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On the motion of particles in non-uniformly vibrating liquid

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The motion of a rigid particle in an inviscid non-uniformly vibrating ambient liquid is considered. The vibrations are caused by a dipole changing its strength in time. This model of a vibrator presents an asymptotic case of vibrations caused by a rigid sphere periodically changing its position when the radius and amplitude are small and the velocity is large. It is found that periodic oscillations with zero mean can cause the directed motion of a submerged particle even if its density equals that of the liquid. The direction of the motion is studied. It is shown that particles of density not less that of the ambient liquid are attracted by a vibrator. The direction of motion of lighter particles depends on their initial position.

1 Introduction

The effect of oscillations of a liquid on the motion of submerged bodies has been the subject of many studies [1–9]. It was established both theoretically and experimentally that vibrations can cause the upward motion of a body whose density exceeds that of the liquid as well as the downward motion of a light particle or a bubble [1–4, 6], and also the directed motion of a particle in the absence of gravity forces [5, 7–9]. These results suggest the possibility of using vibrations to control the motion of particles, for example to extract undesirable admixtures from a liquid. The simplest case of vibrational forcing is that of so-called 'uniform vibrations', when, in the absence of any particle, the liquid moves as a rigid body, driven by the oscillation of the vessel. It is obvious, however, that such vibrations cannot result in the directed motion of an initially stationary body whose density equals that of the liquid, while, as far as liquid purification is concerned, the case of equal or little different densities is of the greatest interest. Thus, it is necessary to consider non-uniform vibrations. The latter can be provided by a vibrator, i.e. some body that moves or changes its form periodically. Marmur and Rubin [7] were probably the first to mention the principal difference between uniform and non-uniform oscillations of liquid as far as the motion of submerged bodies is concerned.

A problem of this type in which a vibrator is a periodically progressive moving rigid sphere was considered by Sennitskii [8]. The asymptotics in which the vibrator and particle radii, and the amplitude of oscillations are small relative to the distance between the body and the vibrator was studied. It was found that a body of density less than that of the

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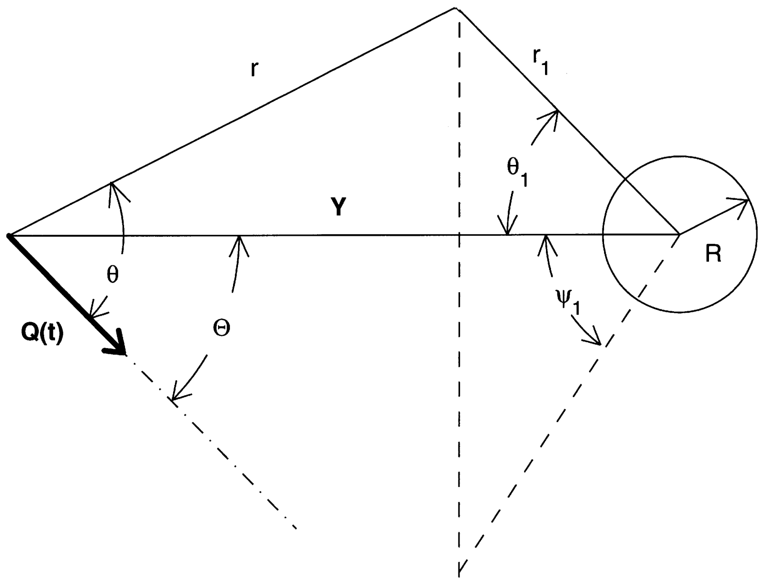

FIGURE 1. Scheme of the process. The dipole of the strength $Q(t)$ is located at the origin of the laboratory coordinate system (r, θ, Ψ). The origin of the moving coordinate system (r_1, θ_1, Ψ_1) is in the centre of the rigid sphere.

liquid moves away from the vibrator, whereas one whose density exceeds that of the liquid moves towards the vibrator.

The directed motion of a rigid body of the same density as the ambient liquid was predicted theoretically by Lavrenteva [5]. It was caused by a hydrodynamic source periodically varying its strength. The motion of a liquid and of a sphere was axisymmetrical. Experimentally such a motion was observed by Stebnovskii [9], the vibrations being caused by a periodically moving rigid sphere. However, the motion was not axisymmetric. The model of a vibrator considered in this work leads to non axisymmetric motion and seems to be more realistic.

All the results of this paper apply only to solid particles. The motion of drops and bubbles may be quite different. Results on the effect of vibrations on deformable interfaces can be found in paper by Lyubimov *et al.* [10].

2 Statement of the problem

The problem of the motion of a rigid sphere in an inviscid liquid in the presence of a vibrator is considered. Initially the liquid and particle are stationary. The vibrator is modelled by a doublet periodically changing its strength.

Let (r, θ, ψ) be a spherical coordinate system such that the centre of the vibrator is located at the point $(0, 0, 0)$ and its axis is directed along the axis $\theta = 0$. We also use a moving spherical coordinate system (r_1, θ_1, ψ_1) with the origin at the centre of the body and oriented so that the axis $\theta_1 = 0$ is directed from the centre of a body to the doublet (see Figure 1). Let the radius vector of the free sphere's centre be $\mathbf{X^0} = (Y^0, \Theta^0, 0)$ at the

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initial time. Since initially the liquid does not move, the flow remains potential, thus

$$\mathbf{v} = \nabla\Phi, \tag{2.1}$$

$$p = -\rho(\Phi_t + \tfrac{1}{2}|\nabla\Phi|^2) + a(t), \tag{2.2}$$

$\mathbf{v}$ being the velocity and p pressure of the flow; ρ denotes the density of a liquid; $a(t)$ is an arbitrary function.

The radius vector of a body's centre $\mathbf{X} = (Y(t), \Theta(t), \Psi(t))$ and the hydrodynamic potential Φ satisfy the following equations:

$$\Delta\Phi = 0, \quad t > 0, \quad \mathbf{x} \in \Omega_t, \tag{2.3}$$

$$\partial_n\Phi = V_n, \quad t > 0, \quad \mathbf{x} \in \Sigma_t, \tag{2.4}$$

$$\Phi = Q(t)\frac{\cos(\theta)}{r^2} + O(1), \quad t > 0, \quad r \to 0, \tag{2.5}$$

$$\lim_{r\to\infty} \nabla\Phi = 0, \quad t > 0, \tag{2.6}$$

$$m\ddot{\mathbf{X}} = \oint_{\Sigma_t} p\mathbf{n}_{\Sigma_t} d\sigma, \quad t > 0, \tag{2.7}$$

$$\mathbf{X}(0) = \mathbf{X}^0, \quad \dot{\mathbf{X}} = (0, 0, 0), \tag{2.8}$$

where $m = 4/3\pi\rho_s R^3$ is a mass of the solid body, '$\dot{} = d/dt$'; Ω_t is the domain occupied by the liquid, the boundary of the free body is $\Sigma_t = \{\mathbf{x} \in \mathbf{R}^3 / |\mathbf{X}(\mathbf{t}) - \mathbf{x}| = R\}, \mathbf{n}_{\Sigma_t}$ denotes an inner normal vector to Σ_t, V_n is the normal velocity of Σ_t, R denotes the radius of the particle, and ρ_s is its density. The doublet strength $Q(t)$ is supposed to be a $T-$periodic function with zero mean, $Q(0) = 0$.

It is easy to see that the potential

$$\begin{aligned}
\Phi = {} & \frac{Q\cos\theta}{r^2} - \frac{R^3\dot{Y}}{2r_1^2}\cos\theta_1 - \frac{R^3Y\dot{\Theta}}{2r_1^2}\sin\theta_1\cos\psi_1 + \frac{QRr_1\cos\Theta(Yr_1\cos\theta - R^2)}{(Y^2r_1^2 - 2YR^2r_1\cos\theta_1 + R^4)^{3/2}} \\
& + \frac{Q\cos\psi_1\sin\Theta}{Y\sqrt{Y^2r_1^2 - 2YR^2r_1\cos\theta_1 + R^4}}\left[\frac{Yr_1 - R^2\cos\theta_1}{Y\sin\theta_1} + \frac{R^4r_1\sin\theta_1}{Y^2r_1^2 - 2YR^2r_1\cos\theta_1 + R^4}\right] \\
& - \frac{Q\cos\psi_1\sin\Theta}{Y^2\sin\theta_1} = \frac{Q\cos\theta}{r^2} - \frac{R^3\dot{Y}}{2r_1^2}\cos\theta_1 + \frac{QR\cos\Theta}{r_1Y^2}\sum_{k=0}^{\infty}k\left(\frac{R^2}{r_1Y}\right)^k P_k(\cos\theta_1) \\
& - \frac{R^3Y\dot{\Theta}}{2r_1^2}\sin\theta_1\cos\psi_1 - \frac{QR\cos\psi_1\sin\Theta}{r_1Y^2}\sum_{k=1}^{\infty}\frac{k}{k+1}\left(\frac{R^2}{r_1Y}\right)^k P_k^1(\cos\theta_1)
\end{aligned} \tag{2.9}$$

satisfies the equations (2.3)–(2.6) for every Y, $V_n = \dot{Y}\cos\theta_1 + Y\dot{\theta}\sin\theta_1\cos\psi_1$. Note that for a stationary particle and radial dipole, $\Theta = 0$, this solution reduces to the classical one, see for example [11]. After substituting this potential into (2.2) and using the resulting pressure in (2.7), the problem reduces to the following Cauchy problem for a fourth order system of ordinary differential equations (see the Appendix):

$$(2\lambda+1)(\ddot{y}-y\dot{\Theta}^2)-\frac{d}{dt}\left(\frac{2q\cos\Theta}{y^3}\right)=\frac{6\dot{y}q}{y^4}\cos\Theta+\frac{2q\dot{\Theta}}{y^3}\sin\Theta$$

$$+\left(3+\frac{4}{y^2}-\frac{1}{y^4}\right)\sin^2\Theta-\frac{q^2}{3y^7(1-y^{-2})^4}\left[12\cos^2\Theta\right], \quad (2.10)$$

$$(2\lambda+1)(y\ddot{\Theta}+2\dot{y}\dot{\Theta})-\frac{d}{dt}\left(\frac{q\sin\Theta}{y^3}\right)=\frac{3q\dot{y}}{y^4}\sin\Theta$$

$$+\frac{5q\dot{\Theta}}{y^3}\cos\Theta-\frac{q^2(3-y^{-2})}{3y^7(1-y^{-2})^3}\sin\Theta\cos\Theta, \quad (2.11)$$

$$y(0)=y^0,\ \dot{y}(0)=0,\ \Theta(0)=\Theta^0,\ \dot{\Theta}(0)=0. \quad (2.12)$$

The following dimensionless variables and parameters are introduced here:

$$y(\tau)=Y(t)/R,\ \tau=t/T,\ \lambda=\rho_s/\rho,$$

$$q(\tau)=3Q(t)T/R^3=3Q(t)/\omega R^3, \quad (2.13)$$

where ω denotes the frequency of the vibrations.

The behaviour of the solutions is investigated analytically for 3 cases: $\Theta_0=0$, $\max|q|\to 0$ and $y_0\to 0$. For other values of parameters the system (2.10)–(2.12) was solved numerically.

Note that, if $\Theta=0$ the last term in the right-hand side of the equation (2.10) is a well-known [11] expression for the force exerted on a fixed sphere by a radial doublet. This force is always directed towards a doublet, and one may anticipate that a free particle moves in the same direction. Yet, as will be shown below, even in the axisymmetric case the direction of motion can be the opposite.

3 Axisymmetric motion

It follows from (2.10)–(2.12) that if $\Theta^0=0$ then $\Theta(\tau)=0$ for all τ and the equation of motion (2.10) can be written in the form

$$\frac{d}{dt}\left((2\lambda+1)\dot{y}-\frac{2q}{y^3}\right)=\frac{6\dot{y}q}{y^4}-\frac{4q^2}{y^7(1-y^{-2})^4}. \quad (3.1)$$

The introduction of a new variable $\eta=\dot{y}-qy^{-3}/(2\lambda+1)$ reduces the problem (2.12), (3.1) to

$$\dot{y}=\eta+\frac{2q}{(2\lambda+1)y^3}, \quad (3.2)$$

$$\dot{\eta}=\frac{6q\eta}{(2\lambda+1)y^4}+\frac{2q^2}{y^7(1-y^{-2})^4}F_\lambda(y), \quad (3.3)$$

$$y(0)=y^0>1,\ \eta(0)=0. \quad (3.4)$$

Here

$$F_\lambda(y)=3(1-y^{-2})^4-(2\lambda+1).$$

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It is easy to see that this function has the following properties:

(1) If $\lambda \geqslant 1$, then $F_\lambda(y)$ is negative for every $y > 1$.

(2) If $\lambda \in (0, 1)$, then $F_\lambda(y)$ is negative for $y > y^*(\lambda)$ and it is positive for $y > y^*(\lambda)$, where

$$y^*(\lambda) = \left(1 - \left(\frac{2\lambda+1}{3}\right)^{1/4}\right)^{-1/2}.$$

Consider now the problem (3.2)–(3.4). The solution makes physical sense only for $y > 1$, as in the opposite case the doublet is inside the body. Therefore consideration is restricted to the time interval $T^*, y(\tau) > 1 \;\; \forall \tau < T^*$.

First, let the density of the body be not less than that of the liquid ($\lambda \geqslant 1$). In this case, $F_\lambda(y)$ is negative for all $y > 1$. As (3.3) is linear with respect to η, it implies that

$$\eta(\tau) = 2e^{\frac{6}{2\lambda+1}\int_0^\tau \frac{q(\xi)}{y^4(\xi)}d\xi} \int_0^\tau e^{-\frac{6}{2\lambda+1}\int_0^\zeta \frac{q(\xi)}{y^4(\xi)}d\xi} \frac{q^2(\zeta)F_\lambda(y(\zeta))}{y^7(\zeta)(1-y^{-2}(\zeta))^4} d\zeta \tag{3.5}$$

if $\eta(0) = 0$. It follows from (3.5) that $\eta \leqslant 0$, since $F_\lambda(y)$ is negative for all $y > 1$.

Equation (3.2) can be rewritten as

$$\frac{d}{dt}\left(y^4 - \frac{2}{2\lambda+1}\int_0^\tau q(\xi)d\xi\right) = \eta y^3 \leqslant 0. \tag{3.6}$$

This means that the differentiated function, say $h(\tau)$, is monotonically decreasing with time. Hence $y^4(\tau)$ is a sum of a 1-periodic and a monotonically decreasing function:

$$y^4 = h(\tau) + h_1(\tau) = h(\tau) + \frac{2}{2\lambda+1}\int_0^\tau q(\xi)d\xi. \tag{3.7}$$

The last summand h_1 is 1-periodic, as q is 1-periodic with zero mean. Hence

$$y(\tau+1) \leqslant y(\tau) \;\; \forall \;\; \tau \leqslant T^* - 1. \tag{3.8}$$

The equality is valid if and only if $q(\tau) \equiv 0$. The particle is attracted by the vibrator.

Now let the density of the body be less than that of the liquid ($\lambda < 1$). The solution can be represented in the form (3.5) and the following propositions are valid:

(1) If $y_0 \in (1, y^*(0))$ and $|h_1| < \min(y_0^4 - 1, y^{*4}(0) - y_0^4)$ then $h(\tau)$ decreases monotonically; $y^*(0) = (1-(1/3)^{1/4})^{-1/2} \simeq 2.05$.

(2) If $1 < y_0^4 < y^{*4} - \max\limits_{\tau\in(0,1)} |h_1|$ then h monotonically decreases.

(3) If $y_0^4 > y^{*4} - \max\limits_{\tau\in(0,1)} |h_1|$ then h monotonically increases.

Let us prove first statement 1. As in the case $\lambda > 1$, it is easy to show that $\eta < 0$, at least in some neighbourhood of the point $\tau = 0$. If h does not decrease monotonically then there exists τ_1 such as $\eta(\tau_1) = 0$ and $\eta(\tau) < 0$ when $\tau < \tau_1$. It follows from the representation (3.5) that $F_\lambda(y(\tau)) < 0$ if $\tau < \tau_2$ and $F_\lambda(y(\tau_2)) = 0$ for some $\tau_2 < \tau_1$. But for negative η, (3.6) implies that $y^4(\tau) < y_0^4 + h_1(\tau) < y^{*4}$, and hence $F_\lambda(y(\tau)) < 0$ for all $\tau < \tau_1$. This contradiction proves the statement. Statements 2 and 3 are proved similarly.

It follows from the above proved statements that (3.8) is valid for the solutions of (2.12), (3.1) under conditions 1 or 2, and the opposite inequality is valid under condition 3. Thus, it is shown that:

(1) If $\lambda \geqslant 1$, i.e. the density of the body exceeds or equals that of the liquid, then the body moves towards the vibrator in the sense of inequality (3.8).

(2) If $\lambda < 1$ the body moves towards the vibrator if originally it is sufficiently near it, and it moves away from the vibrator if it is sufficiently far.

Consider know a fixed particle. The force exerted on it by a dipole with a strength Q was calculated in [11]. It equals the right-hand part of (2.10) with $\Theta = 0$ and $\dot{y} = 0$. One can see that it is always directed towards a dipole. Yet, as it was shown above, a free particle can move in the opposite direction. Similar effect takes place for the motion of a particle in the presence of a pulsating source [5], or in the presence of an oscillating sphere [8].

4 High frequency vibrator

Let A be the amplitude of the dipole strength, then $q = ab(\tau)$, $\max_{\tau\geqslant 1} b(\tau) = 1$, where $a = A/(\omega R^3)$. It is obvious that if the amplitude and radius of the body are fixed, the value of a decays with the growth of frequency ω.

For the asymptotics $a = \varepsilon \to 0$ it is natural to look for a solution in the form of series

$$y = \sum_{k=0}^{\infty} y_k \varepsilon^k, \quad \Theta = \sum_{k=0}^{\infty} \Theta_k \varepsilon^k.$$

It is easy to see that $y_0 = y^0$, $\Theta_0 = \Theta^0$ are constant,

$$y_1 = \frac{1}{2\lambda+1}\int_0^\tau \frac{2b(\zeta)}{y_0^3}\cos\Theta_0 d\zeta, \quad \Theta_1 = \frac{1}{2\lambda+1}\int_0^\tau \frac{b(\zeta)}{y_0^4}\sin\Theta_0 d\zeta$$

are periodic and y_2 satisfies the following equation:

$$(2\lambda+1)\ddot{y}_2 + 2\frac{d}{dt}\left(\frac{3b\cos\Theta_0}{y_0^4}y_1 + \frac{b\sin\Theta_0}{y_0^3}\Theta_1\right) = (2\lambda+1)y_0\dot{\Theta}_1^2$$
$$+ \frac{6b\dot{y}_1}{y_0^4}\cos\Theta_0 + \frac{2b\dot{\Theta}_1}{y_0^3}\sin\Theta_0 - \frac{b^2(12\cos^2\Theta_0 + \sin^2\Theta_0(3+4y_0^{-2} - y_0^{-4}))}{3y_0^7(1-y_0^{-2})^4}.$$

After substituting $\dot{y}_1$, $\dot{\Theta}_1$, we have

$$(2\lambda+1)\ddot{y}_2 = -2\frac{d}{dt}\left(\frac{3b\cos\Theta_0}{y_0^4}y_1 + \frac{b\sin\Theta_0}{y_0^3}\Theta_1\right) + \frac{b^2}{3y_0^7(1-y_0^{-2})^4(2\lambda+1)}H_\lambda(y_0,\Theta_0), \quad (4.1)$$

where

$$H_\lambda(y,\Theta) = -(2\lambda+1)(12\cos^2\Theta + \sin^2\Theta(3+4y^{-2}-y^{-4})) + 9(1-y^{-2})^4(4\cos^2\Theta + \sin^2\Theta),$$

and H_λ has the following properties:

(1) If $\lambda \geqslant 1$ then $H_\lambda(y,\Theta)$ is negative for every Θ, $y > 1$.

(2) For every $\lambda \in (0,1)$ the inequality $H_\lambda(y,\Theta) < 0$ determines a bounded domain

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$\Omega^-(\lambda)$ on the (y_0, Θ_0) plane. If $(y_0, \Theta_0) \notin \Omega^-(\lambda)$, then $H_\lambda(y, \Theta) > 0$. It is easy to see that $\Omega^-(\lambda_1) \subset \Omega^-(\lambda_2)$ if $\lambda_1 < \lambda_2$. The radius of $\Omega^-(\lambda)$ tends to infinity when $\lambda \to 1$. The form of the surface $\Gamma_\lambda : H_\lambda(y, \Theta) = 0$ for different λ is shown on Figure 2. The inner surface marked by '1' is Γ_0. All the particles initially located inside $\Omega^-(0)$ are attracted by the vibrator. The surface 2 is $\Gamma(0.4)$ and the outer surface 3 is $\Gamma(0.8)$.

Integrating (4.1) yields

$$y_2 = -\frac{2(1 + 5\cos^2\Theta_0)}{(2\lambda + 1)^2} \int_0^\tau b(\zeta) \int_0^\zeta b(\xi) d\xi d\zeta + \frac{H_\lambda(y_0, \Theta_0)}{3(2\lambda + 1) y_0^7 (1 - y_0^{-2})^4} \int_0^\tau \int_0^\zeta b^2(\xi) d\xi d\zeta.$$

The first term in this sum is a periodic function and the second is a monotonic one, which decreases if H_λ is negative and increases if it is positive.

Recalling the properties of H_λ, one can conclude that at least for some period of time T_1 (it is easy to see that T_1 is of the order ε^{-1}):

(1) If $\lambda \geqslant 1$, the body moves towards the vibrator in the sense of inequality (3.8).

(2) If $\lambda < 1$, inequality (3.8) is valid for the solutions of (2.10)–(2.12) with $(y_0, \Theta_0) \in \Omega^-$.

(3) For the solution of (2.10)–(2.12) with initial conditions (y_0, Θ_0) outside Ω^- the inequality opposite to (3.8) is valid. $\Omega^-(\lambda)$ is a domain of attraction of particles with the densities larger than $\lambda\rho$. For example, among the particles initially located in a layer between surfaces 2 and 3 on Figure 2, those of density less than 0.4 are repelled, and those of density larger than 0.8 are attracted.

The first and second prepositions are valid together with the proof if $\dot{y}_0 < 0$, $\dot{\Theta}_0 = o(\varepsilon)$; the third is valid for positive $\dot{y}_0$. More accurate estimates are necessary to consider longer time intervals.

5 Small particle

Now let the particle's radius be small compared with the initial distance from the vibrator, $y^0 = \alpha^{-1/2}$, $\alpha \to 0$, q being arbitrary, and look for a solution in the form of series

$$y = y^0 \sum_{k=0}^{\infty} y_k \alpha^k, \ \Theta = \sum_{k=0}^{\infty} \Theta_k \alpha^k.$$

As in the previous case $y_0 = 1$, $\Theta_0 = \Theta^0$, $y_1 = 0$, $\Theta_1 = 0$ are constant,

$$y_2 = \frac{1}{2\lambda + 1} \int_0^\tau 2q(\zeta) \cos\Theta_0 d\zeta, \ \Theta_2 = \frac{1}{2\lambda + 1} \int_0^\tau q(\zeta) \sin\Theta_0 d\zeta$$

are periodic, $y_3 = 0$, $\Theta_3 = 0$, and y_4 satisfies the following equation:

$$(2\lambda + 1)\ddot{y}_4 = +2\frac{d}{dt}(3q\cos\Theta_0 y_2 + q\sin\Theta_0\Theta_2) + 2q\dot{\Theta}_2 \sin\Theta_0$$
$$+(2\lambda + 1)\dot{\Theta}_2^2 + 6q\dot{y}_2 \cos\Theta_0 - q^2(4\cos^2\Theta_0 + \sin^2\theta_0).$$

After substituting y_2, Θ_2, we have

$$(2\lambda + 1)\ddot{y}_4 = -2\frac{d}{dt}(3qy_2\cos\Theta_0 + q\Theta_2\cos\Theta_0) + \frac{2(1 - \lambda)}{2\lambda + 1} q^2 (3\cos^2\Theta_0 + 1). \tag{5.1}$$

It is easy to see that integrating (5.1) gives y_4 in the form of a sum of periodic and monotonic functions. The latter increases if $\lambda < 0$ and decreases if $\lambda > 0$. That means that particles of density less than that of the liquid move away from the vibrator and a particle whose density exceeds that of the liquid is attracted to the vibrator.

These results are valid only if the value of $|1-\lambda|$ is sufficiently large, as if it is of the $O(\alpha)$ the right-hand side of (5.1) does not include the second term which gives the monotonic part of y_4. In this case, it is necessary to consider the next term of the expansion, y_5.

Let $(1-\lambda) = \alpha c$; then $y_0 = 1,\ \Theta_0 = \Theta^0,\ y_1 = 0,\ \Theta_1 = 0$,

$$y_2 = \frac{2}{3}\int_0^\tau q(\zeta)\cos\Theta_0 d\zeta,\ \Theta_2 = \frac{1}{3}\int_0^\tau q(\zeta)\sin\Theta_0 d\zeta,$$

$$y_3 = \frac{4c}{9}\int_0^\tau q(\zeta)\cos\Theta_0 d\zeta,\ \Theta_3 = \frac{2}{9}\int_0^\tau q(\zeta)\sin\Theta_0 d\zeta.$$

Also

$$y_4 = -\tfrac{2}{9}(6\cos^2\Theta_0 + \sin^2\Theta_0)\int_0^\tau q(\zeta)\int_0^\zeta q(\xi)d\xi d\zeta$$

is periodic; lastly y_5 satisfies

$$\ddot{y}_5 = -\frac{8c}{27}(\sin^2\Theta_0 + 6\cos^2\Theta_0)\frac{d}{d\tau}(q(\tau)\int_0^\tau q(\zeta)d\zeta) + \frac{4q^2}{9}[4(c-6)\cos^2\Theta_0 + (c-8)\sin^2\Theta_0].$$

Hence y_5 decreases in the sense of (3.8) if $c > 8$ and increases monotonically if $c < 6$. For $c \in (6,8)$ it increases if $\Theta_0 \in (\Theta^*(c), \pi/2 - \Theta^*(c))$, where $\Theta^*(c) = \arctan 2\sqrt{(6-c)/(8-c)}$.

It is thus shown that at least for some period of time (of order α^{-1})

(1) Particles of density $\rho_s > \rho_1 = \rho(1 - 6(R/Y)^2)$ move towards a pulsating doublet.

(2) Particles of density $\rho_s < \rho_2 = \rho(1 - 8(R/Y)^2)$ move from it.

(3) For particles of density between ρ_2 and ρ_1, the direction of the motion depends upon their initial position with respect to the dipole's axis.

6 General case: Numerical results

For moderate values of y_0 and the amplitude of q, and large time intervals, the problem (2.10)–(2.12) was solved numerically for various initial position of the particle, various values of the parameters λ and $q = a\sin 2\pi\tau$. The calculations show that for $\lambda \geqslant 1$ the particle is always attracted to a vibrator. For the light particle there exists a domain of attraction Ω_λ^- and a domain of repulsion Ω_λ^+. They are separated by a domain Ω_λ^0, which resembles a layer containing a separation surface Γ_λ that separates Ω_λ^- and Ω_λ^+ in the asymptotic case of a 'high frequency vibrator' (small a) shown in Figure 2. Its thickness is proportional to the amplitude of vibrations a and decays with λ. The motion of particles initially located inside it depends not only on λ but on the particular function $q(\tau)$.

Some results of calculations are presented in Figures 3 and 4. Figure 3 shows the dependence on time of the distance between the particle and the vibrator for various densities and initial locations of the particle. On all the graphs, $\lambda = 0$ for curve 1, $\lambda = 0.5$ for curve 2, $\lambda = 1$ for curve 3 and $\lambda = 2$ for curve 4. One can see that light particles move away from the vibrator and particles of larger densities move towards it. The initial

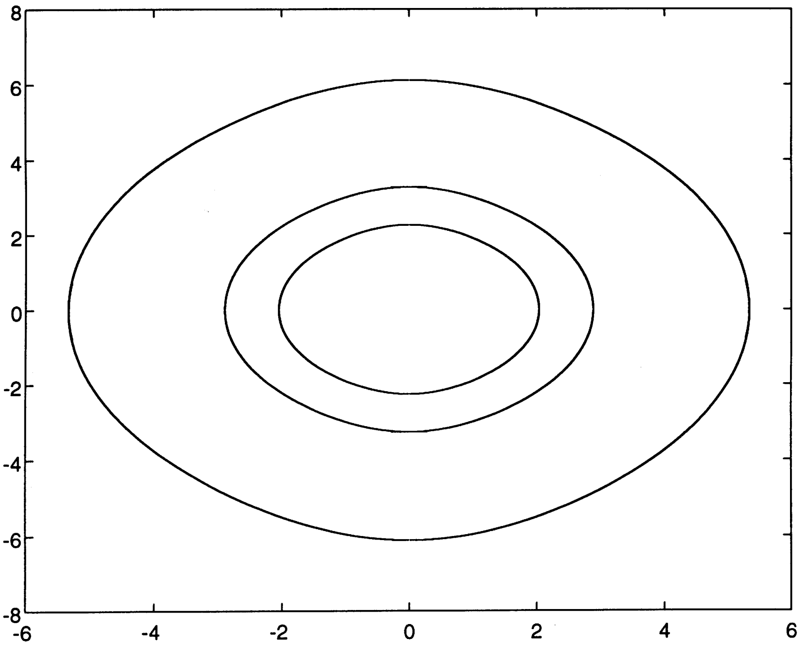

FIGURE 2. Domains of attraction for particles of different densities, meridional section $\psi = const$. Abscissa $r\cos\theta$, ordinate $r\sin\theta$. The vibrator is located in the point $(0,0,0)$. (1) attraction of all particles, (2) $\lambda < 0.4$, (3) $\lambda < 0.8$.

distance for all examples of Figure 3 equals 4 particle radii, the amplitude of vibrations is $a = 0.4$. On the upper graphs the particles were initially located on the line of the dipole's axis. On the lower ones, the line between a centre of a particle and dipole forms initially an angle of $\frac{1}{4}\pi$ with a dipole axis. The directed motion in this case is slower. On the right-hand side graphs the time of motion T_0 is 20 periods of vibration. It is easy to see the oscillating type of motion. On the left-hand-side graphs T_0 is 100 periods. The oscillations become small compared with the directed motion and are not seen.

Some of the trajectories of the particle motion are shown in Figure 4. In Figure 4 (a) the particles were initially located in the domain of attraction for $\lambda \geqslant 0$ and they all move towards a vibrator. The trajectories of heavy particles (curve 4, $\lambda = 10$) are almost straight lines. Lighter particles have more complicated trajectories with distinct angular drift. The trajectories of particles of densities $\lambda = 0$, $\lambda = 0.5$ and $\lambda = 1$ are marked by 1, 2 and 3 correspondingly.

Figure 4 (b) shows the trajectories of particles initially located a little further from a vibrator. Here $\lambda = 0$ for the curve 1, $\lambda = 0.1$ for the curve 2 and $\lambda = 0.2$ for the curve 3. A light particle moves from a vibrator and heavier one is attracted (curve 4, $\lambda = 0.5$). The initial angle between the axis of the dipole and the line of centres was $3\pi/8$ for all the graphs in Figure 4. The time of motion equals 100 periods. For sufficiently large time the trajectories of attracted particles become almost straight lines directed to the vibrator. For the repelled particles trajectories also become straight lines but their directions depend strongly upon densities and initial positions.

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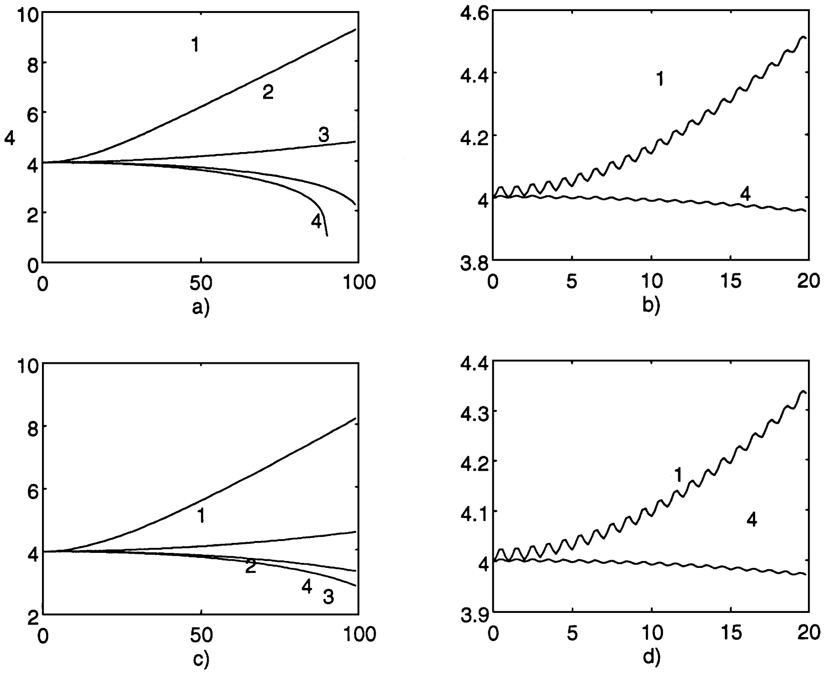

FIGURE 3. Evolution of the distance between particles of different densities and a vibrator. (1) $\lambda = 0$, (2) $\lambda = 0.5$, (3) $\lambda = 0.1$, (4) $\lambda = 2$; $\Theta_0 = 0$ for (a) and (b), $\Theta_0 = \frac{1}{4}\pi$ for (c) and (d).

7 Conclusions

Our results can be summarized as follows

(1) If the density of spherical particle equals or exceeds that of the ambient liquid, the particle is attracted to a vibrating dipole.

(2) A spherical particle of density less than that of the liquid is attracted to a vibrating dipole if initially the ratio of the distance between them and the radius of the body is sufficiently small.

(3) If this ratio is sufficiently large, a spherical body is repelled by a vibrator.

The first proposition means in particular that periodic vibrations with zero mean cause a directed motion even of particles whose densities are equal or little different from that of the liquid.

The last two statements demonstrate the complicated behaviour of particles in vibrating liquid. Even the direction of motion depends strongly on the initial position of the particle with respect to the vibrator, and on its density and dimensions (here, radius of the body). Thus, among particles of the same radii but different densities with initial distance not too small, light enough particles are repelled and heavier ones are attracted. If particles of equal densities initially located on the same distance from a vibrator are considered, then those of small enough radii are repelled, and those of large enough radii are attracted.

These results cannot be obtained by calculating the force acting on a fixed particle, since the latter does not depend on the density of the body. Moreover, in the axisymmmetric

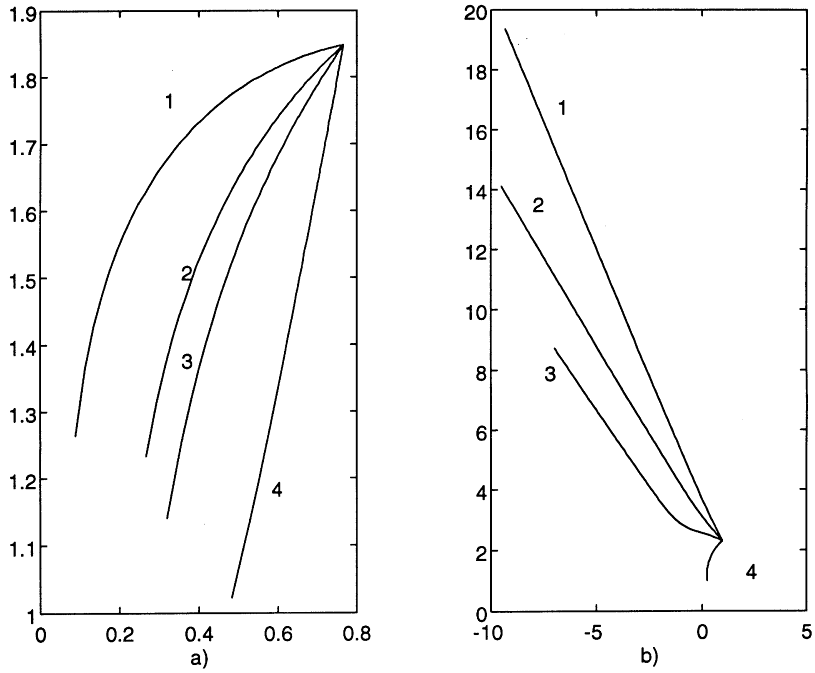

FIGURE 4. Trajectories of particles of different densities with the same initial positions. Abscissa $y\cos\Theta$, ordinate $y\sin\Theta$. $\Theta_0 = 3/8\pi$. (a) (1) $\lambda = 0$, (2) $\lambda = 0.1$, (3) $\lambda = 0.2$, (4) $\lambda = 10$ (b) (1) $\lambda = 0$, (2) $\lambda = 0.1$, (3) $\lambda = 0.2$, (4) $\lambda = 0.5$.

case this force is directed towards a vibrator at any moment, and yet light free particles move in the opposite direction.

The problem under consideration with a vibrator presented by a pulsating singularity is a model one, but it is likely to represent an asymptotic approximation to a physical problem with vibrations caused by an oscillating sphere, radius of this sphere and amplitude of displacement being small, and amplitude of velocity being large.

Appendix A Derivation of the basic equations

Substituting the representation (2.2) into equation of motion of the body (2.7) we have

$$m\ddot{\mathbf{X}} = -\frac{d}{dt}\left(\oint_{\Sigma_t} \Phi \mathbf{n} d\sigma\right) + \frac{1}{2}\oint_{\Sigma_t} |\nabla\Phi_1|^2 \mathbf{n} d\sigma,$$

where

$$\Phi_1 = \Phi - \mathbf{V}\mathbf{x}.$$

It is convenient to use θ_1, ψ_1 for the parametric representation of the surface Σ_t. It follows from representation (2.9) that for $r_1 = R$

$$\Phi = \sum_{n=o}^{\infty}[a_n^0 P_n(\cos\theta_1) + a_n^1 \cos\psi_1 P_n^1(\cos\theta_1)],$$

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where

$$a_0^0 = \frac{q}{y^2},\ \ a_1^0 = \frac{3qR}{y^3}\cos\Theta - \tfrac{3}{2}V_yR,\ \ a_0^1 = 0,\ \ a_1^1 = \frac{3}{2}\left(\frac{qR}{y^3}\sin\Theta - V_\Theta R\right),$$

$$a_n^0 = (2n+1)q\frac{R^n}{y^{n+2}}\cos\Theta,\ \ a_n^1 = \frac{2n+1}{n+1}\frac{qR^n}{y^{n+2}}\sin\Theta,\ \ n > 1.$$

Hence

$$\oint_{\Sigma_t} \Phi\mathbf{n}d\sigma = -\frac{3}{2}\pi R^3\mathbf{V} + \frac{4\pi R^3}{y^3}q\cos\Theta\mathbf{n}_y + \frac{2\pi R^3}{y^3}q\sin\Theta\mathbf{n}_\theta. \qquad \text{(A 1)}$$

Since $\partial\Phi_1/\partial n = \partial\Phi_1/\partial r_1 = 0$ on Σ_t

$$|\nabla\Phi_1|^2_{|r=R} = \frac{1}{R^2}\left|\frac{\partial\Phi_1}{\partial\theta_1}\right|^2 + \frac{1}{R^2\sin^2\theta_1}\left|\frac{\partial\Phi_1}{\partial\theta_1}\right|^2 + \frac{1}{R^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[a_n^0a_m^0\frac{\partial P_n(\cos\theta_1)}{\partial\theta_1}\frac{\partial P_m(\cos\theta_1)}{\partial\theta_1}\right.$$
$$+2a_n^0a_m^1\cos\psi_1\frac{\partial P_n(\cos\theta_1)}{\partial\theta_1}\frac{\partial P_m^1(\cos\theta_1)}{\partial\theta_1}$$
$$\left.+a_n^1a_m^1\cos^2\psi_1\frac{\partial P_n^1(\cos\theta_1)}{\partial\theta_1}\frac{\partial P_m^1(\cos\theta_1)}{\partial\theta_1} + \frac{\sin^2\psi_1}{\sin^2\theta_1}a_n^1a_m^1P_n^1(\cos\theta_1)P_m^1(\cos\theta_1)\right].$$

Hence

$$\oint_{\Sigma_t}|\nabla\Phi|^2\mathbf{n}_y ds = \pi\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_0^\pi\left\{2a_n^0a_m^0\sin^3\theta\cos\theta\frac{\partial P_n(\cos\theta)}{\partial\theta}\frac{\partial P_m(\cos\theta)}{\partial\theta}\right.$$
$$\left.+a_n^1a_m^1\left[\sin^3\theta\cos\theta\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\frac{\partial P_m^1(\cos\theta)}{\partial\theta} + \frac{\cos\theta}{\sin\theta}P_n^1(\cos\theta)P_m^1(\cos\theta)\right]\right\}d\theta; \qquad \text{(A 2)}$$

$$\oint_{\Sigma_t}|\nabla\Phi|^2\mathbf{n}_\Theta ds = 2\pi\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_n^0a_m^1\int_0^\pi P_n^1(\cos\theta)\frac{\partial P_m^1(\cos\theta)}{\partial\theta}\sin^4\theta d\theta. \qquad \text{(A 3)}$$

Recalling some properties of Legendre polynomials and adjoint Legendre functions, it is easy to show that

$$\int_0^\pi \sin^4\theta\frac{\partial P_n(\cos\theta)}{\partial\theta}\frac{\partial P_m^1(\cos\theta)}{\partial\theta}d\theta = \frac{2n(n+1)}{(2n+1)(2m+1)}[n^2\delta_n^{m-1} - m^2\delta_n^{m+1}],$$

$$\int_0^\pi \sin^3\theta\cos\theta\frac{\partial P_n(\cos\theta)}{\partial\theta}\frac{\partial P_m(\cos\theta)}{\partial\theta}d\theta = \int_0^\pi \sin^3\theta\cos\theta P_n^1(\cos\theta)P_m^1(\cos\theta)d\theta$$
$$= \frac{2n(n+1)}{(2n+1)(2m+1)}[m\delta_n^{m+1} + n\delta_n^{m-1}],$$

$$\int_0^\pi \frac{\cos\theta}{\sin\theta}P_n^1(\cos\theta)P_m^1(\cos\theta)d\theta = \begin{cases} 0 & \text{when}\quad m = n+2k \\ n(n+1) & \text{when}\quad m = n+2k+1 \end{cases},$$

and

$$\int_0^\pi\left[\sin^3\theta\cos\theta\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\frac{\partial P_m^1(\cos\theta)}{\partial\theta} + \frac{\cos\theta}{\sin\theta}P_n^1(\cos\theta)P_m^1(\cos\theta)\right]d\theta$$
$$= \frac{2nm}{(2n+1)(2m+1)}[n(m+1)^2\delta_n^{m-1} + m(n+1)^2\delta_n^{m+1}].$$

Using these expressions the series in the right-hand sides of (A 2), (A 3) can be summed explicitly.

Substituting the result into the equation of motion we have in projections on the axes y and Θ equations (2.10), (2.11).

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