Student Problems

Students up to the age of 19 are invited to send solutions to either of the following problems to Stan Dolan, 126A Harpenden Road, St. Albans, Herts AL3 6BZ.

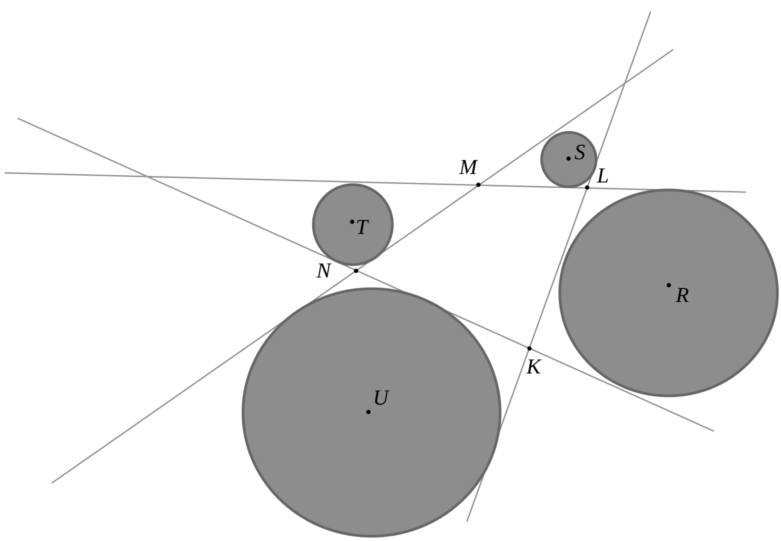
Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by May 20th 2015. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award may be made. Please print your own name clearly! The names of all successful solvers will be published in the July 2015 edition.

Problem 2015.1

Prove that $\tan 20^\circ - \tan 40^\circ + \tan 80^\circ = 3\sqrt{3}$.

Problem 2015.2

In the diagram, *KLMN* is a convex quadrilateral, the sides are extended and circles inscribed as shown.



(a) Prove that points R, S, T and U, the centres of the circles, form a cyclic quadrilateral.

(b) Apply the same construction to quadrilateral $RSTU$ and repeat this process indefinitely. Describe what happens.

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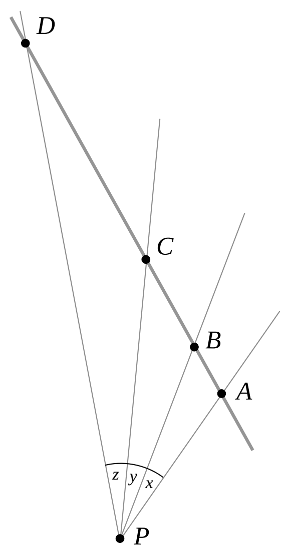
Solutions to 2014.5 and 2014.6

No correct solutions were received for Problem 2014.5. Problem 2014.6 was solved by William Grace (Maidstone Grammar School), James Moore (Sir John Leman High School, Beccles) and Andrew Rout (Maidstone Grammar School).

Problem 2014.5

The distances *AB*, *BC* and *CD* appear equal on this photograph of the Death Valley badlands. If the actual distances are such that $BC : AB = 2 : 1$, then find $CD : AB$.

Solution



This problem was based upon an idea suggested by Tony Robin and can be solved neatly by using the cross-ratio of the four points. A direct proof can be obtained as follows. The point *P* represents the location of the camera.

Let *h* be the perpendicular distance from *P* to the line *ABCD*. Then the area of triangle *PAC* is given by

$$\tfrac{1}{2}AC.h = \tfrac{1}{2}PA.PC \sin(x + y).$$

Using similar results for other triangles, we obtain the following equation.

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$$\frac{AC.BD}{AD.BC} = \frac{PA.PC.PB.PD \sin(x+y)\sin(y+z)}{PA.PD.PB.PC \sin(x+y+z)\sin y}$$
$$= \frac{\sin(x+y)\sin(y+z)}{\sin(x+y+z)\sin y}.$$

This result is remarkable because precisely the same expression would be obtained for *any* set of collinear points on the four lines of sight. In particular, it is the same for the four points on the photograph.

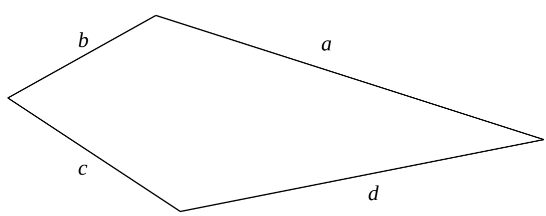
On the ground, let $AB : BC : CD = 1 : 2 : k$, whereas on the photograph $AB : BC : CD = 1 : 1 : 1$. Then

$$\frac{AC.BD}{AD.BC} = \frac{2 \times 2}{1 \times 3} = \frac{3(2 + k)}{2(3 + k)}.$$

Therefore $k = 6$ and $CD : AB = 6 : 1$.

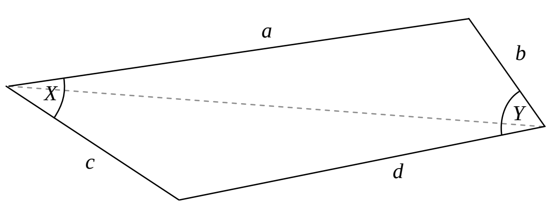
Problem 2014.6

A convex quadrilateral has sides of lengths a, b, c and d as shown. Prove that the area of the quadrilateral cannot exceed $\frac{1}{2}(ac + bd)$. Which quadrilaterals have an area equal to $\frac{1}{2}(ac + bd)$?



Solution

This problem was proposed by Nick Lord. A neat way of proceeding is to first split the quadrilateral into two triangles and replace one of the triangles by its mirror image, as shown below.



This transformation has not changed the area of the quadrilateral. As William, James and Andrew all pointed out, it is now easy to express the area of the quadrilateral in the form

$$\tfrac{1}{2}ac \sin X + \tfrac{1}{2}bd \sin Y \leqslant \tfrac{1}{2}(ac + bd).$$

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For equality, we require $X = Y = 90°$. As expressed by William, this quadrilateral is then ‘any shape formed when two right-angled triangles with equal hypotenuse length are placed together, hypotenuse to hypotenuse’.

It is now an easy application of circle geometry to show that equality holds if and only if the original quadrilateral is a cyclic quadrilateral whose diagonals intersect at right-angles.

The first prize of £25 is awarded to James Moore. The second prize of £20 is awarded to William Grace.

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Dr John Frankland Rigby

(22nd April 1933 - 29th December 2014)

John Rigby, who died in December, was an active member of The Mathematical Association from the day he joined in 1960, serving as President of the Cardiff Branch as well as its meticulous Secretary. He contributed many articles on Euclidean Geometry to the *Gazette*, which were distinguished by their precision, concise style and freedom from jargon. His elegant solutions to the Problem Corner reflected his encyclopaedic knowledge of results that most of us were not even aware of, and his regular talks at MA Conferences were always a magnet of attraction. He was particularly interested in Japanese traditional geometry with its inherent artistic qualities, as well as the mathematical detail of Islamic and Celtic art. John became a world expert in the connection between mathematics and ornamental art, creating his own designs and patterns which appeared in his beautiful Christmas cards as well as on the hassocks in Cardiff's Llandaff Cathedral.

John was a pupil at The Manchester Grammar School and went on to Trinity College, Cambridge, as both undergraduate and postgraduate. After a stint at GCHQ whilst finishing his PhD, he moved to Cardiff in 1959 to teach in the Mathematics Department of the University and remained there until his retirement. At the same time he became well-known internationally, visiting universities in Turkey, Canada, Singapore, the Philippines and Japan. His former Head of Department, Jeff Griffiths, remembers his wonderful lectures on complex analysis, with inimitable freehand sketches of complicated curves on the blackboard. For many years he ran the University's Mathematics Club for talented sixth-formers in nearby schools, along with his colleague and friend James Wiegold. In this context, Jeff Griffiths writes that

‘a worksheet was circulated to the schools a month before each meeting. At the meeting students were invited to present their solution on the blackboard. Although they often produced a technically correct solution, John would inevitably produce a far more elegant one, drawing gasps of admiration from the audience. His perfectly-drawn circles were indeed a wonder to behold’.