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ARTICLE

A proof of a conjecture of Gyárfás, Lehel, Sárközy and Schelp on Berge-cycles[†]

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Abstract
It has been conjectured that, for any fixed $r \geqslant 2$ and sufficiently large n, there is a monochromatic Hamiltonian Berge-cycle in every $(r-1)$-colouring of the edges of K_n^r, the complete r-uniform hypergraph on n vertices. In this paper we prove this conjecture.

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1. Introduction

For a given $r \geqslant 2$ and $n \geqslant r$, an *r-uniform Berge-cycle of length n*, denoted by C_n^r, is an r-uniform hypergraph with the core sequence $v_1, v_2, \ldots, v_n$ as the vertices, and distinct edges $e_1, e_2, \ldots, e_n$ such that e_i contains v_i, v_{i+1}, where addition in indices is modulo n. The case $r = 2$ gives the usual definition of the cycle C_n on n vertices for graphs. A Berge-cycle of length n in a hypergraph with n vertices is called a *Hamiltonian Berge-cycle*.

For an r-uniform hypergraph H, the *Ramsey number $R_k(H)$* is the minimum integer n such that there is a monochromatic copy of H in every k-edge colouring of K_n^r. The existence of such a positive integer is guaranteed by Ramsey's classical result in [9]. The Ramsey numbers of various variations of cycles in uniform hypergraphs have recently been studied; see *e.g.* [5, 6, 8]. In this regard, Gyárfás, Lehel, Sárközy and Schelp proposed the following conjecture for Berge-cycles.

Conjecture 1.1 ([2]). *Assume that $r \geqslant 2$ is fixed and n is sufficiently large. Then every $(r-1)$-edge colouring of K_n^r contains a monochromatic Hamiltonian Berge-cycle.*

Conjecture 1.1 states that for a given $r \geqslant 2$ we have $R_{r-1}(C_n^r) = n$ when n is sufficiently large. The case $r = 2$ is trivial, since for each $n \geqslant 3$ the complete graph K_n has a Hamiltonian cycle. The case $r = 3$ was proved by Gyárfás, Lehel, Sárközy and Schelp [2]. Recently, Maherani and the author gave a proof for the case $r = 4$; see [7]. For general r, the asymptotic form of Conjecture 1.1 was proved by Gyárfás, Sárközy and Szemerédi using the method of the Regularity Lemma; see [4]. To see more results on Conjecture 1.1, we refer the reader to [2, 3, 4] and references therein.

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In this paper we establish Conjecture 1.1. Based on the above results on this conjecture, it only suffices to give a proof for $r \geqslant 5$. The main result of this paper is the following theorem.

Theorem 1.2. *Suppose that $r \geqslant 4$ and $n > 6r\binom{4r}{r-1}$. Then in every $(r-1)$-edge colouring of K_n^r there is a monochromatic Hamiltonian Berge-cycle.*

For a given $r \geqslant 2$, let $p(r)$ be the minimum value of m for which the statement of Conjecture 1.1 holds for any $n \geqslant m$. Theorem 1.2 guarantees the existence of such a function $p(r)$ (in fact it shows that $p(r) \leqslant 6r\binom{4r}{r-1} + 1$). Determining $p(r)$ seems to be an interesting problem, though we will not make any serious attempt in this direction. At present we do not know much about $p(r)$. Our conjecture is that $p(r)$ is much less than $6r\binom{4r}{r-1} + 1$, at least for small values of r. An indication of this is given by $p(3) = 5$ (see [2]) and $p(4) \leqslant 85$ (see [7]).

2. Basic definitions and some preliminaries

Before we give our proof we present some definitions. Assume that H is an r-uniform hypergraph. The *shadow graph* $S(H)$ is a graph with vertex set $V(H)$, where two vertices are adjacent if they are covered by at least one edge of H. Consider an $(r-1)$-edge colouring of $H = K_n^r$ with colours $1, 2, \ldots, r-1$ and assume that $G = S(H)$ (so G is a complete graph). For each edge $e = xy$ of G, we assign a list $L(e)$ of colours of all edges of H containing x and y. For an edge $e \in E(G)$, the colour $i \in L(e)$ is *good* if at least $r-1$ edges (of H) of colour i contain both vertices of e. We consider a new multi-colouring L^* for the edges of G. For each edge $e \in E(G)$, assume that $L^*(e) \subseteq L(e)$ is the set of all good colours for e. Throughout this paper, for each natural number m, assume that $[m] = \{1, 2, \ldots, m\}$. For each vertex $x \in V(G)$ and any $1 \leqslant i \leqslant r-1$, assume that

$$U_i(x) = \{y \in V(G) \setminus \{x\} \mid i \in L^*(xy)\}, \quad \overline{U}_i(x) = \{y \in V(G) \setminus \{x\} \mid i \notin L^*(xy)\},$$

and $d_i(x)$ is the number of edges of colour i containing x in H. For any $I \subseteq [r-1]$, set

$$U_I(x) = \bigcap_{i \in I} U_i(x) \quad \text{and} \quad \overline{U}_I(x) = \bigcap_{i \in I} \overline{U}_i(x).$$

We say that a set of vertices $S \subseteq V(G)$ *avoids* the set of colours $W \subseteq [r-1]$ if, for each $i \in W$, there is a vertex $x \in S$ with $d_i(x) \leqslant \binom{4r}{r-1}$ or an edge $e = xy$ for $x, y \in S$ with $i \notin L^*(e)$. We will use the following lemmas in the proof of Theorem 1.2.

Lemma 2.1 ([7]). *Assume that $r \geqslant 3$ and $H = K_n^r$ is an $(r-1)$-edge coloured complete r-uniform hypergraph on n vertices. Also, suppose that $G = S(H)$ and there is a monochromatic Hamiltonian cycle in G under multi-colouring L^*. Then there is a monochromatic Hamiltonian Berge-cycle in H.*

Lemma 2.2 ([1]). *Let G be a simple graph and let u and v be non-adjacent vertices in G such that $d_G(u) + d_G(v) \geqslant n$. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.*

Lemma 2.3 ([1]). *Let G be a simple graph with degree sequence $0 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n < n$ and $n \geqslant 3$. If, for each $i < n/2$, we have $d_i > i$ or $d_{n-i} \geqslant n - i$, then G is Hamiltonian.*

The following simple remark can be proved by induction on m and it will be used later on.

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Remark 2.1. Assume that $a_m \geqslant a_{m-1} \geqslant \cdots \geqslant a_1 \geqslant a > 0$ are real numbers and $a_1 + \cdots + a_m = l$. Then

$$\prod_{i=1}^{m} a_i \geqslant a^{m-1}(l-(m-1)a).$$

In the rest of this paper, for a real number r, we use $\lfloor r \rfloor$ (resp. $\lceil r \rceil$) to mean the greatest integer not exceeding r (resp. the least integer not less than r).

3. Outline of the proof of Theorem 1.2

Here we sketch the main ideas of our proof of Theorem 1.2. Suppose to the contrary that there is no monochromatic Hamiltonian Berge-cycle in a given $(r-1)$-edge colouring c of $H = K_n^r$ with colours $1, 2, \ldots, r-1$. We will show that (see Claim 4.5), by suitable renaming of colours, for some $0 \leqslant f \leqslant r-2$ there are distinct vertices $x, y_1, y_2, \ldots, y_{r-1}$ such that $|\overline{U}_{r-1}(x)| \geqslant (n-1)/2$, $i \notin L^*(xy_i)$ for any $f+1 \leqslant i \leqslant r-1$ and $\{y_i\}_{i=1}^{f}$ avoids $[f]$. We choose distinct vertices $x, y_1, y_2, \ldots, y_{r-1}$ with these properties and maximum f. Without loss of generality we assume that

$$|\overline{U}_{f+1}(x)| \leqslant |\overline{U}_{f+2}(x)| \leqslant \cdots \leqslant |\overline{U}_{r-1}(x)|.$$

Then we divide our proof into some cases, and in each case, using the distinct vertices x and $\{y_i\}_{i=1}^{r-1}$, we construct a new graph Γ on $V(H)$ so that any Hamiltonian cycle in Γ can be extended to a monochromatic Hamiltonian Berge-cycle of colour $f+1$ in H. In short $V(\Gamma) = V(H)$, and for any two adjacent vertices u and v of Γ there exists an edge $g_{uv} \in E(H)$ of colour $f+1$ containing u and v. Moreover, $g_{uv} \neq g_{u'v'}$ for almost any two distinct edges uv and $u'v'$ in $E(\Gamma)$.

Overall, Γ can be defined as follows. The maximality of f and the choices of the vertices $D = \{y_i\}_{i=1}^{r-1} \cup \{x\}$ imply that for almost all vertices $u \in V(H)$ there are many vertices v such that there is an edge e_{uv} in H of colour $f+1$ containing u, v with $|e_{uv} \cap D| \geqslant r-2$. You can see the reason for the existence of many vertices v with this property in the proof of Claim 4.12 when $f \leqslant r-3$ and in Section 4.3 for $f = r-2$. Now we consider the new graph with vertex set $V(H)$ and edges uv mentioned above. Then for $f < r-2$ we add a few suitable edges (the edges E_2 (4.9), E_3 (4.11) and E_4 (4.12) defined in Section 4.4) to this graph to get a new graph Γ with minimum degree at least $2r+1$.

To complete our proof (in fact to get a contradiction to our incorrect assumption) it suffices to show that Γ is a Hamiltonian graph. To do this, we show that the degree sequence of the graph Γ satisfies Chvátal's condition in Lemma 2.3. More precisely, if $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ are degrees of the vertices of Γ, then for each $i \leqslant n/2$ we have $d_i > i$ or $d_{n-i} \geqslant n-i$. Hence, by Lemma 2.3, Γ is Hamiltonian and we are done.

Finally I would like to mention that in Claim 4.18, the reader can see how we can extend a Hamiltonian cycle in Γ into a monochromatic Hamiltonian cycle in H when $f \leqslant r-3$, as can be seen in Section 4.3 for $r = f-2$.

4. The proof of Theorem 1.2

Suppose to the contrary that there is no monochromatic Hamiltonian Berge-cycle in a given $(r-1)$-edge colouring c of $H = K_n^r$ with colours $1, 2, \ldots, r-1$. We will get a contradiction in this section.

4.1 Useful definitions and facts

For each $1 \leqslant i \leqslant r-1$, let W_i be the set of all edges e of $G = S(H)$ for which $i \notin L^*(e)$. Using Lemma 2.1, we may assume that the subgraph of G with vertex set $V(G)$ and edge set $E(G) \setminus W_i$

is not Hamiltonian. Now consider $S_i \subseteq W_i$ with minimum cardinality, such that the spanning subgraph of G induced by $E(G) \setminus S_i$ is not Hamiltonian. Assume that G_i and G_i^c are the spanning subgraphs of G induced by S_i and $E(G) \setminus S_i$, respectively. For each colour $1 \leqslant i \leqslant r-1$, respectively, assume that T_i and R_i are the sets of all isolated vertices and all vertices with degree at least $(n-1)/2$ in G_i. Also, assume that $Q_i = V(G_i) \setminus (T_i \cup R_i)$. We will frequently need the following fact in our proof.

Fact 4.1. For each $1 \leqslant i \leqslant r-1$, G_i^c is non-Hamiltonian. Moreover, for each $e \in E(G_i)$, we have $i \notin L^*(e)$ and $G_i^c + e$ is Hamiltonian.

For any two non-adjacent vertices x and y of G_i^c, by Fact 4.1 the graph $G_i^c + xy$ is Hamiltonian and so, by Lemma 2.2, we have $d_{G_i^c}(x) + d_{G_i^c}(y) \leqslant n-1$. Therefore we have the following fact on the sums of degrees of adjacent vertices in G_i.

Fact 4.2. For any two adjacent vertices x and y of G_i, we have $d_{G_i}(x) + d_{G_i}(y) \geqslant n-1$.

This fact implies that Q_i is an independent set in G_i. If $R_i = \emptyset$ for some i, then since Q_i is an independent set, the graph G_i has no edge and so G_i^c is a complete graph, a contradiction to the fact that G_i^c is non-Hamiltonian. Hence $R_i \neq \emptyset$ (see Section 2 for the notations that are not defined here). Now we claim that $|R_i| \geqslant |T_i|$ for each $1 \leqslant i \leqslant r-1$. Assume to the contrary that for some i we have $|R_i| < |T_i|$. Let

$$R_i = \{x_1, x_2, \ldots, x_{|R_i|}\}, \quad T_i = \{y_1, y_2, \ldots, y_{|T_i|}\}, \quad Q_i = \{z_1, z_2, \ldots, z_{|Q_i|}\}.$$

Obviously

$$C = y_1 x_1 \ldots y_{|R_i|} x_{|R_i|} y_{|R_i|+1} \ldots y_{|T_i|} z_1 \ldots z_{|Q_i|}$$

is a Hamiltonian cycle in G_i^c, a contradiction. By the same argument, we have $|R_i \cup Q_i| > |T_i|$. Therefore we have the following fact.

Fact 4.3. For each $1 \leqslant i \leqslant r-1$, we have

- Q_i is an independent set in G_i,
- $R_i \neq \emptyset$ and $|R_i| \geqslant |T_i|$,
- $|R_i \cup Q_i| > |T_i|$.

4.2 Vertices avoiding all colours

An argument similar to the proof of Claim 2.3 of Theorem 2.2 in [7] (set $t = 2$ and follow the proof) yields the following result.

Claim 4.4. *Let $P \subseteq [r-1]$ and $|P| = p$. Then there is a set of vertices $Q \subseteq V(G)$ with $|Q| \leqslant p+1$ such that Q avoids P.*

First assume that there is a subset $S \subseteq V(G)$ that avoids a set of colours containing at least $|S|+1$ colours $c_1, c_2, \ldots, c_{|S|+1}$. Using Claim 4.4, there is a subset $S' \subseteq V(G)$ containing at most $r-1-|S|$ vertices that avoids $[r-1] \setminus \{c_1, c_2, \ldots, c_{|S|+1}\}$. Now $S \cup S'$ avoids $[r-1]$, which is impossible since the number of edges in H containing $S \cup S'$ is

$$\binom{n-|S \cup S'|}{r-|S \cup S'|} \geqslant n-r+1,$$

and for each $1 \leqslant i \leqslant r-1$ the number of edges of colour i containing $S \cup S'$ is at most $\binom{4r}{r-1}$ (note that $n > 6r\binom{4r}{r-1}$). Therefore each subset $S \subseteq V(G)$ avoids at most $|S|$ colours in $[r-1]$.

Claim 4.5. *By suitably renaming the colours, there are distinct vertices x and $\{y_i\}_{i=1}^{r-1}$ such that $|\overline{U}_{r-1}(x)| \geqslant (n-1)/2$ and for some $0 \leqslant f \leqslant r-2$, $\{y_i\}_{i=1}^{f} \subseteq \bigcap_{i=f+1}^{r-1} T_i$, the set of vertices $\{y_i\}_{i=1}^{f}$ avoids $[f]$ and $i \notin L^*(xy_i)$ for any $f+1 \leqslant i \leqslant r-1$.*

Proof of Claim 4.5. Let $S = \{y_i\}_{i=1}^{f} \subseteq V(G)$ be the largest subset of vertices with $f \leqslant r-1$ that avoids a set containing f colours. Note that it is possible to have $S = \emptyset$. Without any loss of generality, we may assume that S avoids $[f]$. The case $f = r-1$ is impossible, since the number of edges in H containing S is

$$n - r + 1 > 6r\binom{4r}{r-1} - r + 1,$$

and for each $1 \leqslant i \leqslant r-1$ the number of edges of colour i containing S is at most $\binom{4r}{r-1}$. Hence $f \leqslant r-2$. If $y_i \notin T_j$ for some $1 \leqslant i \leqslant f$ and $f+1 \leqslant j \leqslant r-1$, then there is a vertex $v \in V(G)$ such that $j \notin L^*(vy_i)$ and thus $S \cup \{v\}$ avoids $[f] \cup \{j\}$, a contradiction to the maximality of S. Hence

$$S \subseteq \bigcap_{i=f+1}^{r-1} T_i. \tag{4.1}$$

If $f = r-2$, then choose $x \in R_{r-1}$ and $y_{r-1} \in N_{G_{r-1}}(x)$. Since $d_{G_{r-1}}(x) \geqslant (n-1)/2$ we have $|\overline{U}_{r-1}(x)| \geqslant (n-1)/2$, and there is nothing to prove. Now let $f \leqslant r-3$. If for some $x \in V(G)$ and for some $f+1 \leqslant i, j \leqslant r-1$ with $i \neq j$ we have $\overline{U}_i(x) \cap \overline{U}_j(x) \neq \emptyset$, then for any $v \in \overline{U}_i(x) \cap \overline{U}_j(x)$ the set $S \cup \{x, v\}$ avoids $[f] \cup \{i, j\}$, a contradiction to the maximality of f. Hence the following fact holds. □

Fact 4.6. For any $f+1 \leqslant i, j \leqslant r-1$ with $i \neq j$, and for each $x \in V(G)$, we have $\overline{U}_i(x) \cap \overline{U}_j(x) = \emptyset$.

Now we claim that there is a vertex

$$x \in \bigcup_{i=f+1}^{r-1} R_i \setminus \bigcup_{i=f+1}^{r-1} T_i.$$

If there is such a vertex x, then the proof of Claim 4.5 will be finished by an easy argument. To see this, without any loss of generality assume that $x \in R_{r-1}$. Since x has degree at least $(n-1)/2$ in G_{r-1}, we have $|\overline{U}_{r-1}(x)| \geqslant (n-1)/2$. On the other hand, for each $i = f+1, \ldots, r-1$ we have $x \in R_i \cup Q_i$. Hence, for each $f+1 \leqslant i \leqslant r-1$, there is a vertex y_i with $xy_i \in E(G_i)$, and using Fact 4.1 we have $i \notin L^*(xy_i)$. Therefore the vertices x and $\{y_i\}_{i=1}^{f}$ have the desired properties in Claim 4.5 and we are done. Now, to show that

$$\bigcup_{i=f+1}^{r-1} R_i \setminus \bigcup_{i=f+1}^{r-1} T_i \neq \emptyset,$$

assume to the contrary that

$$\bigcup_{i=f+1}^{r-1} R_i \subseteq \bigcup_{i=f+1}^{r-1} T_i. \tag{4.2}$$

We consider the following cases, and in each case we get a contradiction.

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Case 1. $R_i \cap R_j = \emptyset$ for any $f+1 \leqslant i, j \leqslant r-1$.

By Fact 4.3, for each $i \leqslant r-1$ we have $|R_i| \geqslant |T_i|$. On the other hand, we have $R_i \cap R_j = \emptyset$ for any $f+1 \leqslant i, j \leqslant r-1$, and using (4.2),

$$\bigcup_{i=f+1}^{r-1} R_i \subseteq \bigcup_{i=f+1}^{r-1} T_i.$$

Therefore we have $|R_i| = |T_i|$ for each $f+1 \leqslant i \leqslant r-1$,

$$\bigcup_{i=f+1}^{r-1} R_i = \bigcup_{i=f+1}^{r-1} T_i$$

and $T_i \cap T_j = \emptyset$ for any $f+1 \leqslant i, j \leqslant r-1$ and $i \neq j$. Note that by (4.1) we have $S \subseteq \bigcap_{i=f+1}^{r-1} T_i$, and therefore $f=0$. Using Fact 4.3 for each $1 \leqslant i \leqslant r-1$, we have $R_i \neq \emptyset$. On the other hand, for each $1 \leqslant i \leqslant r-1$ we have $|R_i| = |T_i|$ and the degree of each vertex of R_i in G_i is at least $(n-1)/2$. Hence, for each $1 \leqslant i \leqslant r-1$, $Q_i \neq \emptyset$. For each $1 \leqslant i \leqslant r-1$, we have $d_{G_i}(w) \leqslant n-1-|T_i|$ when $w \in R_i$, and $d_{G_i}(w) \leqslant |R_i|$ when $w \in Q_i$. On the other hand $|R_i| = |T_i|$, and by Fact 4.2 we have $d_{G_i}(x) + d_{G_i}(y) \geqslant n-1$ for any two adjacent vertices x and y of G_i. Therefore, for each i, the bipartite subgraph of G_i with colour classes R_i and Q_i is complete, and also the subgraph of G_i induced by R_i is a complete graph. Without any loss of generality, suppose that for every $1 \leqslant i \leqslant r-2$ we have $|R_{r-1}| \leqslant |R_i|$. Now, for every $1 \leqslant i \leqslant r-2$, set $A_i = R_{r-1} \cap T_i$ and $B_i = R_{r-1} \cap Q_i = R_{r-1} \setminus A_i$ (note that $R_{r-1} \cap R_i = \emptyset$). Also, with no loss of generality, assume that $|A_i| \leqslant |A_j|$ for $i \leqslant j \leqslant r-1$.

First assume A_{r-3} is non-empty. Clearly $R_t \setminus T_{r-1}$ is non-empty for some $t \in \{r-3, r-2\}$, since $|T_{r-1}| = |R_{r-1}| < |R_{r-2} \cup R_{r-3}|$. In the next paragraph we will show that $B_t \neq \emptyset$. Now choose two vertices $u \in B_t$ and $v \in R_t \setminus T_{r-1}$. Since uv is an edge of G_t, using Fact 4.1 we have $t \notin L^*(uv)$. On the other hand $v \in R_t \setminus T_{r-1}$ and $R_t \cap R_{r-1} = \emptyset$, so $v \in Q_{r-1}$. Therefore uv is an edge of G_{r-1}, and again using Fact 4.1 we have $r-1 \notin L^*(uv)$ and thus $\{u, v\}$ avoids $\{t, r-1\}$, which contradicts the fact that $f=0$.

To see the fact $B_t \neq \emptyset$, first suppose that $t = r-3$. If $B_{r-3} = \emptyset$, then $R_{r-1} = A_{r-3} \subseteq T_{r-3}$ and so $R_{r-1} = A_{r-2} \subseteq T_{r-2}$ (note that $|A_{r-3}| \leqslant |A_{r-1}|$ and $A_{r-3} \cup A_{r-2} \subseteq R_{r-1}$). Hence $T_{r-3} \cap T_{r-2} \cap R_{r-1} = R_{r-1} \neq \emptyset$, a contradiction to the fact that $T_i \cap T_j = \emptyset$ for any $1 \leqslant i, j \leqslant r-1$ and $i \neq j$. Now suppose that $t = r-2$. If $B_{r-2} = \emptyset$, then $R_{r-1} = A_{r-2} \subseteq T_{r-2}$ and so $A_{r-3} \subseteq T_{r-2} \cap T_{r-3}$, again a contradiction to the fact that $T_i \cap T_j = \emptyset$ for any $1 \leqslant i, j \leqslant r-1$ and $i \neq j$.

Now assume that $A_{r-3} = \emptyset$. Then $A_1 = \cdots = A_{r-3} = \emptyset$, and therefore $R_{r-1} \subseteq T_{r-2}$, since

$$\bigcup_{i=f+1}^{r-1} R_i = \bigcup_{i=f+1}^{r-1} T_i.$$

If $R_i \setminus T_{r-1}$ is non-empty for some $i \in \{1, \ldots, r-3\}$, then $i, r-1 \notin L^*(uv)$ for all $u \in B_i$ and $v \in R_i \setminus T_{r-1}$ (note that since $R_{r-1} \cap R_i = \emptyset$ and $A_i = \emptyset$, we have $R_{r-1} = B_i \neq \emptyset$) and thus $\{u, v\}$ avoids $\{i, r-1\}$, which is impossible. Otherwise

$$\bigcup_{i=1}^{r-3} R_i \subseteq T_{r-1}.$$

On the other hand, $|T_{r-1}| = |R_{r-1}| \leqslant |R_i|$ for every $1 \leqslant i \leqslant n-1$. Hence $r=4$ and $R_1 = T_3$. Since

$$\bigcup_{i=1}^{3} R_i = \bigcup_{i=1}^{3} T_i \quad \text{and} \quad A_1 = \emptyset,$$

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we have $R_3 = T_2$ and $R_2 = T_1$, and hence $R_1 \subseteq Q_2$, $R_2 \subseteq Q_3$ and $R_3 \subseteq Q_1$. Now since for each $1 \leqslant i \leqslant 3$ the bipartite subgraph of G_i with colour classes R_i and Q_i is complete, for any three vertices $v_i \in R_i$, where $i = 1, 2, 3$, we have $v_1v_3 \in E(G_1)$, $v_1v_2 \in E(G_2)$ and $v_2v_3 \in E(G_3)$, and thus using Fact 4.1 we have $1 \notin L^*(v_1v_3)$, $2 \notin L^*(v_1v_2)$ and $3 \notin L^*(v_2v_3)$. Therefore $\{v_1, v_2, v_3\}$ avoids $[3] = \{1, 2, 3\}$, which is again impossible.

Case 2. $R_i \cap R_j \neq \emptyset$ for some $f + 1 \leqslant i, j \leqslant r - 1$ and $i \neq j$.

Without any loss of generality, assume that $R_{r-2} \cap R_{r-1} \neq \emptyset$, and let $x \in R_{r-2} \cap R_{r-1}$. Note that $f \leqslant r - 3$, and using Fact 4.6 we have $\overline{U}_{r-2}(x) \cap \overline{U}_{r-1}(x) = \emptyset$. Therefore

$$d_{G_{r-2}}(x) = d_{G_{r-1}}(x) = (n-1)/2,$$
$$N_{G_{r-2}}(x) \cap N_{G_{r-1}}(x) = \emptyset,$$
$$V(G) = \{x\} \cup N_{G_{r-2}}(x) \cup N_{G_{r-1}}(x).$$

Hence $T_{r-2} \cap T_{r-1} = \emptyset$ and thus $f = 0$; note that by (4.1) we have

$$S = \{y_i\}_{i=1}^{f} \subseteq \bigcap_{i=f+1}^{r-1} T_i.$$

Recall that $r \geqslant 4$. Without loss of generality assume that

$$|R_{r-3} \cap N_{G_{r-1}}(x)| \geqslant |R_{r-3} \cap N_{G_{r-2}}(x)|.$$

First assume $x \in R_{r-3}$. Then, for each $y \in N_{G_{r-3}}(x)$, the set $\{x, y\}$ clearly avoids a set containing $r - 3$ and one of the colours $r - 2$ or $r - 1$, a contradiction to the fact that $f = 0$. In fact $\{x, y\}$ avoids $\{r - 3, r - 2\}$ if $y \in N_{G_{r-2}}(x)$, and $\{x, y\}$ avoids $\{r - 3, r - 1\}$ if $y \in N_{G_{r-1}}(x)$.

Now assume $x \notin R_{r-3}$. Then since $R_{r-3} \neq \emptyset$, $V(G) = \{x\} \cup N_{G_{r-2}}(x) \cup N_{G_{r-1}}(x)$ and $|R_{r-3} \cap N_{G_{r-1}}(x)| \geqslant |R_{r-3} \cap N_{G_{r-2}}(x)|$, we have $R_{r-3} \cap N_{G_{r-1}}(x) \neq \emptyset$. Now consider $y \in R_{r-3} \cap N_{G_{r-1}}(x)$. If there is a vertex $z \in N_{G_{r-3}}(y) \cap N_{G_{r-2}}(x)$, then $\{x, y, z\}$ avoids $\{r - 3, r - 2, r - 1\}$, again contradicting $f = 0$. Therefore $N_{G_{r-3}}(y) \subseteq N_{G_{r-1}}(x) \cup \{x\}$. Since $y \in N_{G_{r-1}}(x)$, $d_{G_{r-2}}(x) = d_{G_{r-1}}(x) = (n-1)/2$ and $d_{G_{r-3}}(y) \geqslant (n-1)/2$, we have $x \in N_{G_{r-3}}(y)$, and thus $\{x, y\}$ avoids $\{r - 3, r - 1\}$, which is impossible. □

We choose distinct vertices x and $\{y_i\}_{i=1}^{r-1}$ with the desired properties mentioned in Claim 4.5 and maximum f. For simplicity we will denote $U_I(x)$ and $\overline{U}_I(x)$ (for $I \subseteq [r-1]$) by U_I and $\overline{U}_I$, respectively. Also, for simplicity we denote $U_i(x)$ and $\overline{U}_i(x)$ $(1 \leqslant i \leqslant r - 1)$ by U_i and $\overline{U}_i$, respectively. Using Claim 4.5 we have $|\overline{U}_{r-1}| \geqslant (n-1)/2$, and by Fact 4.6 we have $\overline{U}_i \cap \overline{U}_j = \emptyset$ for any $f + 1 \leqslant i, j \leqslant r - 1$ with $i \neq j$. Hence $|U_{r-1}| \geqslant |U_i|$ for each $1 \leqslant i \leqslant r - 1$, and without loss of generality we may assume that

$$|\overline{U}_{f+1}| \leqslant |\overline{U}_{f+2}| \leqslant \cdots \leqslant |\overline{U}_{r-1}|. \tag{4.3}$$

Also, by Claim 4.5 we have $f \leqslant r - 2$. Let

$$Y = \{y_1, y_2, \ldots, y_{r-1}\} \setminus \{y_{f+1}\} \quad \text{and} \quad Y_i = \{y_1, y_2, \ldots, y_{r-1}\} \setminus \{y_{f+1}, y_i\}$$

for every $1 \leqslant i \leqslant r - 1$. We will use the following simple fact in our proof. It follows from the fact that $\{y_i\}_{i=1}^{f}$ avoids $[f]$, and for each $f + 1 \leqslant i \leqslant r - 1$ we have $i \notin L^*(xy_i)$.

Fact 4.7. For every $1 \leqslant i \leqslant r - 1$, the set of vertices $Y_i \cup \{x\}$ avoids the set of colours $[r-1] \setminus \{i, f + 1\}$.

Also, we need the following fact in our proof later on.

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Fact 4.8. For every $1 \leqslant i \leqslant r-1$ and $i \neq f+1$, we have $\overline{U}_i \cap (Y_i \cup \{y_{f+1}\}) = \emptyset$. Moreover, $\overline{U}_{f+1} \cap Y_{f+1} = \emptyset$.

The proof of Fact 4.8 is trivial. In fact, if for $i \neq f+1$ we have $\overline{U}_i \cap (Y_i \cup \{y_{f+1}\}) \neq \emptyset$, then the set of vertices $Y_i \cup \{x, y_{f+1}\}$ avoids all colours $[r-1]$. But this is impossible, since the number of edges in H containing $Y_i \cup \{x, y_{f+1}\}$ is

$$n - r + 1 > 6r\binom{4r}{r-1} - r + 1,$$

and for each $1 \leqslant i \leqslant r-1$ the number of edges of colour i containing $Y_i \cup \{x, y_{f+1}\}$ is at most $\binom{4r}{r-1}$. The proof of the second result in Fact 4.8 is similar.

In the rest of our proof we define a Hamiltonian graph Γ with $V(\Gamma) = V(H)$ in such a way that every Hamiltonian cycle C of Γ can be extended to a monochromatic Hamiltonian Berge-cycle of H. For this, we consider two cases $f = r-2$ and $f \leqslant r-3$, and in each case we first give the definition of the new graph Γ. Then, using Dirac's condition and Lemma 2.3, we show that Γ is Hamiltonian, and finally we prove that every Hamiltonian cycle of Γ can be extended to a monochromatic Hamiltonian Berge-cycle of colour $f+1$ of H. Clearly these results will complete our proof.

4.3 Case $f = r - 2$

In this section we assume $f = r-2$. Consider a graph Γ with vertex set $V(\Gamma) = V(H)$ and edge set $E(\Gamma) = E_1 \cup E_2$, where the sets E_i are defined as follows. Also, the sets F_i are defined and will be used later on.

We define E_1 as follows:

$$E_1 = \{uv \mid u, v \in V(\Gamma) \setminus Y, c(Y \cup \{u, v\}) = r-1\}. \tag{4.4}$$

For each $uv \in E_1$, set $e_{uv} = Y \cup \{u, v\}$ and

$$F_1 = \{e_{uv} \mid uv \in E_1\}. \tag{4.5}$$

We define E_2 as follows:

$$E_2 = \{y_i v \mid 1 \leqslant i \leqslant r-2, v \in V(\Gamma) \setminus Y, \}. \tag{4.6}$$

Since Y avoids $f = [r-2]$, we know that for a fixed $u \in V(\Gamma) \setminus Y$, apart from at most $(r-2)\binom{4r}{r-1}$ choices of $v \in V(\Gamma) \setminus (Y \cup \{u\})$, the edges $e_{uv} = Y \cup \{u, v\}$ of H are of colour $r-1$, so $d_\Gamma(u) \geqslant n - r\binom{4r}{r-1}$. Also, for each $1 \leqslant i \leqslant r-2$, we have $d_\Gamma(y_i) = n - (r-2)$. This observation comes from the definition of the set E_2. One can easily see that Dirac's condition implies that the graph Γ is Hamiltonian; see [1].

Now we show that every Hamiltonian cycle in Γ can be extended to a monochromatic Hamiltonian Berge-cycle of colour $r-1$ in H. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices of a Hamiltonian cycle C in Γ that appear in this order. Now we define the distinct edges $g_1, g_2, \ldots, g_n \in E(H)$ of colour $r-1$ one by one (in the same order as their subscripts appear), such that for each $i = 1, 2, \ldots, n$ we have $\{v_i, v_{i+1}\} \subseteq g_i$, and $g_1, g_2, \ldots, g_n$ form a Hamiltonian Berge-cycle with the core sequence $v_1, v_2, \ldots, v_n$. We choose $g_i = e_{v_i v_{i+1}} \in F_1$ for $v_i v_{i+1} \in E_1$. Now assume $v_i v_{i+1} \in E_2$. Choose $g_i = Y \cup \{v_i, v_{i+1}, u_i\}$ of colour $r-1$ with $u_i \in V(\Gamma) \setminus (Y \cup \{v_{i-1}, v_i, v_{i+1}, v_{i+2}\})$ and $g_i \neq g_j$ for every $j < i$.

Such an edge g_i exists since at least $n - r\binom{4r}{r-1}$ edges of colour $r-1$ contain $Y \cup \{v_i, v_{i+1}\}$, and $\{v_i, v_{i+1}\}$ lies in at most $2(r-3)+2$ edges g_j for $j < i$. Note that for $v_i v_{i+1} \in E_2$, if $\{v_i, v_{i+1}\} \subseteq g_j$ for some $2 \leqslant j \leqslant i-2$, then $v_j v_{j+1} \in E_2$, $|Y \cap \{v_i, v_{i+1}, v_j, v_{j+1}\}| = 2$ and $g_j = Y \cup \{v_i, v_{i+1}, v_j, v_{j+1}\}$. On the other hand, since each edge of E_2 has exactly one vertex y_i, for some $1 \leqslant i \leqslant r-2$ we

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have $|E(C) \cap E_2| = 2(r-2)$ and so $\{v_i, v_{i+1}\} \in E_2$ has been used in at most $2(r-3)$ edges g_j for $2 \leqslant j \leqslant i-2$. Therefore at most $2(r-3)+2$ edges g_j for $1 \leqslant j \leqslant i-1$ contain $\{v_i, v_{i+1}\}$ if $v_i v_{i+1} \in E_2$.

4.4 Case $f \leqslant r-3$

In this section we assume $f \leqslant r-3$. First we prove the following claim.

Claim 4.9. $|\overline{U}_{f+1}| \leqslant r-2$.

Suppose to the contrary that $|\overline{U}_{f+1}| \geqslant r-1$. Now let

$$M = \{xy_1y_2 \ldots y_f u_{f+1} u_{f+2} \ldots u_{r-1} \mid u_i \in \overline{U}_i\}.$$

For each $f+1 \leqslant i \leqslant r-1$, by the definition of $\overline{U}_i$ we have $i \notin L^*(xy)$ for every $y \in \overline{U}_i$, so there are at most $(r-2)|\overline{U}_i|$ edges in M of colour i. On the other hand, $\{y_i\}_{i=1}^{f}$ avoids the set of colours $\{1, 2, \ldots, f\}$, and thus at most $\binom{4r}{r-1}$ edges in M are of colour i for every $1 \leqslant i \leqslant f$. Therefore

$$|M| \leqslant (r-2) \sum_{i=f+1}^{r-1} |\overline{U}_i| + f\binom{4r}{r-1}.$$

The inequalities (4.3), Remark 2.1 and the assumption $|\overline{U}_{f+1}| \geqslant r-1$ imply that

$$\begin{aligned}
&(r-1)^{r-f-3}(s-(r-f-3)(r-1))|\overline{U}_{r-1}| \\
&\leqslant \prod_{i=f+1}^{r-1} |\overline{U}_i| = |M| \\
&\leqslant (r-2)(s+|\overline{U}_{r-1}|) + f\binom{4r}{r-1},
\end{aligned}$$

where

$$s = \sum_{i=f+1}^{r-2} |\overline{U}_i|.$$

Therefore $p(s) \leqslant 0$, where

$$p(x) = ((r-1)^{r-f-3}(x-(r-f-3)(r-1)) - (r-2))|\overline{U}_{r-1}| - (r-2)x - f\binom{4r}{r-1}.$$

Evidently $p(x)$ is an increasing function, its derivative being positive for every x because of our assumption $|\overline{U}_{f+1}| \geqslant r-1$. By Claim 4.5 we have $|\overline{U}_{r-1}| \geqslant (n-1)/2$. On the other hand, $s \geqslant (r-f-2)(r-1), f \leqslant r-3$ and $n > 6r\binom{4r}{r-1}$. Hence we have

$$\begin{aligned}
p(s) &\geqslant p((r-f-2)(r-1)) \\
&= ((r-1)^{r-f-2} - (r-2))|\overline{U}_{r-1}| - (r-f-2)(r-2)(r-1) - f\binom{4r}{r-1} \\
&> 0,
\end{aligned}$$

a contradiction. Hence $|\overline{U}_{f+1}| \leqslant r-2$.

Assume that

$$B_i = \overline{U}_i \setminus ((\cup_{j>i} \overline{U}_j) \cup \{y_i\})$$

for every $1 \leqslant i \leqslant r-1$ and

$$\overline{U}_{f+1} = \{u_1(=y_{f+1}), u_2, \ldots, u_l\}.$$

According to Claim 4.9, we have $l \leqslant r-2$. Let U be the set of all vertices $y \notin Y \cup \{x, y_{f+1}\}$ for which the edge $Y \cup \{x, y\}$ is of colour $f+1$. We have the following claim.

Claim 4.10. $|U| \geqslant n - r\binom{4r}{r-1}$.

To give a proof of Claim 4.10, we know that $\{y_i\}_{i=1}^{f}$ avoids the set of colours $\{1, 2, \ldots, f\}$. Hence, for each $1 \leqslant i \leqslant f$, the number of vertices $y \notin Y \cup \{x, y_{f+1}\}$ for which the edge $Y \cup \{x, y\}$ is of colour i is at most $\binom{4r}{r-1}$. On the other hand, $i \notin L^*(xy_i)$ for every $f+1 \leqslant i \leqslant r-1$, so for each $f+1 \leqslant i \leqslant r-1$, the number of vertices $y \notin Y \cup \{x, y_{f+1}\}$ for which the edge $Y \cup \{x, y\}$ is of colour i is at most $r-2$. Therefore we have $|U| \geqslant n - r\binom{4r}{r-1}$.

Let U be partitioned into $A_1, A_2, \ldots, A_{r-1}$, where $|A_{r-1}| = \lfloor n/2 \rfloor + 1$, $A_{f+1} = \emptyset$ and $||A_i| - |A_j|| \leqslant 1$ for every $1 \leqslant i, j \leqslant r-2$ with $i, j \neq f+1$. Based on Claim 4.10, we have

$$|A_i| \geqslant n/(2r) - \binom{4r}{r-1} - 1$$

for every $1 \leqslant i \leqslant r-2$. This fact will be used later on.

Now consider a graph Γ with vertex set $V(\Gamma) = V(H)$ and edge set $E(\Gamma) = \bigcup_{i=1}^{5} E_i$, where the sets E_i are defined as follows. Also, the sets F_i are defined and will be used later on.

We define E_1 as follows:

$$E_1 = \{uv \mid u \in B_i, i \neq f+1, v \notin Y \cup \{x, u\}, c(Y_i \cup \{x, u, v\}) = f+1\}. \tag{4.7}$$

For each $uv \in E_1$, we set $e_{uv} = Y_i \cup \{x, u, v\}$, where i is the minimum number such that $i \neq f+1$, $B_i \cap \{u, v\} \neq \emptyset$ and $c(Y_i \cup \{x, u, v\}) = f+1$. Now we let

$$F_1 = \{e_{uv} \mid uv \in E_1\}. \tag{4.8}$$

Note that for every $1 \leqslant i \leqslant r-1$ we have

$$B_i = \overline{U}_i \setminus ((\cup_{j>i} \overline{U}_j) \cup \{y_i\}),$$

and by Fact 4.8 we have $B_i \cap Y = \emptyset$. Therefore, in the subgraph of $G = S(H)$ induced by the edges E_1, the vertices Y are isolated vertices. Now we define the edges crossing the vertices Y.

We define E_2 as follows:

$$E_2 = \{y_i v \mid v \in A_i, 1 \leqslant i \leqslant r-1, i \neq f+1\}. \tag{4.9}$$

For each $y_i v \in E_2$ we set $e_{y_i v} = Y \cup \{x, v\}$. Also, we let

$$F_2 = \{e_{y_i v} \mid y_i v \in E_2\}. \tag{4.10}$$

Now we define new edges to increase the degrees of vertices in $\overline{U}_{f+1}$ with small degrees in the subgraph of $G = S(H)$ induced by the edges $E_1 \cup E_2$. In fact we define a set of new edges E_3 such that the degree of each vertex u_i for $1 \leqslant i \leqslant l$ in the subgraph of $G = S(H)$ with vertex set $V(H)$ and edge set $E_1 \cup E_2 \cup E_3$ is at least $2r+1$. To define E_3, we do the following. Let Γ_1 be the graph with vertex set $V(H)$ and edge set $E_1 \cup E_2$. For each $1 \leqslant i \leqslant l$ assume that $\overline{N}_i = Y \cup \overline{U}_{f+1} \cup N_{\Gamma_1}(u_i) \cup \{x\}$ and set $t_i = 0$ if $d_{\Gamma_1}(u_i) > 2r$ and $t_i = 2r + 1 - d_{\Gamma_1}(u_i)$ otherwise. Now we show that there are $\sum_{i=1}^{l} t_i$ distinct edges $e_{ij} \notin F_1 \cup F_2$ (where $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant t_i$) of colour $f+1$ with

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$u_i \in e_{ij}$ such that for each $1 \leqslant i \leqslant l$ there exist t_i distinct vertices $v_{ij} \in e_{ij} \setminus \overline{N}_i$. For this, set $r_{11} = 0$, $N_{11} = \overline{N}_1$ and $E_{11} = F_1 \cup F_2$ and repeat the following step for $i = 1, 2, \ldots, l$ if $t_i > 0$.

Step i. For each $1 \leqslant j \leqslant t_i$, since

$$d_{f+1}(u_i) > \binom{4r}{r-1} \geqslant \binom{|N_{ij}| - 1}{r-1} + r_{ij},$$

there is an edge $e_{ij} \notin E_{ij}$ of colour $f+1$ which contains u_i and a vertex $v_{ij} \in e_{ij} \setminus N_{ij}$. Note that since $\{y_i\}_{i=1}^{f}$ avoids $[f]$ and f is maximum subject to this property, we have $d_{f+1}(u_i) > \binom{4r}{r-1}$. Now set $r_{i(j+1)} = r_{ij} + 1$, $N_{i(j+1)} = N_{ij} \cup \{v_{ij}\}$ and $E_{i(j+1)} = E_{ij} \cup \{e_{ij}\}$ and continue the above procedure. We apply the above procedure t_i times to find the edges e_{ij} and the vertices v_{ij} for $1 \leqslant j \leqslant t_i$ with desired properties. Finally, let $r_{(i+1)1} = r_{i(t_i+1)}$, $N_{(i+1)1} = \overline{N}_{i+1}$ and $E_{(i+1)1} = E_{i(t_i+1)}$ and go to step $i+1$.

Clearly $E_{l(t_l+1)} \setminus E_{11}$ contains $\sum_{i=1}^{l} t_i$ distinct edges e_{ij} with desired properties. Now set

$$A = \bigcup_{i=1}^{l} \bigcup_{j=1}^{t_i} e_{ij}, \quad \overline{E}_i = \{u_i v_{ij} \mid 1 \leqslant j \leqslant t_i\}, \quad \overline{F}_i = \{e_{ij} \mid 1 \leqslant j \leqslant t_i\}, \quad E_3 = \bigcup_{i=1}^{l} \overline{E}_i, \quad F_3 = \bigcup_{i=1}^{l} \overline{F}_i. \tag{4.11}$$

The set of edges E_4 is defined in a more or less similar way. Here we define these edges to increase the degrees of vertices in $U_{\{1,2,\ldots,r-1\}}$ with small degrees in the subgraph of $G = S(H)$ induced by the edges $E_1 \cup E_2 \cup E_3$, where

$$U_{\{1,2,\ldots,r-1\}} = \bigcap_{i=1}^{r-1} U_i.$$

In fact we define a set of new edges E_4 such that the degree of each vertex in $U_{\{1,2,\ldots,r-1\}}$ in the subgraph of G with vertex set $V(H)$ and edge set $\bigcup_{i=1}^{4} E_i$ is at least $2r+1$. We will see this result in Fact 4.16. To define E_4, we do the following.

Assume that

$$U_{\{1,2,\ldots,r-1\}} = \{w_1, w_2, \ldots, w_m\} \quad \text{and} \quad d_{\Gamma_2}(w_1) \leqslant d_{\Gamma_2}(w_2) \leqslant \cdots \leqslant d_{\Gamma_2}(w_m),$$

where Γ_2 is the graph with vertex set $V(H)$ and edge set $\bigcup_{i=1}^{3} E_i$. For each $1 \leqslant i \leqslant r' = \min\{r, m\}$, set $t'_i = 0$ when $d_{\Gamma_2}(w_i) > 2r$. Otherwise set $t'_i = 2r + 1 - d_{\Gamma_2}(w_i)$. Also, set

$$N'_i = Y \cup \overline{U}_{f+1} \cup N_{\Gamma_2}(w_i) \cup \{x\}.$$

An argument similar to that used in the definition of E_3 shows that there are $\sum_{i=1}^{r'} t'_i$ distinct edges $e'_{ij} \notin F_1 \cup F_2 \cup F_3$ (where $1 \leqslant i \leqslant r'$ and $1 \leqslant j \leqslant t'_i$) of colour $f+1$ with $w_i \in e'_{ij}$ such that for each $1 \leqslant i \leqslant r'$ there exist t'_i distinct vertices $v'_{ij} \in e'_{ij} \setminus N'_i$. Now set

$$B = \bigcup_{i=1}^{r'} \bigcup_{j=1}^{t'_i} e'_{ij}, \quad E'_i = \{w_i v'_{ij} \mid 1 \leqslant j \leqslant t'_i\}, \quad F'_i = \{e'_{ij} \mid 1 \leqslant j \leqslant t'_i\}, \quad E_4 = \bigcup_{i=1}^{r'} E'_i, \quad F_4 = \bigcup_{i=1}^{r'} F'_i. \tag{4.12}$$

We define E_5 as follows:

$$E_5 = \{xv \mid v \in V(\Gamma) \setminus (Y \cup \overline{U}_{f+1} \cup A \cup B)\}. \tag{4.13}$$

In the following fact, using the above definitions, we see that the set of edges F_1, F_1, F_1, F_1 are pairwise disjoint.

Fact 4.11. For each $1 \leqslant i, j \leqslant 4$ and $i \neq j$, we have $F_i \cap F_j = \emptyset$.

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First we show that $F_1 \cap F_2 = \emptyset$. To the contrary assume that $f \in F_1 \cap F_2$. Since $f \in F_1$ from the definition of F_1 we have $f = e_{uv} = Y_i \cup \{x, u, v\}$, where $u \in B_i$, $i \neq f+1$, $v \notin Y \cup \{x, u\}$ and $c(Y_i \cup \{x, u, v\}) = f+1$. One can easily see that $y_i \notin f$. On the other hand, $f \in F_2$. Hence $f = e_{y_j z} = Y \cup \{x, z\}$ for some $z \in A_j$, where $1 \leqslant j \leqslant r-1$ and $j \neq f+1$. Hence $y_i \in Y \subseteq f$, a contradiction. Therefore $F_1 \cap F_2 = \emptyset$. Now, using the definition of E_3, we have $F_3 = \bigcup_{i=1}^{l} \overline{F}_i$ and $\overline{F}_i = \{e_{ij} \mid 1 \leqslant j \leqslant t_i\}$. On the other hand, $e_{ij} \notin F_1 \cup F_2$ for every $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant t_i$. Therefore $F_3 \cap (F_1 \cup F_2) = \emptyset$. Again, from the definition of E_4, we have $F_4 = \bigcup_{i=1}^{r'} F'_i$ and $F'_i = \{e'_{ij} \mid 1 \leqslant j \leqslant t'_i\}$. Moreover, $e'_{ij} \notin F_1 \cup F_2 \cup F_3$ for every $1 \leqslant i \leqslant r'$ and $1 \leqslant j \leqslant t'_i$. Therefore $F_4 \cap (F_1 \cup F_2 \cup F_3) = \emptyset$.

Claim 4.12. *The graph* Γ *is Hamiltonian.*

Proof of Claim 4.12. Assume that $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ are the degrees of the vertices of Γ. Our aim is to show that $d_1 > 2r$ and $d_{n-i} \geqslant n-i$ for each $2r-1 \leqslant i \leqslant n/2$. Then Lemma 2.3 will imply the existence of a Hamiltonian cycle in Γ. Now we give the following facts about the degrees of vertices of Γ.

Fact 4.13. $d_\Gamma(x) \geqslant n - 4r^3$.

To see Fact 4.13, note that using the definitions A and B (in the definitions of E_3 and E_4) and Claim 4.9 (which indicates $l \leqslant r-2$) and the fact $r' \leqslant r$, we have

$$|A| \leqslant r(t_1 + t_2 + \cdots + t_l) \leqslant r(2r+1)l \leqslant r(r-2)(2r+1)$$

and

$$|B| \leqslant r(t'_1 + t'_2 + \cdots + t'_{r'}) \leqslant r^2(2r+1).$$

Therefore

$$d_\Gamma(x) = n - |Y \cup \overline{U}_{f+1} \cup A \cup B| \geqslant n - 4r^3.$$

Fact 4.14. For each $1 \leqslant i \leqslant r-1$ with $i \neq f+1$ and each $u \in \overline{U}_i \setminus \{y_i\}$, we have

$$d_\Gamma(u) > n - r\binom{4r}{r-1}.$$

Moreover, for every $u \in \overline{U}_{f+1}$, we have $d_\Gamma(u) > 2r$.

To show Fact 4.14, note that Fact 4.7 implies that the set of vertices $Y_i \cup \{x\}$ avoids the set of colours $[r-1] \setminus \{i, f+1\}$. On the other hand $i \notin L^*(xu)$ for $u \in \overline{U}_i \setminus \{y_i\}$, and thus $(Y_i \cup \{x, u\})$ avoids all colours $[r-1] \setminus \{f+1\}$. Therefore, apart from at most $(r-2)\binom{4r}{r-1}$ choices of $v \in V(\Gamma) \setminus (Y \cup \{x, u\})$, we have $uv \in E_1$ and so $d_\Gamma(u) > n - r\binom{4r}{r-1}$. Moreover, for every $u_i \in \overline{U}_{f+1}$, we have $d_\Gamma(u_i) \geqslant d_{\Gamma_1}(u_i) + t_i > 2r$ (see the definition of E_3).

Fact 4.15. $d_\Gamma(y_{r-1}) > n/2$ and $d_\Gamma(y_i) > 2r$ for each $1 \leqslant i \leqslant r-1$ and $i \neq f+1$.

Fact 4.15 follows from the fact that $y_i v \in E(\Gamma)$ for each $v \in A_i$ and $|A_{r-1}| > n/2$ and $|A_i| > 2r$ for each $1 \leqslant i \leqslant r-1$ and $i \neq f+1$.

Fact 4.16. $d_\Gamma(u) > 2r$ for each $u \in U_{\{1,2,\ldots,(r-1)\}}$.

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To see Fact 4.16 assume that

$$U_{12\ldots(r-1)} = \{w_1, w_2, \ldots, w_m\} \neq \emptyset.$$

We claim that

$$\min\{d_\Gamma(w_i) \mid 1 \leqslant i \leqslant m\} > 2r.$$

First assume that $m \leqslant r$. According to the definition of E_4, for each $1 \leqslant i \leqslant m$ we have $d_\Gamma(w_i) \geqslant d_{\Gamma_2}(w_i) + t'_i > 2r$, where Γ_2 is the graph with vertex set $V(\Gamma)$ and edge set $\bigcup_{i=1}^{3} E_i$. Now let $m \geqslant r+1$, $|\overline{U}_{r-1} \setminus \{y_{r-1}\}| = k$ and

$$d_{\Gamma_2}(w_1) \leqslant d_{\Gamma_2}(w_2) \leqslant \cdots \leqslant d_{\Gamma_2}(w_m).$$

Again, according to the definition of the edges E_4, we have $d_\Gamma(w_i) > 2r$ for $1 \leqslant i \leqslant r$ and so it suffices to show that $d_\Gamma(w_{r+1}) \geqslant d_{\Gamma_2}(w_{r+1}) > 2r$. For $i = 1, \ldots, m$, consider

$$N_i = \{\{x, y_1, y_2, \ldots, y_{r-2}, v, w_i\} \setminus \{y_{f+1}\} \mid v \in \overline{U}_{r-1} \setminus \{y_{r-1}\}\}.$$

For every $1 \leqslant i \leqslant m$, suppose that n_i is the number of edges of colour $f+1$ in N_i. Clearly, for each $1 \leqslant i \leqslant m$, the edges of colour $f+1$ in N_i belong to F_1 and so $d_{\Gamma_2}(w_i) \geqslant n_i$. Moreover, the set $\{x, y_1, y_2, \ldots, y_{r-2}\} \setminus \{y_{f+1}\}$ avoids the colours $[r-1] \setminus \{f+1, r-1\}$ and $r-1 \notin L^*(xv)$ for each $v \in \overline{U}_{r-1} \setminus \{y_{r-1}\}$. Therefore, among all mk edges in $\bigcup_{i=1}^{m} N_i$, there are at most $\binom{4r}{r-1}$ edges of colour i for each $i \neq f+1, r-1$ and at most $(r-2)k$ edges of colour $r-1$. Thus

$$\sum_{i=1}^{m} n_i \geqslant (m-r+2)k - (r-3)\binom{4r}{r-1}.$$

If $d_{\Gamma_2}(w_{r+1}) \leqslant 2r$, then

$$\sum_{i=1}^{r+1} n_i \leqslant \sum_{i=1}^{r+1} d_{\Gamma_2}(w_i) \leqslant 2r(r+1).$$

Therefore

$$\sum_{i=r+2}^{m} n_i \geqslant (m-r+2)k - (r-3)\binom{4r}{r-1} - 2r(r+1) > (m-r+1)k,$$

which is impossible since $|\bigcup_{i=r+2}^{m} N_i| = (m-r-1)k$. Thus $d_\Gamma(w_{r+1}) \geqslant d_{\Gamma_2}(w_{r+1}) > 2r$ and consequently $d_\Gamma(w_i) > 2r$ for $r+1 \leqslant i \leqslant m$. On the other hand, according to the definition of Γ, we have $d_\Gamma(w_i) \geqslant d_{\Gamma_2}(w_i) + t'_i > 2r$ for each $1 \leqslant i \leqslant r$, and thus $\min\{d_\Gamma(w_i) \mid 1 \leqslant i \leqslant m\} > 2r$.

Clearly

$$V(H) = V(\Gamma) = (\cup_{i=1}^{r-1} \overline{U}_i) \cup \{y_i\}_{i=1}^{f} \cup U_{\{1,2,\ldots,(r-1)\}} \cup \{x\}.$$

Therefore Facts 4.13–4.16 imply that the minimum degree of Γ is greater than $2r$, so $d_1 > 2r$. Now we are going to show that $d_{n-i} \geqslant n-i$ for each $2r-1 \leqslant i \leqslant n/2$. To see this, first we show that most of the vertices of $\overline{U}_{r-1}$ have degree greater than $n-2r$ in Γ. For this, let D_i be the set of all edges of colour i containing the vertices of $Y_{r-1} \cup \{x\}$ for each $i \neq f+1, r-1$, and let

$$W = \bigcup_{i \neq f+1, r-1} \bigcup_{e \in D_i} (e \setminus (Y_{r-1} \cup \{x\})).$$

Using Fact 4.7, $Y_{r-1} \cup \{x\}$ avoids each colour $i \neq f+1, r-1$, so $|D_i| \leqslant \binom{4r}{r-1}$. On the other hand, for each $i \neq f+1, r-1$ and each $e \in D_i$ we have $|e \setminus (Y_{r-1} \cup \{x\})| = 2$, and thus $|W| \leqslant 2(r-3)\binom{4r}{r-1}$. For every $u \in \overline{U}_{r-1} \setminus (W \cup \{y_{r-1}\})$, $r-1 \notin L^*(xu)$, and we have $uv \in E_1$, apart from at most $r-2$ choices of $v \in V(\Gamma) \setminus (Y \cup \{x, u\})$. Moreover, for every $u \in \overline{U}_{r-1} \cap W \setminus \{y_{r-1}\}$, apart

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from at most $(r-2)\binom{4r}{r-1}$ choices of $v \in V(\Gamma) \setminus (Y \cup \{x, u\})$, we have $uv \in E_1$ and so $d_\Gamma(u) > n - r\binom{4r}{r-1}$. Hence we have the following fact.

Fact 4.17. $d_\Gamma(u) > n - 2r$, where $u \in \overline{U}_{r-1} \setminus (W \cup \{y_{r-1}\})$. Moreover, for each $u \in \overline{U}_{r-1} \cap W \setminus \{y_{r-1}\}$, we have $d_\Gamma(u) > n - r\binom{4r}{r-1}$.

By Fact 4.17, for each vertex $u \in \overline{U}_{r-1} \setminus (W \cup \{y_{r-1}\})$ we have $d_\Gamma(u) > n - 2r$. Moreover, since $|\overline{U}_{r-1}| \geqslant (n-1)/2$ and $|W| \leqslant 2(r-3)\binom{4r}{r-1}$, we have

$$|\overline{U}_{r-1} \setminus (W \cup \{y_{r-1}\})| \geqslant \frac{n-3}{2} - 2(r-3)\binom{4r}{r-1},$$

and hence at least

$$\left\lceil \frac{n-3}{2} \right\rceil - 2(r-3)\binom{4r}{r-1}$$

vertices of Γ have degree greater than $n - 2r$. This means that

$$d_i > n - 2r \quad \text{for } i \geqslant \left\lfloor \frac{n+5}{2} \right\rfloor + 2(r-3)\binom{4r}{r-1}. \tag{4.14}$$

Fact 4.14 implies that for each $1 \leqslant i \leqslant r-1$ and $i \neq f+1$ and for every $u \in \overline{U}_i \setminus \{y_i\}$, we have $d_\Gamma(u) > n - r\binom{4r}{r-1}$. Now, using Fact 4.13, we have $d_\Gamma(x) \geqslant n - 4r^3$. On the other hand, $|\overline{U}_{r-1}| \geqslant (n-1)/2$ and $n - 4r^3 > n - r\binom{4r}{r-1}$, and thus at least $\lceil (n-1)/2 \rceil$ vertices of Γ have degree greater than $n - r\binom{4r}{r-1}$. This means that

$$d_i > n - r\binom{4r}{r-1} \quad \text{for } i \geqslant \left\lfloor \frac{n+3}{2} \right\rfloor. \tag{4.15}$$

Now, using Fact 4.15, we have $d_\Gamma(y_{r-1}) > n/2$. Therefore we have

$$d_i > n/2 \quad \text{for } i \geqslant \left\lfloor \frac{n+1}{2} \right\rfloor. \tag{4.16}$$

Since $n > 6r\binom{4r}{r-1}$, using (4.14), (4.15) and (4.16) we conclude that $d_{n-i} \geqslant n - i$ for each $2r - 1 \leqslant i \leqslant n/2$. Moreover, $d_1 > 2r$. Now, Lemma 2.3 implies the existence of a Hamiltonian cycle in Γ. □

Claim 4.18. *There is a monochromatic Hamiltonian Berge-cycle of colour $f+1$ in H.*

Proof of Claim 4.18. We show that every Hamiltonian cycle in Γ can be extended to a monochromatic Hamiltonian Berge-cycle of colour $f+1$ in H. Suppose that $v_1, v_2, \ldots, v_n = x$ are the vertices of a Hamiltonian cycle C in Γ. Now, for $i = 1, 2, \ldots, n$, we define the edges $g_i \in E(H)$ of colour $f+1$ one by one (in the same order as their subscripts appear), so that $\{v_i, v_{i+1}\} \subseteq g_i$ and $g_1, g_2, \ldots, g_n$ form a Hamiltonian Berge-cycle with the core vertices $v_1, v_2, \ldots, v_n$. First we repeat the following step for $i = 1, 2, \ldots, n-2$ to define the edges $g_1, g_2, \ldots, g_{n-2}$.

Step i. If $v_i v_{i+1} \in E_j$ for some $j \in \{1, 2\}$, then set $g_i = e_{v_i v_{i+1}} \in F_j$. Let $g_i = e_{kj} \in F_3$ if $\{v_i, v_{i+1}\} = \{u_k, v_{kj}\}$ and $u_k v_{kj} \in E_3$, where $k \in \{1, 2, \ldots, l\}$ and $1 \leqslant j \leqslant t_k$. Finally, let $g_i = e'_{kj} \in F_4$ if $\{v_i, v_{i+1}\} = \{w_k, v'_{kj}\}$ and $w_k v'_{kj} \in E_4$, where $k \in \{1, 2, \ldots, r'\}$ and $1 \leqslant j \leqslant t'_k$. Then go to step $i+1$.

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According to the definitions of F_1, F_2, F_3 and F_4, for each $1 \leqslant i \leqslant n-2$ the edge $g_i \in \bigcup_{i=1}^{4} F_i$ is of colour $f+1$ and $\{v_i, v_{i+1}\} \subseteq g_i$. Now we claim that $g_i \neq g_j$ for every $i \neq j$ with $1 \leqslant i, j \leqslant n-2$. It suffices to prove the following fact.

Fact 4.19. For each $1 \leqslant i \leqslant n-2$ and $1 \leqslant j < i$, we have $g_i \neq g_j$.

Proof of Claim 4.19. Assume that $g_i \in F_{r_i}$ and $g_j \in F_{r_j}$, where $r_i, r_j \in \{1, 2, 3, 4\}$. Using Fact 4.11, $F_{r_i} \cap F_{r_j} = \emptyset$ if $r_i \neq r_j$. Hence $g_i \neq g_j$ when $r_i \neq r_j$. Therefore we may assume that $r_i = r_j$. First assume that $j = i-1$. We divide our proof of this case into some subcases.

Subcase 1. First let $r_{i-1} = r_i = 1$. Then

$$g_{i-1} = e_{v_{i-1}v_i} = Y_p \cup \{x, v_{i-1}, v_i\} \quad \text{and} \quad g_i = e_{v_i v_{i+1}} = Y_q \cup \{x, v_i, v_{i+1}\},$$

where p, q are the minimum numbers such that $p, q \neq f+1$,

$$B_p \cap \{v_{i-1}, v_i\} \neq \emptyset, \quad B_q \cap \{v_i, v_{i+1}\} \neq \emptyset$$

and

$$c(Y_p \cup \{x, v_{i-1}, v_i\}) = c(Y_q \cup \{x, v_i, v_{i+1}\}) = f+1.$$

One can easily see that $\{v_{i-1}, v_i\} \nsubseteq g_i$ and thus $g_i \neq g_{i-1}$.

Subcase 2. Now let $r_{i-1} = r_i = 2$. Then $\{v_{i-1}, v_i\} = \{y_t, v\}$ for some $1 \leqslant t \leqslant r-1$, $t \neq f+1$, $v \in A_t$ and $g_{i-1} = e_{v_{i-1}v_i} = e_{y_t v} = Y \cup \{x, v\}$. Since $A_p \cap A_q = \emptyset$ for $p \neq q$ and $r_i = 2$, we have $v_i = y_t$, $v_{i-1}, v_{i+1} \in A_t$ and $g_i = e_{v_i v_{i+1}} = e_{y_t v_{i+1}} = Y \cup \{x, v_{i+1}\}$. Clearly $v_{i+1} \notin g_{i-1}$ and thus $g_i \neq g_{i-1}$.

Subcase 3. Now let $r_{i-1} = r_i = 3$. Then, by the definitions of E_3 and F_3 (see (4.11)), we have $g_{i-1} = e_{k_1 j_1} \in F_3$ and $g_i = e_{k_2 j_2} \in F_3$, where $\{v_{i-1}, v_i\} = \{u_{k_1}, v_{k_1 j_1}\}$ and $\{v_i, v_{i+1}\} = \{u_{k_2}, v_{k_2 j_2}\}$ for some $k_1, k_2 \in \{1, 2, \ldots, l\}$, $1 \leqslant j_1 \leqslant t_{k_1}$ and $1 \leqslant j_2 \leqslant t_{k_2}$. Now assume to the contrary that $g_{i-1} = g_i$. Using the definitions of E_3 and F_3, we have $v_{k_1 j_1}, v_{k_2 j_2} \notin \overline{U}_{f+1}$ and so $v_i = u_{k_1} = u_{k_2}$, $k_1 = k_2$, $v_{i-1} = v_{k_1 j_1}$ and $v_{i+1} = v_{k_2 j_2}$. On the other hand $v_{i-1} \neq v_{i+1}$, and thus $j_1 \neq j_2$. Hence, from the definition of F_3, we have $e_{k_1 j_1} \neq e_{k_1 j_2}$ and thus $g_{i-1} \neq g_i$, a contradiction to our assumption.

Subcase 4. Finally, let $r_{i-1} = r_i = 4$, and then using the definitions of E_4 and F_4 (see (4.12)) we have $g_{i-1} = e'_{k_1 j_1} \in F_4$ and $g_i = e'_{k_2 j_2} \in F_4$, where $\{v_{i-1}, v_i\} = \{w_{k_1}, v'_{k_1 j_1}\}$ and $\{v_i, v_{i+1}\} = \{w_{k_2}, v'_{k_2 j_2}\}$ for some $k_1, k_2 \in \{1, 2, \ldots, r'\}$, $1 \leqslant j_1 \leqslant t'_{k_1}$ and $1 \leqslant j_2 \leqslant t'_{k_2}$. With the same argument we can see that $k_1 \neq k_2$ or $j_1 \neq j_2$. Therefore, from the definition of F_4, we have $e'_{k_1 j_1} \neq e'_{k_1 j_2}$ and thus $g_{i-1} \neq g_i$.

Now assume $j \leqslant i-2$. In this case, by the definitions of F_1, F_2, F_3 and F_4, one can easily see that $\{v_i, v_{i+1}\} \nsubseteq g_j$ or $\{v_j, v_{j+1}\} \nsubseteq g_i$ and so again $g_i \neq g_j$. This completes the proof of Claim 4.19. □

Now we are going to give the definitions of g_{n-1} and g_n with desired properties. First let $i = n-1$. Since $\{v_{n-1}, x\}$ has been used in at most one of the edges g_i, with $1 \leqslant i \leqslant n-2$ (only possibly in g_{n-2}) and $f+1 \in L^*(v_{n-1}x)$, we can choose an appropriate edge g_{n-1} of colour $f+1$, where $g_{n-1} \neq g_i$ for each $1 \leqslant i \leqslant n-2$. Similarly, for $i = n$, since $\{x, v_1\}$ has been used in at most two edges g_i, with $1 \leqslant i \leqslant n-1$ (only possibly in g_1 and g_{n-1}) and $f+1 \in L^*(xv_1)$, then we can choose an appropriate edge g_n of colour $f+1$, where $g_n \neq g_i$ for each $1 \leqslant i \leqslant n-1$. This completes the proof of Claim 4.18. □

This finishes the proof of Theorem 1.2. □

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