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Schnorr dimension

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Following Lutz's approach to effective (constructive) dimension, we define a notion of dimension for individual sequences based on Schnorr's concept(s) of randomness. In contrast to computable randomness and Schnorr randomness, the dimension concepts defined via computable martingales and Schnorr tests coincide, that is, the Schnorr Hausdorff dimension of a sequence always equals its computable Hausdorff dimension. Furthermore, we give a machine characterisation of the Schnorr dimension, based on prefix-free machines whose domain has computable measure. Finally, we show that there exist computably enumerable sets that are Schnorr (computably) irregular: while every c.e. set has Schnorr Hausdorff dimension 0, there are c.e. sets of computable packing dimension 1, which is, from Barzdiņš' Theorem, an impossible property for the case of effective (constructive) dimension. In fact, we prove that every hyperimmune Turing degree contains a set of computable packing dimension 1.

1. Introduction

Schnorr (Schnorr 1971) issued a fundamental criticism concerning the notion of effective null sets introduced by Martin-Löf. He argued that, although we know how fast a Martin-Löf test converges to zero, it is not effectively given, in the sense that the measure of the test sets U_n is not computable, but only enumerable from below, so, in general, we cannot decide whether a given cylinder belongs to the nth level of some test.

Schnorr presented two alternatives, both clearly closer to what one would call a *computable approach to randomness*. One is based on the idea of randomness as an unpredictable event in the sense that it should not be possible to win in a betting game (martingale) against a truly random sequence of outcomes. The other sticks to Martin-Löf's approach, but requires the tests defining a null set to be *a uniformly computable sequence* of open sets having uniformly *computable measure*, not merely a sequence of uniformly computably enumerable sets such that the nth set has measure less than 2^{-n}.

Schnorr was able to show that both approaches yield reasonable notions of randomness, that is, random sequences according to his concepts exhibit most of the robust properties one would expect from a random object. However, his suggestions have some serious drawbacks. They are harder to deal with technically, which is mainly due to the absence of universal tests. Besides, a machine characterisation of randomness like the elegant coincidence of Martin Löf-random sequences with those incompressible by a universal prefix-free machine is technically more involved and was only recently given in Downey and Griffiths (2004).

Recently, Lutz (Lutz 2000a; 2000b; 2003) introduced an effective notion of Hausdorff dimension. As (classical) Hausdorff dimension can be seen as a refinement of Lebesgue measure on 2^{ω}, in the sense that it further distinguishes between classes of measure 0, the effective Hausdorff dimension of an individual sequence can be interpreted as a *degree of randomness* of the sequence. This viewpoint is supported by a series of results due to Ryabko (Ryabko 1984; 1986), Staiger (Staiger 1993; 2005), Cai and Hartmanis (Cai and Hartmanis 1994) and Mayordomo (Mayordomo 2002), which establish that the effective Hausdorff dimension of a sequence equals its lower asymptotic Kolmogorov complexity (plain or prefix-free).

Lutz's framework of martingales (gales) is very flexible with respect to the level of effectivisation one wishes to obtain, see Lutz (2000a). This makes it easy to define a version of algorithmic dimension based on computable martingales, *computable dimension*, in analogy to computable randomness. This was done in Lutz (2000a), and was treated briefly in Terwijn (2003).

In this paper we will study the Schnorr-style approach to algorithmic dimension in more detail. We will define a notion of dimension based on Schnorr's test concept. We will see that the technical difficulties mentioned above also apply to dimension. Furthermore, Schnorr dimension behaves in many respects like effective (constructive) dimension, which was introduced in Lutz (2000b; 2003) – see also Reimann (2004) and Reimann and Stephan (2005). However, we will also see that for dimension, Schnorr's two approaches coincide, in contrast to Schnorr randomness and computable randomness: the Schnorr Hausdorff dimension of a sequence always equals its computable Hausdorff dimension. Furthermore, it turns out that, with respect to Schnorr/computable dimension, computably enumerable sets can exhibit a complex behaviour, to some extent. Namely, we will show that there are c.e. sets of high computable packing dimension, which is impossible in the effective case, due to a result in Barzdiņš (1968). In fact, every hyperimmune Turing degree contains a set of computable packing dimension 1, and this set can be chosen to be c.e. in the special case of a c.e. Turing degree. On the other hand, we prove that the computable Hausdorff dimension of the characteristic sequence of a c.e. set is 0. Thus, the class of computably enumerable sets contains *irregular* sequences – sequences for which Hausdorff and packing dimension do not coincide.

The paper is structured as follows. In Section 2 we give a short introduction to the classical theory of Hausdorff measures and dimension, as well as packing dimension. In Section 3 we will define algorithmic variants of these concepts based on Schnorr's test approach to randomness.

In Section 4 we prove that the dimension concepts based on Schnorr tests, on the one hand, and computable martingales, on the other hand, coincide, in contrast to Schnorr randomness and computable randomness. We also present two basic examples of sequences of non-integral dimension (Section 5). In Section 6 we derive a machine characterisation of Schnorr/computable Hausdorff and packing dimension. Finally, in

Section 7, we study the Schnorr/computable dimension of computably enumerable sets. The main result here will be that on those sets computable Hausdorff dimension and computable packing dimension can differ as largely as possible.

We will use fairly standard notation. 2^ω will denote the set of infinite binary sequences. Sequences will be denoted by upper case letters like A, B, C, or X, Y, Z. We will refer to the nth bit ($n \geqslant 0$) in a sequence B by either B_n or $B(n)$, that is, $B = B_0B_1B_2\dots = B(0)\,B(1)\,B(2)\dots$.

Strings, that is, finite sequences of 0s and 1s, will be denoted by lower case letters from the end of the alphabet, u, v, w, x, y, z along with some lower case Greek letters like σ and τ. $2^{<\omega}$ will denote the set of all strings. ϵ denotes the empty string. The *initial segment of length n*, $A\restriction_n$, of a sequence A is the string of length n corresponding to the first n bits of A. More generally, if $Z \subseteq \mathbb{N}$, we let $A\restriction_Z$ denote the restriction of A to the elements of Z. Formally, if $Z = \{z_0, z_1, \dots\}$ with $z_n < z_{n+1}$,

$$A\restriction_Z(n) = A(z_n).$$

In this way, $A\restriction_n = A\restriction_{\{0,\dots,n-1\}}$.

Given two strings v and w, the string v is called a *prefix* of w, $v \sqsubseteq w$ for short, if there exists a string x such that $vx = w$, where vx is the concatenation of v and x. If w is strictly longer than v, we write $v \sqsubset w$, and we extend this notation in a natural way to pairs of a string and a sequence. A set of strings is called *prefix-free* if no element of the set has a prefix (other than itself) in the set.

Initial segments induce a standard topology on 2^ω. The basis of the topology is formed by the *basic open cylinders* (or just *cylinders*, for short). Given a string $w = w_0 \dots w_{n-1}$ of length n, the basic open cylinder corresponding to w is defined by

$$[w] \;=\; \{A \in 2^\omega \,:\, A\restriction_n = w\}.$$

We extend this notation to sets of strings as follows: given $C \subseteq 2^{<\omega}$, we define

$$[C] = \bigcup_{w \in C} [w].$$

Throughout the paper we assume a familiarity with the basic concepts of computability theory such as Turing machines, computably enumerable sets, computable and left-computable (or left-c.e. or just c.e.) reals, and some central concepts of algorithmic information theory, in particular, Kolmogorov complexity and the Kraft–Chaitin Theorem. A standard reference for computability theory is Soare (1987), while a comprehensive treatise of algorithmic information theory is Li and Vitányi (1997). The forthcoming book Downey and Hirschfeldt (2006) will cover both areas.

2. Hausdorff measures and dimension

The basic idea behind Hausdorff dimension is to determine which 'scaling factor' best reflects the geometry of a set. One devises a family of (outer) measures, the so-called Hausdorff measures, which are generalisations of Lebesgue measure in the sense that they introduce a parameter with which the open sets (or rather their diameters) used in

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a covering are scaled. These measures are linearly ordered by the scaling parameter, and the Hausdorff dimension of a set picks out the parameter that induces the most 'suitable' measure.

Definition 2.1. Let $\mathcal{X} \subseteq 2^\omega$.

(1) Given $\delta > 0$, a set $C \subseteq 2^{<\omega}$ is a *δ-cover* of $\mathcal{X}$ if

$$(\forall w \in C)\,[2^{-|w|} \leqslant \delta] \quad \text{and} \quad \mathcal{X} \subseteq [C].$$

(2) For $s \geqslant 0$, define

$$\mathscr{H}^s_\delta(\mathcal{X}) = \inf\left\{\sum_{w\in C} 2^{-|w|s} : C \text{ is a } \delta\text{-cover of } \mathcal{X}\right\}.$$

The *s-dimensional Hausdorff measure* of $\mathcal{X}$ is defined by

$$\mathscr{H}^s(\mathcal{X}) = \lim_{\delta\to 0} \mathscr{H}^s_\delta(\mathcal{X}).$$

Remark. Given a string w, the cylinder $[w]$ has diameter $2^{-|w|}$ according to a standard metric compatible with the cylinder topology. Hence, the s-Hausdorff measure is obtained by restricting the admissible covers to finer and finer diameters. This is a geometric condition, and Hausdorff measures form an essential part of the theory of *fractal geometry*. Note further that $\mathscr{H}^s(\mathcal{X})$ is well defined, since, as δ decreases, there are fewer δ-covers available, hence $\mathscr{H}^s_\delta$ is non-decreasing. However, the value may be infinite. It can be shown that $\mathscr{H}^s$ is an outer measure and that the Borel sets of 2^ω are $\mathscr{H}^s$-measurable. For $s = 1$, one obtains the Lebesgue measure on 2^ω.

The outer measures $\mathscr{H}^s$ have an important property.

Proposition 2.2. Let $\mathcal{X} \subseteq 2^\omega$. If, for some $s \geqslant 0$, $\mathscr{H}^s(\mathcal{X}) < \infty$, then $\mathscr{H}^t(\mathcal{X}) = 0$ for all $t > s$.

Proof. Let $\mathscr{H}^s(\mathcal{X}) < \infty$, $t > s$. If $C \subseteq 2^{<\omega}$ is a δ-cover of $\mathcal{X}$, $\delta > 0$, we have

$$\sum_{w\in C} 2^{-|w|t} \leqslant \delta^{t-s} \sum_{w\in C} 2^{-|w|s},$$

so, taking infima, $\mathscr{H}^t_\delta(\mathcal{X}) \leqslant \delta^{t-s}\mathscr{H}^s_\delta(\mathcal{X})$. As $\delta \to 0$, the result follows. □

This means that there exists a point $s \geqslant 0$ where the s-dimensional Hausdorff measure drops from a positive (possibly infinite) value to zero. This point is the *Hausdorff dimension* of the class.

Definition 2.3. For a class $\mathcal{X} \subseteq 2^\omega$, we define the *Hausdorff dimension* of $\mathcal{X}$ by

$$\dim_{\mathrm{H}} \mathcal{X} = \inf\{s \geqslant 0 : \mathscr{H}^s(\mathcal{X}) = 0\}.$$

It is not hard to show that the notion of Hausdorff dimension is well behaved: it is *monotone* (that is, $\mathcal{X} \subseteq \mathcal{Y}$ implies $\dim_{\mathrm{H}}(\mathcal{X}) \leqslant \dim_{\mathrm{H}}(\mathcal{Y})$), and *stable* – if $\{\mathcal{X}_i\}_{i\in\mathbb{N}}$ is a countable family of classes, then

$$\dim_{\mathrm{H}}\Big(\bigcup_{i\in\mathbb{N}} \mathcal{X}_i\Big) = \sup_{i\in\mathbb{N}}\{\dim_{\mathrm{H}} \mathcal{X}_i\}.$$

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Furthermore, it can be seen as a *refinement of measure* 0. If $\mathcal{X}$ has a positive (outer) Lebesgue measure, then $\dim_{\mathrm{H}}(\mathcal{X}) = 1$ (as Lebesgue measure λ corresponds to $\mathscr{H}^1$). In particular, $\mathscr{H}^1(2^\omega) = \lambda(2^\omega) = 1$. On the other hand, no $\mathcal{X} \subseteq 2^\omega$ can have a Hausdorff dimension greater than 1, as $\mathscr{H}^s(\mathcal{X}) = 0$ for all $s > 1$. Hence, classes of non-integral Hausdorff dimension are necessarily Lebesgue null classes.

We give two examples of classes of non-integral dimension.

Theorem 2.4.

(1) Let $Z \subseteq \mathbb{N}$ be such that $\lim_n |Z \cap \{0, \dots, n-1\}|/n = \delta$, and define

$$\mathcal{D}_Z = \{A \in 2^\omega : A(n) = 0 \text{ for all } n \notin Z\}.$$

Then $\dim_{\mathrm{H}} \mathcal{D}_Z = \delta$.

(2) For $s \in [0,1]$, define $\mathcal{B}_s \subseteq 2^\omega$ as

$$\mathcal{B}_s = \left\{A \in 2^\omega : \lim_{n\to\infty} \frac{|\{k < n : A(k) = 1\}|}{n} = s\right\}.$$

Then $\dim_{\mathrm{H}} \mathcal{B}_s = -[s \log s + (1-s)\log(1-s)]$.

The first assertion is a special case of a general behaviour of Hausdorff dimension under Hölder transformations, see Falconer (1990). The second result is from Eggleston (1949). For more on Hausdorff measures and dimension refer to Falconer (1990).

2.1. *Packing dimension*

Packing dimension can be seen as a dual to Hausdorff dimension. While Hausdorff measures are defined in terms of coverings, that is, enclosing a set from outside, packing measures approximate from the inside, by packing the set 'as densely as possible' with disjoint sets of small size.

For this purpose, we say that a prefix-free set $P \subseteq 2^{<\omega}$ is a *packing* in $\mathcal{X} \subseteq 2^\omega$ if for every $\sigma \in P$ there is an $X \in \mathcal{X}$ such that $\sigma \sqsubset X$. Geometrically speaking, a packing in $\mathcal{X}$ is a collection of mutually disjoint open balls with centres in $\mathcal{X}$. If the balls all have radius $\leqslant \delta$, we call it a δ-packing in $\mathcal{X}$.

A possible suggestion for a packing that is 'as dense as possible' is as follows: given $s \geqslant 0$, $\delta > 0$, let

$$\mathscr{P}^s_\delta(\mathcal{X}) = \sup\left\{\sum_{w\in P} 2^{-|w|s} : P \text{ is a } \delta\text{-packing in } \mathcal{X}\right\}. \tag{1}$$

As $\mathscr{P}^s_\delta(\mathcal{X})$ decreases with δ, the limit

$$\mathscr{P}^s_0(\mathcal{X}) = \lim_{\delta\to 0} \mathscr{P}^s_\delta(\mathcal{X})$$

exists. However, this definition leads to problems concerning stability: taking, for instance, the rational numbers in the unit interval, we can find denser and denser packings yielding $\mathscr{P}^s_0(\mathbb{Q} \cap [0,1]) = \infty$ for every $0 \leqslant s < 1$, so this notion lacks countable additivity, in particular, it is not a measure. This can be overcome by applying a Caratheodory-type

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process to $\mathscr{P}^s_0$. Define

$$\mathscr{P}^s(\mathcal{X}) = \inf\left\{\sum \mathscr{P}^s_0(\mathcal{X}_i) : \mathcal{X} \subseteq \bigcup_{i\in\mathbb{N}} \mathcal{X}_i\right\}, \tag{2}$$

where the infimum is taken over arbitrary countable covers of $\mathcal{X}$. $\mathscr{P}^s$ is an (outer) measure on 2^ω, and it is Borel regular. $\mathscr{P}^s$ is called, in correspondence to Hausdorff measures, the *s-dimensional packing measure* on 2^ω. Packing measures were introduced in Tricot (1982) and Taylor and Tricot (1985). They can be seen as a dual concept to Hausdorff measures, and behave in many ways similarly to them. In particular, one may define the *packing dimension* in the same way as the Hausdorff dimension.

Definition 2.5. The *packing dimension* of a set $\mathcal{X} \subseteq 2^\omega$ is defined as

$$\dim_{\mathrm{P}} \mathcal{X} = \inf\{s : \mathscr{P}^s(\mathcal{X}) = 0\} = \sup\{s : \mathscr{P}^s(\mathcal{X}) = \infty\}. \tag{3}$$

It is not hard to see that the packing dimension of a set is always at least as large as its Hausdorff dimension. Once more we refer to Falconer (1990) for details on packing measures and dimension.

2.2. *Martingales*

It is also possible to characterise Hausdorff and packing dimension via *martingales*. This fundamental observation was first made in Lutz (2000a; 2000b) in the case of the Hausdorff dimension, and then in Athreya *et al.* (2004) for packing dimension.

Martingales have become a fundamental tool in probability theory. In Cantor space 2^ω, they can be understood as simple betting games, which is reflected in the following definition.

Definition 2.6.

(a) A *betting strategy* b is a function $b : 2^{<\omega} \to [0, 1] \times \{0, 1\}$.

(b) Given a betting strategy b and a positive real number $\alpha > 0$, the *martingale* $d^\alpha_b : 2^{<\omega} \to [0, \infty)$ *induced by* b and α is inductively defined by $d^\alpha_b(\epsilon) = \alpha$, where ϵ denotes the empty string, and

$$d^\alpha_b(wi_w) = d^\alpha_b(w)(1 + q_w), \tag{4}$$

$$d^\alpha_b(w(1 - i_w)) = d^\alpha_b(w)(1 - q_w) \tag{5}$$

for $w \in 2^{<\omega}$ and $b(w) = (q_w, i_w)$. If $\alpha = 1$, the martingale d^α_b is *normed*.

(c) A *martingale* d is a function $\{0, 1\}^* \to [0, \infty)$ that is induced by a betting strategy and some number $\alpha > 0$.

Martingales can be interpreted as capital functions of the accordant betting strategy, when applied to a binary sequence: $d(w)$ is equal to the player's capital after bits $w(0), \ldots, w(|w| - 1)$ have been revealed to him.

It is easy to check that every martingale satisfies a *fairness condition*:

$$d(w) = \frac{d(w0) + d(w1)}{2} \quad \text{for all } w \in 2^{<\omega}.$$

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This means that the betting game underlying a martingale is *fair* in the sense that the expected payoff is equal to the current capital.

Later we will use the fact that martingales are *additive*: if d_1 and d_2 are martingales, so is $d_1 + d_2$. Furthermore, if $(d_n)_{n\in\mathbb{N}}$ is a countable sequence of martingales, then $\sum_{n\in\mathbb{N}} 2^{-n} d_n(\epsilon)^{-1} d_n$ is a martingale too.

The notions of Hausdorff and packing measure zero on 2^ω can be characterised through martingales. The smaller s gets, the harder it is to cover a given set in terms of an s-dimensional Hausdorff/packing measure. This is reflected by the following winning condition for martingales.

Definition 2.7. Let $s \geqslant 0$, and d be a martingale.

(a) d is *s-successful* (*s-succeeds*) on a sequence $B \in 2^\omega$ if

$$d(B\restriction_n) \geqslant 2^{(1-s)n} \quad \text{for infinitely many } n. \tag{6}$$

(b) d is *strongly s-successful* (or *s-succeeds strongly*) on a sequence $B \in 2^\omega$ if

$$d(B\restriction_n) \geqslant 2^{(1-s)n} \quad \text{for all but finitely many } n. \tag{7}$$

The next theorem states that the relation between $\mathscr{H}^s$-null sets and s-successful martingales is indeed very close.

Theorem 2.8. Let $\mathcal{X} \subseteq 2^\omega$. Then

$$\dim_{\mathrm{H}} \mathcal{X} = \inf\{s : \text{some martingale } d \text{ is } s\text{-successful on all } B \in \mathcal{X}\}. \tag{8}$$

$$\dim_{\mathrm{P}} \mathcal{X} = \inf\{s : \text{some martingale } d \text{ is strongly } s\text{-successful on all } B \in \mathcal{X}\}. \tag{9}$$

In the form presented here, Equation (8) was first proved in Lutz (2000a). However, a close connection between Hausdorff dimension and winning conditions on martingales had been remarked on in Ryabko (1993) and Staiger (1998). Equation (9) was given in Athreya *et al.* (2004).

Note that if a martingale s-succeeds on a sequence A, for any $t > s$, we have

$$\limsup_{n\to\infty} \frac{d(A\restriction_n)}{2^{(1-t)n}} = \infty. \tag{10}$$

So, when it comes to dimension, we will, if convenient, use (10) and the original definition interchangeably. Furthermore, a martingale that satisfies (10) for $s = 1$ is simply called *successful on A*.

3. Schnorr null sets and Schnorr dimension

We now define a notion of dimension based on Schnorr's test approach to randomness. The basic idea is to extend the concept of a Schnorr test to Hausdorff measures and show that an effective version of Proposition 2.2 holds. Then the definition of the Schnorr Hausdorff dimension follows in a straightforward way. A definition of the Schnorr packing dimension based on packing measures defined via coverings would be rather involved. However, in the next section we will see that the Schnorr Hausdorff dimension can be characterised via martingales. In fact, the Schnorr Hausdorff and computable

Hausdorff dimensions coincide. Hence we can regard computable packing dimension, defined in terms of strongly successful computable martingales, as the dual to the Schnorr Hausdorff dimension.

As we are mostly interested in algorithmic notions of dimension, we only need to consider rational valued dimensions s. In this way we do not have to worry about problems of effectivity concerning real numbers.

Definition 3.1. Let $s \geqslant 0$ be a rational number.

(a) A *Schnorr s-test* is a uniformly c.e. sequence $(S_n)_{n\in\mathbb{N}}$ of sets of strings that satisfies the following conditions for all n:

(1)

$$\sum_{w\in S_n} 2^{-|w|s} \leqslant 2^{-n}. \tag{11}$$

(2) The real number $\sum_{w\in S_n} 2^{-|w|s}$ is uniformly computable in n; that is, there exists a computable function f such that for each n, i, $\left|f(n,i) - \sum_{w\in S_n} 2^{-|w|s}\right| \leqslant 2^{-i}$.

(b) A class $\mathcal{A} \subseteq 2^\omega$ is *Schnorr s-null* if there exists a Schnorr s-test (S_n) such that

$$\mathcal{A} \subseteq \bigcap_{n\in\mathbb{N}} [S_n].$$

To be compatible with the conventional notation, we say the Schnorr 1-null sets are *Schnorr null* for short. The *Schnorr random* sequences are those that are (as a singleton in 2^ω) not Schnorr null.

In Downey and Griffiths (2004), it is observed that, by adding elements, one can replace any Schnorr 1-test by an equivalent one (that is, one defining the same Schnorr null sets) in which each level of the test has measure exactly 2^{-n}. We can apply the same argument in the case of arbitrary rational s, and hence we may, if appropriate, assume that (11) holds with equality. (In this case, condition (2) in Definition 3.1 is automatically satisfied.)

Note further that, for rational s, the sets S_n in a Schnorr s-test are actually uniformly computable, since to determine whether $w \in S_n$, all we need to do is enumerate S_n until the accumulated sum given by $\sum 2^{-|w|s}$ exceeds $2^{-n} - 2^{|w|s}$ (assuming the measure of the n-th level of the test is in fact 2^{-n}). If w has not been enumerated yet, it cannot be in S_n. The converse, however, does not hold: if $W \subseteq 2^{<\omega}$ is computable, this does not necessarily imply that the measure of $[W]$ is computable.

One can also describe Schnorr s-null sets in terms of *Solovay tests*. Solovay tests were introduced in Solovay (1975) and allowed for a characterisation of Martin-Löf null sets via a single test set, instead of a uniformly computable sequence of test sets.

Definition 3.2. Let $s \geqslant 0$ be rational.

(a) A *Solovay s-test* is a c.e. set $D \subseteq 2^{<\omega}$ such that

$$\sum_{w\in D} 2^{-|w|s} \leqslant 1\,.$$

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(b) A Solovay s-test is *total* if

$$\sum_{w \in D} 2^{-|w|s}$$

is a computable real number.

(c) A Solovay s-test D *covers* a sequence $A \in 2^\omega$ if

$$(\exists^\infty w \in D)\ [w \sqsubset A].$$

In this case we also say that A *fails* the test D.

Theorem 3.3. For any rational $s \geqslant 0$, a class $\mathcal{X} \subseteq 2^\omega$ is Schnorr s-null if and only if there is a total Solovay s-test that covers every sequence $A \in \mathcal{X}$.

Proof.

($\Rightarrow$) Let $\mathcal{X}$ be Schnorr s-null via a test $(U_n)_{n\in\mathbb{N}}$. Let

$$C = \bigcup_{n\geqslant 1} U_n.$$

Obviously, C is a Solovay s-test that covers all of $\mathcal{X}$, so it remains to show that C is total. But in order to compute $c = \sum_{v\in C} 2^{-|v|s}$ with precision 2^{-n}, it is enough to compute, for $i = 1, \ldots, n+1$, the measure of U_i up to precision $2^{-(i+n+1)}$.

($\Leftarrow$) Let C be a total Solovay s-cover of $\mathcal{X}$. Given n, compute $c = \sum_{v\in C} 2^{-|v|s}$ up to precision 2^{-n-2}, say as a value $\widetilde{c}$. Now find a finite subset $\widetilde{C} \subseteq C$ such that

$$\widetilde{c} - 2^{-n-1} \leqslant \sum_{w\in\widetilde{C}} 2^{-|w|s} \leqslant \widetilde{c} - 2^{-n-2}.$$

Then $C \setminus \widetilde{C}$ covers every sequence $A \in \mathcal{X}$. Furthermore, we have

$$\sum_{w\in C\setminus\widetilde{C}} 2^{-|w|s} \leqslant 3/2^{n+2} \leqslant 1/2^n.$$

Hence, if we define $U_n = C \setminus \widetilde{C}$, the (U_n) will form a Schnorr s-test for $\mathcal{X}$. □

Note that the equivalence between Solovay and Schnorr s-tests does not extend to Martin-Löf s-tests in general. For a Martin-Löf s-test we only require the first condition (1) in Definition 3.1, but not the second one. Martin-Löf s-tests and the corresponding dimension notions have been explicitly studied in Tadaki (2002), Reimann (2004) and Calude *et al.* (2005). Implicitly, via martingales, they were already present in Lutz's introduction of effective dimension (Lutz 2000b). Solovay showed that a set $\mathcal{X} \subseteq 2^\omega$ is covered by a Martin-Löf 1-test if and only if it is covered by a Solovay 1-test. However, Reimann and Stephan (2005a) recently showed that for any rational $0 < s < 1$ there exists a sequence A that is not Martin-Löf s-null but is covered by a Solovay s-test.

3.1. *Schnorr dimension*

As in the classical case, for each class, the family of Schnorr s-measures possesses a critical value.

Proposition 3.4. Let $\mathcal{X} \subseteq 2^\omega$. Then for any rational $s \geqslant 0$, if $\mathcal{X}$ is Schnorr s-null, it is also Schnorr t-null for any rational $t \geqslant s$.

Proof. It is enough to show that if $s \leqslant t$, every Schnorr s-test (U_n) is also a Schnorr t-test. So we assume $\{U_n\}$ is a Schnorr s-test. Given any real $\alpha \geqslant 0$ and $l \in \mathbb{N}$, let

$$m_n(\alpha) := \sum_{w \in U_n} 2^{-|w|\alpha} \quad \text{and} \quad m_n^l(\alpha) := \sum_{\substack{w \in U_n \\ |w| \leqslant l}} 2^{-|w|\alpha}.$$

It is easy to check that

$$m_n^l(t) \leqslant m_n(t) \leqslant m_n^l(t) + m_n(s)2^{(s-t)l}.$$

Now $m_n(s)$ is computable, as is $2^{(s-t)l}$, and $2^{(s-t)l}$ goes to zero as l gets larger. Therefore, we can effectively approximate $m_n(t)$ to any desired degree of precision. □

The definition of the Schnorr Hausdorff dimension now follows in a straightforward way.

Definition 3.5. The *Schnorr Hausdorff dimension* of a class $\mathcal{X} \subseteq 2^\omega$ is defined as

$$\dim_{\mathrm{H}}^{\mathrm{S}} \mathcal{X} = \inf\{s \geqslant 0 : \mathcal{X} \text{ is Schnorr } s\text{-null}\}.$$

For a sequence $A \in 2^\omega$, we write $\dim_{\mathrm{H}}^{\mathrm{S}} A$ for $\dim_{\mathrm{H}}^{\mathrm{S}}\{A\}$ and refer to $\dim_{\mathrm{H}}^{\mathrm{S}} A$ as the Schnorr Hausdorff dimension of A.

Note that the Schnorr Hausdorff dimension of any sequence is at most 1, since for any $\varepsilon > 0$ the 'trivial' $1+\varepsilon$-test $W_n = \{w : |w| = l_n\}$, $l_n \geqslant \lceil n/\varepsilon \rceil$, will cover all of 2^ω.

3.2. *Schnorr packing dimension*

Because of the more involved definition of packing dimension, it is not immediately clear how we should define a Schnorr-type version of packing dimension. However, we will see in the next section that, by building on Theorem 2.8, Schnorr Hausdorff dimension allows an elegant characterisation in terms of martingales.

4. Schnorr dimension and martingales

Schnorr, in line with his unpredictability paradigm for algorithmic randomness, suggested a notion of randomness based on *computable* martingales (Schnorr 1971). According to this notion, which is nowadays referred to as *computable randomness*, a sequence is computably random if no computable martingale succeeds on it.

Schnorr proved that a sequence is Martin-Löf random if and only if no left-computable martingale succeeds on it. Therefore, one might be tempted to derive a similar relation between Schnorr random sequences and *computable* martingales. However, Schnorr pointed out that the increase in capital of a successful computable martingale can be so slow that it cannot be computably detected. Therefore, he introduced *order functions* ('Ordnungsfunktionen'), which ensure an effective control of the growth of the capital infinitely often.

In general, any non-negative, real-valued, non-decreasing unbounded function g will be called an *order function*. (It should be noted that, in Schnorr's terminology, an 'Ordnungsfunktion' is always computable.)

Definition 4.1. Let $g : \mathbb{N} \to \mathbb{R}$ be a computable order function. A martingale is g-*successful* on a sequence $B \in 2^\omega$ if

$$d(B\restriction_n) \geqslant g(n) \ \text{ for infinitely many } n.$$

Schnorr showed that Schnorr null sets can be characterised via computable martingales successful against computable orders.

Theorem 4.2 (Schnorr). A set $\mathcal{X} \subseteq 2^\omega$ is Schnorr null if and only if there exists a computable martingale d and a computable order function g such that d is g-successful on all $B \in \mathcal{X}$.

Observe that, in light of Theorem 2.8 (and the remark following (10)), a martingale being s-successful means it is g-successful for the order function $g(n) = 2^{(1-s)n}$. These are precisely what Schnorr calls *exponential orders*, so much of effective dimension is already, though apparently without explicit reference, present in Schnorr's treatment of algorithmic randomness (Schnorr 1971).

Definition 4.3. Given $B \in 2^\omega$, the *computable Hausdorff dimension* $\dim_{\mathrm{H}}^{\mathrm{comp}} B$ and the *computable packing dimension* $\dim_{\mathrm{P}}^{\mathrm{comp}} B$ are defined as follows:

$$\dim_{\mathrm{H}}^{\mathrm{comp}} B = \inf\{s : \text{ some computable martingale } d \text{ is } s\text{-successful on } B\}.$$
$$\dim_{\mathrm{P}}^{\mathrm{comp}} B = \inf\{s : \text{ some computable martingale } d \text{ is strongly } s\text{-successful on } B\}.$$

Computable Hausdorff dimension was first explicitly defined in Lutz (2000a), and computable packing dimension in Athreya *et al.* (2004).

A sequence is *computably random* if no computable martingale succeeds on it. Wang (Wang 1999) showed that the concepts of computable randomness and Schnorr randomness do not coincide. There are Schnorr random sequences on which some computable martingale succeeds. However, the differences vanish when it comes to dimension.

Theorem 4.4. For any sequence $B \in 2^\omega$,

$$\dim_{\mathrm{H}}^{\mathrm{S}} B = \dim_{\mathrm{H}}^{\mathrm{comp}} B.$$

Proof.

($\leqslant$) Suppose a computable martingale d is s-successful on B. (We may assume that $s < 1$. The case $s = 1$ is trivial. We may also assume that the d is normed.) It is enough to show that for any t such that $1 > t > s$ we can find a Schnorr t-test that covers B. We define

$$U_k^{(t)} = \left\{ \sigma : \sigma \text{ is minimal such that } \frac{d(\sigma)}{2^{(1-t)|\sigma|}} \geqslant 2^k \right\}.$$

It is easy to see that the $(U_k^{(t)})_{k\in\mathbb{N}}$ cover B. Since d is computable, the cover is effective. To show that the measure of each $U_k^{(t)}$ is at most 2^{-k}, note that an easy

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induction based on the fairness property of martingales shows that for all prefix-free sets $V \subseteq 2^{<\omega}$,

$$\sum_{\sigma \in V} d(\sigma) 2^{|\sigma|} \leqslant 1.$$

Therefore,

$$\sum_{\sigma \in U_k^{(t)}} 2^{(1-t)|\sigma|} 2^k 2^{|\sigma|} \leqslant 1,$$

and hence

$$\sum_{\sigma \in U_k^{(t)}} 2^{-t|\sigma|} \leqslant 2^{-k}.$$

The only thing left to prove is that $\sum_{w \in U_k^{(t)}} 2^{-s|w|}$ is a computable real number.

To approximate $\sum_{w \in U_k^{(t)}} 2^{-s|w|}$ within 2^{-r}, effectively find a number n such that $2^{(1-t)n} \geqslant 2^r d(\epsilon)$. If we enumerate only those strings σ into $U_k^{(t)}$ for which $|\sigma| \leqslant n$, we may conclude for the remaining strings $\tau \in U_k^{(t)}$ that $d(\tau) \geqslant 2^{(1-t)n} 2^k \geqslant 2^{r+k} d(\epsilon)$.

We now employ an inequality for martingales, which is sometimes referred to as *Kolmogorov's inequality*, but was first shown in Ville (1939). If d is a martingale, then for every $k > 0$,

$$\lambda\{B \in 2^\omega : d(B \upharpoonright_n) \geqslant k \text{ for some } n\} \leqslant \frac{d(\epsilon)}{k},$$

where λ denotes the Lebesgue measure on 2^ω. It follows that the measure induced by the strings not enumerated is at most $2^{-(r+k)}$.

($\geqslant$) Suppose $\dim_{\mathrm{H}}^{\mathrm{S}} B < s < 1$. (Again the case $s = 1$ is trivial.) We show that for any $t > s$, there exists a computable martingale d that is s-successful on B.

Let $(V_k)_{k \in \mathbb{N}}$ be a Schnorr t-test for B. Since each V_k is computable, we may assume each V_k is prefix-free. Let

$$d_k(\sigma) = \begin{cases} 2^{(1-s)|v|} & \text{if } \sigma \sqsupset v \text{ for some } v \in V_k, \\ \sum_{\sigma w \in V_k} 2^{-|w| + (1-s)(|\sigma| + |w|)} & \text{otherwise.} \end{cases}$$

We verify that d_k is a martingale. Given $\sigma \in 2^{<\omega}$, if there is a $v \in V_k$ such that $v \sqsubseteq \sigma$, we have

$$d_k(\sigma 0) + d_k(\sigma 1) = 2^{1 + (1-s)|v|}.$$

If $v \sqsubset \sigma$, then $d_k(\sigma) = 2^{(1-s)|v|}$, so $d_k(\sigma 0) + d_k(\sigma 1) = 2 d_k(\sigma)$. If $v = \sigma$, then, by definition of d_k and the fact that V_k is prefix-free, $d_k(\sigma) = 2^{(1-s)|\sigma|}$, and thus $d_k(\sigma 0) + d_k(\sigma 1) = 2 d_k(\sigma)$ holds in this case too.

If such v does not exist,

$$\begin{aligned} d_k(\sigma 0) + d_k(\sigma 1) &= \sum_{\sigma 0 w \in V_k} 2^{-|w| + (1-s)(|\sigma| + |w| + 1)} + \sum_{\sigma 1 w \in V_k} 2^{-|w| + (1-s)(|\sigma| + |w| + 1)} \\ &= \sum_{\sigma u \in V_k} 2^{(-|u| + 1) + (1-s)(|\sigma| + |u|)} = 2 d_k(\sigma). \end{aligned}$$

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Besides,

$$d_k(\epsilon) = \sum_{w \in V_k} 2^{-|w|+(1-s)|w|} = \sum_{w \in V_k} 2^{-s|w|} \leqslant 2^{-k},$$

so the function

$$d = \sum_k d_k$$

defines a martingale as well (using additivity). Finally, note that for $w \in V_k$, we have $d(w) \geqslant d_k(w) = 2^{(1-s)|w|}$. So if $B \in \bigcap_k [V_k]$, then $d(B \restriction_n) \geqslant 2^{(1-s)n}$ infinitely often, which means that d is s-successful on all $B \in \mathcal{X}$.

Since each $d_k(\epsilon) \leqslant 2^{-k}$, the computability of d follows easily from the uniform computability of each d_k, which is easily verified based on the fact that the measure of the V_k is uniformly computable. (Note that each σ can be in at most finitely many V_k.) □

An alternative proof of Theorem 4.4 could have been obtained by showing that a sequence B is Schnorr s-null if and only if there exists a computable order function g such that $d(B \restriction_n) \geqslant 2^{(1-s)n} g(n)$ infinitely often, that is, by transferring Schnorr's characterisation of Schnorr random sequences via computable martingales to the case of Hausdorff measures. Then, using (10), Theorem 4.4 follows easily.

So, in contrast to randomness, the approaches to dimension via Schnorr tests and via computable martingales yield the same concept.

In the following, we will use both names, $\dim_H^S$ and $\dim_H^{comp}$, stressing whether the reasoning follows the test or martingale approach, respectively. Theorem 4.4 justifies regarding the computable packing dimension as the dual to Schnorr Hausdorff dimension.

It follows from the definitions that for any sequence $A \in 2^\omega$, we have $\dim_H^{comp} A \leqslant \dim_P^{comp} A$. Following Tricot (1982) and Athreya *et al.* (2004), we say sequences for which the computable Hausdorff and computable packing dimensions coincide are *computably regular*. It is easy to construct a non-computably regular sequence; however, in Section 7 we will see that such sequences already occur among the class of c.e. sets.

5. Examples of Schnorr/computable dimensions

The previous results allow us to exhibit two typical examples of Schnorr dimensions. They can be seen as 'pointwise' versions of Theorem 2.4 and are Schnorr dimension analogues of two canonical examples of sequences having non-integral effective (constructive) dimension (Athreya *et al.* 2004; Reimann 2004). The first example is obtained by 'inserting' zeroes into a sequence of dimension 1. Note that it easily follows from the definitions that every Schnorr random sequence has Schnorr Hausdorff dimension one. On the other hand, it is not hard to show that not every sequence of Schnorr Hausdorff dimension 1 is also Schnorr random.

The second class of examples is based on the fact that Schnorr random sequences satisfy the law of large numbers, not only with respect to Lebesgue measure (which corresponds to the uniform Bernoulli measure on 2^ω), but also with respect to other computable

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Bernoulli distributions. Given a sequence $\vec{p} = (p_n)_{n\in\mathbb{N}}$ of real numbers, where $p_n \in (0, 1)$ for all n, the *Bernoulli measure* $\mu_{\vec{p}}$ is defined by setting

$$\mu_{\vec{p}}[\sigma] = \prod_{\sigma(i)=1} p_i \prod_{\sigma(i)=0} 1 - p_i.$$

The sequence $\vec{p}$ is called the *bias sequence* of $\mu_{\vec{p}}$. The measure $\mu_{\vec{p}}$ is *computable* if the bias sequence is a uniformly computable sequence of real numbers.

One can modify the definition of Schnorr tests to obtain randomness notions for arbitrary computable measures μ. Given a computable measure μ, a sequence is called Schnorr μ-random if it is not covered by any μ-Schnorr test.

Theorem 5.1.

(1) Let $S \in 2^\omega$ be Schnorr random and Z be a computable, infinite, co-infinite set of natural numbers such that $\delta_Z = \lim_n |\{0, \dots, n-1\} \cap Z|/n$ exists. Define a new sequence S_Z by

$$S_Z \restriction_Z = S \quad \text{and} \quad S_Z \restriction_{\overline{Z}} = 0,$$

where we use 0 here to denote the sequence consisting of zeroes only. Then,

$$\dim_{\mathrm{H}}^{\mathrm{S}} S_Z = \delta_Z.$$

(2) Let $\mu_{\vec{p}}$ be a computable Bernoulli measure on 2^ω with bias sequence $(p_0, p_1, \dots)$ such that $\lim_n p_n = p$. Then, for any Schnorr $\mu_{\vec{p}}$-random sequence B,

$$\dim_{\mathrm{H}}^{\mathrm{S}} B = -[p \log p + (1-p)\log(1-p)].$$

Part (1) of the theorem is straightforward (using for instance the martingale characterisation of Theorem 4.4); part (2) is an easy adaption of the corresponding theorem for effective (that is, Martin-Löf style) dimension (as, for example, in Reimann (2004)).

It is not hard to see that for the examples given in Theorem 5.1, the Hausdorff and packing dimensions coincide, so they describe regular sequences. In Section 7, we will see that there are highly irregular c.e. sets of natural numbers: while all c.e. sets have computable Hausdorff dimension 0, there are c.e. sets of computable packing dimension 1.

6. A machine characterisation of Schnorr dimension

One of the most cogent arguments in favour of Martin-Löf's approach to randomness is the coincidence of the Martin-Löf random sequences with the sequences that are incompressible in terms of prefix-free Kolmogorov complexity K. Furthermore, there exists a fundamental correspondence between effective Hausdorff and packing dimension, $\dim_{\mathrm{H}}^1$ and $\dim_{\mathrm{P}}^1$, respectively, and Kolmogorov complexity: for any sequence A,

$$\dim_{\mathrm{H}}^1 A = \liminf_{n\to\infty} \frac{\mathrm{K}(A\restriction_n)}{n} \quad \text{and} \quad \dim_{\mathrm{P}}^1 A = \limsup_{n\to\infty} \frac{\mathrm{K}(A\restriction_n)}{n}.$$

The first equation was first proved explicitly in Mayordomo (2002), but much of it is already present in earlier work on Kolmogorov complexity and Hausdorff dimension, such as Ryabko (1984) and Staiger (1993). The second identity is from Athreya *et al.* (2004).

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Note that in both equations one could replace prefix-free complexity K by plain Kolmogorov complexity C, since the two complexities differ only by a logarithmic factor.

To obtain a machine characterisation of Schnorr dimension, we have to restrict the admissible machines to those with domains having computable measure. Recall that a Turing machine is *prefix-free* if its domain is.

Definition 6.1. A prefix-free machine M is *computable* if

$$\sum_{w \in \mathrm{dom}(M)} 2^{-|w|} \tag{12}$$

is a computable real number.

Note that, as in the case of Schnorr tests, if a machine is computable then its domain is computable (but not *vice versa*). To determine whether $M(w)\downarrow$, we enumerate $\mathrm{dom}(M)$ until the value of $\sum_{w \in \mathrm{dom}(M)} 2^{-|w|}$ is approximated with precision 2^{-N}, where $N > |w|$. If $M(w)\downarrow$, then w must have been enumerated by this point.

The definition of machine complexity follows the standard scheme. We restrict ourselves to prefix-free machines.

Definition 6.2. Given a Turing machine M with prefix-free domain, the *M-complexity* of a string x is defined as

$$\mathrm{K}_M(x) = \min\{|p| : M(p) = x\},$$

where $\mathrm{K}_M(x) = \infty$ if there does not exist a $p \in 2^{<\omega}$ such that $M(p) = x$.

We refer to the books Li and Vitányi (1997) and Downey and Hirschfeldt (2006) for comprehensive treatments of machine (Kolmogorov) complexity. Furthermore, following Downey and Griffiths (2004), we may assume that the measure of the domain of a computable machine is 1. Namely, for each computable prefix-free machine M there exists a prefix-free machine $\widetilde{M}$ such that $\lambda(\mathrm{dom}(\widetilde{M})) = 1$, and for all $\sigma \in \mathrm{range}(M)$, $\mathrm{K}_M(\sigma) = \mathrm{K}_{\widetilde{M}}(\sigma) + \mathrm{O}(1)$ (that is, there exists a constant c such that for all $\sigma \in \mathrm{range}(M)$, $\mathrm{K}_M(\sigma)$ and $\mathrm{K}_{\widetilde{M}}(\sigma)$ differ by at most c). This can be justified by adding 'superfluous' strings to the domain and applying the Kraft–Chaitin Theorem.

Our machine characterisation of Schnorr dimension will be based on the following characterisation of Schnorr randomness by Downey and Griffiths.

Theorem 6.3 (Downey and Griffiths 2004). A sequence A is Schnorr random if and only if for every computable machine M,

$$(\exists c)\,(\forall n)\;\; \mathrm{K}_M(A\restriction_n) \geqslant n - c.$$

Building on this characterisation, we can go on to describe the Schnorr dimension as asymptotic entropy with respect to computable machines.

Theorem 6.4. For any sequence A,

$$\dim_{\mathrm{H}}^{\mathrm{S}} A = \inf_M \underline{\mathrm{K}}_M(A) \quad \text{where} \quad \underline{\mathrm{K}}_M(A) = \liminf_{n\to\infty} \frac{\mathrm{K}_M(A\restriction_n)}{n},$$

where the infimum is taken over all computable prefix-free machines M.

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Proof.

($\geqslant$) Let $s > \dim_{\mathrm{H}}^{\mathrm{S}} A$. We show that this implies $s \geqslant \underline{\mathrm{K}}_M(A)$ for some computable machine M, which yields $\dim_{\mathrm{H}}^{\mathrm{S}} A \geqslant \inf_M \underline{\mathrm{K}}_M(A)$.
As $s > \dim_{\mathrm{H}}^{\mathrm{S}} A$, there exists a Schnorr s-test $\{U_i\}$ such that $A \in \bigcap_i [U_i]$. Assume each set in the test is given as $U_n = \{\sigma_{n,1}, \sigma_{n,2}, \dots\}$. Note that the Kraft–Chaitin Theorem is applicable to the set of axioms

$$\langle \lceil s|\sigma_{n,i}|\rceil - 1, \sigma_{n,i}\rangle \quad (n \geqslant 2,\, i \geqslant 1).$$

Hence, there exists a prefix-free machine M such that for $n \geqslant 2$ and all i, we have $\mathrm{K}_M(\sigma_{n,i}) = \lceil s|\sigma_{n,i}|\rceil - 1$. Furthermore, M is computable since $\sum_{n,i} 2^{-\lceil s|\sigma_{n,i}|\rceil - 1}$ is computable.
We know that for all n there is an i_n such that $\sigma_{n,i_n} \sqsubset A$, and it is easy to see that the length of these σ_{n,i_n} goes to infinity. Hence, there must be infinitely many $m = |\sigma_{n,i_n}|$ such that

$$\mathrm{K}_M(A \upharpoonright_m) \leqslant \lceil s|\sigma_{n,i_n}|\rceil - 1 \leqslant sm,$$

which in turn implies that

$$\liminf_{n\to\infty} \frac{\mathrm{K}_M(A \upharpoonright_n)}{n} \leqslant s.$$

($\leqslant$) Suppose $s > \inf_M \underline{\mathrm{K}}_M(A)$. So there exists a computable prefix-free machine M such that $s > \underline{\mathrm{K}}_M(A)$. Define the set

$$S_M = \{w \in 2^{<\omega} : \mathrm{K}_M(w) < |w|s\}.$$

We claim that this is a total Solovay s-cover for A. It is obvious that the set covers A infinitely often, so it remains to show that

$$\sum_{w\in S_M} 2^{-|w|s}$$

is a computable real number less than or equal to 1. The latter follows from

$$\sum_{w\in S_M} 2^{-|w|s} < \sum_{w\in S_M} 2^{-K_M(w)} \leqslant 1,$$

by Kraft's inequality and the fact that M is a prefix-free machine. To show computability, given ε, compute the measure induced by $\mathrm{dom}(M)$ up to precision ε, so all strings not enumerated by that stage (call it t) will add in total at most ε to the measure of $\mathrm{dom}(M)$, which means they will also add at most ε to $\sum_{w\in S_M} 2^{-|w|s}$, and hence

$$\sum_{w\in S_{M_t}} 2^{-|w|s} \leqslant \sum_{w\in S_M} 2^{-|w|s} \leqslant \sum_{w\in S_{M_t}} 2^{-|w|s} + \varepsilon,$$

since a v contributes to S_M only if $\mathrm{K}(v) < |v|s$. But, obviously, this only happens if $v \in \mathrm{dom}(M)$.

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Hitchcock independently obtained a similar machine characterisation of Schnorr Hausdorff dimension (Hitchcock 2003). One can use a similar argument to obtain a machine characterisation of the computable packing dimension.

Theorem 6.5. For any sequence A,

$$\dim_{\mathrm{P}}^{\mathrm{comp}} A = \inf_{M} \overline{\mathrm{K}}_M(A) \quad \text{where} \quad \overline{\mathrm{K}}_M(A) = \limsup_{n\to\infty} \frac{\mathrm{K}_M(A\restriction_n)}{n},$$

where the infimum is taken over all computable prefix-free machines M.

Proof.

($\geqslant$) Let $s > \dim_{\mathrm{P}}^{\mathrm{comp}} A$. We show that this implies $s \geqslant \overline{\mathrm{K}}_M(A)$ for some computable machine M, which yields $\dim_{\mathrm{P}}^{\mathrm{comp}} A \geqslant \inf_M \overline{\mathrm{K}}_M(A)$.
So we assume d is a computable martingale that is strongly t-successful on A for some $t < s$. For each n, consider the set

$$U_n = \{w \in \{0,1\}^n : d(w) \geqslant 2^{(1-t)n}\}.$$

Then A is covered by all but finitely many U_n. Furthermore, the U_n are uniformly computable, as is the measure of each $[U_n]$. It follows from Kolmogorov's inequality that $|U_n| \leqslant 2^{nt}$. Hence

$$\sum_{w\in U_n} 2^{-|w|s} \leqslant 2^{n(t-s)}.$$

Since $t - s < 0$, we can choose an n_0 such that $\sum_{n\geqslant n_0}\sum_{w\in U_n} 2^{-|w|s} \leqslant 1/2$. Let $U = \bigcup_{n\geqslant n_0} U_n$. We can build a Kraft–Chaitin set based on the axioms

$$\langle \lceil s|w|\rceil - 1, w\rangle, \quad w \in U.$$

Then there exists a prefix-free machine M such that for all $w \in U$, we have $\mathrm{K}_M(w) \leqslant s|w|$. Furthermore, M is computable since $\sum_{w\in U} 2^{-|w|s}$ is computable. But for $n \geqslant n_0$, every prefix $A\restriction_n$ is in U, and hence

$$\limsup_{n\to\infty} \frac{\mathrm{K}_M(A\restriction_n)}{n} \leqslant s.$$

($\leqslant$) Suppose $s > \inf_M \overline{\mathrm{K}}_M(A)$. So there exists a computable prefix-free machine M such that $s > \overline{\mathrm{K}}_M(A)$. For each n, define the set

$$U_n = \{w \in \{0,1\}^n : \mathrm{K}_M(w) < |w|s\}.$$

Again, A is covered by all but finitely many U_n. For each n, define a martingale d_n as in the $\geqslant$-part of the proof of Theorem 4.4. The d_n are uniformly computable since the U_n are. We use a fundamental result by Chaitin (Chaitin 1976): for any $n,k \in \mathbb{N}$,

$$|\{w \in \{0,1\}^n : \mathrm{K}(w) \leqslant k\}| \leqslant 2^{k-\mathrm{K}(n)+\mathrm{O}(1)}.$$

Since M is prefix-free, $\mathrm{K} \leqslant \mathrm{K}_M + \mathrm{O}(1)$, and hence

$$|\{w \in \{0,1\}^n : \mathrm{K}_M(w) \leqslant k\}| \leqslant 2^{k-\mathrm{K}(n)+\mathrm{O}(1)}.$$

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It follows that for some constant c and each n,

$$d_n(\epsilon) = \sum_{w \in U_n} 2^{-s|w|} = |U_n| 2^{-sn} \leqslant 2^{-\mathrm{K}(n)+c}.$$

So $d = \sum_n d_n$ is well defined, since $\sum_n 2^{-\mathrm{K}(n)}$ is finite. A is covered by all but finitely many U_n, and for $w \in U_n$, $d(w) \geqslant d_n(w) = 2^{(1-s)|w|}$, so d is strongly s-successful on A. □

7. Schnorr dimension and computable enumerability

Usually, when studying algorithmic randomness, interest focuses on *left-computable real numbers* (also known as *c.e. reals*) rather than on *c.e. sets* (of natural numbers). The reason is that c.e. sets exhibit a trivial behaviour with respect to most randomness notions, while there are c.e. reals that are random, such as Chaitin's Ω.

As regards left-computable reals, with respect to computability, so far all notions of effective dimension show mostly the same behaviour as the corresponding notions of randomness. For instance, it has been shown in Reimann (2004) and Terwijn (2003) that every left-computable real of positive effective dimension is Turing-complete, a result that was previously known to hold for left-computable Martin-Löf random reals. For Schnorr dimension, a straightforward generalisation of a proof in Downey and Griffiths (2004), which showed that every left-computable Schnorr random real is of high degree, shows that the same holds true for left-computable reals of positive Schnorr Hausdorff dimension. That is, if A is left-computable and $\dim_{\mathrm{H}}^{\mathrm{S}} A > 0$, then $A' \equiv_{\mathrm{T}} 0''$.

As regards *computably enumerable sets* (of natural numbers), they are usually, in the context of algorithmic randomness, of marginal interest, since they exhibit a rather non-random behaviour. For instance, it is easy to see that no computably enumerable set can be Schnorr random.

Proposition 7.1. No computably enumerable set is Schnorr random.

Proof. Every infinite c.e. set contains an infinite computable subset. So, given an infinite c.e. set $A \subseteq \mathbb{N}$, we choose some computable infinite subset B. Assume $B = \{b_1, b_2, \dots\}$, with $b_i < b_{i+1}$.

Define a Schnorr test $\{V_n\}$ for A as follows: at level n, put all those strings v of length $b_n + 1$ into V_n for which

$$v(b_i) = 1 \quad \text{for all } i \leqslant n + 1.$$

Then, surely, $A \in [V_n]$ for all n, and $\lambda[V_n] = 2^{-n}$. □

It is not clear how we can improve the preceding result to Schnorr dimension zero. Indeed, defining coverings from the enumeration of a set directly might not work, because, due to the dimension factor in Hausdorff measures, longer strings will be weighted higher. Depending on how the enumeration is distributed, this might not lead to a Schnorr s-covering at all.

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However, one can exploit the somewhat predictable nature of a c.e. set to define a computable martingale that is, for any $s > 0$, s-successful on the characteristic sequence of the enumerable set, thereby ensuring that each c.e. set has computable dimension 0.

Theorem 7.2. Every computably enumerable set $A \subseteq \mathbb{N}$ has Schnorr Hausdorff dimension zero.

Proof. Given rational $s > 0$, we show that there exists a computable martingale d such that d is s-successful on A.

First, we partition the natural numbers into effectively given, disjoint intervals I_n such that $|I_n| \ll |I_{n+1}|$, for instance, $|I_n| = 2^{|I_0|+\cdots+|I_{n-1}|}$. Set $i_n = |I_n|$ and $j_n = i_0 + i_1 + \cdots + i_n$.

We use δ to denote the upper density of A on I_n, that is,

$$\delta = \limsup_{n\to\infty} \frac{|A \cap I_n|}{i_n}.$$

Without loss of generality, we may assume that $\delta > 0$. For any $\varepsilon > 0$ with $\varepsilon < \delta$ there is a rational number r such that $\delta - \varepsilon < r < \delta$. Given such an r, there must be infinitely many n_k for which

$$|A \cap I_{n_k}| > r i_{n_k}.$$

We now define a computable martingale d by describing an accordant betting strategy as follows. At stage 0, initialise with $d(\epsilon) = 1$. At stage $k + 1$, assume d is defined for all τ with $|\tau| \leqslant l_k$ for some $l_k \in \mathbb{N}$. Enumerate A until we know, for some interval I_{n_k} with $j_{n_k-1} > l_k$ (that is, I_{n_k} has not been bet on before), that

$$|A \cap I_{n_k}| > r i_{n_k}.$$

For all strings σ with $l_k < |\sigma| \leqslant j_{n_k-1}$, bet nothing (that is, d remains constant here). Fix a (rational) stake $\gamma > 2^{1-s} - 1$. On I_{n_k}, bet γ on the mth bit being 1 ($j_{n_k-1} < m \leqslant j_{n_k}$) if m has already been enumerated into A. Otherwise, bet γ on the mth bit being 0. Set $l_{k+1} = j_{n_k}$.

When betting against A, it is obvious that this strategy will lose at most $\lceil 2\varepsilon \rceil |I_{n_k}|$ times on I_{n_k}. Thus, for all sufficiently large n_k,

$$\begin{aligned} d(A \restriction_{l_{k+1}}) &\geqslant d(A \restriction_{l_k})(1+\gamma)^{i_{n_k} - \lceil 2\varepsilon \rceil i_{n_k}} (1-\gamma)^{\lceil 2\varepsilon \rceil i_{n_k}} \\ &= d(A \restriction_{l_k})(1+\gamma)^{i_{n_k}} \left(\frac{1-\gamma}{1+\gamma}\right)^{\lceil 2\varepsilon \rceil i_{n_k}} > 2^{(1-s) i_{n_k}} \left(\frac{1-\gamma}{1+\gamma}\right)^{\lceil 2\varepsilon \rceil i_{n_k}}. \end{aligned}$$

Choosing ε small enough and n large enough, we see that d is s-successful on A. □

On the other hand, it is not hard to see that for every Schnorr 1-test there is a c.e. set that is not covered by it. This means that the class of all c.e. sets has Schnorr Hausdorff dimension 1. For effective Hausdorff dimension, Lutz (Lutz 2003) showed that for any class $\mathcal{X} \subseteq 2^\omega$,

$$\dim^1_{\mathrm{H}} \mathcal{X} = \sup\{\dim^1_{\mathrm{H}} A : A \in \mathcal{X}\}.$$

This means that effective dimension has a strong *stability* property. The class of c.e. sets yields an example where stability fails for Schnorr dimension.

In contrast to Theorem 7.2, and perhaps somewhat surprisingly, the upper Schnorr entropy of c.e. sets can be as high as possible, namely, there exist c.e. sets with computable packing dimension 1. This stands in sharp contrast to the case of effective dimension, where J. M. Barzdiņš' Theorem (Barzdiņš 1968) ensures that all c.e. sets have effective packing dimension 0. Namely, if A is a c.e. set, there exists a c such that for all n, we have $C(A\restriction_n) \leqslant \log n + c$.

In fact, it can be shown that every hyperimmune degree contains a set of computable packing dimension 1. As the proof of the theorem shows, this holds mainly because of the requirement that all machines involved in the determination of Schnorr dimension are total.

Before giving the proof, however, it should be mentioned that there are degrees that do not contain any sequence of high computable packing dimension. This can be shown by a straightforward construction.

Theorem 7.3. For any hyperimmune set B there exists a set $A \equiv_T B$ such that

$$\dim_P^{\mathrm{comp}} A = 1.$$

Furthermore, if the set B is c.e., A can be chosen to be c.e. also.

Proof. For given B, it suffices to construct a set $C \leqslant_T B$ such that $\dim_P^{\mathrm{comp}} C = 1$, and to let, for some computable set of places Z of sublinear density, the set A be a join of B and C where B is coded into the places in Z in the sense that

$$A\restriction_Z = B \quad \text{and} \quad A\restriction_{\overline{Z}} = C;$$

a similar argument works for the case of c.e. sets.

So fix any hyperimmune set B. Then there is a function g computable in B such that for any computable function f there are infinitely many n such that $f(n) < g(n)$. We now partition the natural numbers into effectively given, pairwise disjoint intervals

$$\mathbb{N} = I_0 \cup I_1 \cup I_2 \cup \dots$$

such that $|I_0| + \ldots + |I_n| \ll |I_{n+1}|$ for all n; for instance, choose I_n such that $|I_{n+1}| = 2^{|I_0|+\cdots+|I_n|}$, and let $i_n = |I_n|$. Furthermore, let $M_0, M_1, \ldots$ be a standard enumeration of all prefix-free (not necessarily computable) Turing machines with uniformly computable approximations $M_e[s]$.

For any pair of indices e and n, when

$$\sum_{M_e[g(n)](w)\downarrow} 2^{-|w|} \leqslant 1 - 2^{-i_{\langle e,n\rangle}}, \tag{13}$$

we let C have an empty intersection with the interval $I_{\langle e,n\rangle}$.

Otherwise, when (13) is false, any string of length $i_{\langle e,n\rangle}$ not output by M_e at stage $g(n)$ via an M_e-program of length at most $i_{\langle e,n\rangle}$ is M_e-incompressible in the sense that the string has M_e-complexity of at least $i_{\langle e,n\rangle}$; we pick such a string σ and let $C\restriction_{I_{\langle e,n\rangle}} = \sigma$ (if there is no such string, the domain of the prefix-free machine M_e contains exactly the finitely many strings of length $i_{\langle e,n\rangle}$ and we do not have to worry about M_e). Observe that $C \leqslant_T B$ because g is computable in B.

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For any M_e with domain of measure one, the function f_e that maps n to the first stage t such that

$$\sum_{M_e[t](w)\downarrow} 2^{-|w|} > 1 - 2^{-i_{\langle e,n\rangle}} \tag{14}$$

is total, and, in fact, computable; hence there are infinitely many n such that $f_e(n) < g(n)$, and for all these n, the restriction of C to $I_{\langle e,n\rangle}$ is M_e-incompressible. To see that this ensures computable packing dimension 1, suppose

$$\dim_{\mathrm{P}}^{\mathrm{comp}} C < 1.$$

Then there exists a computable machine M, an $\varepsilon > 0$ and some $n_\varepsilon \in \mathbb{N}$ such that

$$(\forall n \geqslant n_\varepsilon)\ [\mathrm{K}_M(C\restriction_n) \leqslant (1-\varepsilon)n].$$

We define another total machine $\widetilde{M}$ with the same domain as M: given x, we compute $M(x)$. If $M(x)\downarrow$, check whether $|M(x)| = i_0 + i_1 + \cdots + i_k$ for some k. If it does, output the last i_k bits, otherwise output 0. Let e be an index of $\widetilde{M}$. By choice of the i_k, for all sufficiently large n, the $\widetilde{M}$-complexity of $C\restriction_{I_{\langle e,n\rangle}}$ can be bounded as follows:

$$\mathrm{K}_{\widetilde{M}}(C\restriction_{I_{\langle e,n\rangle}}) \leqslant \mathrm{K}_M(C\restriction_{I_{\langle e,0\rangle}\cup\ldots\cup I_{\langle e,n\rangle}}) \leqslant (1-\varepsilon)(i_{\langle e,0\rangle} + \cdots + i_{\langle e,n\rangle}) \leqslant (1-\frac{\varepsilon}{2})i_{\langle e,n\rangle}\,.$$

This contradicts the fact that, by construction, there are infinitely many n such that the restriction of C to the interval $I_{\langle e,n\rangle}$ is M_e-incompressible, that is, $\widetilde{M}$-incompressible.

In the case of a non-computable c.e. set B, it is not hard to see that we obtain a function g as above if we let $g(n)$ be equal to the least stage such that some fixed effective approximation to B agrees with B at place n. Using this function g in the construction above, the set C becomes c.e. because for any index e and for all n, when n is not in B, the restriction of C to the interval $I_{\langle e,n\rangle}$ is empty, but when it is in B, we just wait for the stage $g(n)$ such that n enters B, then compute from $g(n)$ the restriction of C to the interval $I_{\langle e,n\rangle}$, and, finally, enumerate all the elements of C in this interval. □

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