-\sigma q}{\sqrt{\bar{\chi}-\chi}}.$$

Hence,

$$\begin{aligned}q > \frac{4+2\breve{\mu}}{n-2} &= \frac{n+a-2\sqrt{\bar{\chi}-\chi}-\sigma q}{\sqrt{\bar{\chi}-\chi}}\\ &> \frac{n+a-2\sqrt{\bar{\chi}-\chi}}{\sqrt{\bar{\chi}}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}n+2+2\breve{\mu} &= n+2+\frac{(n-2)[a-\sigma(q-2)+\alpha]-n\alpha}{\sqrt{\bar{\chi}-\chi}}\\ &= \frac{(n+2)\sqrt{\bar{\chi}-\chi}+(n-2)a-(n-2)\sigma q-n\alpha}{\sqrt{\bar{\chi}-\chi}}\\ &= \frac{(n-2)(n+a)-(n-2)\sqrt{\bar{\chi}-\chi}-(n-2)\sigma q}{\sqrt{\bar{\chi}-\chi}}.\end{aligned}$$

Hence, we have

$$\frac{n+2+2\breve{\mu}}{n-2} = \frac{(n+a)-\sqrt{\bar{\chi}-\chi}-\sigma q}{\sqrt{\bar{\chi}-\chi}}$$

and

$$\begin{aligned}q > \frac{n+2+2\breve{\mu}}{n-2} &= \frac{(n+a)-\sqrt{\bar{\chi}-\chi}-\sigma q}{\sqrt{\bar{\chi}-\chi}}\\ &> \frac{n+a-\sqrt{\bar{\chi}-\chi}}{\sqrt{\bar{\chi}}}.\end{aligned}$$

(ii) Let $q < (2(n+\breve{\mu}))/(n-2)$. Since

$$\begin{aligned}2n+2\breve{\mu} &= 2n+\frac{(n-2)[a-\sigma(q-2)+\alpha]-n\alpha}{\sqrt{\bar{\chi}-\chi}}\\ &= \frac{2n\sqrt{\bar{\chi}-\chi}+(n-2)a-(n-2)\sigma q-n\alpha}{\sqrt{\bar{\chi}-\chi}}\\ &= \frac{(n-2)(n+a)-(n-2)\sigma q}{\sqrt{\bar{\chi}-\chi}},\end{aligned}$$

we have

$$q < \frac{2n + 2\check{\mu}}{n-2} = \frac{n + a - \sigma q}{\sqrt{\bar{\chi} - \chi}} < \frac{n+a}{\sqrt{\bar{\chi}}}.$$

(iii) Let $q = 2$. By

$$\check{\mu} = \frac{(n-2)(a+\alpha) - n\alpha}{2\sqrt{\bar{\chi} - \chi}} = \frac{(n-2)a - 2\alpha}{2\sqrt{\bar{\chi} - \chi}} = \frac{(n-2)(a+2) - 4\sqrt{\bar{\chi} - \chi}}{2\sqrt{\bar{\chi} - \chi}},$$

we have

$$n \geqslant \check{\mu} + 4 = \frac{(n-2)(a+2) - 4\sqrt{\bar{\chi} - \chi}}{2\sqrt{\bar{\chi} - \chi}} + 4 \geqslant \frac{(n-2)(a+2) + 4\sqrt{\bar{\chi} - \chi}}{2\sqrt{\bar{\chi} - \chi}}$$

and

$$\sqrt{\bar{\chi} - \chi} \geqslant \frac{a+2}{2}, \qquad \chi \leqslant \bar{\chi} - \left(\frac{a+2}{2}\right)^2.$$

On the other hand,

$$n > 2\check{\mu} + 6 = \frac{(n-2)(a+2) - 4\sqrt{\bar{\chi} - \chi}}{\sqrt{\bar{\chi} - \chi}} + 6 > \frac{(n-2)(a+2) + 2\sqrt{\bar{\chi} - \chi}}{\sqrt{\bar{\chi} - \chi}}$$

and

$$\sqrt{\bar{\chi} - \chi} > a + 2, \qquad \chi < \bar{\chi} - (a+2)^2.$$

Therefore, the conditions in theorem 1.1 are transformed into the conditions in theorem 1.3.

5. Final remarks

Under the same hypotheses of theorem 1.1, Tarantello's approach in [14] also leads to the existence of nodal solutions $\pm u$ of (1.1). Similarly, there is a pair of sign-changing solutions $\pm w$ of (1.3) satisfying

$$\int_{B_1} (|y|^{\nu}|w|^{p-2} + \tilde{\lambda}|y|^{a}|w|^{q-2})v(w)w = 0,$$

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where $v(w)$ is the first eigenfunction of the weighted eigenvalue problem

$$-\left(\Delta + \frac{\chi}{|y|^2}\right)v = \gamma(\tilde{\lambda}|y|^a|w|^{q-2} + |y|^\nu|w|^{p-2})v \quad \text{in } B_1,$$
$$v = 0 \quad \text{on } \partial B_1.$$

For $q = 2$, we have

$$\int_{B_1} |y|^\nu|w|^{p-2}v(w)w = 0,$$

where $v(w)$ is the first eigenfunction of the weighted eigenvalue problem

$$-\left(\Delta + \frac{\chi}{|y|^2} + \tilde{\lambda}\right)v = \gamma|y|^\nu|w|^{p-2}v \text{ in } B_1, \quad v = 0 \text{ on } \partial B_1.$$

In the case $0 \geqslant \mu > -2$ and $0 \geqslant \nu > -2$ in theorem 1.1, we may consider the problem in $H_0^{1,2}(\Omega)$ in a bounded domain Ω containing 0, and proceed to obtain nodal solutions of (1.1) in a similar way.

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