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Characterizing asymptotic randomization in abelian cellular automata

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Abstract. Abelian cellular automata (CAs) are CAs which are group endomorphisms of the full group shift when endowing the alphabet with an abelian group structure. A CA randomizes an initial probability measure if its iterated images have weak*-convergence towards the uniform Bernoulli measure (the Haar measure in this setting). We are interested in structural phenomena, i.e., randomization for a wide class of initial measures (under some mixing hypotheses). First, we prove that an abelian CA randomizes in Cesàro mean if and only if it has no soliton, i.e., a non-zero finite configuration whose time evolution remains bounded in space. This characterization generalizes previously known sufficient conditions for abelian CAs with scalar or commuting coefficients. Second, we exhibit examples of strong randomizers, i.e., abelian CAs randomizing in simple convergence; this is the first proof of this behaviour to our knowledge. We show, however, that no CA with commuting coefficients can be strongly randomizing. Finally, we show that some abelian CAs achieve partial randomization without being randomizing: the distribution of short finite words tends to the uniform distribution up to some threshold, but this convergence fails for larger words. Again this phenomenon cannot happen for abelian CAs with commuting coefficients.

Key words: cellular automata, classical ergodic theory, symbolic dynamics
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1. *Introduction*

Cellular automata (CAs), although extremely simple to define, provide a rich source of examples of dynamical systems which are not yet well understood. This is particularly true when taking a measure theoretic point of view and studying the evolution of a probability measure under iterations of CAs. The situation can be roughly depicted as follows: for non-surjective CAs, essentially all behaviours that are not prohibited by immediate computability restrictions can happen [**1**, **5**, **12**]; for the surjective case, various forms of rigidity are observed (see [**21**] for an overview). In particular, since the pioneering work of Lind and Miyamoto on the 'addition modulo 2' CA [**16**, **19**], many CAs of algebraic origin were shown to behave like randomizers [**7**, **13**, **18**, **22**, **23**], i.e., they converge in Cesàro mean or in density to the uniform Bernoulli measure from any initial probability measure from a large class $\mathcal{C}$. In [**16**, **19**], the class $\mathcal{C}$ is Bernoulli measures of full support. It was later extended to full support Markov measures or n-step Markov processes, measures with complete connections and summable decay of correlations and harmonically mixing measures [**21**] and more [**24**, **28**]. Apart from specific examples (as in [**17**]), the class of CAs where randomizing behaviour has been shown is essentially contained in that of 'linear' CAs defined on an abelian group alphabet by

$$F(x)_i = \sum_{j \in V} \theta_j(x_{i+j}),$$

where θ_j are *commuting* endomorphisms (most of the time automorphisms or scalar coefficients). Furthermore, the type of convergence considered has always been Cesàro mean or convergence in density.

In this paper, we consider the class of harmonically mixing measures and the class of abelian CAs that are like the 'linear' CAs described above but *without* the assumption of commutation of endomorphisms. Our first main result is a complete characterization of randomization in density in that setting.

THEOREM. (See Theorem 2) *An abelian CA F randomizes in density any harmonically mixing measure if and only if it does not possess a soliton, i.e., a non-zero finite configuration whose set of non-zero cells stays within a bounded diameter under iterations of the CA.*

We show that this theorem extends the most general previous result [**23**] and allows us to easily prove randomization for particular examples, even in the setting of non-commutative coefficients [**17**], and hence answers a question of [**21**].

Our approach uses tools from harmonic analysis using a similar approach to the work of Pivato and Yassawi on diffusion of characters [**22**, **23**]. We rely on the abelian structure of the considered CA to reduce randomization to a combinatorial property of diffusivity. More precisely, we define a dual CA on the (Pontryagin) dual group and show that diffusion in the dual is equivalent to randomization and that the diffusion property is preserved by duality. Finally, we prove the equivalence between diffusivity and the absence of solitons, not by using the abelian structure but by general combinatorial properties of surjective CAs. This allows us to go beyond the commuting coefficient case, which was treated in previous works by a careful analysis of binomial coefficients of the iterates of F.

We also prove the existence of a stronger form of randomization where taking subsequences of density one or Cesàro mean is not necessary.

THEOREM. (See Theorem 4) *There exist abelian CAs that randomize in simple convergence any harmonically mixing measure.*

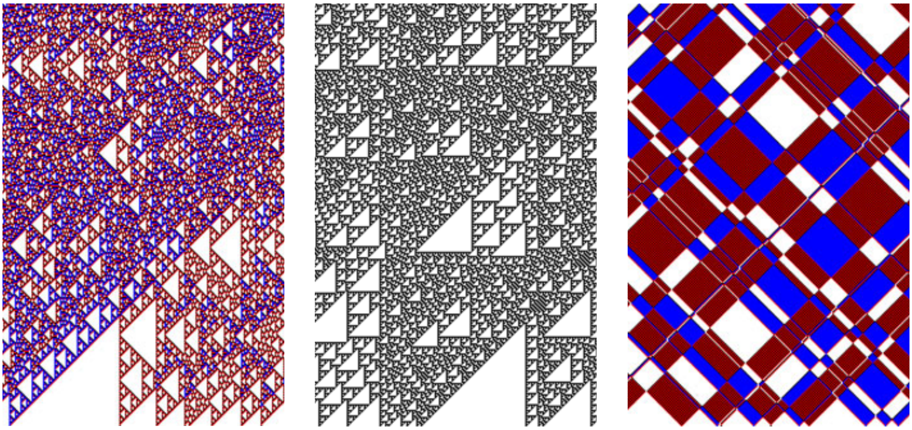

FIGURE 1. Three forms of randomization: F_2 (defined in §6.2), addition modulo 2 and $I_{\mathbb{Z}_2}$ (defined in §6.3). The direction of time is upward. The CAs are iterated on an initial configuration drawn according to a Bernoulli measure with 95% white (state 0). The addition modulo 2 does not converge directly because the image measure is far from the uniform measure around times $t = 2^n$ (see Theorem 3). $I_{\mathbb{Z}_2}$ randomizes individual cells but not cylinders of length 2 (see Proposition 8).

This answers a question of [**14**, Question 59]. Experiments on small surjective CAs [**11**, **29**] suggest that this strong form of randomization is the most common, that it occurs as well on non-abelian CAs, and even that randomization occurring only in density (or Cesàro mean) might be an artefact of abelian CAs. This confirms the importance of the non-commutative coefficients case, since we also prove that abelian CAs with commuting coefficients cannot achieve such a strong form of randomization.

The results above are stated as randomization for the class of harmonically mixing measures. We do not investigate when there are more randomized measures (as in [**24**]). However, we show that there cannot be fewer: if an abelian CA randomizes in density full-support Bernoulli measures, then it randomizes in density all harmonically mixing measures. Interestingly, the rigidity is even stronger for abelian CAs with commutative coefficients: we prove that if the frequency of individual states is randomized (in density), then the CA is fully randomizing in density. In the case of non-commutative coefficients, we can have partial randomization: we give examples for any K of abelian CAs which do randomize all cylinders up to size K but fail to randomize completely. This suggests that experimental work on randomization in general CAs should be done with care: randomization might fail in non-obvious ways on long-range correlations.

The paper is organized as follows. In §2, we recall basic definitions and tools about measure theoretic aspects of cellular automata; in §3, we study the evolution of ranks of characters under iterations of abelian CAs, the property of character diffusion and its link with randomization; in §4, we define the dual of an abelian CA, show that duality preserves diffusivity in density and link this property with the absence of solitons; in §5, we establish our main result, which is a characterization of randomization in density through the absence of solitons; in §6, we exhibit a class of examples of strong randomization and

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randomization up to a fixed-length cylinder, and we show that these behaviours are specific to CAs with non-commuting coefficients; and, finally, in §7 we give some directions for further research on this topic.

2. *Definitions and tools*

Throughout this paper, we will state our results for dimension one, but they extend straightforwardly to the d-dimensional case. Our convention on natural number is $0 \in \mathbb{N}$.

Let $\mathcal{A}$ be a finite alphabet. We define $\mathcal{A}^* = \bigcup_{n\in\mathbb{N}} \mathcal{A}^n$ to be the set of finite *words* and $\mathcal{A}^{\mathbb{Z}}$ to be the set of (one-dimensional) *configurations*. For a finite set $U \subset \mathbb{Z}$ and $u \in \mathcal{A}^U$, define the *cylinder*

$$[u]_U = \{x \in \mathcal{A}^{\mathbb{Z}} : x|_U = u\}.$$

For $u \in \mathcal{A}^n$ and $k \in \mathbb{Z}$, also define $[u]_k = [u]_{\{k,\ldots,k+n-1\}}$ and $[u] = [u]_0$.

We endow $\mathcal{A}^{\mathbb{Z}}$ with the product topology, which is metrizable using the *Cantor distance*: i.e.,

$$\text{for all } x, y \in \mathcal{A}^{\mathbb{Z}}, d(x, y) = 2^{-\Delta(x,y)} \quad \text{where } \Delta(x, y) = \min\{|i| : x_i \neq y_i\}.$$

The *shift map* is defined by

$$\text{for all } x \in \mathcal{A}^{\mathbb{Z}}, \quad \sigma(x) = (x_{i+1})_{i\in\mathbb{Z}}.$$

A *CA* is a pair $(\mathcal{A}, F)$, where $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a continuous function that commutes with the shift map (i.e., $F \circ \sigma = \sigma \circ F$). Equivalently, F is defined by a finite neighborhood $\mathcal{N} \subset \mathbb{Z}$ and a *local rule* $f : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}$ in the sense that

$$\text{for all } x \in \mathcal{A}^{\mathbb{Z}} \quad \text{and} \quad \text{for all } i \in \mathbb{Z}, \quad F(x)_i = f(x_{i+\mathcal{N}}).$$

Let $(\mathcal{A}, +)$ be an abelian group and let 0 its neutral element. A *finite configuration* is a configuration $x \in \mathcal{A}^{\mathbb{Z}}$ such that $x(i) = 0$ for all $i \in \mathbb{Z}$ except on a finite set. If x is a finite configuration, we define its *support* by $\mathrm{supp}(x) = \{i \in \mathbb{Z} : x(i) \neq 0\}$ and its *rank* by $\mathrm{rank}(x) = |\mathrm{supp}(x)|$. Note that the set of finite configurations is dense in $\mathcal{A}^{\mathbb{Z}}$. The notion of finite points makes sense also when $\mathcal{A}$ has no group structure, assuming that a choice of zero element $0 \in \mathcal{A}$ has been made.

An *abelian CA* F is a CA that is an endomorphism for $(\mathcal{A}^{\mathbb{Z}}, +)$ (componentwise addition): i.e.,

$$\text{for all } x, y \in \mathcal{A}^{\mathbb{Z}}, \quad F(x + y) = F(x) + F(y).$$

Equivalently, F is a finite sum of shifts composed of endomorphisms of $(\mathcal{A}, +)$. More precisely, there is a finite $\mathcal{N} \subset \mathbb{Z}$ and a collection $(\phi_i)_{i\in\mathcal{N}}$ of endomorphisms of $(\mathcal{A}, +)$ such that

$$F = \sum_{i\in\mathcal{N}} \overline{\phi}_i \circ \sigma^i \quad \text{where } \overline{\phi}_i : \begin{array}{ccc} \mathcal{A}^{\mathbb{Z}} & \to & \mathcal{A}^{\mathbb{Z}} \\ x & \mapsto & (\phi_i(x(j)))_j. \end{array}$$

Note that the image of a finite configuration is always a finite configuration. In particular, $0 \in \mathcal{A}$ is a *quiescent state*, meaning that $F(0^{\mathbb{Z}}) = 0^{\mathbb{Z}}$, where $0^{\mathbb{Z}}$ denotes the constant-0 configuration.

We also define addition on abelian CAs by $F + F' : x \mapsto F(x) + F'(x)$.

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We say that F has *commuting endomorphisms* if the endomorphisms ϕ_i commute pairwise. Supposing $\mathcal{A}$ to be a vector space over a finite field $\mathbb{F}_p$ turns out to be a source of simple yet illustrative examples. As said above, we are particularly interested in the non-commuting case. We will illustrate our results with the two representatives F_2 and H_2 defined over $\mathcal{A} = \mathbb{F}_2^2$ by

$$F_2(x)_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot x_i + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot x_{i+1},$$

$$H_2(x)_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot x_{i-1} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot x_i + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot x_{i+1},$$

where elements of $\mathcal{A}$ are seen as vectors and matrix notation is used to denote endomorphisms of $\mathcal{A}$.

2.1. *CA acting on probability measures.* Let $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ be the space of probability measures on the Borel sigma-algebra of $\mathcal{A}^{\mathbb{Z}}$. In particular, we consider $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$, the subset of all σ-invariant measures. Here are a few examples that we mention throughout the paper.

- *Bernoulli measure.* Take $(\beta_i)_{i\in\mathcal{A}} \in [0, 1]^{\mathcal{A}}$ such that $\sum_i \beta_i = 1$. Let β be the usual Bernoulli measure of parameters (β_i) on $\mathcal{A}$. The *Bernoulli measure* of parameters (β_i) on $\mathcal{A}^{\mathbb{Z}}$ is defined as $\mu = \otimes_{\mathbb{Z}}\beta$: that is, each cell is drawn in an independent and identically distributed manner and distributed as a Bernoulli measure. In other words,

$$\text{for all } u \in \mathcal{A}^*,\ \mu([u]) = \prod_{0\le i<|u|} \beta_{u_i}.$$

  A particularly important example is the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$, denoted by λ, which is the Bernoulli measure of parameters $(1/|\mathcal{A}|)_{i\in\mathcal{A}}$.
- *Markov measure.* Let $(p_{i,j})_{i,j\in\mathcal{A}}$ be a non-negative matrix satisfying $\sum_j p_{ij} = 1$ for all i, and let $(\mu_i)_{i\in\mathcal{A}}$ be an eigenvector associated with the eigenvalue 1 (the choice being unique if the matrix is irreducible). The associated *two-step Markov measure* is defined as

$$\text{for all } u \in \mathcal{A}^*,\ \mu([u]) = \mu_{u_0} \prod_{0\le i<|u|} p_{u_i u_{i+1}}.$$

  This can be extended to n-step Markov measures.

The weak-* topology on $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ is metrizable. A possible metric is given by the distance

$$d_{\mathcal{M}}(\mu, \nu) = \sum_{k\in\mathbb{N}} \frac{1}{2^k} \max_{u\in\mathcal{A}^{2k+1}} |\mu([u]_{-k}) - \nu([u]_{-k})|.$$

A CA $(\mathcal{A}, F)$ yields a continuous action on the space of probability measures $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$.

$$\text{For any Borel set } U,\ F\mu(U) = \mu(F^{-1}U).$$

Given a subset $X \subseteq \mathbb{N}$ of natural numbers, its *lower density* is defined as $\liminf_n (|\{i \in X, i \le n\}|/n)$. We prove a diagonalization lemma for sequences of lower density one, stating, informally, that the intersection of countably many sequences of lower density one 'eventually has lower density one'.

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LEMMA 1. *Suppose that I is countable and, for all $n \in I$, $N_n \subset \mathbb{N}$ is a set with lower density one. Then there there exists a set $N \subset \mathbb{N}$ of lower density one such that, for all $n \in I$, $N \cap [k, \infty) \subset N_n \cap [k, \infty)$ for all large enough k.*

Proof. We can assume that $I = \mathbb{N}$. The intersection of finitely many sets of lower density one has lower density one. Thus, we may assume that the N_n form a decreasing sequence (with respect to inclusion) by replacing each N_n by $N_0 \cap N_1 \cap \cdots \cap N_n$.

Now let $m_0 = 0$ and pick an increasing sequence of natural numbers $(m_n)_{n \geq 1}$ such that $|N_n \cap [0, m)| \geq m(1 - 1/n)$ for all $m \geq m_n$, using the fact that N_n has lower density one. Define $N \cap [m_{n-1}, m_n) = N_n \cap [m_{n-1}, m_n)$ for all n. Then

$$\begin{aligned} |N \cap [0, m)| &= |N_1 \cap [0, m_1)| + \cdots + |N_\ell \cap [m_{\ell-1}, m_\ell)| + |N_{\ell+1} \cap [m_\ell, m)| \\ &\geq |N_\ell \cap [0, m)| \\ &\geq m(1 - 1/\ell), \end{aligned}$$

where ℓ is maximal such that $m_\ell < m$, and where the first inequality follows because the N_n form a decreasing sequence under inclusion. □

Considering the iterated action of a CA on an initial measure μ, we distinguish various forms of convergence.

- $(F^t\mu)_{t\in\mathbb{N}}$ *converges* to ν if $F^t\mu \to \nu$ (for the weak-$*$ convergence); equivalently, $F^t\mu([u]) \to \nu([u])$ for every finite word u.
- $(F^t\mu)_{t\in\mathbb{N}}$ *converges in Cesàro mean* to ν if $(1/T)\sum_{t=0}^{T-1} F^t\mu \to \nu$; equivalently, if $(1/T)\sum_{t=0}^{T-1} F^t\mu([u]) \to \nu([u])$ for every finite word u.
- $(F^t\mu)_{t\in\mathbb{N}}$ *converges in density* to ν if there exists an increasing sequence $(\varphi(t))_{t\in\mathbb{N}}$ of lower density one such that $F^{\varphi(t)}\mu \to \nu$; equivalently by Lemma 1, if for every finite word u there exists an increasing sequence $(\varphi_u(t))_{t\in\mathbb{N}}$ of lower density one such that $F^{\varphi_u(t)}\mu([u]) \to \nu([u])$.
- $(F^t\mu)_{t\in\mathbb{N}}$ *converges on cylinders of support* $\subset \mathbb{U}$ to ν if $\mu(\cdot \mid \mathfrak{B}_{\mathbb{U}}) \to \nu(\cdot \mid \mathfrak{B}_{\mathbb{U}})$, where $\mathfrak{B}_{\mathbb{U}}$ is the Borel σ-algebra generated by the cylinders of support $\subset \mathbb{U}$ (this can be seen as convergence of measures of $\mathcal{M}(\mathcal{A}^{\mathbb{U}})$); equivalently, $F^t\mu([u]) \to \nu([u])$ for every word u with $\mathrm{supp}(u) \subset \mathbb{U}$.

Recall that, in a context where the alphabet is $\mathcal{A}$, λ is the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$.

Definition 1. (Randomization) Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a CA and let $\mathcal{M} \subset \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ be a class of initial measures.

F strongly randomizes $\mathcal{M}$ (respectively, in Cesàro mean, in density, on cylinders of support $\mathbb{U}$) if, for all $\mu \in \mathcal{M}$, $(F^t\mu)_{t\in\mathbb{N}}$ converges to λ (respectively, in Cesàro mean, in density, on cylinders of support $\mathbb{U}$).

PROPOSITION 1. *Strong randomization implies all other forms of randomization, and randomization in Cesàro mean is equivalent to randomization in density for σ-invariant measures.*

Proof. The first point is clear. The second point stems from the fact that the uniform Bernoulli measure λ is an extremal point of $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and that $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is compact.

We prove this point by contraposition. Assume that $(F^t\mu)_{t\in\mathbb{N}}$ does not converge to λ in density. Then there must exist some $\varepsilon > 0$ and some sequence $(\varphi(t))$ of upper density $\alpha > 0$ such that $F^{\varphi(t)}\mu \notin B(\lambda, \varepsilon)$, where $B(\lambda, \varepsilon)$ is the open ball of radius ε centered on λ (otherwise, for any $n > 0$, the set of times t with $F^t\mu \in B(\lambda, 1/n)$ would be of density one and, by Lemma 1, $(F^t\mu)_{t\in\mathbb{N}}$ would converge to λ in density). Therefore there exists a sequence of times $(T_i)_{i\in\mathbb{N}}$ such that $T_i/\varphi(T_i) \to \alpha$. Then

$$\frac{1}{\varphi(T_i)+1}\sum_{t=0}^{\varphi(T_i)} F^t\mu = \frac{1}{\varphi(T_i)+1}\sum_{t=0}^{T_i} F^{\varphi(t)}\mu + \frac{1}{\varphi(T_i)+1}\sum_{\substack{t=0\\ t\notin\varphi(\mathbb{N})}}^{\varphi(T_i)} F^t\mu.$$

Let $\mathcal{C}$ be the convex hull of $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})\backslash B(\lambda, \varepsilon)$. By compactness, this sequence admits accumulation points that must be of the form $\alpha\nu + (1-\alpha)\eta$ for some $\nu \in \mathcal{C}$ and $\eta \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$. However, since λ is extremal in $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$, $\lambda \notin \mathcal{C}$, so that $\lambda \neq \nu$ and $\lambda \neq \alpha\nu + (1-\alpha)\eta$. In other words, the sequence $((1/(T+1))\sum F^t\mu)_{t\in\mathbb{N}}$ admits some accumulation point which is not λ, and the proof is complete. □

2.2. *Fourier theory.*

Definition 2. (Character) A *character* of a topological group G is a continuous group homomorphism $G \to \mathbb{T}^1$, where $\mathbb{T}^1$ is the unit circle group (under multiplication). Denote by $\widehat{G}$ the group of characters of G under elementwise multiplication.

The following result is well known (see, e.g., [**4**, Lemma 4.1.3]).

PROPOSITION 2. *Any finite abelian group G is isomorphic to its dual $\widehat{G}$.*

If $\mathcal{A}$ is a finite abelian group, $\widehat{\mathcal{A}^\mathbb{Z}}$ is in bijective correspondance with the sequences of $(\widehat{\mathcal{A}})^\mathbb{Z}$ whose elements are all **1** except for a finite number. That is, $\chi \in \widehat{\mathcal{A}^\mathbb{Z}}$ can be written as $\chi(x) = \prod_{k\in\mathbb{Z}} \chi_k(x_k)$, where all but finitely many elements are equal to one. In this context, we call the elements of $\widehat{\mathcal{A}}$ *elementary characters*.

Definition 3. Let $\chi \in \widehat{\mathcal{A}^\mathbb{Z}}$ and $(\chi_i)_{i\in\mathbb{Z}}$ be its decomposition in elementary characters. The *support* of χ is $\mathrm{supp}(\chi) = \{i \in \mathbb{Z} : \chi_i \neq 1\}$. Its *rank* is $\mathrm{rank}(\chi) = |\mathrm{supp}(\chi)|$.

Definition 4. (Fourier coefficients, or Fourier–Stieltjes transform) The *Fourier coefficients* of a measure $\mu \in \mathcal{M}(\mathcal{A}^\mathbb{Z})$ are given by

$$\widehat{\mu}[\chi] = \int_{\mathcal{A}^\mathbb{Z}} \chi\, d\mu$$

for all characters $\chi \in \widehat{\mathcal{A}^\mathbb{Z}}$. For a character $\chi = \prod_{k\in S}\chi_k$ (where $S = \mathrm{supp}(\chi)$), this can be rewritten as a finite sum

$$\widehat{\mu}[\chi] = \sum_{u\in\mathcal{A}^S}\prod_{k\in S}\chi_k(u_k)\cdot\mu([u]_S).$$

The Fourier coefficients of μ completely characterize it. They also behave well with regard to convergence in (weak-*) topology.

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THEOREM 1. (Lévy's continuity theorem) *Let G be a locally compact abelian group and let $\mu_1, \mu_2, \ldots \mu_\infty \in \mathcal{M}(G)$. Then*

$$\mu_n \to \mu_\infty \quad \text{in the weak-* topology} \quad \iff \quad \text{for all } \chi \in \widehat{G}, \quad \widehat{\mu_n}[\chi] \to \widehat{\mu_\infty}[\chi].$$

This theorem was first introduced in [**15**] (in French). It has been extended to locally compact abelian groups in [**20**], and to more general settings which are out of the scope of this article.

Definition 5. (Harmonically mixing measure) $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ is harmonically mixing if, for all $\varepsilon > 0$, there exists $R > 0$ such that $\text{rank}(\chi) > R \implies \widehat{\mu}[\chi] < \varepsilon$.

Throughout this paper, we sometimes omit to specify the class of initial measures, which is always the class of harmonically mixing measures.

PROPOSITION 3. *Let $\mathcal{A}$ be any finite abelian group. Any Bernoulli or (n-step) Markov measure on $\mathcal{A}^{\mathbb{Z}}$ with non-zero parameters is harmonically mixing.*

This is in [**22**, Propositions 6 and 8 and Corollary 10].

3. *Character-diffusivity*

Let $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$ and let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be an abelian CA. Then $\chi \circ F$ is a character (composition of continuous group homomorphisms). One of the central ideas introduced in [**22**] is to focus on the evolution of the rank of characters under the action of F in order to establish randomization in density of harmonically mixing measures. They introduce the following notion of diffusivity over characters. We call it character-diffusivity to clearly distinguish it from the notion of diffusivity that we will introduce later.

Definition 6. (Character diffusion) Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be an abelian CA. We say that F *strongly diffuses* a character $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$ if $\text{rank}(\chi \circ F^t) \to \infty$ and that it *diffuses* χ *in density* if the convergence occurs along an increasing sequence of times of lower density one. We say that F is *strongly character-diffusive* if it strongly diffuses every non-trivial character, and we define *character-diffusivity in density* analogously.

Definition 7. A measure μ is *strongly non-uniform* if $\mu[\chi] \neq 0$ for all characters $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$.

Example 1. A Bernoulli measure $\mu = \otimes\beta$ whose parameters are all equal except one is strongly non-uniform. Assume that $\beta(a) = c$ for all $a \in \mathcal{A}$ except for $\beta(a') \neq c$. Let χ_k be a non-trivial elementary character, i.e., a character of $\mathcal{A}$.

$$\mu[\chi_k] = \sum_{a\in\mathcal{A}} \beta(a)\chi_k(a) = (\beta(a') - c)\chi_k(0) + c\sum_{a\in\mathcal{A}} \chi_k(a) = \beta(a') - c \neq 0$$

by hypothesis. It follows that, for every character $\chi = \prod_k \chi_k$, we have $\mu[\chi] = \prod_k \mu[\chi_k] \neq 0$, where we are using the fact that μ is a Bernoulli measure.

An example of a non-strongly non-uniform measure is any measure of the form $\lambda \times \mu$ on $(\mathcal{A} \times \mathcal{B})^{\mathbb{Z}}$, where λ is the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$.

The following proposition completes [**22**, Theorem 12] by giving an equivalence between character-diffusivity and randomization. It also shows that randomization is a structural phenomenon, in the sense that it cannot happen on individual initial measures without happening on a large class.

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Definition 8. A Bernoulli measure $\otimes\beta$ is *non-degenerate* if the support of β has non-trivial intersection with at least two cosets of every proper subgroup of $\mathcal{A}$.

PROPOSITION 4. *Let F be an abelian CA. The following are equivalent:*

(i) *F is character-diffusive;*

(ii) *F randomizes the class of harmonically mixing measures;*

(iii) *F randomizes the class of non-degenerate Bernoulli measures; and*

(iv) *F randomizes some strongly non-uniform Bernoulli measure.*

This equivalence holds for all three kinds of character-diffusivity and randomization: that is, strong character-diffusivity/randomization, character-diffusivity/randomization in density, and character-diffusivity for characters of support $\subset \mathbb{U}$/randomization on cylinders of support $\subset \mathbb{U}$ for any $\mathbb{U} \subset \mathbb{Z}$.

Proof. (i) $\Rightarrow$ (ii) Assume that F is strongly character-diffusive. Let μ be a harmonically mixing measure and let χ be any non-trivial character of $\mathcal{A}^{\mathbb{Z}}$. Since F is strongly character-diffusive, $\operatorname{rank}(\chi \circ F^t) \underset{t\to\infty}{\longrightarrow} \infty$. Since μ is harmonically mixing, it follows that $F^t\mu[\chi] = \mu[\chi \circ F^t] \underset{t\to\infty}{\longrightarrow} 0 = \lambda[\chi]$. Since this is true for any character χ, we have by Lévy's continuity theorem that $F^t\mu \underset{t\to\infty}{\longrightarrow} \lambda$.

For randomization in density, the proof is the same, where each convergence is taken along a subsequence of upper density one.

For randomization for characters of support $\subset \mathbb{U}$, the same argument shows that $F^t\mu[\chi] = \mu[\chi \circ F^t] \underset{t\to\infty}{\longrightarrow} 0 = \lambda[\chi]$ for any non-trivial character χ with support in $\mathbb{U}$. There is a bijection between the characters of support $\subset \mathbb{U}$ and $\widehat{\mathcal{A}^{\mathbb{U}}}$, and a conditional measure $\mu(\cdot \mid \mathfrak{B}_{\mathbb{U}})$ can be seen as a measure of $\mathcal{M}(\mathcal{A}^{\mathbb{U}})$. Applying Lévy's continuity theorem to $\mathcal{A}^{\mathbb{U}}$, it follows that $F^t\mu(\cdot \mid \mathfrak{B}_{\mathbb{U}}) \to \lambda(\cdot \mid \mathfrak{B}_{\mathbb{U}})$.

(ii) $\Rightarrow$ (iii): We prove that any non-degenerate Bernoulli measure $\mu = \otimes\beta$ is harmonically mixing. First note that, for any elementary character $\chi_0 \neq \mathbf{1}$,

$$\mu[\chi_0] = \int_{\mathcal{A}^{\mathbb{Z}}} \chi \, d\mu = \sum_{a\in\mathcal{A}} \chi_0(a)\beta(a).$$

We claim that there exist $a, b \in \mathcal{A}$ such that $\beta(a), \beta(b) > 0$ and $\chi_0(a) \neq \chi_0(b)$. To see this, let $K \leq \mathcal{A}$ be the kernel of χ_0. Since $\chi_0 \neq \mathbf{1}$, $\chi_0(g) \neq 1$ for some $g \in \mathcal{A}$, and thus $K < \mathcal{A}$. Then, by the assumption that β is non-degenerate, there exist a, b in the support of β such that $aK \neq bK$. Thus $\beta(a), \beta(b) > 0$ and $\chi_0(a) \neq \chi_0(b)$.

Now the existence of a, b shows that $\sum_{a\in\mathcal{A}} \chi_0(a)\beta(a)$ is a non-trivial convex combination of points on the unit circle, so by the strict convexity of the unit circle, $\mu[\chi_0]$ is a non-extremal point of the unit disk, that is, $|\mu[\chi_0]| < 1$.

Define $m = \max\{|\mu[\chi_0]| : \chi_0 \in \widehat{\mathcal{A}}\backslash\mathbf{1}\} < 1$. For any character $\chi = \prod_{i\in\mathbb{Z}} \chi_i$,

$$|\mu[\chi]| = \left|\prod_{i\in\mathbb{Z}} \mu[\chi_i]\right| \leq m^{\operatorname{rank}(\chi)},$$

where the first equality comes from the fact that μ is a Bernoulli measure. This implies that μ is harmonically mixing.

(iii) $\Rightarrow$ (iv) See, e.g., the first measure in Example 1.

(iv) $\Rightarrow$ (i) Assume that F is not strongly character-diffusive and take a character $\chi \neq \mathbf{1}$ such that $\mathrm{rank}(F^t \circ \chi) \not\to \infty$. This means that there exists $C \in \mathbb{N}$ and a subsequence φ such that $\mathrm{rank}(\chi \circ F^{\varphi(t)}) \leq C$. Since there is a finite number of elementary characters, we have, for any strongly non-uniform Bernoulli measure μ,

$$|F^{\varphi(t)}\mu[\chi]| \geq m^C \quad \text{where } m = \min\{|\mu[\chi_0]| : \chi_0 \in \widehat{A}\} > 0.$$

Therefore $F^t\mu[\chi] \not\to 0$, which implies that $F^t\mu \not\to \lambda$. We conclude by contraposition.

For randomization in density, carry out the same proof along a sequence of times with positive upper density.

For randomization of cylinders of support $\subset \mathbb{U}$, we can carry out the same proof for any character of support $\subset \mathbb{U}$. □

Remark 1. The strongly non-uniform hypothesis is necessary to prevent the following kind of counterexample. Take $\mathcal{A} = (\mathbb{Z}/2\mathbb{Z})^2$, $F' = F \times \mathrm{Id}$, where F is a strongly randomizing CA on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ (such as F_2, as we prove later) and $\mu = \nu \times \lambda$, where ν is any harmonically mixing measure and λ is the uniform Bernoulli measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$. Then F' strongly randomizes μ, but does not strongly randomize any measure whose second component is non-uniform.

Definition 9. (Dependency function) To any abelian CA $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, we associate a *dependency function*

$$\text{for all } (t, i) \in \mathbb{N} \times \mathbb{Z}, \quad \Delta_F(t, i) = \begin{cases} \mathcal{A} & \to \quad \mathcal{A}, \\ q & \mapsto \quad F^t(x_q)_i, \end{cases}$$

where x_q is the configuration worth q at position 0 and is 0 everywhere else.

Notice that, by linearity,

$$\text{for all } x \in \mathcal{A}^{\mathbb{Z}} \quad \text{and} \quad \text{for all } t \in \mathbb{N}, \quad F^t(x)_z = \sum_{j \in \mathbb{Z}} \Delta_F(t, z - j)(x_j), \tag{1}$$

where only a finite number of terms are non-zero.

In the following lemma, we prove that the support of the image of a fixed character at time t is entirely determined by the local dependency diagram.

LEMMA 2. *Let F be an abelian CA, and let χ be a character whose support is included in $[0, m]$ for $m \geq 0$. If there are (t_1, z_1) and (t_2, z_2) such that*

$$\textit{for all } z \in [0, m], \quad \Delta_F(t_1, z_1 + z) = \Delta_F(t_2, z_2 + z),$$

then

$$-z_1 \in \mathrm{supp}(\chi \circ F^{t_1}) \Leftrightarrow -z_2 \in \mathrm{supp}(\chi \circ F^{t_2}).$$

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Proof. By equation (1),

$$\begin{aligned}\chi \circ F^t(x) &= \prod_{i\in\mathbb{Z}} \chi_i\bigg(\sum_{j\in\mathbb{Z}} \Delta_F(t, i-j)(x_j)\bigg)\\ &= \prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}} \chi_i \circ \Delta_F(t, i-j)(x_j)\\ &= \prod_{j\in\mathbb{Z}}\bigg(\prod_{k\in\mathbb{Z}} \chi_{k+j}\circ \Delta_F(t,k)\bigg)(x_j),\end{aligned}$$

where the last step is obtained by rewriting the sum: $k = i - j$. It follows that

$$\begin{aligned}-z_1 \in \operatorname{supp}(\chi \circ F^{t_1}) &\Leftrightarrow \prod_{k\in\mathbb{Z}} \chi_{k-z_1} \circ \Delta_F(t_1, k) \neq 0\\ &\Leftrightarrow \prod_{k\in\mathbb{Z}} \chi_{k-z_2} \circ \Delta_F(t_2, k) \neq 0\\ &\Leftrightarrow -z_2 \in \operatorname{supp}(\chi \circ F^{t_2}),\end{aligned}$$

where the second step uses the hypothesis of the lemma and the fact that $\chi_{k-z_1} = 0$ whenever $k - z_1 \notin [0, m]$. □

Given an abelian CA F and $t \in \mathbb{N}$, denote by $d(t)$ the number of non-trivial dependencies of F at time t by

$$d(t) = |\{z \in \mathbb{Z} : \Delta_F(t, z) \neq 0\}| \quad \text{(the zero map)}.$$

Following [**22**, **23**], we introduce *isolated bijective dependencies* which provide useful lower-bounds on the rank of the image of characters under the action of F.

Definition 10. (Isolated dependency) For $k \geq 1$, a k-isolated dependency is a pair $(t, z) \in \mathbb{N} \times \mathbb{Z}$ such that:
(1) $\Delta_F(t, z)$ is a bijective dependency; and
(2) $\Delta_F(t, z + i) = 0$ for $1 \leq i \leq k$.
We denote by $S_k(t)$ the set of k-isolated dependencies at time t (i.e., of the form (t, z)) and $s_k(t) = |S_k(t)|$.

This concept of k-isolated dependencies is also used in [**24**] to define dispersion mixing measures and dispersive CAs. The main technique of [**22**] (proof of Theorem 15) and [**23**] (V-separating sets) is essentially to use s_k as a lower bound for the rank of characters under the iteration of an abelian CA.

PROPOSITION 5. *Let F be an abelian CA, let $\chi \neq \mathbf{1}$ be a character and let k be the diameter of* $\operatorname{supp}(\chi)$. *Then*

$$\textit{for all } t \in \mathbb{N}, \quad s_{k-1}(t) \leq \operatorname{rank}(\chi \circ F^t) \leq \operatorname{rank}(\chi) \cdot d(t).$$

Proof. The rank being invariant by translation, we can suppose that $\operatorname{supp}(\chi) \subset [0, k-1]$ and that $\chi_0 \neq \mathbf{1}$. By equation (1),

$$\chi \circ F^t(x) = \prod_{i=0}^{k-1}\prod_{j\in\mathbb{Z}} \chi_i \circ \Delta_F(t, i-j)(x_j) = \prod_{j\in\mathbb{Z}}\bigg(\prod_{i=0}^{k-1} \chi_i \circ \Delta_F(t, i-j)\bigg)(x_j).$$

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First,

$$j \in \text{supp}(\chi \circ F^t) \Rightarrow \text{there exists } i,\ \chi_i \circ \Delta_F(t, i-j) \neq 1$$
$$\Rightarrow \text{there exists } i \in \text{supp}(\chi),\ \Delta_F(t, i-j) \neq 0.$$

Therefore we get the upper bound $\text{rank}(\chi \circ F^t) \leq \text{rank}(\chi) \cdot d(t)$.

Second, if $(t, -j)$ is $k-1$-isolated, then $j \in \text{supp}(\chi \circ F^t)$. Indeed,

$$(\chi \circ F^t)_j = \prod_{i=0}^{k-1} \chi_i \circ \Delta_F(t, i-j) = \chi_0 \circ \Delta_F(t, -j).$$

We deduce that $s_{k-1}(t) \leq \text{rank}(\chi \circ F^t)$. □

Example 2. By Propositions 4 and 5, having $s_k(t) \longrightarrow_t +\infty$ for all k is a sufficient condition for randomization, but it is not necessary. For instance, take $F' = F \times (\sigma^N \circ F)$, where F is any abelian CA that randomizes in density. By Corollary 1 below, F' is randomizing in density. However, when N is large enough that F and $\sigma^N \circ F$ have disjoint neighborhoods, F' has no bijective dependency, and therefore $s_k(t) = 0$ for all k and t.

On the other hand, having many bijective dependencies is not enough if they are not well isolated. For example, one can check that H_2 satisfies $s_1(t) = t - 2$ but it is not randomizing, as shown in §5.

4. *Duality, diffusivity and solitons*

In the last section, we saw that the iterated images of characters under the action of a CA is key to understanding its action on probability measures. It turns out that the action of abelian CAs on characters can be seen as a CA on the dual group, and that, furthermore, this dual CA shares many properties with the original CA.

Remember that any character χ of $\widehat{G^{\mathbb{Z}}}$ can be written as a finite product of cellwise (elementary) characters: i.e., $\chi(x) = \prod_{z \in \mathcal{N}} \chi_z(x_z)$ for some finite set $\mathcal{N} \subset \mathbb{Z}$ and $\chi_z \in \widehat{G}$. To such a χ, we associate $\Psi(\chi)$, the configuration of $\widehat{G}^{\mathbb{Z}}$ defined by

$$\Psi(\chi)(z) = \begin{cases} \chi_z & \text{if } z \in \mathcal{N}, \\ 1 & \text{otherwise.} \end{cases}$$

Note that $\Psi(\widehat{G^{\mathbb{Z}}})$ is exactly the set of finite configurations of $\widehat{G}^{\mathbb{Z}}$.

Definition 11. (Dual CA) Let F be an abelian CA over $G^{\mathbb{Z}}$. It can be written as

$$F(x)_z = \sum_{i \in \mathcal{N}} \phi_i(x_{z+i}),$$

where $\mathcal{N} \subset \mathbb{Z}$ is finite and ϕ_i are endomorphisms of G. We define $\widehat{F}$ over the finite configurations of $\widehat{G}^{\mathbb{Z}}$ by

$$\widehat{F}(\Psi(\chi)) = \Psi(\chi \circ F).$$

Since $\widehat{F}$ is uniformly continuous and shift-invariant on finite configurations, it can be extended by continuity to a cellular automaton $\widehat{G}^{\mathbb{Z}} \to \widehat{G}^{\mathbb{Z}}$; this is the dual CA of F, and is an abelian CA for the group $(\widehat{G}, \times)$.

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More concretely, if $\chi(x)=\prod_{z\in A}\chi_z(x_z)$, then

$$\widehat{F}(\Psi(\chi))=\begin{cases} x_z\mapsto \prod_{i\in\mathcal{N}}\chi_{z-i}(\phi_i(x_z)) & \text{if } z\in A+\mathcal{N},\\ 1 & \text{otherwise.}\end{cases}$$

Then, for any $c\in\widehat{G}^{\mathbb{Z}}$, we can define

$$\widehat{F}(c)_z=\prod_{i\in\mathcal{N}}\gamma_i(c_{z-i}),\tag{2}$$

where γ_i is the endomorphism of $\widehat{G}$ defined by

$$\gamma_i(\chi)=g\mapsto\chi\circ\phi_i(g).$$

When $G=\mathbb{F}_p^d$, the dual of a CA is obtained (up to conjugacy) by applying a mirror operation and transposing the matrix corresponding to each coefficient. Indeed, the map

$$\begin{aligned} G&\to\widehat{G}\\ a&\mapsto\chi_a\end{aligned}\quad\text{where }\chi_a: b\in\mathcal{A}\mapsto e^{(2i\pi/p)\langle a,b\rangle},$$

where $\langle a,b\rangle$ denotes the scalar product of a and b seen as d-dimensional vectors, is an isomorphism. Through that isomorphism, we have that $\chi_a\circ M=\chi_{M^ta}$ for any endomorphism $M:\mathcal{A}\to\mathcal{A}$, and the result comes from equation (2).

In particular, our examples F_2 and H_2 are flip conjugate to their own dual since all their coefficients are symmetric matrices. H_2 is actually conjugate to its dual since it is left-right symmetric.

We do not know whether an abelian CA F is always flip conjugate to its dual $\widehat{F}$; however, we show in the remainder of this section that they are dynamically close enough that properties like randomization or diffusion are preserved by duality.

LEMMA 3. *Let Φ_1 and Φ_2 be two abelian CA over $G^{\mathbb{Z}}$. Then $\widehat{\Phi_1\circ\Phi_2}=\widehat{\Phi_2}\circ\widehat{\Phi_1}$. As a consequence:*

- *$\widehat{F^t}=(\widehat{F})^t$ for any $t>0$;*
- *$\widehat{F\circ\sigma}=\widehat{F}\circ\sigma^{-1}$; and*
- *if F is reversible, then $\widehat{F}$ is also reversible and $\widehat{F^{-1}}=(\widehat{F})^{-1}$.*

Furthermore, $\widehat{\widehat{F}}=F$ up to a canonical isomorphism.

Proof. By definition of dual CAs, for any $\chi\in\widehat{G^{\mathbb{Z}}}$,

$$\widehat{\Phi_1\circ\Phi_2}(\Psi(\chi))=\Psi(\chi\circ\Phi_1\circ\Phi_2)=\widehat{\Phi_2}(\Psi(\chi\circ\Phi_1))=\widehat{\Phi_2}\circ\widehat{\Phi_1}(\Psi(\chi)).$$

Since $\Psi(\widehat{G^{\mathbb{Z}}})$ is dense in $\widehat{G}^{\mathbb{Z}}$, we deduce that $\widehat{\Phi_1\circ\Phi_2}=\widehat{\Phi_2}\circ\widehat{\Phi_1}$ on the whole space.

For the last point, it is well known that $\widehat{\widehat{G^{\mathbb{Z}}}}\simeq G^{\mathbb{Z}}$ through the canonical isomorphism $\psi: g\mapsto(\chi\mapsto\chi(g))$ (see, e.g., [**4**, Lemma 4.1.4]). Then we check that $\widehat{\widehat{F}}:\psi(g)\mapsto\psi(g)\circ\widehat{F}=\psi(F(g))$, so that $\widehat{\widehat{F}}\simeq F$ up to this isomorphism. □

In the following two technical lemmas and in the remainder of the section, we stress when our results do not require the CA to be abelian, even though we will only apply them to abelian CAs.

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LEMMA 4. *Let F be a CA with quiescent state zero. Then for any finite configuration x,* $\mathrm{rank}(F(x)) \leq |\mathcal{N}| \cdot \mathrm{rank}(x)$.

In particular, if F is abelian and χ is a character, then $\mathrm{rank}(\chi \circ F) \leq |\mathcal{N}| \cdot \mathrm{rank}(\chi)$.

Proof. Since zero is quiescent, the only non-zero cells in $F(x)$ belong to $\mathrm{supp}(x) + \mathcal{N}$. It follows that $\mathrm{rank}(F(x)) \leq |\mathcal{N}| \cdot \mathrm{rank}(x)$. The second statement follows by applying the result to $\widehat{F}$, noticing that F and $\widehat{F}$ have the same neighborhood size by equation (2). □

The converse lemma holds for reversible CAs.

LEMMA 5. *Let F be a reversible CA with quiescent state zero. There exists a constant $C > 0$ such that, for any finite configuration x,* $\mathrm{rank}(F(x)) \geq C \cdot \mathrm{rank}(x)$.

In particular, if F is abelian, there exists $C > 0$ such that $\mathrm{rank}(\chi \circ F) \geq C \cdot \mathrm{rank}(\chi)$ *for any character χ.*

Proof. Apply Lemma 4 on F^{-1}. The second point uses the last point of Lemma 3. □

Definition 12. A CA F with a quiescent state zero is *strongly diffusive* (respectively, diffusive in density) if, for any finite configuration c, we have $\mathrm{rank}(F^t(c)) \to \infty$ (respectively, on a subsequence of density one).

LEMMA 6. *An abelian CA F is strongly character-diffusive (respectively, character-diffusive in density) if and only if $\widehat{F}$ is strongly diffusive (respectively, diffusive in density).*

Proof. $\widehat{F}(\Psi(\chi)) = \Psi(\chi \circ F)$ and $\mathrm{rank}(\chi) = \mathrm{rank}(\Psi(\chi))$. □

Definition 13. (Soliton) Let F be an abelian CA. A *soliton* is a finite configuration $c \neq \overline{0}$ such that $F^p(c) = \sigma^q(c)$ for some $p \geq 1$ and $q \in \mathbb{Z}$.

Intuitively, having a soliton is the opposite of being diffusive (even in density). In the remainder of the section, we will develop this intuition and prove a series of technical results about solitons that will culminate in the characterization of randomization in density in the next section.

First note that all the configurations in the orbit of a soliton have bounded rank. Conversely, we can extract a soliton from any orbit of finite configurations whose rank is bounded on a set of time steps of positive density.

PROPOSITION 6. *Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a surjective CA with a quiescent state zero. Assume that F is not diffusive in density. Then F admits a soliton.*

Proof. There is a finite initial configuration x and an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of positive upper density such that $\mathrm{rank}(F^{T_n}(x))$ is bounded. Without loss of generality, assume that $\mathrm{rank}(F^{T_n}(x)) = k$ for all n. Denote by $i_1(T_n), \ldots, i_k(T_n)$ the non-zero coordinates at time T_n.

Now let $m \in \{1, \ldots, k\}$ be the maximum integer such that there exists an integer M such that the subsequence

$$(T_{\varphi(n)})_{n\in\mathbb{N}} = \{n \in \mathbb{N} : i_m(T_n) - i_1(T_n) \leq M\}$$

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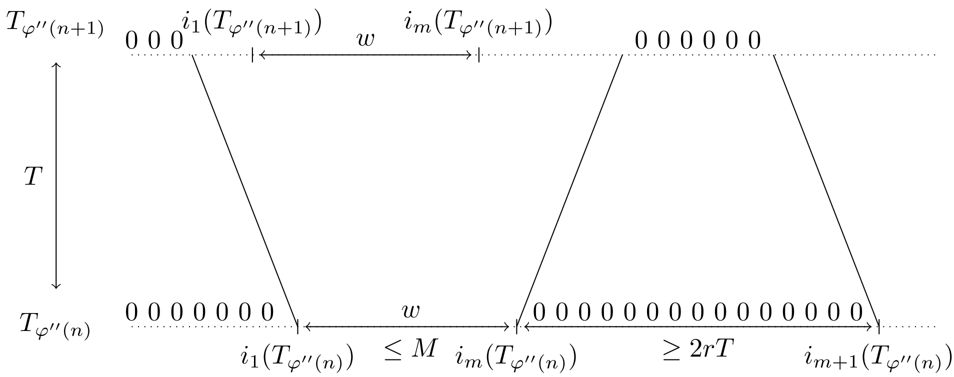

FIGURE 2. The finite configuration defined by w is a soliton.

has positive upper density. In particular, $(T_{\varphi(n)})_{n\in\mathbb{N}}$ has positive upper density. We distinguish two cases.

$m = k$: $F^{T_{\varphi(n)}}(x)_{[i_1(T_{\varphi(n)}),i_k(T_{\varphi(n)})]}$ can take at most $|\bigcup_{j=0}^{M}\mathcal{A}^j| = \sum_{j=0}^{M}|\mathcal{A}|^j$ different values. By the pigeonhole principle, we can find two integers $a < b$ such that

$$F^{T_{\varphi(a)}}(x)_{[i_1(T_{\varphi(a)}),i_k(T_{\varphi(a)})]} = F^{T_{\varphi(b)}}(x)_{[i_1(T_{\varphi(b)}),i_k(T_{\varphi(b)})]}.$$

But this means that $F^{T_{\varphi(b)}}(x) = \sigma^{i_1(T_{\varphi(b)})-i_1(T_{\varphi(a)})} \circ F^{T_{\varphi(a)}}(x)$, so we have found a soliton.

$m < k$: First, by the same argument as above, $F^{T_{\varphi(n)}}(x)_{[i_1(T_{\varphi(n)}),i_m(T_{\varphi(n)})]}$ can only take a finite number of values, so at least one of these words appear with positive density. Denote by w the corresponding word and by $(T_{\varphi'(n)})_{n\in\mathbb{N}}$ the corresponding subsequence. We now prove that the configuration

$$\cdots 0\cdot 0\cdot 0\cdot w\cdot 0\cdot 0\cdot 0\cdots$$

is a soliton.

Take $N \in \mathbb{N}$ such that $1/N$ is a lower bound on the density of $(T_{\varphi'(n)})_{n\in\mathbb{N}}$ and let r be the radius of F. By construction of m, the times $T_{\varphi'(n)}$, where $i_{m+1}(T_{\varphi'(n)}) - i_m(T_{\varphi'(n)}) \leq 2rN$, have upper density zero. Therefore we extract from the sequence $(T_{\varphi'(n)})_{n\in\mathbb{N}}$ a new subsequence $(T_{\varphi''(n)})_{n\in\mathbb{N}}$ corresponding to times t, where $i_{m+1}(t) - i_m(t) > 2rN$ with the same upper density. In particular, we can find some n such that $T = T_{\varphi''(n+1)} - T_{\varphi''(n)} \leq N$.

As shown in Figure 2, only two disjoint areas can contain non-zero values in $F^{T_{\varphi''(n+1)}}(x)$:

- the interval $[i_1(T_{\varphi''(n)}) - rT, i_m(T_{\varphi''(n)}) + rT]$, which contains $f^T(0^{2rT}\cdot w\cdot 0^{2rT})$; and
- the interval $[i_{m+1}(T_{\varphi''(n)}) - rT, i_k(T_{\varphi''(n)}) + rT]$, which contains $f^T(0^{2rT}\cdot F^{T_{\varphi''(n)}}(x)_{[i_{m+1}(T_{\varphi''(n)}),i_k(T_{\varphi''(n)})]}\cdot 0^{2rT})$.

Indeed, any cell outside these regions can be written as $f^T(0^{2rT+1}) = 0$ since zero is quiescent. Consider the different possibilities for the value of $i_1(T_{\varphi''(n+1)})$.

- If $i_1(T_{\varphi''(n+1)}) < i_1(T_{\varphi''(n)}) - rT$, then $F^{T_{\varphi''(n+1)}}(x)_{i_1(T_{\varphi''(n+1)})} = f^T(0^{2rT+1}) = 0$ which is a contradiction.

- If $i_1(T_{\varphi''(n+1)}) > i_1(T_{\varphi''(n)}) + rT$, then $F^{T_{\varphi''(n)}}(x)_{[i_1(T_{\varphi''(n)}) \pm rT]}$ is a non-zero word whose image under f^T is zero. This means that F is not preinjective (two different configurations that differ on a finite subset of cells have the same image), so by the Garden-of-Eden theorem [**3**, **10**] it is not surjective, which a contradiction.

Therefore $i_1(T_{\varphi''(n+1)}) \in [i_1(T_{\varphi''(n)}) \pm rT]$. In particular, the interval $I = [i_1(T_{\varphi''(n)}) - rT,\ i_m(T_{\varphi''(n)}) + rT]$ contains all $i_\ell(T_{\varphi''(n+1)})$ for $\ell \leq m$. Using a similar argument, $i_{m+1}(T_{\varphi''(n+1)}) \in [i_{m+1}(T_{\varphi''(n)}) \pm rT]$, so that I does not contain any $i_\ell(T_{\varphi''(n+1)})$ for $\ell > m$.

From this we conclude that, for some constant C,

$$F^T(0^{2rT} \cdot w \cdot 0^{2rT}) = 0^C \cdot w \cdot 0^{2rT-C}$$

and therefore we have found a soliton. □

The remainder of this section is dedicated to proving the following proposition.

PROPOSITION 7. *Let F be an abelian CA. Then F has a soliton if and only if $\widehat{F}$ has a soliton.*

To prove this proposition, we need a series of lemmas. For an abelian CA F, a finite fixed point is just a fixed point that is also a finite configuration. A finite fixed point is non-trivial if it is not the configuration everywhere equal to zero.

LEMMA 7. *Let F be an abelian CA and denote by X_F the set of spatially periodic fixed points*

$$X_{F,n} = \{x : \sigma^n(x) = x \text{ and } F(x) = x\}, \qquad X_F = \bigcup_n X_{F,n}.$$

F has a non-trivial finite fixed point if and only if X_F is infinite. In this case, we actually have $|X_{F,n}| = 2^{\Omega(n)}$.

Remark 2. This remark holds in dimension d if one replaces the infiniteness assumption by $|X_{F,n}| = 2^{\Omega(n^d)}$.

Proof. First suppose that $F(x) = x$, where x is a non-trivial finite configuration, and denote by u a finite word containing the non-zero part of x. Let r be the radius of F and let $k = |u| + 2r$. Consider the set of finite words

$$W_n = \left\{ w \in \mathcal{A}^n : \begin{array}{l} \text{for all } i,\ 0 \leq i < \left\lfloor \dfrac{n}{k} \right\rfloor \Rightarrow w_{[ki,k(i+1)-1]} = 0^k \text{ or } w_{[ki,k(i+1)-1]} = 0^r u 0^r \\ \text{and} \\ w_{k\cdot\lfloor n/k \rfloor, n} = 0 \end{array} \right\}.$$

Any periodic configuration made of concatenated copies of some word in W_n (except for 0^n) is a non-trivial fixed point and $|W_n| = 2^{\lfloor n/k \rfloor}$. Therefore $|X_{F,n}| = 2^{\Omega(n)}$.

Conversely, suppose that X_F is infinite. Then, since $X_{F,n} \subset X_{F,kn}$ for $k \in \mathbb{N}^+$, $n \mapsto |X_{F,n}|$ is not bounded from above. For some n, $X_{F,n}$ must contain at least two distinct configurations x_1 and x_2 such that $x_1|_{[1,r]} = x_2|_{[1,r]}$ and $x_1|_{[n-r+1,n]} = x_2|_{[n-r+1,n]}$ by the pigeonhole principle. It follows that the configuration

$$x : z \in \mathbb{Z} \mapsto \begin{cases} 0 & \text{if } z \leq 0 \text{ or } z > n, \\ x_1(z) - x_2(z) & \text{otherwise} \end{cases}$$

is a non-trivial finite fixed point. □

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LEMMA 8. *Let G be an abelian group. For any endomorphism $h : G \to G$, define its dual $\widehat{h}$ by $\widehat{h}(\chi) = \chi \circ h$ for any character $\chi \in \widehat{G}$. Then $|\ker(h)| = |\ker(\widehat{h})|$.*

Proof. We have $\chi \in \ker(\widehat{h}) \Leftrightarrow \chi \circ h = 1 \Leftrightarrow \mathrm{Im}(h) \subset \ker(\chi)$. Therefore the restriction

$$\chi \mapsto \chi|_{(G/\mathrm{Im}\, h)}$$

is a bijection between $\ker(\widehat{h})$ and $(\widehat{G/\mathrm{Im}\, h})$. Since $|(\widehat{G/\mathrm{Im}\, h})| = |G/\mathrm{Im}\, h| = |\ker h|$ by Proposition 2, the proof is complete. □

LEMMA 9. *Let F be an abelian CA over alphabet G. F has a non-trivial finite fixed point if and only if $\widehat{F}$ has a non-trivial finite fixed point.*

Proof. For any $n \in \mathbb{N}$ with $n \geq 1$, let h_n be the endomorphism of G^n defined as follows. For any $u \in G^n$, let $x^u \in G^{\mathbb{Z}}$ be the (spatially) periodic point of period u, i.e., $x^u_i = u_{i \bmod n}$ for all i. For each $u \in G^n$, define

$$h_n(u) = (F(x^u) - x^u)_{[0,n-1]}.$$

h_n captures the action of the CA $F - \mathrm{Id}$ on spatially periodic configurations of period n. In particular, $|\ker(h_n)| = |X_{F,n}|$. Now consider its dual $\widehat{h_n}$, which is an endomorphism of $\widehat{G^n} = (\widehat{G})^n$. For any $\chi = \prod_{1 \leq i \leq n} \chi_i$,

$$\begin{aligned}\widehat{h_n}(\chi) : u \mapsto \prod_{i=1}^{n} \chi_i(F(x^u) - x^u)_i &= \prod_{i=1}^{n} \chi_i \circ F(x^u)_i \cdot (\chi_i(x^u)_i)^{-1} \\ &= (\widehat{F}(x^\chi) \cdot x^{1/\chi})(x^u)\end{aligned}$$

by definition of $\widehat{F}$, and where again we define the periodic point $x^\chi \in \widehat{G}^{\mathbb{Z}}$ by $x^\chi_i = \chi_{i \bmod n}$. Therefore $|\ker(\widehat{h_n})| = |X_{\widehat{F},n}|$, similarly as above.

Now, by Lemma 8, we have $|\ker(h_n)| = |\ker(\widehat{h_n})|$ and therefore $|X_{F,n}| = |X_{\widehat{F},n}|$. We conclude by Lemma 7. □

Proof of Proposition 7. Suppose that F has a soliton x, that is, $F^p(x) = \sigma^q(x)$ for $p \geq 1$ and $q \in \mathbb{Z}$. Then x is a finite fixed point for the abelian CA $\sigma^{-q} \circ F^p$. By Lemma 9, we deduce that $\widehat{\sigma^{-q} \circ F^p}$ also has a finite fixed point y. Using Lemma 3, we can rewrite this as $(\widehat{F})^p(y) = \sigma^{-q}(y)$, which shows that y is actually a soliton of $\widehat{F}$.

Applying the same reasoning, a soliton in $\widehat{F}$ implies a soliton in $\widehat{\widehat{F}}$, and $\widehat{\widehat{F}} = F$ up to a canonical isomorphism by Lemma 3. □

Example 3. Note that the smallest solitons of F and $\widehat{F}$ need not be of the same size. For example, consider the CA F defined over the alphabet $\mathbb{F}_2^2$ by

$$F(x)_z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot x_z + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot x_{z+1}.$$

F acts like the identity on any configuration whose second components are all zero. In particular, it admits solitons of rank one. However, its dual, obtained up to conjugacy by mirroring and transposing the coefficients,

$$\widehat{F}(x)_z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot x_{z-1} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot x_z,$$

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has no soliton of rank one. Indeed, take any finite configuration x supported in $[i, j]$ for $i \leq j$ and such that $x_i \neq 0$, $x_j \neq 0$. Notice that

$$F(x)_i = \begin{pmatrix} (x_i)_1 \\ (x_i)_1 + (x_i)_2 \end{pmatrix} \neq 0 \quad \text{and} \quad F(x)_{j+1} = \begin{pmatrix} 0 \\ (x_j)_2 \end{pmatrix}.$$

In particular, if $(x_j)_2 \neq 0$, then x cannot be a soliton.

Now take any finite x of rank one, assuming that $x_0 \neq 0$. If $(x_0)_2 \neq 0$, then x is not a soliton by the previous argument. If $(x_0)_2 = 0$, then $\widehat{F}(x)$ is a finite configuration of rank one such that $\widehat{F}(x)_0 = \begin{pmatrix} (x_0)_1 \\ (x_0)_1 \end{pmatrix}$ and is not a soliton.

Finally, it is easy to check that the configuration

$$y = \cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdots$$

is a fixed point, and therefore a soliton, of $\widehat{F}$.

5. *Characterization of randomization in density*

We are now ready to prove our first main result, which is a combinatorial characterization of randomization in density by the absence of solitons.

THEOREM 2. *Let F be an abelian CA. The following are equivalent:*
(i) *F randomizes in density any harmonically mixing measure;*
(ii) *F has no soliton;*
(iii) *F is diffusive in density; and*
(iv) *for some strongly non-uniform Bernoulli measure μ, the sequence $(F^t\mu)_{t\in\mathbb{N}}$ admits λ as an accumulation point.*

Proof. Write the dual claims of (i), (ii), (iii) and (iv) for $\hat{F}$ as (j), (jj), (jjj), (jv), respectively. Then

$$\text{(i)} \overset{\text{Prop. 4}}{\Longleftrightarrow} (jjj) \overset{\text{Prop. 6}}{\Longleftrightarrow} (jj) \overset{\text{Prop. 7}}{\Longleftrightarrow} \text{(ii)} \overset{\text{Prop. 6}}{\Longleftrightarrow} \text{(iii)},$$

so the first three claims are equivalent, and are also equivalent to (jj). Moreover, it is clear that (i) $\Rightarrow$ (iv). We prove that (iv) $\Rightarrow (jj)$. Suppose that (jj) does not hold, and let $\chi \in \widehat{G}^{\mathbb{Z}}$ be a soliton for $\widehat{F}$. Solitons are finite non-zero configurations, so we can consider $\chi \in \widehat{G}^{\mathbb{Z}}$ to be a non-trivial character satisfying that $\operatorname{rank}(\widehat{F}^t(\chi))$ is bounded from above by some m. Since μ is strongly non-uniform and Bernoulli, there exists $\varepsilon > 0$ such that $\mu[\chi'] \geq \varepsilon^{\operatorname{rank}\chi'}$. In particular, $F^t\mu[\chi] = \mu[\widehat{F}^t(\chi)] \geq \varepsilon^m$ for all t. Since $\lambda[\chi] = 0$, $F^t\mu[\chi]$ does not have λ as an accumulation point. □

Before giving some general consequences of this theorem and applying it to the commutative case, we will use it on our examples.

Example 4. The CA H_2 admits a soliton

$$\cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdots$$

and therefore it is not randomizing in density.

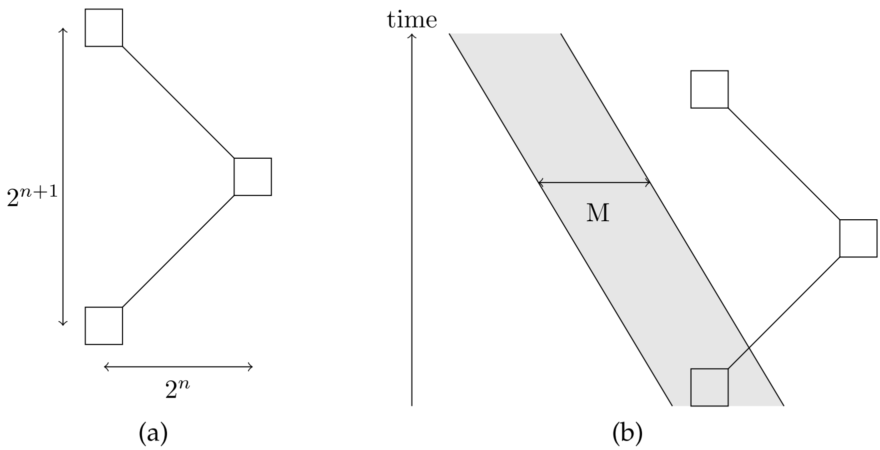

FIGURE 3. The direction of time is upward. (a) Any three cells in these relative positions in the space–time diagram of F_2 sum to zero (cancelling shape). (b) The space–time diagram of a soliton is zero except around a space–time line, and we can always position a cancelling shape that does not sum to zero.

On the contrary, we show that F_2 has no soliton and therefore is randomizing in density (it is actually strongly randomizing; see §6.2). This is an alternative proof to the result of [**17**], where it was proved through a delicate analysis of F_2 using binomial coefficients and Lucas' lemma. Our proof is illustrated in Figure 3. It is easy to show by induction that

$$\text{for all } c \in \mathcal{A}^{\mathbb{Z}} \quad \text{for all } n \in \mathbb{N} \quad \text{for all } z \in \mathbb{Z}, \quad F_2^{2^{n+1}}(c)_z + F_2^{2^n}(c)_{z+2^n} + c_z = 0 \bmod 2.$$

Suppose, by contradiction, that F_2 has a soliton c such that $F^p(c) = \sigma^q(c)$. Therefore there is a constant M such that any non-zero cell of the space–time diagram $(F^t(c))_{t\in\mathbb{N}}$ is at horizontal distance of at most M of the real line $L = \{(z, t) : pz + qt = 0\}$. In other words, $F^t(c)_z \neq 0 \Rightarrow |pz + qt| < pM$. Now take n such that $2^n > 2M$ and any $|z| \leq M$ and distinguish these three cases.

$q = 0$: Since $|z - 2^n| > M$, we have $c_{z-2^n} = 0$ and $F_2^{2^{n+1}}(c)_{z-2^n} = 0$, so that $F_2^{2^n}(c)_z = 0$. This is true for every z such that $|z| \leq M$, so $F_2^{2^n}(c) = 0$, which is a contradiction.

$q = -p$: Assume that $p = -q = 1$. $c_{z+2^{n+1}} = 0$ and $F_2^{2^n}(c)_{z+2^{n+1}+2^n} = 0$, so that $F_2^{2^{n+1}}(c)_{z+2^{n+1}} = 0$. At time 2^{n+1}, this applies to every z such that $|p(z + 2^{n+1}) - q(2^{n+1})| = |z| \leq M$, so $F_2^{2^{n+1}}(c) = 0$, which is a contradiction.

Otherwise: $F_2^{2^{n+1}}(c)_z = F_2^{2^n}(c)_{z+2^n} = 0$ when n is large enough, so that $c_z = 0$ for all $|z| \leq M$, which is a contradiction.

Definition 14. (Positive expansiveness) A CA is *positively expansive* if there is some finite $W \subseteq \mathbb{Z}$ such that, for any pair of distinct configurations $x, y \in \mathcal{A}^{\mathbb{Z}}$,

$$\text{there exists } t \in \mathbb{N} \quad \text{there exists } z \in W, \quad F^t(x)_z \neq F^t(y)_z.$$

More generally, for $\alpha \in \mathbb{R}$, we say that F is *positively expansive in direction* α if there is some finite $W \subseteq \mathbb{Z}$ such that, for any pair of distinct configurations $x, y \in \mathcal{A}^{\mathbb{Z}}$,

$$\text{there exists } t \in \mathbb{N} \quad \text{there exists } z \in W, \quad F^t(x)_{z+\lceil \alpha t \rceil} \neq F^t(y)_{z+\lceil \alpha t \rceil}.$$

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See [**6**, **25**] for further developments on directional dynamics in CAs.

In the next result, for a CA $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ and a subalphabet $\mathcal{B} \subset \mathcal{A}$ such that $F(\mathcal{B}^{\mathbb{Z}}) \subseteq \mathcal{B}^{\mathbb{Z}}$, the corresponding *subautomaton* of F is $F' = F|_{\mathcal{B}^{\mathbb{Z}}}$.

COROLLARY 1. *Let F and G be abelian CA. Then:*

- *if F and G are randomizing, then so is $F \times G$;*
- *if F is randomizing, then so are all its subautomata;*
- *if F is randomizing and reversible, then so is F^{-1}; and*
- *if F has a direction of positive expansivity then it is randomizing,*

where randomizing means randomizing in density any harmonically mixing measure.

Proof. This corollary follows from Theorem 2 by the following elementary observations on solitons.

- A soliton in $F \times G$ implies a soliton in either F or G.
- A soliton in a subautomaton of F is a soliton for F.
- A soliton for F^{-1} is a soliton for F.
- A positively expansive CA cannot admit a soliton. Indeed, take a CA F with a direction of positive expansiveness α and assume, for the sake of contradiction, that it admits a soliton c: $F^p(c) = \sigma^q(c)$. For any finite $W \subset \mathbb{Z}$, take the two distinct configurations $x = 0$ and $y = \sigma^k(c)$ for $|k|$ large enough and $\mathrm{sign}(k) = \mathrm{sign}(q/p - \alpha)$, and check that $F^t(x)_{z+\lceil \alpha t \rceil} = F^t(y)_{z+\lceil \alpha t \rceil} = 0$ for every $t \in \mathbb{N}$ and $z \in W$. □

Remark 3. A CA with local rule $f : \mathcal{A}^m \to \mathcal{A}$ is bipermutive if $m \geq 2$ and the maps $x \mapsto f(x, a_1, \ldots, a_{m-1})$ and $x \mapsto f(a_1, \ldots, a_{m-1}, x)$ are permutations of $\mathcal{A}$ for any $a_1, \ldots, a_{m-1} \in \mathcal{A}$. Since bipermutivity implies the existence of a direction of positive expansivity [**2**], the above corollary implies that any bipermutive abelian CA randomizes in density any harmonically mixing measure. This generalizes [**23**, Theorem 9], where the authors consider abelian CA of the form

$$F = \sum_{i \in \mathcal{N}} \overline{\phi}_i \circ \sigma^i,$$

where $|\mathcal{N}| \geq 2$ and $\overline{\phi}_i$ are commuting automorphisms. We do not need this hypothesis here, and, for instance, we prove that the following CA over $\mathbb{F}_2^2$ is randomizing in density: i.e.,

$$F(c)_z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot c_z + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot c_{z+1}.$$

6. *Other forms of randomization*

In this section, we consider other forms of randomization that have been less studied in the literature. First, we prove that, in the case of abelian CAs whose coefficients are commuting endomorphisms, only randomization in density can happen. Then we provide examples of abelian CAs that exhibit strong randomization and randomization for cylinders up to some fixed length.

6.1. *Abelian CAs with commuting coefficients.* The case of abelian CAs with commuting coefficients is in many regards similar to the case of scalar coefficients. These CAs have more rigidity in their time evolution than general abelian CAs: the image of a single cell at time t can be determined directly through the use of the binomial theorem and modular arithmetic of binomial coefficients. In particular, when t is some power of the order of the group, the number of bijective dependencies is bounded, which explains why these CAs cannot randomize strongly.

LEMMA 10. *Let p be a prime number and let $l \geq 0$. Let $(\mathcal{X}, +, \times)$ be a commutative ring of characteristic p^l, i.e., such that, for any $X \in \mathcal{X}$,*

$$p^l \cdot X = 0,$$

where 0 is the neutral element for $+$.

Then, for any $n \geq 0$ and any elements $X_i \in \mathcal{X}$, $1 \leq i \leq k$, we have, in $\mathcal{X}$,

$$\left(\sum_{i=1}^{k} X_i\right)^{p^{n+l-1}} = \left(\sum_{i=1}^{k} X_i^{p^n}\right)^{p^{l-1}}.$$

Proof. First, by the binomial theorem,

$$\left(\sum_{i=1}^{k} X_i\right)^{p^n} = \left(X_1 + \sum_{i=2}^{k} X_i\right)^{p^n} = X_1^{p^n} + \left(\sum_{i=2}^{k} X_i\right)^{p^n} + p \cdot Y$$

for some Y because, by Kummer's theorem, p divides $\binom{p^n}{i}$ for $0 < i < p^n$. By a direct induction, we deduce that

$$\left(\sum_{i=1}^{k} X_i\right)^{p^n} = \sum_{i=1}^{k} X_i^{p^n} + p \cdot Y'$$

for some Y'. Now, applying the binomial theorem again, we get

$$\begin{aligned}
\left(\sum_{i=1}^{k} X_i\right)^{p^{n+l-1}} &= \left(\sum_{i=1}^{k} X_i^{p^n} + p \cdot Y'\right)^{p^{l-1}} \\
&= \left(\sum_{i=1}^{k} X_i^{p^n}\right)^{p^{l-1}} + \sum_{j=1}^{p^{l-1}} \binom{p^{l-1}}{j} \cdot p^j \cdot (Y')^j \cdot \left(\sum_{i=1}^{k} X_i^{p^n}\right)^{p^{l-1}-j} \\
&= \left(\sum_{i=1}^{k} X_i^{p^n}\right)^{p^{l-1}} + p^l \cdot Z
\end{aligned}$$

because, by Kummer's theorem, p^l divides $\binom{p^{l-1}}{j} \cdot p^j$ for any $1 \leq j \leq p$. Then the desired equality is shown since $p^l \cdot Z = 0$. □

Now we prove that abelian CAs cannot randomize strongly and cannot randomize some cylinders without randomizing in density.

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THEOREM 3. *There is no strongly randomizing abelian CA with commuting endomorphisms. Moreover, for any abelian group G, there exists an $N \in \mathbb{N}$ such that, for any abelian CA F over G with commuting endomorphisms, the following are equivalent:*
(i) *F randomizes in density;*
(ii) $\mathrm{rank}(F^N(c)) \geq 2$ *for any finite configuration c of rank one; and*
(iii) *F randomizes in density on cylinders of length one,*
where, as usual, the class of initial measures is the set of harmonically mixing measures.

Proof. If p is a prime number, a *p-group* is a group G' where the order of every element $g \in G'$ is a power of p. By the decomposition theorem for finite abelian groups, they can be written as a direct product of finite p-groups for distinct primes p. Using this fact together with Corollary 1, it is enough to consider the case where G is a p-group (because an abelian CA on $G_1 \times G_2$, where G_1 and G_2 have relatively prime orders, is a Cartesian product of abelian CAs on G_1 and G_2, respectively).

As usual, we write F as

$$F = \sum_{i \in \mathcal{N}} \overline{\phi}_i \circ \sigma^i.$$

Consider the commutative ring generated by the $\overline{\phi}_i$ and the shift map under addition and composition. This ring has characteristic p^l for some l because we considered a p-group as the alphabet. By Lemma 10, we get

$$\text{for all } n \in \mathbb{N}, \quad F^{p^{n+l-1}} = \left(\sum_{i \in \mathcal{N}} (\overline{\phi}_i)^{p^n} \circ \sigma^{ip^n} \right)^{p^{l-1}} = \sum_{j \in \mathcal{N}'} (\gamma_j)^{p^n} \circ \sigma^{jp^n}, \tag{3}$$

where

$$\mathcal{N}' = \{n_1 + \cdots + n_{p^{l-1}} : n_i \in \mathcal{N}\}$$

and each γ_j is a sum of compositions of some ϕ_i that do not depend on n. The number of terms in the right-hand expression is bounded independently of n, so the number of dependencies of F^t is bounded on an infinite sequence of times and so F cannot be strongly randomizing by Proposition 5 (it cannot even strongly randomize cylinders of size one).

For the second part of the proposition, consider $N = p^{n_0+l-1}$ for some n_0 such that $p^{n_0} > |G|$. This choice of N guaranties that, for any endomorphism h of G, we have $\ker(h^{p^{n_0}}) = \ker(h^{p^n})$ for any $n \geq n_0$ (because the sequence $\ker(h^i)$ increases strictly until it stabilizes). We have the following alternatives.

(a) *There is c of rank one such that* $\mathrm{rank}(F^N(c)) = 0$. In particular, F is not surjective. We claim that there is some $g \in G$ such that $F^{-t}([g])$ is empty for any large enough t. From the claim, we deduce that F does not randomize cylinders of length one starting from the uniform Bernoulli measure. We now prove the claim. Suppose that $F^{-t}([g])$ is never empty, whatever g and t. Then consider any finite word $g_1 \cdots g_k$ and take n large enough so that $p^n > k$. By equation (3) and by choice of n, we get that, for any configuration c, $F^{p^{n+l-1}}(c)_i$ depends only on $c_{|i+V_n}$, where $V_n \subseteq \mathbb{Z}$ is such that $1 + V_n$, $2 + V_n, \ldots, k + V_n$ are two-by-two disjoint sets. Therefore, from the assumption that $F^{-p^{n+l-1}}([g_i]) \neq \emptyset$ for any $1 \leq i \leq k$ and by linearity of F, we deduce that there is some c

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with $F^{p^{n+l-1}}(c) \in [g_1 \cdots g_k]$. Since the choice of $g_1 \cdots g_k$ was arbitrary, we proved that F is surjective, which is a contradiction.

(b) $\operatorname{rank}\big(F^N(c)\big) > 0$ *for any c of rank one, but there is d of rank one such that* $\operatorname{rank}(F^N(d)) = 1$. By equation (3), there is some $g \in G$ and a $j \in \mathcal{N}'$ such that $g \notin \ker(\gamma_j^{p^{n_0}})$ but $g \in \ker(\gamma_{j'}^{p^{n_0}})$ for all $j' \neq j$. As said before, the choice of n_0 ensures that, for any $n \geq n_0$, we have $g \notin \ker(\gamma_j^{p^n})$ but $g \in \ker(\gamma_{j'}^{p^n})$ for all $j' \neq j$. Hence, using the formula for $F^{p^{n+l-1}}$, we deduce that $\operatorname{rank}(F^{p^{n+l-1}}(d)) = 1$ for any $n \geq n_0$. In that case, F admits a soliton of size one.

(c) *For any c of rank one,* $\operatorname{rank}(F^N(c)) \geq 2$. For the same reason as in the previous case, we deduce that $\operatorname{rank}(F^{p^{n+l-1}}(c)) \geq 2$ for any $n \geq n_0$. But equation (3) above shows that the non-zero cells in $F^{p^{n+l-1}}(c')$ belong to the set $\mathcal{N}'_n = \{jp^n : j \in \mathcal{N}'\}$ for any c' of rank one. We deduce that, for any d of rank m and n large enough, $F^{p^{n+l-1}}(d)$ contains two non-zero cells distant from each other by at least $p^n - m$ cells. Therefore F does not have any soliton and it randomizes harmonically mixing measures in density by Theorem 2.

To summarize, we have (c) $\Rightarrow$ (i), while (a) and (b) are both incompatible with (i). Since (c) corresponds to (ii), we have shown that (i) $\Leftrightarrow$ (ii)($\Leftrightarrow$ (c)). Since, clearly, (i) $\Rightarrow$ (iii), we now prove that (iii) $\Rightarrow$ (c).

If F randomizes in density on cylinders of length one, then F is character-diffusive on characters or rank one (by Proposition 5) which means that $\widehat{F}$ has no soliton of size one. It is straightforward to check by equation (2) that an abelian CA with commuting endomorphisms has a dual with commuting endomorphisms, and therefore the above alternative (a)/(b)/(c) applies also to $\widehat{F}$. In other words, $\widehat{F}$ satisfies (c), which implies that it admits no solitons. This means, in turn, that F must satisfy (c) as well. The theorem follows. □

Remark 4. In [**23**] the authors consider abelian CAs with integer coefficients (i.e., endomorphisms of the form $a \mapsto n \cdot a$); such a CA is called *proper* if, for any prime divisor p of the order of the alphabet, there are at least two coefficients not divisible by p. Theorem 6 of [**23**] shows that proper CAs are randomizing in density. This is a particular case of the above theorem. Indeed, if F is proper and taking N from the theorem, it is easy to check that F^N is proper and that this is equivalent to $\operatorname{rank}(F^N(c)) \geq 2$ for any finite configuration of rank one.

6.2. *Strong randomization.* We now give examples of strongly randomizing abelian CAs. They are all defined over the alphabet $\mathcal{A} = \mathbb{Z}_p^2$, where p is a prime number. In this section, we denote by π_1 and π_2 the projections on the first and second component of the alphabet, respectively.

We also denote by $n \cdot g = \underbrace{g + \cdots + g}_{n \text{ times}}$ and $\overline{0}$ the neutral element of the group $(\mathcal{A}^{\mathbb{Z}}, +)$ (i.e., the configuration everywhere equal to (0, 0)).

LEMMA 11. *If two abelian CAs F and G over $\mathcal{A}^{\mathbb{Z}}$ commute, then, for any $n \geq 0$,*

$$(F + G)^{p^n} = F^{p^n} + G^{p^n}.$$

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Proof. Due to the structure of the group $\mathcal{A} = \mathbb{Z}_p^2$, for any configuration $c \in \mathcal{A}^{\mathbb{Z}}$, we have $p \cdot c = \overline{0}$. Therefore, for any CA F, we have that $p \cdot F$ is the constant map equal to $\overline{0}$.

Now, from the binomial formula and from the fact that p divides $\binom{p}{n}$ for any $1 < n < p$, we deduce that

$$(F + G)^p = F^p + G^p.$$

The lemma follows by an easy induction. □

Extending example F_2 from §2, we now consider, for each prime $p \geq 2$ and all $c \in \mathcal{A}^{\mathbb{Z}}$,

$$F_p(c)_z = (\pi_1(c_{z+1}) + \pi_2(c_z), \pi_1(c_z)),$$
$$G_p(c)_z = (\pi_1(c_{z+1}) + \pi_1(c_z) + \pi_2(c_z), \pi_1(c_z)).$$

In the remainder of the section, we prove that F_p and G_p are strongly diffusive for any prime $p \geq 2$.

These CAs are reversible and are, in fact, also time symmetric, i.e., the product of two involutions [**8**]. For instance the inverse of F_p is

$$F_p^{-1}(c)_z = (\pi_2(c_z), \pi_1(c_z) - \pi_2(c_{z+1})).$$

Since they are reversible, we extend their dependency diagrams to negative times, which means that $\Delta_\Phi(t, z)$ is defined for any $(t, z) \in \mathbb{Z}^2$, where Φ denotes F_p or G_p.

Note that both F_p and its inverse can be defined with neighborhood $\{0, 1\}$. The same is true for G_p.

LEMMA 12. *For any $n \geq 0$, any $t \in \mathbb{Z}$ and any $z \in \mathbb{Z}$,*

$$\Delta_{F_p}(2p^n + t, z) = \Delta_{F_p}(p^n + t, p^n + z) + \Delta_{F_p}(t, z),$$
$$\Delta_{G_p}(2p^n + t, z) = \Delta_{G_p}(p^n + t, p^n + z) + \Delta_{G_p}(p^n + t, z) + \Delta_{G_p}(t, z).$$

Proof. First, it is straightforward to check that $F_p^2 = (\sigma \circ F_p) + I$ (where I denotes the identity map over $\mathcal{A}^{\mathbb{Z}}$). Hence, using Lemma 11, we get the identity $F_p^{2p^n} = (\sigma_{p^n} \circ F_p^{p^n}) + I$. For every configuration c, $t \in \mathbb{Z}$ and $z \in \mathbb{Z}$, we have $F_p^{2p^n+t}(c)_z = F_p^{p^n+t}(c)_{p^n+z} + c_z$, which proves the Lemma.

The same proof applies to Δ_{G_p}. □

LEMMA 13. *Let Φ be either F_p or G_p. For any $t \in \mathbb{Z}$:*

(1) $\Delta_\Phi(t, z)$ *is the constant map equal to* $(0, 0)$ *when $z > 0$ or $z < -|t|$; and*

(2) $\Delta_\Phi(t, 0)$ *is a bijection.*

Proof. First, both Φ and Φ^{-1} have neighborhood $\{0, 1\}$. So the first item is straightforward by induction.

Second, by definition, $\Delta_\Phi(0, 0)$ is a bijection. We can check that

$$\Delta_{F_p}(1, 0) : (g, h) \mapsto (h, g), \quad \Delta_{G_p}(1, 0) : (g, h) \mapsto (g + h, g),$$
$$\Delta_{G_p}(2, 0) : (g, h) \mapsto (h, g + h),$$

which are bijections. Applying Lemma 12 with $n = 0$, we get, by straightforward induction, that $\Delta_{F_p}(t + 2, 0) = \Delta_{F_p}(t, 0)$ and $\Delta_{G_p}(t + 3, 0) = \Delta_{G_p}(t, 0)$. We have proved the second item. □

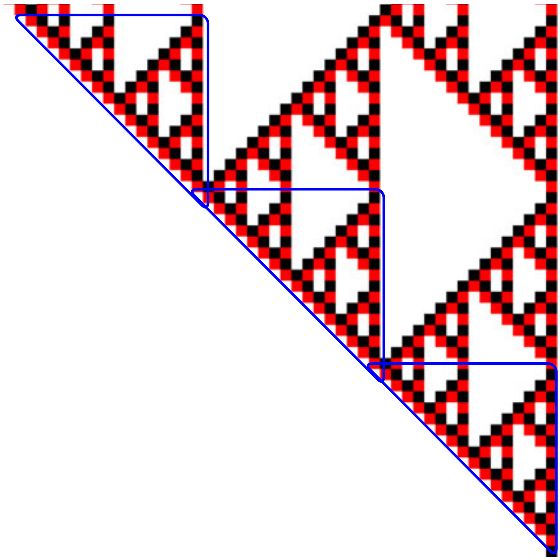

FIGURE 4. A representation of $\Delta_{F_2}(i, t)$ for $i \in \mathbb{Z}$ and $t \in \mathbb{N}$. The direction of time is upward. Thick blue lines indicate triangular zones $T_{4,0}$ and its translations by $(j \cdot 2^4, -j \cdot 2^4)$ for $j = 1, 2$.

Much of the structure of Δ_Φ can be understood when focusing on particular 'triangular' zones of $\mathbb{Z}^2$ at various scales. For $k \geq 0$ and n large enough so that $p^n - k > k$, we define the corresponding zone as

$$T_{n,k} = \{(t, z) : k < t < p^n - k \text{ and } -t < z < k\}.$$

LEMMA 14. *Let Φ be either F_p or G_p. Let $k \geq 0$ and let n be such that $p^n - k > k$. Then, for any $(t, z) \in T_{n,k}$ and any $j \geq 1$,*

$$\Delta_\Phi(t, z) = \Delta_\Phi(t + j \cdot p^n, z - j \cdot p^n).$$

In particular, if χ is a character whose support is of diameter at most k, then, for any t with $k < t < p^n - k$,

$$\operatorname{rank}(\chi \circ \Phi^{t+j\cdot p^n}) \geq \operatorname{rank}(\chi \circ \Phi^t) + 1.$$

The lemma is illustrated in Figure 4.

Proof. For the first assertion consider some $(t, z) \in T_{n,k}$. From Lemma 13, we have that both $\Delta_\Phi(t - p^n, z - p^n)$ and $\Delta_\Phi(t, z - p^n)$ are the constant map equal to $(0, 0)$. Therefore the following identities obtained from Lemma 12, i.e.,

$$\Delta_{F_p}(t + p^n, z - p^n) = \Delta_{F_p}(t - p^n, z - p^n) + \Delta_{F_p}(t, z),$$
$$\Delta_{G_p}(t + p^n, z - p^n) = \Delta_{G_p}(t - p^n, z - p^n) + \Delta_{G_p}(t, z - p^n) + \Delta_{G_p}(t, z)$$

can, in both cases, be simplified to

$$\Delta_\Phi(t, z) = \Delta_\Phi(t + p^n, z - p^n).$$

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This proves the case when $j = 1$. The same idea shows the induction step on j: i.e.,

$$\Delta_\Phi(t + j \cdot p^n, z - j \cdot p^n) = \Delta_\Phi(t + (j + 1) \cdot p^n, z - (j + 1) \cdot p^n).$$

For the second assertion, apply Lemma 2 on the first assertion. We get that $-z \in \mathrm{supp}(\chi \circ \Phi^t)$ if and only if $-z + j \cdot p_n \in \mathrm{supp}(\chi \circ \Phi^{t+j\cdot p^n})$. Furthermore, we also have $0 \in \mathrm{supp}(\chi \circ \Phi^{t+j\cdot p^n})$ since 0 is a k-isolated bijective dependency by Lemma 13. Accounting for both contributions, we get $\mathrm{rank}(\chi \circ \Phi^{t+j\cdot p^n}) \geq \mathrm{rank}(\chi \circ \Phi^t) + 1$. □

THEOREM 4. *F_p and G_p strongly randomize the harmonically mixing measures.*

Proof. We prove that F_p and G_p are strongly character-diffusive, and Proposition 4 implies the result.

Denote by Φ either F_p or G_p. Φ is reversible and denote by C the diameter of the neighborhood of Φ^{-1}. By Lemma 5, $\mathrm{rank}(\chi \circ \Phi) \geq C \cdot \mathrm{rank}(\chi)$ for any character χ. Since Φ^{-t} can be defined by a neighborhood of diameter $C \cdot t$, $\mathrm{rank}(\chi \circ \Phi^t) \geq C \cdot t \cdot \mathrm{rank}(\chi)$ for any character χ. In particular, for any $m \geq 0$, any $T \geq 0$ and any character χ_0 of rank at least $C \cdot m \cdot T$, we have rank $(\chi_0 \circ \Phi^t) \geq m$ for $1 \leq t \leq T$.

Let χ be any non-trivial character and let k be the diameter of its support. Denote by $R(t) = \mathrm{rank}(\chi \circ \Phi^t)$. We are going to show that $R(t) \to \infty$, which implies the claim since the choice of χ is arbitrary.

First, let n_0 be large enough and let t_0 be such that $k < t_0 < p^{n_0} - k$. By successive applications of Lemma 14, we get

$$R(t_0 + (p-1)p^{n_0} + (p-1)p^{n_0+1} + \cdots + (p-1)p^{n_0+m}) \geq m.$$

Since $p^{n_0+m+1} = \sum_{j=0}^{m} (p-1)p^{n_0+j} + p^{n_0}$, we deduce that

$$p^{n_0+m+1} - \left(t_0 + \sum_{j=0}^{m} (p-1)p^{n_0+j}\right) \leq p^{n_0} - t_0$$

so that as soon as $m \geq C \cdot M \cdot (p^{n_0} - t_0)$ it follows $R(p^{n_0+m+1}) \geq M$ by Lemma 5. Therefore

$$R(p^n) \to_n \infty. \tag{4}$$

Now define the predicate $P_{t_0,n,m}$ as the conjunction of the following conditions:

(1) $k \leq t_0 \leq p^n - k$;
(2) for all t, $t_0 \leq t \leq p^n - k : R(t) \geq m$; and
(3) $R(p^n - k) \geq C \cdot (m + 1) \cdot (t_0 + k)$.

First, since $R(t) \geq 1$ for any t (from Lemma 13), we have by equation (4) above and Lemma 4 that $P_{k,n,1}$ holds for n large enough.

Furthermore, if $P_{t_0,n,m}$, then $P_{t_0+p^n,n',m+1}$ for all large enough n'. Indeed, condition 1 is obviously true for $n' > n$ and condition 3 is true for any large enough n' from (4) above and Lemma 4. Finally, condition 2 is obtained from Lemma 14. For any $j \geq 1$:

(1) for $j \cdot p^n + t_0 \leq t \leq (j + 1) \cdot p^n - k$, $R(t) \geq R(t - j \cdot p^n) + 1 \geq m + 1$; and
(2) for $(j + 1) \cdot p^n - k \leq t \leq (j + 1) \cdot p^n + t_0$, we have $R(t) \geq m + 1$ because $R((j + 1) \cdot p^n - k) \geq R(p^n - k) \geq C \cdot (m + 1) \cdot (t_0 + k)$.

We have shown that, for any m, there exists t_0 such that, for any large enough n', we have $P_{t_0,n',m}$. This, in particular, implies that $R(t) \geq m$ for all $t \geq t_0$. We conclude that $R(t) \to \infty$. □

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6.3. *Randomizing only up to fixed-length cylinders.* We now define a family of CA that randomize finite cylinders up to a certain length, but no further. The alphabet is G^2, where G is any finite abelian group. Define

$$I_G(c)_z = (-[\pi_1(c_{z-1}) + \pi_1(c_{z+1}) + \pi_2(c_z)], \pi_1(c_z)).$$

Notice that $I_{\mathbb{Z}_2} = H_2$.

PROPOSITION 8. *I_G randomizes cylinders of length one, but not cylinders of length two.*

LEMMA 15. *For $t > 0$,*

$$\Delta_{I_G}(t, z) = (g, h) \mapsto \begin{cases} 0 & \textit{if } |z| > t, \\ ((-1)^t g, 0) & \textit{if } |z| = t, \\ ((-1)^t g, (-1)^{t+1} h) & \textit{if } |z| < t, t + z = 0 \bmod 2, \\ ((-1)^t h, (-1)^{t+1} g) & \textit{if } |z| < t, t + z = 1 \bmod 2. \end{cases}$$

Proof. This is by straightforward induction. □

Now we use Proposition 4 in conjunction with the following proposition to prove the announced result.

PROPOSITION 9. *I_G is strongly diffusive on characters of rank one, but not on characters of length two.*

Proof. Let χ be a character of rank one, i.e., $\chi(x) = \chi_0(x_0)$ with $\chi_0 \neq \mathbf{1}$. For $t > 0$,

$$\chi \circ I_G^t(x) = \chi_0\bigg(\sum_{z=-t}^{t} \Delta_{I_G}(t, z)(x_{-z})\bigg)$$
$$= \prod_{z=-t}^{t} \chi_0 \circ \Delta_{I_G}(t, z)(x_{-z}),$$

where χ_0 is non-trivial, and by the previous Lemma, $\Delta_{I_G}(t, z)$ is an isomorphism when $|z| < t$. We deduce that $\chi_0 \circ \Delta_{I_G}(t, z)$ is non-trivial whenever $|z| < t$ and therefore $\operatorname{rank}(\chi \circ I_G^t(x)) \geq 2t - 1 \to \infty$: I_G strongly diffuses characters of rank one.

For the second point, take any elementary character $\eta_0 \in \widehat{G}$ and define another character $\eta : x \mapsto \eta_0(\pi_1(x_0) + \pi_2(x_1))$. Then, by a straightforward computation,

$$\eta \circ I_G(x) = \eta_0(-[\pi_1(x_{-1}) + \pi_1(x_1) + \pi_2(x_0)] + \pi_1(x_1))$$
$$= \eta_0(-\pi_1(x_{-1}) - \pi_2(x_0)) = \sigma^{-1} \circ \eta(x)^{-1},$$

which means that η is a soliton for $\widehat{F}$ of rank two. □

Now we introduce the CA $I_{G,n}$, which consists of applying the local rule of I_G on the neighborhood $\{z_{-n}, z_0, z_n\}$: i.e.,

$$I_{G,n}(c)_z = (-[\pi_1(c_{z-n}) + \pi_1(c_{z+n}) + \pi_2(c_z)], \pi_1(c_z)).$$

Intuitively, a space–time diagram for $I_{G,n}$ consists of n intertwined space–time diagrams for I_G.

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THEOREM 5. *For any* $n \geq 1$, $I_{G,n}$ *randomizes cylinders of length n, but does not randomize cylinders of length* $n + 1$.

The next lemma is obvious (by straightforward induction).

LEMMA 16.

$$\Delta_{I_{G,n}}(t, z) = \begin{cases} \Delta_{I_G}(t, z/n) & \text{if } z = 0 \bmod n, \\ 0 & \text{otherwise}, \end{cases}$$

so that we can use Lemma 15. As in the previous case, we use Proposition 4 in conjunction with the following proposition.

PROPOSITION 10. $I_{G,n}$ *is strongly character-diffusive on characters of support* $\subset [0, n-1]$*, but not on characters of support* $\{0, n\}$.

Proof. Let χ be a non-zero character of support $\subset [0, n-1]$, that is, $\chi = \prod_{i=0}^{n-1} \chi_i$; without loss of generality assume that $\chi_0 \neq \mathbf{1}$. For any $t > 0$,

$$\begin{aligned} \chi \circ I_{G,n}^t(x) &= \prod_{i=0}^{n-1} \chi_i \left(\sum_{z=-t}^{t} \Delta_{I_{G,n}}(t, z+i)(x_{-z}) \right) \\ &= \prod_{z=-t}^{t} \left(\prod_{i=0}^{n-1} \chi_i \circ \Delta_{I_{G,n}}(t, z+i) \right)(x_{-z}). \end{aligned}$$

Now, for any z such that $z = 0 \mod n$, the corresponding term in the previous equation is $\chi_0 \circ \Delta_{I_{G,n}}(t, z)$ by Lemma 16, and this term is an isomorphism when $t < z < t$ by Lemma 15. Therefore $\text{rank}(\chi \circ I_{G,n}^t) \geq 2\lfloor t/n \rfloor - 1 \to \infty$.

For the second point, take any non-trivial elementary character $\eta_0 \in \widehat{G}$ and define another character $\eta : x \mapsto \eta_0(\pi_1(x_0) + \pi_2(x_n))$. This is a soliton for $\widehat{F}$ of support $\{0, n\}$ by Lemma 16 and the same proof as that of Proposition 9. □

7. *Open problems*

Building upon the approach of [**22**, **23**] we completely characterized randomization in density for abelian CAs. Furthermore, we provided examples of other forms of randomization, most notably strong randomization (in simple convergence), that can only happen for abelian CAs whose coefficients are non-commuting endomorphisms.

As mentioned by several authors, the most important research direction for randomization in CAs is to develop tools and techniques to go beyond the abelian case: i.e., CAs with a non-abelian group structure or nonlinear CAs [**13**, **14**]. There is some experimental evidence pointing at nonlinear randomization candidates [**11**, **29**]. The class of bipermutive CAs, although it does not encompass all candidates, has some relevant related work; the set of invariant well-behaved (i.e. Gibbs) measures is limited to the uniform Bernoulli measure [**14**, Corollary 46] and it exhibits a topological analogue to randomization [**27**]. Apart from the existence of examples, one can ask, in the general case, whether randomization is a structural property (see Proposition 4), whether it is preserved by elementary operations and whether it is implied by some topological properties such as positive expansivity (see Corollary 1).

We believe that it is worth considering several intermediate questions raised by the present work.

- Is strong randomization of an abelian CA equivalent to strong randomization of its dual?
- What is the importance of reversibility in the above examples of strong randomization? Our proof relies on reversibility and the smoothness it implies on the evolution of the rank of characters. Can a reversible abelian CA be randomizing in density but not strongly randomizing? What are the examples of strongly randomizing non-reversible abelian CAs?
- The notions of soliton and diffusivity can be defined for arbitrary CAs using diamonds (as in the notion of pre-injectivity): a pair (x, y) such that $\Delta(x, y) \subseteq \mathbb{Z}$ (the set of positions where x and y differ) is finite. We can define a soliton as a diamond (x, y) such that $\Delta(F^t(x), F^t(y))$ has a bounded diameter (independent of t). On the contrary, diffusivity can be defined as the property of having $|\Delta(F^t(x), F^t(y))| \to \infty$ for any diamond (x, y). What are the links with topological properties such as positive expansivity, pre-expansivity, mixing or transitivity? Are there links with randomization beyond abelian CAs? The results of [**26**] can be useful for that line of research. For reference, positive pre-expansivity, introduced in [**9**], is the property of being positively expansive on diamonds. A reversible CA can be positively pre-expansive (but never positively expansive) like the example F_2 of this paper. Corollary 1 states that, for abelian CAs, a direction of positive expansivity implies randomization in density; this implication actually holds for directions of positive pre-expansivity.
- Theorem 3 gives a procedure for testing randomization in density for abelian CAs with commuting coefficients because the constant N can be explicitly computed. Is there an algorithm to decide the presence of a soliton in abelian CAs? What about solitons in general CAs (again formalizing through diamonds as above)?

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