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NUMERICAL RANGES IN II_1 FACTORS

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Abstract In this paper we generalize the notion of the C-numerical range of a matrix to operators in arbitrary tracial von Neumann algebras. For each self-adjoint operator C, the C-numerical range of such an operator is defined; it is a compact, convex subset of $\mathbb{C}$. We explicitly describe the C-numerical ranges of several operators and classes of operators.

Keywords: II_1 Factors; Numerical Range; Generalized Numerical Range

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1. Introduction

A rich invariant of an operator is its numerical range. Given a Hilbert space $\mathcal{H}$ and a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$, the *numerical range* of T is the set of complex numbers

$$W_1(T) = \{\langle T\xi, \xi\rangle_{\mathcal{H}} \mid \xi \in \mathcal{H}, \|\xi\|_{\mathcal{H}} = 1\}.$$

The Hausdorff-Toeplitz Theorem (see [**16**] for a short and delightful proof; see references therein for historical background) states that the numerical range of an operator is always a convex subset. Furthermore, when restricting to finite dimensional $\mathcal{H}$, the numerical range of a matrix is compact and can be used to obtain several interesting structural results, such as that a matrix of trace zero is always unitarily equivalent to a matrix with zeros along the diagonal.

The numerical range of a matrix is often substantially larger than the spectrum and yields cruder information about the matrix. For example, if N is a normal matrix, then $W_1(N)$ is the convex hull of the eigenvalues of N. Therefore, precise information about the eigenvalues of N cannot be obtained from $W_1(N)$.

In [**17**], Paul Halmos proposed a generalization of the numerical range of a matrix. For each $\xi \in \mathbb{C}^n$ with $\|\xi\|_2 = 1$ and $T \in \mathcal{M}_n(\mathbb{C})$, we have

$$\langle T\xi, \xi\rangle_{\mathbb{C}^n} = \text{Tr}(TP_\xi),$$

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where Tr is the (unnormalized) trace and $P_\xi \in \mathcal{M}_n(\mathbb{C})$ is the rank one projection onto $\mathbb{C}\xi$. Thus, for $T \in \mathcal{M}_n(\mathbb{C})$ and $k \in \{1, \ldots, n\}$, the *k-numerical range of T* is defined as

$$W_k(T) = \left\{ \frac{1}{k}\mathrm{Tr}(TP) \mid P \in \mathcal{M}_n(\mathbb{C}) \text{ a projection of rank } k \right\}.$$

C. A. Berger showed, using the Hausdorff-Toeplitz Theorem and the fact that $W_1(T)$ is convex, that each $W_k(T)$ is a convex set (see [**17**, Solution 211]). Operators' k-numerical ranges have been extensively studied and much is known. For example [**13**, Theorem 1.2] shows that

$$W_k(T) = \frac{1}{k}\{\mathrm{Tr}(TX) \mid 0 \leq X \leq I_n, \mathrm{Tr}(X) = k\}.$$

It is clear that the set on the right-hand-side of the above equation is a convex set, yet this did not produce an new proof of Berger's result as [**13**, Theorem 1.2] relied on Berger's result. These k-numerical ranges provide substantially more information about a matrix than the numerical range alone. Indeed, if $N \in \mathcal{M}_n(\mathbb{C})$ is a normal matrix with eigenvalues $\{\lambda_j\}_{j=1}^n$ listed according to their multiplicities, then, by [**13**, Theorem 1.5], the k-numerical range of N is the convex hull of the set

$$\left\{ \frac{1}{k} \sum_{j \in K} \lambda_j \mid J \subseteq \{1, \ldots, n\}, |J| = k \right\}.$$

By varying k, these sets provide enough information to determine the eigenvalues of N and, thus, to determine N up to unitary equivalence.

In [**36**], Westwick analyzed a generalization of the k-numerical ranges of a matrix which was later further generalized by Golberg and Straus in [**15**]. Given two matrices $C, T \in \mathcal{M}_n(\mathbb{C})$, the *$C$-numerical range of T* is defined to be the set

$$W_C(T) = \{\mathrm{Tr}(TU^*CU) \mid U \in \mathcal{M}_n(\mathbb{C}) \text{ a unitary}\}. \tag{1.1}$$

It is not difficult to see that if $C_k \in \mathcal{M}_n(\mathbb{C})$ is a matrix with $1/k$ along the diagonal precisely k times and zeros elsewhere, then $W_{C_k}(T) = W_k(T)$. Thus, the C-numerical ranges are indeed generalizations of the k-numerical ranges.

Using ideas from [**19**], Westwick in [**36**] demonstrated that if $C \in \mathcal{M}_n(\mathbb{C})$ is self-adjoint, then $W_C(T)$ is a convex set. However, Westwick also showed that if $C = \mathrm{diag}(0, 1, i) \in \mathcal{M}_3(\mathbb{C})$, then $W_C(C)$ is not convex. Based on [**36**] and [**15**], in [**33**] Poon gave another proof that the C-numerical ranges are convex for self-adjoint $C \in \mathcal{M}_n(\mathbb{C})$. Poon's work gave an alternate description of the C-numerical range based on a notion of majorization for n-tuples of real numbers. This notion of majorization is the one appearing in a classical theorem of Schur ([**34**]) and Horn ([**23**]) characterizing the possible diagonal n-tuples of a self-adjoint matrix based on its eigenvalues.

As the notion of majorization has an analogue in arbitrary tracial von Neumann algebras, the goal of this paper is to examine C-numerical ranges in arbitrary von Neumann algebras. In light of the example of Westwick given above, we will restrict our attention to self-adjoint C. Furthermore, we note that analogues of the k-numerical ranges inside

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diffuse von Neumann algebras have been previously studied in [**1**–**4**]. Consequently, the results contained in this paper are a mixture of generalizations of results from [**1**–**4**], new proofs of results in [**1**–**4**], and additional results. This paper contains a total of six sections, including this one, and is structured as follows.

Section 2 begins by recalling a notion of majorization for elements of $L^\infty[0,1]$. The generalization of C-numerical ranges to tracial von Neumann algebras is then obtained by applying majorization to eigenvalue functions of self-adjoint operators. After many basic properties of C-numerical ranges are demonstrated, several important results, such as the fact that C-numerical ranges are independent of the von Neumann algebra under consideration, are obtained. Of importance are the results that C-numerical ranges are always compact, convex sets of $\mathbb{C}$ and, if one restricts to type II_1 factors, one can define C-numerical ranges using the closed unitary orbit of C instead of the notion of majorization. In addition, we demonstrate the C-numerical range of T is continuous in both C and T, and we demonstrate results from [**1**–**4**] that follow immediately from this different view.

Section 3 is dedicated to describing the C-numerical ranges of self-adjoint operators via eigenvalue functions. This is particularly important for Section §4 which demonstrates a method for computing C-numerical ranges of operators based on knowledge of C-numerical ranges of self-adjoint operators. This is significant as numerical ranges of matrices are often difficult to compute (see [**27**] for the 3×3 case).

Section 5 computes α-numerical ranges (i.e. the generalization of the k-numerical range of a matrix) for several operators. Although computing the k-numerical ranges of a matrix is generally a hard task, there are several interesting examples of operators in II_1 factors whose α-numerical ranges can be explicitly described. In particular, we demonstrate the existence of normal and non-normal operators whose α-numerical ranges agree for all α.

Section 6 concludes the paper by examining the relationship between α-numerical ranges and conditional expectations of operators onto subalgebras. In particular, we demonstrate that a scalar λ is in the α-numerical range of an operator T in a II_1 factor if and only if there exists diffuse abelian von Neumann subalgebra $\mathcal{A}$ such that the trace of the spectral projection of the expectation of T onto $\mathcal{A}$ corresponding to the set $\{\lambda\}$ is at least α.

2. Definitions and basic results

In this section we generalize the notion of the C-numerical range of a matrix to tracial von Neumann algebras (for self-adjoint C) thereby obtaining more general numerical ranges than those considered in [**1**–**4**]. The C-numerical range of an operator is a compact, convex set defined using a notion of majorization for eigenvalue functions of self-adjoint operators and is described via an equation like equation (1.1) inside II_1 factors. Many properties of C-numerical ranges will be demonstrated including continuity results and lack of dependence on the von Neumann algebra considered.

Throughout this paper, $(\mathfrak{M}, \tau)$ will denote a von Neumann algebra $\mathfrak{M}$ possessing a normal, faithful, tracial state, with τ such a state. We will call such a pair *a tracial von Neumann algebra*. Furthermore, $\mathrm{Proj}(\mathfrak{M})$ will denote the set of projections in $\mathfrak{M}$ and $\mathfrak{M}_{\mathrm{sa}}$ will be used to denote the set of self-adjoint elements of $\mathfrak{M}$.

To begin, we will need a concept whose origin is due to Hardy, Littlewood, and Pólya.

Definition 2.1 (see [18]). Let $f, g \in L^\infty[0,1]$. It is said that f *majorizes* g, denoted $g \prec f$, if

$$\int_0^t g^*(x)\,dx \leq \int_0^t f^*(x)\,dx \text{ for all } t \in [0,1] \qquad \text{and} \qquad \int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx,$$

where g^* and f^* are the nonincreasing rearrangements of g and f (see Definition 3.3).

Note if $g \prec f$ and $h \prec g$, one clearly has $h \prec f$.

We now review an analogue of eigenvalues for self-adjoint operators in tracial von Neumann algebras that was introduced by Murray and von Neumann [31]. For this section and the rest of the paper, given an normal operator N in a von Neumann algebra, we will use $1_X(N)$ to denote the spectral projection of N corresponding to a Borel set $X \subseteq \mathbb{C}$.

Definition 2.2. Let $(\mathfrak{M}, \tau)$ be a diffuse, tracial von Neumann algebra and let $T \in \mathfrak{M}$ be self-adjoint. The *eigenvalue function of* T (also called its *spectral scale*) is defined for $s \in [0,1)$ by

$$\lambda_T(s) = \inf\{t \in \mathbb{R} \mid \tau(1_{(t,\infty)}(T)) \leq s\}.$$

A related notion we will use is that of the *spectral distribution* of a normal element $N \in \mathfrak{M}$, which is the Borel probability measure $X \mapsto \tau(1_X(N))$ supported on the spectrum of N.

It is elementary to verify that the eigenvalue function of T is a bounded, non-increasing, right continuous function from $[0,1)$ to $\mathbb{R}$. By [32 Theorem 1], if $\mathfrak{M}$ is represented on a Hilbert space $\mathcal{H}$, then we have

$$\lambda_T(s) = \sup_e\{\langle T\xi, \xi\rangle \mid \xi \in \mathcal{H},\ \|\xi\| = 1,\ e\xi = \xi\},$$

where the supremum is taken over all projections $e \in \mathfrak{M}$ such that $\tau(1-e) \leq s$. The following results are easily proved.

Proposition 2.3. *Let $T \in \mathfrak{M}$. Then*

(i) *if $a \geq 0$, then $\lambda_{aT}(s) = a\lambda_T(s)$ for all $s \in [0,1)$,*

(ii) *if $a \leq 0$, then $\lambda_{aT}(s) = a\lambda_T(1-s)$ for all but at most countably many $s \in (0,1)$,*

(iii) *if $a \in \mathbb{R}$, then $\lambda_{aI+T}(s) = a + \lambda_T(s)$ for all $s \in [0,1)$.*

The following result is seemingly folklore, and a proof may be found in [6, Proposition 2.3].

Proposition 2.4. *Let $(\mathfrak{M}, \tau)$ be a diffuse, tracial von Neumann algebra and let $T \in \mathfrak{M}$ be self-adjoint. Then there is a projection-valued measure e_T on $[0,1)$ valued in $\mathfrak{M}$ such that $\tau(e_T([0,t))) = t$ for every $t \in [0,1)$ and*

$$T = \int_0^1 \lambda_T(s)\,de_T(s).$$

In particular $\tau(T) = \int_0^1 \lambda_T(s)\,ds$.

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Remark 2.5. Note the von Neumann algebra generated by $\{e_T([0,t))\}_{t\in[0,1)}$ is isomorphic to a copy of $L^\infty[0,1]$ inside $\mathfrak{M}$ in such a way that T corresponds to the L^∞-function $s \mapsto \lambda_T(s)$ and τ restricts to integration against the Lebesgue measure m.

Using the above definitions, we may now define the main objects of study in this paper.

Definition 2.6. Let $(\mathfrak{M}, \tau)$ be a tracial von Neumann algebra and let $C \in \mathfrak{M}_{\mathrm{sa}}$. The C*-numerical range* of an element $T \in \mathfrak{M}$ is the set

$$V_C(T) := \{\tau(TX) \mid X \in \mathfrak{M}_{\mathrm{sa}}, \lambda_X \prec \lambda_C\}.$$

Remark 2.7. Thereom 2.14 gives an equivalent characterization of $V_C(T)$ that is analogous to the classical definition (1.1) in the case of matrices.

Remark 2.8. It is not difficult to verify that if $(\mathfrak{M}, \tau)$ is a tracial von Neumann algebra, if $T, S \in \mathfrak{M}_{\mathrm{sa}}$ with T positive, and if $\lambda_S \prec \lambda_T$, then S must be positive. In addition, it is not difficult to show that if $P \in \mathfrak{M}$ is a projection with $\tau(P) = \alpha \in [0,1]$, then

$$\{X \in \mathfrak{M}_{\mathrm{sa}} \mid \lambda_X \prec \lambda_P\} = \{X \in \mathfrak{M} \mid 0 \leq X \leq I_{\mathfrak{M}}, \tau(X) = \alpha\}.$$

In analogy, for $\alpha \in (0,1]$ and $T \in \mathfrak{M}$, we define the α*-numerical range of* T to be the set

$$\widetilde{V}_\alpha(T) := \frac{1}{\alpha}\{\tau(TX) \mid X \in \mathfrak{M}, 0 \leq X \leq I_{\mathfrak{M}}, \tau(X) = \alpha\}.$$

Thus, we have $\widetilde{V}_\alpha(T) = (1/\alpha)V_P(T)$, where P is as described above. The α-numerical ranges were originally studied (through a multivariate analogue for commuting n-tuples of self-adjoint operators) in the papers [**1**–**4**] and the $1/\alpha$ factor is included so that if $0 < \alpha < \beta \leq 1$ then $\widetilde{V}_\beta(T) \subseteq \widetilde{V}_\alpha(T)$.

The following contains a collection of important properties of C-numerical ranges that mainly follow from properties of eigenvalue functions contained in [**11**, **12**, **32**]. Note for two subsets X, Y of $\mathbb{C}$ and $\omega \in \mathbb{C}$, we define

$$\omega X = \{\omega z \mid z \in X\},$$
$$\omega + X = \{\omega + z \mid z \in X\}$$

and

$$X + Y = \{z + w \mid z \in X, w \in Y\}.$$

Proposition 2.9. *Let $(\mathfrak{M}, \tau)$ be a tracial von Neumann algebra, let $T, S \in \mathfrak{M}$, and let $C, C_1, C_2 \in \mathfrak{M}_{\mathrm{sa}}$. Then*

(i) *$V_C(T)$ is a convex set for all $T \in \mathfrak{M}$,*

(ii) *$V_C(T^*)$ equals the complex conjugate of $V_C(T)$,*

(iii) *$V_C(\mathrm{Re}(T)) = \{\mathrm{Re}(z) \mid z \in V_C(T)\}$ and $V_C(\mathrm{Im}(T)) = \{\mathrm{Im}(z) \mid z \in V_C(T)\}$,*

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(iv) $V_C(T+S) \subseteq V_C(T) + V_C(S)$,

(v) $V_C(zI_{\mathfrak{M}} + wT) = z\tau(C) + wV_C(T)$ *for all* $z, w \in \mathbb{C}$,

(vi) $V_C(U^*TU) = V_C(T)$ *for all unitaries* $U \in \mathfrak{M}$,

(vii) $V_{C_1}(T) \subseteq V_{C_2}(T)$ *whenever* $\lambda_{C_1} \prec \lambda_{C_2}$, *and*

(viii) $V_{aC+bI_{\mathfrak{M}}}(T) = aV_C(T) + b\tau(T)$ *for all* $a, b \in \mathbb{R}$.

Proof. For part (2.9), notice that if $X_1, X_2 \in \mathfrak{M}_{\mathrm{sa}}$ are such that $\lambda_{X_1}, \lambda_{X_2} \prec \lambda_C$, then

$$\lambda_{tX_1+(1-t)X_2} \prec t\lambda_{X_1} + (1-t)\lambda_{X_2} \prec \lambda_C$$

for all $t \in [0,1]$ by [**12**, Lemma 2.5(ii)] and [**12**, Theorem 4.4(ii)], where we have used Proposition 2.3 in order to assume all three operators are positive. Hence it trivially follows that

$$\{X \in \mathfrak{M}_{\mathrm{sa}} \mid \lambda_X \prec \lambda_C\}$$

is a convex set; thus, $V_C(T)$ is convex (being the image under a linear map of a convex set).

Except for parts (2.9) and (2.9), the other parts are trivial computations. To see part (2.9), note $\lambda_{U^*CU} = \lambda_C$ for all unitaries $U \in \mathfrak{M}$ and all $C \in \mathfrak{M}_{\mathrm{sa}}$. Part (2.9) follows easily from the properties in Proposition 2.3. □

Our next goal is to show the very useful property that the C-numerical ranges of an operator do not depend on the ambient von Neumann algebra. To do so, we recall the following result, which is a consequence of any of [**25**, Theorem 3], [**20**, Theorem 4.5] and [**21**, Theorem 2.1].

Theorem 2.10. *Let* $(\mathfrak{M}, \tau)$ *be a tracial von Neumann algebra, let* $\mathfrak{N}$ *be a von Neumann subalgebra of* $\mathfrak{M}$, *and let* $E_{\mathfrak{N}} : \mathfrak{M} \to \mathfrak{N}$ *be the trace-preserving conditional expectation of* $\mathfrak{M}$ *onto* $\mathfrak{N}$. *Then* $\lambda_{E_{\mathfrak{N}}(X)} \prec \lambda_X$ *for all* $X \in \mathfrak{M}_{\mathrm{sa}}$.

Proposition 2.11. *Let* $(\mathfrak{M}, \tau)$ *be a tracial von Neumann algebra and let* $C \in \mathfrak{M}_{\mathrm{sa}}$. *For* $T \in \mathfrak{M}$ *let* $V_C(T)$ *denote the* C*-numerical range as given in Definition 2.6. Let* $\mathfrak{N}$ *be a von Neumann subalgebra of* $\mathfrak{M}$ *such that* $T \in \mathfrak{N}$. *Then*

$$V_C(T) = \{\tau(TX) \mid X \in \mathfrak{N}_{\mathrm{sa}}, \lambda_X \prec \lambda_C\}. \tag{2.1}$$

In particular, $V_C(T)$ *does not depend on the diffuse tracial von Neumann algebra considered.*

Proof. The inclusion $\supseteq$ in (2.1) is clear. For the reverse inclusion, let $E_{\mathfrak{N}} : \mathfrak{M} \to \mathfrak{N}$ denote the trace-preserving conditional expectation of $\mathfrak{M}$ onto $\mathfrak{N}$. If $X \in \mathfrak{M}_{\mathrm{sa}}$ is such that $\lambda_X \prec \lambda_C$, then $E_{\mathfrak{N}}(X) \in \mathfrak{N}$, $\lambda_{E_{\mathfrak{N}}(X)} \prec \lambda_X \prec \lambda_C$ by Theorem 2.10, and

$$\tau(TE_{\mathfrak{N}}(X)) = \tau(E_{\mathfrak{N}}(TX)) = \tau(TX).$$

This proves (2.1). □

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By Proposition 2.11, we may compute the C-numerical ranges in any tracial von Neumann algebra we like. In particular, as every tracial von Neumann algebra embeds in a trace-preserving way into a type II_1 factor, we may restrict our attention to type II_1 factors when considering C-numerical ranges. By doing so, we will obtain an alternate description of C-numerical ranges that is a direct analogue of equation (1.1) and produces many corollaries. We begin with the following.

Definition 2.12. Let $\mathfrak{A}$ be an arbitrary C*-algebra and let $\mathcal{U}(\mathfrak{A})$ denote the unitary group of $\mathfrak{A}$. For $T \in \mathfrak{A}$, the *unitary orbit of* T is the set

$$\mathcal{U}(T) = \{U^*TU \mid U \in \mathcal{U}(\mathfrak{A})\}$$

and the (norm-)closed unitary orbit of T is the set $\mathcal{O}(T) = \overline{\mathcal{U}(T)}^{\|\cdot\|}$.

Remark 2.13. Notice if $T, S \in \mathfrak{M}$ are self-adjoint operators then $\lambda_T \prec \lambda_S$ and $\lambda_S \prec \lambda_T$ if and only of $\lambda_T(s) = \lambda_S(s)$ for all $s \in [0, 1)$. By Definition 2.2, these are equivalent to T and S having the same spectral distribution. It is well-known that these are all equivalent to $T \in \mathcal{O}(S)$, provided $\mathfrak{M}$ is a type II_1 factor.

Notice that if $\mathfrak{A}$ is a finite dimensional C*-algebra, then $\mathcal{U}(T) = \mathcal{O}(T)$. In general, $\mathcal{O}(T)$ is the correct object to consider when studying infinite dimensional C*-algebras. In particular, we will use $\mathcal{O}(T)$ to generalize equation (1.1) to type II_1 factors. In particular, the work of [**15**, **33**] proves the following result when $\mathfrak{M}$ is a matrix algebra.

Theorem 2.14. *Let $(\mathfrak{M}, \tau)$ be a type II_1 factor and let $C \in \mathfrak{M}_{\mathrm{sa}}$. Then for all $T \in \mathfrak{M}$,*

$$V_C(T) = \{\tau(TX) \mid X \in \mathfrak{M}_{\mathrm{sa}}, X \in \mathcal{O}(C)\}.$$

To prove Theorem 2.14, we will need two results. The first is the following connection between majorization of eigenvalue functions and convex hulls of unitary orbits.

Theorem 2.15 (see [5, 7, 20, 22, 24–26]). *Let $(\mathfrak{M}, \tau)$ be a factor and let $X, T \in \mathfrak{M}_{\mathrm{sa}}$. Then the following are equivalent:*

(1) $\lambda_X \prec \lambda_T$;

(2) $X \in \overline{\mathrm{conv}(\mathcal{U}(T))}^{\|\cdot\|}$ *and*

(3) $X \in \overline{\mathrm{conv}(\mathcal{U}(T))}^{w^*}$.

The second result required to prove Theorem 2.14 is the following technical result, whose proof is contained in the proof of [**10**, Theorem 5.3] and follows by simple manipulations of functions.

Proposition 2.16 ([8, Theorem 5.3]). *Let $(\mathfrak{M}, \tau)$ be a type II_1 factor and let $A, C \in \mathfrak{M}$ be self-adjoint operators such that $\lambda_A \prec \lambda_C$ and $A \notin \mathcal{O}(C)$. Then there exists a non-zero projection $P \in \mathfrak{M}$ and an $\epsilon > 0$ such that $\lambda_{A+S} \prec \lambda_C$ for all self-adjoint operators $S \in \mathfrak{M}$ satisfying $\|S\| < \epsilon$, $S = PS = SP$, and $\tau(S) = 0$.*

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Proof of Theorem 2.14. Fix $C \in \mathfrak{M}_{\mathrm{sa}}$ and $T \in \mathfrak{M}$. Then

$$\{\tau(TX) \mid X \in \mathfrak{M}_{\mathrm{sa}}, X \in \mathcal{O}(C)\} \subseteq V_C(T)$$

by Remark 2.13 and Definition 2.6.

For the other inclusion, fix $X \in \mathfrak{M}_{\mathrm{sa}}$ with $\lambda_X \prec \lambda_C$ and define

$$Q_{X,C} = \{Y \in \mathfrak{M}_{\mathrm{sa}} \mid \tau(TY) = \tau(TX),\ \lambda_Y \prec \lambda_C\}.$$

Since the linear map $Z \mapsto \tau(TZ)$ is weak*-continuous, by Theorem 2.15 $Q_{X,C}$ is a non-empty (as $X \in Q_{X,C}$), convex, weak*-compact subset. Hence, by the Krein–Milman Theorem, $Q_{X,C}$ has an extreme point, say A.

We will show $A \in \mathcal{O}(C)$ to complete the proof. To see this, suppose to the contrary that $A \notin \mathcal{O}(C)$. Since $A \in Q_{X,C}$, $\lambda_A \prec \lambda_C$ so by Proposition 2.16 there exists a non-zero projection $P \in \mathfrak{M}$ and an $\epsilon > 0$ such that $\lambda_{A+S} \prec \lambda_C$ for all self-adjoint operators $S \in \mathfrak{M}$ with $\|S\| < \epsilon$, $S = PS = SP$ and $\tau(S) = 0$.

Consider the linear map

$$\psi : \{S \in \mathfrak{M}_{\mathrm{sa}} \mid S = PS = SP, \tau(S) = 0\} \to \mathbb{C}$$

defined by $\psi(S) = \tau(TS)$. By dimension requirements, there exists a $S \in \ker(\psi) \setminus \{0\}$. By scaling, we obtain a non-zero $S \in \mathfrak{M}_{\mathrm{sa}}$ such that $\|S\| < \epsilon$, $S = PS = SP$, $\tau(S) = 0$, and $\tau(TS) = 0$. By construction $A \pm S \in Q_{X,C}$ and, since

$$A = \frac{1}{2}(A+S) + \frac{1}{2}(A-S),$$

we obtain a contradiction to the fact that A was an extreme point of $Q_{X,C}$. □

With Proposition 2.11 and Theorem 2.14 complete, we obtain several important corollaries. In fact, [**1**] went to great lengths to obtain a (multivariate) version of the following result, for which our techniques provide a quicker proof.

Corollary 2.17. *Let $(\mathfrak{M}, \tau)$ be a type II_1 factor, let $T \in \mathfrak{M}$, and let $\alpha \in (0, 1]$. Then*

$$\widetilde{V}_\alpha(T) = \frac{1}{\alpha}\{\tau(TP) \mid P \in \mathrm{Proj}(\mathfrak{M}), \tau(P) = \alpha\}.$$

Corollary 2.18. *Let $(\mathfrak{M}, \tau)$ be a tracial von Neumann algebra, let $T \in \mathfrak{M}$, and let $C \in \mathfrak{M}_{sa}$. Then $V_C(T)$ is a compact set.*

Proof. By Proposition 2.11 we may assume that $\mathfrak{M}$ is a type II_1 factor. Hence Theorem 2.10 implies that

$$V_C(T) = \left\{\tau(TX) \,\middle|\, X \in \overline{\mathrm{conv}(\mathcal{U}(T))}^{w^*}\right\}.$$

As $\overline{\mathrm{conv}(\mathcal{U}(T))}^{w^*}$ is weak*-compact and τ is a weak*-continuous linear functional, we obtain that $V_C(T)$ is compact. □

Corollary 2.19. *Let $(\mathfrak{M}, \tau)$ be a tracial von Neumann algebra and let $T, C \in \mathfrak{M}_{sa}$. Then $V_C(T) = V_T(C)$.*

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Proof. By Proposition 2.11 we may assume that $\mathfrak{M}$ is a type II_1 factor. As $\mathcal{U}(T)$ is (norm-)dense in $\mathcal{O}(T)$ and $\mathcal{U}(C)$ is (norm-)dense in $\mathcal{O}(C)$, we obtain that

$$\{\tau(TU^*CU) \mid U \in \mathfrak{M}, U \text{ a unitary}\}$$

is dense in both $V_C(T)$ and $V_T(C)$ by Theorem 2.14. Hence $V_C(T) = V_T(C)$ as both sets are compact by Corollary 2.18. □

Another important corollary is the continuity of the C-numerical range of T as both C and T vary. For this discussion, recall that for compact subsets X and Y of $\mathbb{C}$, the Hausdorff distance between X and Y is defined to be

$$d_H(X, Y) = \max\left\{\sup_{x\in X} \operatorname{dist}(x, Y), \sup_{y\in Y} \operatorname{dist}(y, X)\right\}.$$

Proposition 2.20. *Let $(\mathfrak{M}, \tau)$ be a tracial von Neumann algebra and let $T \in \mathfrak{M}$. If $C_1, C_2 \in \mathfrak{M}_{\mathrm{sa}}$, then*

$$d_H(V_{C_1}(T), V_{C_2}(T)) \leq \|T\| \, \|C_1 - C_2\| .$$

In particular, the map $C \mapsto V_C(T)$ is a continuous map from $\mathfrak{M}_{\mathrm{sa}}$ (equipped with the operator norm) to the compact, convex subsets of $\mathbb{C}$ equipped with the Hausdorff distance.

Proof. To begin we may assume that $\mathfrak{M}$ is a type II_1 factor by Proposition 2.11. Note for all $X \in \mathcal{O}(C_1)$ and $\epsilon > 0$ there exists an $X' \in \mathcal{O}(C_2)$ such that

$$\|X - X'\| \leq \epsilon + \|C_1 - C_2\|$$

and thus

$$|\tau(TX) - \tau(TX')| \leq \|T\| \, \|X - X'\| \leq \|T\| \, \|C_1 - C_2\| + \epsilon \, \|T\| .$$

As one may also interchange the roles of C_1 and C_2, the result follows by Theorem 2.14. □

Proposition 2.21. *Let $(\mathfrak{M}, \tau)$ be a tracial von Neumann algebra, let $T, S \in \mathfrak{M}$, and let $C \in \mathfrak{M}_{\mathrm{sa}}$. Then*

$$d_H(V_C(T), V_C(S)) \leq \|C\| \, \|T - S\| .$$

Thus, for any fixed $C \in \mathfrak{M}_{sa}$, the map $T \mapsto V_C(T)$ is continuous from $\mathfrak{M}$ (equipped with the operator norm) to the compact, convex subsets of $\mathbb{C}$ equipped with the Hausdorff distance.

Proof. To begin we may assume that $\mathfrak{M}$ is a type II_1 factor by Proposition 2.11. For all $X \in \mathcal{O}(C)$, note that

$$|\tau(TX) - \tau(SX)| \leq \|T - S\| \, \|X\| = \|T - S\| \, \|C\| .$$

Hence the result follows by Theorem 2.14. □

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Corollary 2.22. *Let $(\mathfrak{M}, \tau)$ be a tracial von Neumann algebra and let $T, S \in \mathfrak{M}$. If T and S are approximately unitarily equivalent, that is $S \in \mathcal{O}(T)$, then $V_C(T) = V_C(S)$ for all $C \in \mathfrak{M}_{\mathrm{sa}}$.*

Proof. The result follows from part (2.9) of Proposition 2.9 and Proposition 2.21. □

3. C-numerical ranges of self-adjoint operators

In this section, we will use eigenvalue functions to describe $V_C(T)$ when $C, T \in \mathfrak{M}_{\mathrm{sa}}$. This will be of use in the subsequent section when developing a method for computing C-numerical ranges of an arbitrary operator T.

To begin our description of $V_C(T)$ for all $C, T \in \mathfrak{M}_{\mathrm{sa}}$, we will assume that C and T are positive operators. From the description of such $V_C(T)$, Proposition 2.9 will yield descriptions of $V_C(T)$ for all $C, T \in \mathfrak{M}_{\mathrm{sa}}$.

Theorem 3.1. *Let $(\mathfrak{M}, \tau)$ be a tracial von Neumann algebra and let $T, C \in \mathfrak{M}$ be positive. Then*

$$V_C(T) = \left[\int_0^1 \lambda_T(s)\lambda_C(1-s)\, ds, \int_0^1 \lambda_T(s)\lambda_C(s)\, ds \right].$$

Remark 3.2. Note if $T, C \in \mathfrak{M}_{\mathrm{sa}}$ with C positive, then we still have

$$V_C(T) = \left[\int_0^1 \lambda_T(s)\lambda_C(1-s)\, ds, \int_0^1 \lambda_T(s)\lambda_C(s)\, ds \right]$$

by Proposition 2.9 and the fact that $\lambda_{aI_{\mathfrak{M}}+T}(s) = a + \lambda_T(s)$ for all $s \in [0,1)$ and $a \in \mathbb{R}$. Taking C to be a projection of trace α, this yields (see Remark 2.8)

$$\widetilde{V}_\alpha(T) = \left[\frac{1}{\alpha} \int_{1-\alpha}^1 \lambda_T(s)\, ds, \frac{1}{\alpha} \int_0^\alpha \lambda_T(s)\, ds \right] \tag{3.1}$$

To begin the proof of Theorem 3.1, we note by Remark 2.5 and Proposition 2.11 that we may assume $\mathfrak{M} = L^\infty[0,1]$ equipped with the trace given by integration against Lebesgue measure m and that $T = \lambda_T$ as a function on $[0,1]$.

To understand C-numerical ranges inside $L^\infty[0,1]$, we need to understand which functions have the same eigenvalue functions. This returns us to the work of Hardy, Littlewood, and Pólya.

Definition 3.3 (Hardy *et al.* [18, §10.12]). For a real-valued function $f \in L^\infty[0,1]$, the *non-increasing rearrangement* of f is the function

$$f^*(s) = \inf\{x \mid m(\{t \mid f(t) \geq x\}) \leq s\} \text{ for all } s \in [0,1).$$

Comparing to Definition 2.2, we immediately see that if $f \in L^\infty[0,1]$, then $\lambda_f = f^*$. Furthermore, if $f = 1_E$ is a characteristic function, then $f^* = 1_{[0,m(E))}$. We begin the demonstration of Theorem 3.1 by proving some preliminary observations.

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Lemma 3.4. *Suppose $w \in \mathbb{R}^n$ is such that $w_1 + \cdots + w_m \geq 0$ for all $m \in \{1, \dots, n\}$. Suppose $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then $\sum_{k=1}^n a_k w_k \geq 0$*

Proof. Setting $a_{n+1} = 0$, we have

$$\sum_{k=1}^n a_k w_k = \sum_{k=1}^n (a_k - a_{k+1}) \sum_{j=1}^k w_j \geq 0.$$

□

Lemma 3.5. *Let $f, g \in L^\infty[0,1]$ be non-increasing, positive, right continuous functions where g is a step function. Then*

$$\int_0^1 f(x)g(x)\,dx = \sup\left\{ \int_0^1 f(x)h(x)\,dx \,\middle|\, h^* = g \right\}$$
$$\int_0^1 f(1-x)g(x)\,dx = \inf\left\{ \int_0^1 f(x)h(x)\,dx \,\middle|\, h^* = g \right\}.$$

Proof. By the assumptions on g, there exists $0 = x_0 < x_1 < \cdots < x_n = 1$ and $a_1 > a_2 > \cdots > a_n \geq 0$ such that

$$g = \sum_{k=1}^n a_k 1_{[x_{k-1}, x_k)}.$$

Suppose $h \in L^\infty[0,1]$ is such that $h^* = g$. It will suffice to show

$$\int_0^1 f(x)g(x)\,dx \leq \int_0^1 f(x)h(x)\,dx, \tag{3.2}$$
$$\int_0^1 f(1-x)g(x)\,dx \geq \int_0^1 f(1-x)h(x)\,dx. \tag{3.3}$$

By the definition of the non-increasing rearrangement (also see Remark 2.13), there exists disjoint Borel subsets $\{X_k\}_{k=1}^n$ of $[0,1]$ such that $m\left(\bigcup_{k=1}^n X_k\right) = 1$, $m(X_k) = x_k - x_{k-1}$ for all k, and

$$h = \sum_{k=1}^n a_k 1_{X_k}.$$

Define $y, z \in \mathbb{R}^n$ by

$$y_k = \int_{X_k} f, \qquad z_k = \int_{[x_{k-1}, x_k]} f.$$

Since f is nonincreasing and $m(X_k) = x_k - x_{k-1}$, we have

$$\sum_{k=1}^m (z_k - y_k) \geq 0, \qquad (m \in \{1, \dots, n\}).$$

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Invoking Lemma 3.4, we get

$$\int_0^1 f(x)g(x)\,dx - \int_0^1 f(x)h(x)\,dx = \sum_{k=1}^{n} a_k(z_k - y_k) \geq 0.$$

This implies (3.2).

Now (3.3) follows from (3.2). Indeed, letting $\gamma = \int_0^1 g(x)\,dx = \int_0^1 h(x)\,dx$, we have

$$\begin{aligned}\|f\|_\infty\gamma - \int_0^1 f(1-x)g(x)\,dx &= \int_0^1 \Big(\|f\|_\infty - f(1-x)\Big)g(x)\,dx \\ &\leq \int_0^1 \Big(\|f\|_\infty - f(1-x)\Big)h(x)\,dx \\ &= \|f\|_\infty\gamma - \int_0^1 f(1-x)h(x)\,dx.\end{aligned}$$

□

Proof of Theorem 3.1. As remarked above, we may assume $\mathfrak{M} = L^\infty[0,1]$ and $T = \lambda_T$ under this identification. Since the map $X \mapsto \lambda_X$ is operator-norm to $L^\infty[0,1]$-norm continuous, and since $T \mapsto V_C(T)$ and $C \mapsto V_C(T)$ are operator-norm to Hausdorff distance continuous, we may assume without loss of generality that T and C have finite spectrum. Consequently, there exists $0 = x_0 < x_1 < \cdots < x_n = 1$, $t_1 \geq t_2 > \cdots > t_n \geq 0$, and $c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$ such that

$$T = \sum_{k=1}^{n} a_k 1_{[x_{k-1},x_k)} \qquad \text{and} \qquad \lambda_C = \sum_{k=1}^{n} c_k 1_{[x_{k-1},x_k)}.$$

As $\lambda_C \in \mathfrak{M}$ and

$$\tau(T\lambda_C) = \int_0^1 \lambda_T(x)\lambda_C(x)\,dx$$

by definition, we clearly have $\int_0^1 \lambda_T(x)\lambda_C(x)\,dx \in V_C(T)$. Similarly, letting $f(x) = \lambda_C$ $(1-x)$, we have $f \in \mathfrak{M}$, $f^* = \lambda_C$ and

$$\tau(Tf) = \int_0^1 \lambda_T(x)\lambda_C(1-x)\,dx.$$

Thus, we have $\int_0^1 \lambda_T(x)\lambda_C(1-x)\,dx \in V_C(T)$. Since $V_C(T)$ is a compact, convex subset of $\mathbb{R}$ (as C and T are positive), to complete the proof, it suffices so show that

$$\sup(V_C(T)) = \int_0^1 \lambda_T(x)\lambda_C(x)\,dx \quad \text{and} \quad \inf(V_C(T)) = \int_0^1 \lambda_T(x)\lambda_C(1-x)\,dx.$$

Suppose that $g \in \mathfrak{M}$ is such that $\lambda_g \prec \lambda_C$ (thus g is positive). We desire to show that $\tau(Tg) \leq \tau(T\lambda_C)$. Let $\mathfrak{N}$ be the von Neumann subalgebra of $\mathfrak{M}$ generated by the projections $\{1_{[x_{k-1},x_k)}\}_{k=1}^n$ and let $E_{\mathfrak{N}} : \mathfrak{M} \to \mathfrak{N}$ be the trace-preserving conditional expectation

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onto $\mathfrak{N}$. By Theorem 2.10, $h = E_{\mathfrak{N}}(g) \in \mathfrak{N}$ is a positive operator with finite spectrum such that $\lambda_h \prec \lambda_g \prec \lambda_C$ and $\tau(Th) = \tau(Tg)$. Hence it suffices to show $\tau(Tg) \leq \tau(T\lambda_C)$ for all $g \in \mathfrak{M}$ with finite spectrum and $\lambda_g \prec \lambda_C$.

For such a g, we may without loss of generality assume $g = g^*$ by Lemma 3.5. Consequently, we may assume there exists $0 = x'_0 < x'_1 < \cdots < x'_m = 1$, $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$, $c'_1 \geq c'_2 \geq \cdots \geq c'_n \geq 0$, and $b_1 \geq b_2 \geq \cdots \geq b_m \geq 0$ such that

$$T = \sum_{k=1}^{m} a'_k 1_{[x'_{k-1}, x'_k)}, \quad \lambda_C = \sum_{k=1}^{m} c'_k 1_{[x'_{k-1}, x'_k)}, \quad \text{and} \quad g = \sum_{k=1}^{m} b_k 1_{[x'_{k-1}, x'_k)}.$$

Since $g \prec \lambda_C$, we obtain that

$$\sum_{k=1}^{q} b_k (x'_k - x'_{k-1}) \leq \sum_{k=1}^{q} c'_k (x'_k - x'_{k-1}) \tag{3.4}$$

for all q with equality when $q = m$. Therefore, setting $a'_{m+1} = 0$, we have

$$\begin{aligned} \tau(T(\lambda_C - g)) &= \sum_{k=1}^{m} a'_k (c'_k - b_k)(x'_k - x'_{k-1}) \\ &= \sum_{q=1}^{m} \sum_{j=1}^{q} \left(a'_q - a'_{q+1}\right) \left(c'_j - b_j\right)(x'_j - x'_{j-1}). \end{aligned}$$

Since $a'_q - q'_{q+1} \geq 0$ for all q and $\sum_{j=1}^{q} (c'_j - b_j)(x'_j - x'_{j-1}) \geq 0$ by (3.4), we obtain $\tau(T(\lambda_C - g)) \geq 0$ as desired.

The proof that

$$\inf(V_C(T)) = \int_0^1 \lambda_T(x) \lambda_C(1 - x)\, dx$$

follows from similar arguments. □

4. A method for computing C-numerical ranges

In this section, we will use Theorem 3.1 together with some additional arguments to develop a method for computing $V_C(T)$ for general $T \in \mathfrak{M}$. This will enable us to show that if one knows all α-numerical ranges of an operator T, one also knows all C-numerical ranges of T.

Given an operator T, the main idea is to reduce the computation of the C-numerical range of T to the C-numerical ranges of the real parts of rotations of T, which are described in terms of eigenvalue functions by Theorem 3.1. This is motivated by [**28**] (or see the English translation [**29**]). To begin, we will require the following functions.

Notation 4.1. For a non-empty, bounded subset $E \subseteq \mathbb{C}$, let

$$\sup(\mathrm{Re}(E)) = \sup\{\mathrm{Re}(z) \mid z \in E\}$$

and define $g_E : [0, 2\pi) \to \mathbb{R}$ by

$$g_E(\theta) = \sup(\mathrm{Re}(e^{i\theta} E)).$$

Proposition 4.2. *For a non-empty, compact, convex set $K \subseteq \mathbb{C}$, the function g_K completely determines K. Concretely,*

$$K = \{z \in \mathbb{C} \mid \mathrm{Re}(e^{i\theta} z) \leq g_K(\theta) \text{ for all } \theta \in [0, 2\pi)\}.$$

Proof. Let $\Psi(K)$ denote the set on the right-hand-side of the above equation. Since $g_{w+K}(\theta) = \mathrm{Re}(e^{i\theta} w) + g_K(\theta)$ for all $w \in \mathbb{C}$, we have

$$\Psi(w + K) = w + \Psi(K).$$

Thus, we may assume without loss of generality that $0 \in K$.

By definition, it is clear that $K \subseteq \Psi(K)$. For the other inclusion, suppose $w \in K^c$. Choose a line separating w from K (for example, the line that bisects the line segment from w to the point of K closest to w). This line is the solution set in $\mathbb{C}$ of the equation $\mathrm{Re}(e^{-i\theta} z) = c$ for some $\theta \in [0, 2\pi)$ and some $c \geq 0$. Thus, the line $\mathrm{Re}(z) = c$ separates $e^{i\theta} K$ from $e^{i\theta} w$. Since $0 \in K$, we have that $0 \leq g_K(\theta) < c < \mathrm{Re}(e^{i\theta} w)$, so $w \notin \Psi(K)$. □

Example 4.3. For $a, b \in \mathbb{R}$ with $a, b > 0$, consider the solid ellipse

$$K = \left\{ x + iy \,\middle|\, x, y \in \mathbb{R}, \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

The parametrization of the boundary of K in polar coordinates is defined by the map

$$\theta \mapsto a\cos(\theta) + ib\sin(\theta),$$

and from this it is elementary to verify that

$$g_K(\theta) = \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}.$$

As the C-numerical ranges of an operator are compact, convex subsets of $\mathbb{C}$, in order to determine them it suffices to describe the functions $g_{V_C(T)}(\theta)$. Furthermore, it suffices to describe $V_C(T)$ for C positive by part (2.9) of Proposition 2.9 (otherwise we translate C to be a positive operator C', compute $V_{C'}(T)$, and then translate back).

Method 4.4. Given a tracial von Neumann algebra $(\mathfrak{M}, \tau)$, $T \in \mathfrak{M}$, and a positive $C \in \mathfrak{M}$, by combining Propositions 3.1 and 4.2 we obtain a method of computing $V_C(T)$, provided we can obtain sufficient information about the distributions of the operators $\mathrm{Re}(e^{i\theta} T)$ for $\theta \in [0, 2\pi)$. Indeed, by Theorem 3.1 (or, more specifically, Remark 3.2), we have

$$g_{V_C(T)}(\theta) = \int_0^1 \lambda_{\mathrm{Re}(e^{i\theta}T)}(s) \lambda_C(s)\, ds.$$

Thus, Proposition 4.2 implies that

$$V_C(T) = \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{Re}(e^{i\theta} z) \leq \int_0^1 \lambda_{\mathrm{Re}(e^{i\theta}T)}(s) \lambda_C(s)\, ds \text{ for all } \theta \in [0, 2\pi) \right\}.$$

In particular, the above method works provided we can describe λ_C and $\lambda_{\mathrm{Re}(e^{i\theta}T)}$ for all $\theta \in [0, 2\pi)$.

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Method 4.5. We now show how to find $V_C(T)$ for arbtrary $C \in \mathfrak{M}_{sa}$ and $T \in \mathfrak{M}$ in terms of the the spectral distribution of C and the collection of α-numerical ranges

$$\left(\widetilde{V}_\alpha(\mathrm{Re}(e^{i\theta}T))\right)_{\alpha\in(0,1],\,\theta\in[0,2\pi)} \tag{4.1}$$

Recall (see Remark 2.8) that the α-numerical range $\widetilde{V}_\alpha(S)$ is equal to $\frac{1}{\alpha}V_P(S)$ where $P \in \mathfrak{M}$ is a projection of trace α.

By part (2.9) of Proposition 2.9, $V_C(T) = -r\tau(T) + V_{C+rI_{\mathfrak{M}}}(T)$ for every $r \in \mathbb{R}$, so it will suffice to show how to find $V_C(T)$ in terms of the spectral distribution of C and collection (4.1) when C is positive. Let M be an integer greater than $\|C\|$. For each integer $n \geq 1$, let

$$C_n = \sum_{k=1}^{nM} \frac{k}{n} 1_{[\frac{k-1}{n},\frac{k}{n})}(C),$$

where $1_{[(k-1)/n,k/n)}(C)$ denotes the spectral projection of C for the indicated interval. Note that C_n converges in norm to C as $n \to \infty$. By Proposition 2.20, $V_C(T)$ is the limit in Hausdorff metric of $V_{C_n}(T)$, as $n \to \infty$, and, thus, it will suffice to show how each $V_{C_n}(T)$ can be described in terms of the collection (4.1).

Method 4.4 describes $V_{C_n}(T)$ in terms of the function

$$\theta \mapsto g_{V_{C_n}(T)}(\theta) = \int_0^1 \lambda_{\mathrm{Re}(e^{i\theta}T)}(s)\lambda_{C_n}(s)\,ds.$$

But

$$\lambda_{C_n} = \sum_{j=1}^{nM} \frac{j}{n} 1_{[1-x_j,1-x_{j-1})},$$

where $x_j = \tau(1_{[0,\frac{j}{n})}(C))$. Thus,

$$g_{V_{C_n}(T)}(\theta) = \sum_{j=1}^{nM} \frac{j}{n} \int_{1-x_j}^{1-x_{j-1}} \lambda_{\mathrm{Re}(e^{i\theta}T)}(s)\,ds.$$

But for any $0 \leq \alpha \leq \beta \leq 1$ and any $X \in \mathfrak{M}_{sa}$, by equation (3.1) in Remark 3.2, we have

$$\int_\alpha^\beta \lambda_X(s)\,ds = \beta \sup\left(\widetilde{V}_\beta(X)\right) - \alpha \sup\left(\widetilde{V}_\alpha(X)\right).$$

This completes the description of how $V_C(T)$ can be determined from the spectral scale of C and the family (4.1).

5. Further Examples

Method 4.5 shows how the α-numerical ranges determine all C-numerical ranges. In this section, we compute the α-numerical ranges of several operators. Although computing

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the k-numerical ranges of a matrix is generally a hard task, there are several interesting examples of operators in II_1 factor whose α-numerical ranges can be explicitly described.

We begin by noting the following.

Proposition 5.1. *Let $(\mathfrak{M}_1, \tau_1)$ and $(\mathfrak{M}_2, \tau_2)$ be tracial von Neumann algebras, let $T_1 \in \mathfrak{M}_1$, and let $T_2 \in \mathfrak{M}_2$. If T_1 and T_2 have the same $*$-distributions, then $\widetilde{V}_\alpha(T_1) = \widetilde{V}_\alpha(T_2)$ for all $\alpha \in (0, 1]$.*

Proof. By Proposition 2.11, we may assume, without loss of generality, that $\mathfrak{M}_k = W^*(T_k)$ for $k = 1, 2$. Since T_1 and T_2 have the same $*$-distributions, there exists a trace-preserving isomorphism of $W^*(T_1)$ and $W^*(T_2)$ that sends T_1 to T_2. This clearly implies $\widetilde{V}_\alpha(T_1) = \widetilde{V}_\alpha(T_2)$ for all $\alpha \in (0, 1]$, by Definition 2.6. $\square$

Recall from the introduction that the k-numerical range of a normal matrix $N \in \mathcal{M}_n(\mathbb{C})$ with eigenvalues $\{\lambda_j\}_{j=1}^n$ is

$$W_k(N) = \operatorname{conv}\left(\left\{ \frac{1}{k} \sum_{j \in K} \lambda_j \;\middle|\; J \subseteq \{1, \ldots, n\}, |J| = k \right\}\right).$$

The following generalizes this result to normal operators with finite spectrum in a tracial von Neumann algebra.

Proposition 5.2. *Let $(\mathfrak{M}, \tau)$ be a tracial von Neumann algebra, let $N \in \mathfrak{M}$ be a normal operator such that $\sigma(N) = \{\lambda_k\}_{k=1}^n$, and let $w_k = \tau(1_{\{\lambda_k\}}(N))$ for each $k \in \{1, \ldots, n\}$. Then for each $\alpha \in (0, 1]$, we have*

$$\widetilde{V}_\alpha(N) = \left\{ \frac{1}{\alpha} \sum_{k=1}^n \lambda_k t_k \;\middle|\; 0 \leq t_k \leq w_k, \sum_{k=1}^n t_k = \alpha \right\}.$$

Proof. Using Proposition 5.1, we may without loss of generality assume $\mathfrak{M} = L^\infty[0, 1]$ and

$$N = \sum_{k=1}^n \lambda_k 1_{X_k},$$

where $\{X_k\}_{k=1}^n$ are disjoint Borel measurable sets such that $\bigcup_{k=1}^n X_k = [0, 1]$ and $m(X_k) = w_k$ for all k (m the Lebesgue measure).

Consider the surjection

$$\psi : \{X \subseteq [0, 1] \mid X \text{ Borel}, m(X) = \alpha\} \to \left\{ (t_1, \ldots, t_n) \;\middle|\; 0 \leq t_k \leq w_k, \sum_{k=1}^n t_k = \alpha \right\}$$

defined by

$$\psi(X) = (m(X \cap X_1), \ldots, m(X \cap X_n)).$$

If $X \subseteq [0, 1]$ is Borel measurable with $m(X) = \alpha$, then

$$\tau(N 1_X) = \int_X \sum_{k=1}^n \lambda_k 1_{X_k}(s)\, ds = \sum_{k=1}^n \lambda_k t_k$$

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where $(t_1, \ldots, t_n) = \psi(X)$. Since every $P \in \mathrm{Proj}(L^\infty[0,1])$ is of the form $P = 1_X$ where $X \subseteq [0,1]$ and $\tau(P) = m(X)$, the result follows, using Corollary 2.17. □

For our next example, recall that a Haar unitary is a unitary element U whose spectral distribution is Haar measure on the unit circle, or, equivalently, such that $\tau(U^k) = 0$ for all integers $k \geq 1$.

Example 5.3. Let $(\mathfrak{M}, \tau)$ be a tracial von Neuman algebra, let $U \in \mathfrak{M}$ be a Haar unitary, and let $\mathbb{D}$ denote the closed unit disk. For every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, λU and U have the same spectral distribution. Therefore, Proposition 5.1 implies

$$\widetilde{V}_\alpha(U) = \widetilde{V}_\alpha(\lambda U) = \lambda \widetilde{V}_\alpha(U)$$

for every $\alpha \in (0,1]$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Since each $\widetilde{V}_\alpha(U)$ is a compact, convex set, this implies

$$\widetilde{V}_\alpha(U) = r(\alpha)\mathbb{D},$$

where $r : (0,1] \to [0,1]$ is such that $r(\alpha) = \sup\{\mathrm{Re}(z) \mid z \in \widetilde{V}_\alpha(U)\} = \sup \widetilde{V}_\alpha(\mathrm{Re}(U))$ where the last equality is part (2.9) of Proposition 2.9.

To compute $r(\alpha)$, note that by Proposition 5.1 we may assume that $U = (s \mapsto e^{is}) \in L^\infty[-\pi, \pi]$, so $\mathrm{Re}(U) = (s \mapsto \cos(s))$ and, arguing as in the proof of Theorem 3.1, we deduce that

$$r(\alpha) = \frac{1}{2\pi\alpha} \int_{-\pi\alpha}^{\pi\alpha} \cos(s)\, ds = \frac{1}{\pi\alpha} \sin(\pi\alpha).$$

Thus

$$\widetilde{V}_\alpha(U) = \frac{1}{\pi\alpha} \sin(\pi\alpha)\mathbb{D}$$

for all $\alpha \in (0,1]$.

The above example exhibits a method for computing α-numerical ranges, provided there exists sufficient symmetry.

Corollary 5.4. *Let $(\mathfrak{M}, \tau)$ be a diffuse tracial von Neumann algebra and suppose $T \in \mathfrak{M}$ is such that*

$$\widetilde{V}_\alpha(T) = e^{i\theta}\widetilde{V}_\alpha(T) \text{ for all } \theta \in [0, 2\pi). \tag{5.1}$$

Then $\widetilde{V}_\alpha(T)$ is the closed disk centered at the origin of radius $r_\alpha(T)$, where

$$r_\alpha(T) = \frac{1}{\alpha} \int_0^\alpha \lambda_{\mathrm{Re}(T)}(s)\, ds = \sup \widetilde{V}_\alpha(\mathrm{Re}(T)).$$

Recall that the $*$-distribution of an element $T \in \mathfrak{M}$ is the collecton of its $*$-moments, $\tau(T^{\epsilon(1)} \cdots T^{\epsilon(n)})$ over all $n \geq 1$ and all $\epsilon(1), \ldots, \epsilon(n) \in \{1, *\}$. Of course, the hypothesis (5.1) of the above corollary is satisfied whenever the $*$-distribution of T is the same as the $*$-distribution of $e^{i\theta}T$ for all $\theta \in \mathbb{R}$.

Using Method 4.4, we may compute the α-numerical ranges of several interesting operators.

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Example 5.5. Consider the infinite tensor view of the hyperfinite II_1 factor

$$\mathfrak{R} = \bigotimes_{n\geq 1} \mathcal{M}_2(\mathbb{C})$$

and consider the Tucci operator [**35**]

$$T = \sum_{n\geq 1} \frac{1}{2^n}(\underbrace{I_2 \otimes \cdots \otimes I_2}_{n-1 \text{ times}} \otimes Q \otimes I_2 \otimes \cdots)$$

where $Q = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. This operator is quasinilpotent and generates $\mathfrak{R}$. To compute $\widetilde{V}_\alpha(T)$ for every $\alpha \in (0, 1]$, we first notice that T and $e^{i\theta}T$ are approximately unitarily equivalent via the unitaries

$$U_{n,\theta} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \otimes I_2 \otimes I_2 \otimes \cdots,$$

as $U^*_{n,\theta}(e^{i\theta}T)U_{n,\theta}$ approximate T in norm. Therefore, Corollary 2.22 and Corollary 5.4 imply

$$\widetilde{V}_\alpha(T) = r_\alpha(T)\mathbb{D}$$

where $\mathbb{D}$ denotes the closed unit disk and $r_\alpha(T)$ may be computed by as

$$r_\alpha(T) = \sup(\widetilde{V}_\alpha(\mathrm{Re}(T))).$$

Let

$$A_0 = \mathrm{Re}(Q) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\mathrm{Re}(T) = \sum_{n\geq 1} \frac{1}{2^n}(\underbrace{I_2 \otimes \cdots \otimes I_2}_{n-1 \text{ times}} \otimes A_0 \otimes I_2 \otimes \cdots).$$

However, since $2A_0$ is unitarily equivalent to

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we obtain that $\mathrm{Re}(T)$ is approximately unitarily equivalent to

$$S = \frac{1}{2}\sum_{n\geq 1} \frac{1}{2^n}(\underbrace{I_2 \otimes \cdots \otimes I_2}_{n-1 \text{ times}} \otimes A \otimes I_2 \otimes \cdots).$$

Thus, Corollary 2.22 implies

$$r_\alpha(T) = \sup(\widetilde{V}_\alpha(S)).$$

Notice

$$\sum_{n=1}^{2} \frac{1}{2^n}(\underbrace{I_2 \otimes \cdots \otimes I_2}_{n-1 \text{ times}} \otimes A \otimes I_2 \otimes \cdots) = \mathrm{diag}\left(\frac{3}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{3}{4}\right).$$

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Furthermore,

$$\sum_{n=1}^{3} \frac{1}{2^n} (\underbrace{I_2 \otimes \cdots \otimes I_2}_{n-1 \text{ times}} \otimes A \otimes I_2 \otimes \cdots) = \operatorname{diag}\left(\frac{7}{8}, \frac{5}{8}, \frac{3}{8}, \frac{1}{8}, -\frac{1}{8}, -\frac{3}{8}, -\frac{5}{8}, -\frac{7}{8}\right).$$

This pattern continues and thus we see that the spectral scale of S is

$$\lambda_S(s) = \frac{1}{2}(1-2s).$$

Thus,

$$r_\alpha(T) = \frac{1}{2\alpha}\int_0^\alpha (1-2s)\, ds = \frac{1}{2}(1-\alpha)$$

so

$$\widetilde{V}_\alpha(T) = \frac{1}{2}(1-\alpha)\mathbb{D}.$$

It is not very difficult to construct a normal operator N satisfying $\widetilde{V}_\alpha(N) = \widetilde{V}_\alpha(T)$ for all $\alpha \in (0,1]$, namely, having the same numerical ranges as the quasinilpotent operator T. Indeed, considering the radially symmetric distribution ν on the unit disk such that $\nu(r\mathbb{D}) = 1 - \sqrt{1-r^2}$ for $0 < r < 1$, one can show that the marginal distribution of ν is uniform measure on $[-1,1]$. It follows that the normal operator N whose spectral distribution is ν satisfies $\lambda_{\mathrm{Re}(N)}(s) = \frac{1}{2}(1-2s)$ for all $s \in [0,1)$ and this implies $\widetilde{V}_\alpha(N) = \widetilde{V}_\alpha(T)$ for all $\alpha \in (0,1]$.

Example 5.6. Recall a $(0,1)$-circular operator is an element Z of a tracial von Neumann algebra of the form

$$Z = \frac{1}{\sqrt{2}}(X + iY),$$

where X and Y are freely independent $(0,1)$-semicircular operators. As the $*$-distribution of Z is the same as the $*$-distribution of $e^{i\theta}Z$ for all $\theta \in \mathbb{R}$, Corollary 5.4 implies that

$$\widetilde{V}_\alpha(Z) = r_\alpha(Z)\mathbb{D}$$

where $r_\alpha(Z) = \sup(\widetilde{V}_\alpha(\mathrm{Re}(Z)))$. Since the spectral distribution of $\mathrm{Re}(Z) = \frac{1}{\sqrt{2}}X$ is given by the semicircular law

$$\frac{1}{\pi}1_{[-\sqrt{2},\sqrt{2}]}(x)\sqrt{2-x^2},$$

we obtain that

$$r_\alpha(Z) = \frac{1}{\pi}\int_{h(\alpha)}^{\sqrt{2}} x\sqrt{2-x^2}\, dx = \frac{1}{3\pi\alpha}\left(2 - h(\alpha)^2\right)^{3/2},$$

where $h(\alpha) \in \left[-\sqrt{2}, \sqrt{2}\right)$ is such that

$$\frac{1}{\pi}\int_{h(\alpha)}^{\sqrt{2}} \sqrt{2-x^2}\, dx = \alpha.$$

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Thus, h is the inverse with respect to composition of the monotone decreasing function $f : [-\sqrt{2}, \sqrt{2}] \to [0,1]$ given by

$$f(y) = \frac{1}{\pi}\int_y^{\sqrt{2}} \sqrt{2-x^2}\,dx = \frac{1}{2} - \frac{1}{2\pi}y\sqrt{2-y^2} - \frac{1}{\pi}\arcsin\left(\frac{y}{\sqrt{2}}\right).$$

We note the asymptotic expansions

$$f(\sqrt{2}-x) = \frac{2^{7/4}}{3\pi}x^{3/2} - \frac{1}{5\pi 2^{3/4}}x^{5/2} + O(x^{7/2}) \qquad (\text{as } x \to 0^+),$$

$$h(\alpha) = \sqrt{2} - \frac{(3\pi)^{2/3}}{2^{7/6}}\alpha^{2/3} - \frac{(3\pi)^{4/3}}{5(2^{23/6})}\alpha^{4/3} + O(\alpha^2) \quad (\text{as } \alpha \to 0^+),$$

$$r_\alpha(Z) = \sqrt{2} - \frac{3^{5/3}\pi^{2/3}}{5(2^{7/6})}\alpha^{2/3} + O(\alpha) \qquad (\text{as } \alpha \to 0^+).$$

For comparison, a $(0,1)$-circular element has norm 2 and spectrum equal to the disk centred at the origin of radius 1. Note that, since the push-forward measure of the spectral distribution of the normalized Lebesgue measure on the disk of radius $\sqrt{2}$ onto the real axis produces the semicircular law $(1/\sqrt{2})X$, Z is an easy example of a non-normal operator such that there exists a normal operator N with $\widetilde{V}_\alpha(Z) = \widetilde{V}_\alpha(N)$ for all $\alpha \in (0,1]$.

Example 5.7. The quasinilpotent DT-operator S was introduced in [8] as one of an interesting class of operators in the free group factor $L(\mathbb{F}_2)$, that can be realized as limits of upper triangular random matrices. As the name suggests, its spectrum is $\{0\}$, and it satisfies $\|S\| = \sqrt{e}$ and $\tau(S^*S) = 1/2$. In [9], it was shown that S generates $L(\mathbb{F}_2)$ and that S has many non-trivial hyperinvariant subspaces. Moreover, $\mathrm{Re}(S) = \frac{1}{2}X$, where X is a $(0,1)$-semicircular operator and the $*$-distribution of S is the same as that of $e^{i\theta}S$ for all $\theta \in \mathbb{R}$. Thus, the method of Corollary 5.4 applies, exactly as in Example 5.6, to yield

$$\widetilde{V}_\alpha(S) = r_\alpha(S)\mathbb{D},$$

where $r_\alpha(S) = \frac{1}{\sqrt{2}}r_\alpha(Z)$, where $r_\alpha(Z)$ is the function as defined in Example 5.6. Note that the normal measure whose distribution is uniform measure on the disk of radius $1/\sqrt{2}$ has the same α-numerical ranges as the quasinilpotent operator S.

Example 5.8. As a generalization of Example 5.6, consider the operator

$$T = \cos(\psi)X + i\sin(\psi)Y$$

where $\psi \in (0, \frac{\pi}{2})$ and X and Y are freely independent $(0,1)$-semicircular operators. In particular, the case $\psi = \frac{\pi}{4}$ produces the circular operator studied in Example 5.6.

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These elliptic variants of circular operators were studied by Larsen in [30], where he showed

- $\|T\| = 2$,
- the spectrum of T is $\left\{z \in \mathbb{C} \;\middle|\; \frac{\mathrm{Re}(z)^2}{\cos^4(\psi)} + \frac{\mathrm{Im}(z)^2}{\sin^4(\psi)} \leq 4\right\}$,

  and
- the Brown measure of T is uniform distribution on its spectrum.

To determine $\widetilde{V}_\alpha(T)$, we apply Method 4.4. Note that $\mathrm{Re}(e^{i\theta}T)$ is

$$\cos(\psi)\cos(\theta)X - \sin(\psi)\sin(\theta)Y,$$

which is $(0, b(\theta)^2)$-semicircular where

$$b(\theta) = \sqrt{\cos^2(\psi)\cos^2(\theta) + \sin^2(\psi)\sin^2(\theta)}.$$

Thus the spectral distribution of $\mathrm{Re}(e^{i\theta}T)$ is the same as the spectral distribution of $\sqrt{2}\, b(\theta)\mathrm{Re}(Z)$, where Z is the $(0,1)$-circular operator from Example 5.6. Hence

$$g_{\widetilde{V}_\alpha(T)}(\theta) = \sqrt{2}\, r_\alpha(Z) b(\theta).$$

Therefore, by Proposition 4.2 and Example 4.3, we find

$$\widetilde{V}_\alpha(T) = \left\{ z \in \mathbb{C} \;\middle|\; \frac{\mathrm{Re}(z)^2}{\cos^2(\psi)} + \frac{\mathrm{Im}(z)^2}{\sin^2(\psi)} \leq 2 r_\alpha(Z)^2 \right\}.$$

It is curious, although not surprising, that the eccentricity of the ellipse bounding $\widetilde{V}_\alpha(T)$ is (except in the circular case $\psi = \pi/4$) different from the eccentricity of the ellipse bounding the spectrum $\sigma(T)$.

To complete this section, we note the following interpolation result that generalizes [14, Corollary 1]. This enables one to obtain knowledge pertaining to one α-numerical range based on others. We note that further results in [14] also have immediate generalizations to α-numerical ranges.

Proposition 5.9. *Let $(\mathfrak{M}, \tau)$ be a diffuse, tracial von Neumann algebra and let $T \in \mathfrak{M}$. If $0 < \alpha < \beta < \gamma \leq 1$, then*

$$\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)} \widetilde{V}_\alpha(T) + \frac{\gamma(\beta - \alpha)}{\beta(\gamma - \alpha)} \widetilde{V}_\gamma(T) \subseteq \widetilde{V}_\beta(T).$$

Proof. Let $\lambda \in \widetilde{V}_\alpha(T)$ and let $\mu \in \widetilde{V}_\gamma(T)$. By definition, there exist positive contractions $X, Y \in \mathfrak{M}$ such that $\tau(X) = \alpha$, $\tau(Y) = \gamma$,

$$\lambda = \frac{1}{\alpha}\tau(TX), \quad \text{and} \quad \mu = \frac{1}{\gamma}\tau(TY).$$

Let

$$Z = \frac{\gamma - \beta}{\gamma - \alpha}X + \frac{\beta - \alpha}{\gamma - \alpha}Y \in \mathfrak{M}.$$

It is clear that Z is a positive operator such that

$$Z \leq \frac{\gamma - \beta}{\gamma - \alpha}I_{\mathfrak{M}} + \frac{\beta - \alpha}{\gamma - \alpha}I_{\mathfrak{M}} = I_{\mathfrak{M}}$$

and

$$\tau(Z) = \frac{\gamma - \beta}{\gamma - \alpha}\alpha + \frac{\beta - \alpha}{\gamma - \alpha}\gamma = \beta.$$

Finally,

$$\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)}\lambda + \frac{\gamma(\beta - \alpha)}{\beta(\gamma - \alpha)}\mu = \frac{1}{\beta}\frac{\gamma - \beta}{\gamma - \alpha}\tau(TX) + \frac{1}{\beta}\frac{\beta - \alpha}{\gamma - \alpha}\tau(TY) = \frac{1}{\beta}\tau(TZ) \in \widetilde{V}_\beta(T),$$

completing the proof. □

Remark 5.10. One may ask whether set equality must occur in Proposition 5.9. Taking $T \in \mathfrak{M}$ to be a Haar unitary, Example 5.3 implies that this question asks (by letting $\gamma = 1$) whether

$$\frac{1 - \beta}{\pi(\beta - \alpha\beta)}\sin(\pi\alpha)\mathbb{D} + 0 = \frac{1}{\pi\beta}\sin(\pi\beta)\mathbb{D}$$

holds for all $0 < \alpha < \beta < 1$. As this is clearly not the case, equality need not hold in Proposition 5.9. However, one may use [**3**] to demonstrate that equality does hold in Proposition 5.9 when T is an $n \times n$ matrix, $\alpha = k/n$, and $\gamma = (k+1)/n$ for some $k \in \{1, \ldots, n\}$.

6. Numerical ranges and diagonals

In this our final section, we desire description of when a scalar belongs to the α-numerical range of an operator based on the possible 'diagonals' of an operator. Our characterization is similar to that for k-numerical ranges of matrices found in [**13**, Theorem 2.4]. Unfortunately, we do not obtain true 'diagonals' as we do not know if one can guarantee $\mathcal{A}$ in the following technical lemma (whose proof is a generalization of a matricial result) is a MASA.

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Lemma 6.1. *Let $(\mathfrak{M}, \tau)$ be a type II_1 factor and let $T \in \mathfrak{M}$ be such that $\tau(T) = 0$. Then there exists a diffuse abelian von Neumann subalgebra $\mathcal{A}$ of $\mathfrak{M}$ such that $E_{\mathcal{A}}(T) = 0$, where $E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$ is the normal conditional expectation.*

Proof. Notice that $0 \in \widetilde{V}_1(T) \subseteq \widetilde{V}_{\frac{1}{2}}(T)$. Hence there exists a projection $P \in \mathfrak{M}$ such that $\tau(P) = \frac{1}{2}$ and $\tau(TP) = 0$. Then, of course, $\tau(T(I_{\mathfrak{M}} - P)) = 0$. By repeating this argument in $P\mathfrak{M}P$ and $(I_{\mathfrak{M}} - P)\mathfrak{M}(I_{\mathfrak{M}} - P)$, we obtain four projections $\{P_k\}_{k=1}^4$ such that P_k commutes with P and $I_{\mathfrak{M}} - P$, $\tau(P_k) = \frac{1}{4}$, and $\tau(TP_k) = 0$ for all k. By continuing to repeat the first argument on each compression and by taking the von Neumann algebra generated by these projections, the desired diffuse abelian von Neumann subalgebra of $\mathfrak{M}$ is obtained. □

Proposition 6.2. *Let $(\mathfrak{M}, \tau)$ be a type II_1 factor, let $T \in \mathfrak{M}$, and let $\alpha \in (0, 1]$. Then $\lambda \in \widetilde{V}_\alpha(T)$ if and only if there exists a diffuse abelian von Neumann subalgebra $\mathcal{A}$ of $\mathfrak{M}$ such that $\tau(1_{\{\lambda\}}(E_{\mathcal{A}}(T))) \geq \alpha$, where $E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$ is the normal,* trace preserving *conditional expectation.*

Proof. Suppose $\mathcal{A}$ a diffuse abelian von Neumann subalgebra of $\mathfrak{M}$ such that $\beta := \tau(1_{\{\lambda\}}(E_{\mathcal{A}}(T))) \geq \alpha$. Thus

$$\lambda = \tau(E_{\mathcal{A}}(T)1_{\{\lambda\}}(E_{\mathcal{A}}(T))) = \tau(T1_{\{\lambda\}}(E_{\mathcal{A}}(T))) \in \widetilde{V}_\beta(T) \subseteq \widetilde{V}_\alpha(T).$$

(See Remark 2.8.)

For the converse direction, suppose $\lambda \in \widetilde{V}_\alpha(T)$. By part (2.9) of Proposition 2.9, we may without loss of generality assume that $\lambda = 0$. Since $0 \in \widetilde{V}_\alpha(T)$, by Corollary 2.17 there exists a projection P of trace α such that $\frac{1}{\alpha}\tau(TP) = 0$. Hence $\tau_{P\mathfrak{M}P}(PTP) = 0$ where $\tau_{P\mathfrak{M}P}$ is the trace for $P\mathfrak{M}P$. By Lemma 6.1 there exists a diffuse abelian von Neumann subalgebra $\mathcal{A}_0$ of $P\mathfrak{M}P$ such that $E_{\mathcal{A}_0}(PTP) = 0$. If $\mathcal{A}'$ is any diffuse abelian von Neumann subalgebra of $(I_{\mathfrak{M}} - P)\mathfrak{M}(I_{\mathfrak{M}} - P)$, then $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}' \subseteq \mathfrak{M}$ is a diffuse abelian von Neumann subalgebra containing P such that $E_{\mathcal{A}}(T)P = 0$. Hence $\tau(1_{\{\lambda\}}(E_{\mathcal{A}}(T))) \geq \alpha$ as desired. □

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References

1. C. Akemann and J. Anderson, A geometrical spectral theory for n-tuples of self-adjoint operators in finite von Neumann algebras: II, *Pacific J. Math.* **205** (2002), 257–285.
2. C. Akemann and J. Anderson, The spectral scale and the k-numerical range, *Glasgow Math. J.* **45** (2003), 225–238.
3. C. Akemann and J. Anderson, The spectral scale and the numerical range, *Internat. J. Math.* **14** (2003), 171–189.
4. C. Akemann, J. Anderson and N. Weaver, A geometrical spectral theory for n-tuples of self-adjoint operators in finite von Neumann algebras, *J. Funct. Anal.* **165** (1999), 258–292.

https://doi.org/10.1017/S0013091516000572 Published online by Cambridge University Press

5. T. Ando, Majorization, doubly stochastic matrices, and comparison of eigenvalues, *Linear Alg. Applc.* **118** (1989), 163–248.
6. M. Argerami and P. Massey, A Schur–Horn theorem in II_1 factors, *Indiana Univ. Math. J.* **56** (2007), 2051–2060.
7. M. Argerami and P. Massey, The local form of doubly stochastic maps and joint majorization in II_1 factors, *Integral Equations Operator Theory* **61**(1) (2008), 1–19.
8. K. Dykema, J. Fang, D. Hadwin and R. Smith, The carpenter and Schur–Horn problems for MASAs in finite factors, *Illinois J. Math.* **56** (2012), 1313–1329.
9. K. Dykema and U. Haagerup, DT-operators and decomposability of Voiculescu's circular operator, *Amer. J. Math.* **126** (2004), 121–189.
10. K. Dykema and U. Haagerup, Invariant subspaces of the quasinilpotent DT-operator, *J. Funct. Anal.* **209** (2004), 332–366.
11. T. Fack, Sur la notion de valuer caractéristique, *J. Operator Theory* **7** (1982), 207–333.
12. T. Fack and H. Kosaki, Generalized s-numbers of τ-measurable operators, *Pacific J. Math* **123** (1986), 269–300.
13. P. A. Fillmore and J. P. Williams, Some convexity theorems for matrices, *Glasgow Math J.* **11**(2) (1971), 110–117.
14. M. Goldberg and E. Straus, Inclusion relations involving k-numerical ranges, *Linear Algebra Appl.* **15**(3) (1976), 261–270.
15. M. Goldberg and E. Straus, Elementary inclusion relations for generalized numerical ranges, *Linear Algebra Appl.* **18**(1) (1977), 1–24.
16. K. Gustafson, The Toeplitz-Hausdorff theorem for linear operators, *Proc. Amer. Math. Soc.* **25** (1970), 203–204.
17. P. R. Halmos, A Hilbert Space Problem Book, **1** (Princeton: van Nostrand, 1967).
18. G. H. Hardy, J. E. Littlewood and G. Pólya, Some simple inequalities satisfied by convex functions, *Messenger Math.* **58** (1929) 145–152.
19. F. Hausdorff, Der Wertvorrat einer Bilinearform, *Math. Z.* **3**(1) (1919), 314–316.
20. F. Hiai, Majorization and stochastic maps in von Neumann algebras, *Journal of Mathematical Analysis and Applications.* **127** (1987), 18–48.
21. F. Hiai, Spectral majorization between normal operators in von Neumann algebras, *Operator algebras and operator theory, Craiova*, 1989, Pitman Res. Notes Math. Ser., **271**, Longman Sci. Tech., Harlow, 1992, 78–115.
22. F. Hiai and Y. Nakamura, Closed convex hulls of unitary orbits in von Neumann algebras, *Trans. Amer. Math. Soc.* **323** (1991), 1–38.
23. A. Horn, Doubly stochastic matrices and the diagonal of a rotation matrix, *Amer. J. Math.* **76** (1954), 620–630.
24. E. Kamei, Majorization in finite factors, *Math. Japon.* **28** (1983), 495–499.
25. E. Kamei, Double stochasticity in finite factors, *Math. Japon.* **29** (1984), 903–907.
26. E. Kamei, An order on statistical operators implicitly introduced by von Neumann, *Math. Japon.* **30** (1985), 891–895.
27. D. Keeler, L. Rodman and I. Spitkovsky, The numerical range of 3×3 matrices, *Linear Algebra Appl.* **252**(1) (1997), 115–139.
28. R. Kippenhahn, Über den Wertevorrat einer Matrix, *Math. Nachr.* **6** (1951), 193–228.
29. R. Kippenhahn, On the numerical range of a matrix, Translated from the German by Paul F. Zachlin and Michiel E. Hochstenbach, *Linear and Multilinear Algebra* **56** (2008), 185–225.
30. F. Larsen, Brown Measures and R-diagonal Elements in Finite von Neumann Algebras, 1999, Ph.D. Thesis, University of Southern Denmark.
31. F. J. Murray and J. von Neumann, On rings of operators, *Ann. of Math. (2)* **37** (1936), 116–229.

https://doi.org/10.1017/S0013091516000572 Published online by Cambridge University Press

32. D. Petz, Spectral scale of self-adjoint operators and trace inequalities, *J. Math. Anal. Appl.* **109** (1985), 74–82.

33. Y. T. Poon, Another proof of a result of Westwick, *Linear Algebra Appl.* **9** (1980), 35–37.

34. I. Schur, Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinantentheorie, *Sitzungsber. Berl. Math. Ges.* **22** (1923), 9–20.

35. G. Tucci, Some quasinilpotent generators of the hyperfinite II_1 factor, *J. Funct. Anal.* **254** (2008), 2969–2994.

36. R. Westwick, A theorem on numerical range, *Linear Algebra Appl.* **2** (1975), 311–315.