ANZIAM J. 49(2008), 495–501
doi:10.1017/S1446181108000187

EXPANSION IN BELL POLYNOMIALS OF THE DISTRIBUTION OF THE TOTAL CLAIM AMOUNT WITH WEIBULL-DISTRIBUTED CLAIM SIZES

RAMON LACAYO[1]

(Received 31 August, 2006)

Abstract

The total claim amount for a fixed period of time is, by definition, a sum of a random number of claims of random size. In this paper we explore the probabilistic distribution of the total claim amount for claims that follow a Weibull distribution, which can serve as a satisfactory model for both small and large claims. As models for the number of claims we use the geometric, Poisson, logarithmic and negative binomial distributions. In all these cases, the densities of the total claim amount are obtained via Laplace transform of a density function, an expansion in Bell polynomials of a convolution and a subsequent Laplace inversion.

2000 *Mathematics subject classification*: 91B30.

Keywords and phrases: Laplace transform, Bell polynomial, total claim amount, claim size process, claim number process, probability density function.

1. Introduction

The distribution of the aggregate claim amount, together with ruin theory, is one of the main research areas in actuarial mathematics. Despite researchers' efforts, however, at present only a few distributions of the total claim amount are known in closed form. The computational difficulties are great enough that most of the advances have been made only on recursive methods (see, for example, [6]) or purely numerical approximation techniques such as the fast Fourier transform (see the works of Grübel and Hermesmeier [2, 3]). Some techniques are based on the central limit theorem and Monte Carlo methods (see [5]).

In this paper, we depart from numerical approaches and obtain some distributions of the total claim amount in closed form. Specifically, we shall obtain the densities of the total claim amount in the form of very rapidly converging double series that contain

[1]Escuela de Ingeniería Comercial, Universidad Católica del Norte, Larrondo 1281, Coquimbo, Chile; e-mail: rlacayo@ucn.cl.

Bell polynomials. These polynomials lend themselves to simple recursive definition and implementation with common mathematical software.

By definition, for fixed time t the total claim amount S is a random variable given by a compound sum; in other words, it is the random sum of N randomly sized positive claims X_i,

$$S = \sum_{i=1}^{N} X_i,$$

with partial sums

$$S_0 = 0 \quad \text{and} \quad S_n = X_1 + \cdots + X_n \text{ for } n \geq 1.$$

The claims X_i are assumed to be independent and identically distributed, with common density $f_X(x)$ and independent of N.

If U is a random variable with density $f_U(x)$, then we denote by $L_U(z)$ the ordinary Laplace transform of $f_U(x)$; that is, $L_U(z) = \int_0^\infty e^{-zx} f_U(x)\, dx$. The correspondence between the original function and the image function under the Laplace transform is written as $f_U(x) \risingdotseq L_U(z)$.

Let $P(N = n)$ be the probability function of N, $F_X(x)$ the common cumulative distribution function of the X_i, and $F_{S_n}(x)$ the cumulative distribution function of the nth partial sum of S.

The cumulative distribution function of S is given by

$$P(S \leq x) = E[P(S \leq x|N)] = \sum_{n=0}^{\infty} P(S_n \leq x) P(N = n)$$

(see [5]) or, equivalently,

$$F_S(x) = \sum_{n=0}^{\infty} F_{S_n}(x) P(N = n).$$

In terms of the densities, after differentiation we obtain

$$f_S(x) = \sum_{n=0}^{\infty} f_{S_n}(x) P(N = n).$$

From the above equality, in view of the linearity of the Laplace transformation, we obtain

$$L_S(z) = \sum_{n=0}^{\infty} L_{S_n}(z) P(N = n) \tag{1.1}$$

where, given that the X_i are independent and that $f_{S_n}(x)$ is the n-fold convolution of $f_X(x)$ with itself,

$$L_{S_n}(z) = (L_X(z))^n. \tag{1.2}$$

The function $f_S(x)$ can now be obtained by complex inversion of (1.1).

https://doi.org/10.1017/S1446181108000187 Published online by Cambridge University Press

According to a well-known theorem [1], if the series

$$\sum_{\nu=0}^{\infty} \frac{a_\nu}{s^{\lambda_\nu}} \quad (0 < \lambda_0 < \lambda_1 < \cdots) \tag{1.3}$$

converges absolutely for $|s| > \rho \geq 0$, then it is the Laplace transform of the series

$$f(t) = \sum_{\nu=0}^{\infty} a_\nu \frac{t^{\lambda_\nu - 1}}{\Gamma(\lambda_\nu)}, \tag{1.4}$$

which is obtained by termwise transformation of the former; the latter series converges absolutely for all $t \neq 0$. This is the main tool that we shall employ to find the density $f_S(x)$ from the Laplace transform $L_S(z)$. It is straightforward to verify that all such series appearing in our paper satisfy the conditions of this theorem.

In the following sections we will derive closed-form expressions, in terms of Bell polynomials, for the distribution of the total claim amount when the claim size has a Weibull distribution and the number process follows a Poisson, negative binomial, geometric or logarithmic distribution.

The paper is organized as follows. Section 2 summarizes the specifics of the Weibull distribution in the context of this paper. Sections 3, 4 and 5 deal with the computation of the densities for the Poisson, negative binomial and logarithmic cases, respectively. The Weibull–geometric case is considered in Section 4 as a special case of the Weibull–negative binomial combination.

2. Preliminaries

Let X be any one of the claims. If it follows a Weibull distribution with parameters α and λ (α, $\lambda > 0$), then we write

$$f_X(x) = \alpha\lambda x^{\alpha-1} e^{-\lambda x^\alpha}, \quad x > 0. \tag{2.1}$$

It is understood that this density vanishes for non-positive x.

The Weibull distribution appears to model "small claims" fairly well for $a \geq 1$, whereas for $\alpha < 1$ it provides a satisfactory model for "large claims".

We now obtain the Laplace transform of $f_X(x)$,

$$f_X(x) \risingdotseq L_X(z) = \int_0^\infty e^{-zx} f_X(x)\, dx = \alpha\lambda \int_0^\infty x^{\alpha-1} e^{-\lambda x^\alpha} e^{-zx}\, dx. \tag{2.2}$$

From (1.2) and (1.3), it is clear that $L_X(z)$ should be written as a series of inverse powers of z. To this end, we expand the $\exp(-\lambda x^\alpha)$ in (2.2) in powers of $-\lambda x^\alpha$, and then integrate term by term. As a result we find that

$$L_X(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\Gamma[k\alpha + 1]\lambda^k}{k!} \frac{1}{z^{k\alpha}}, \tag{2.3}$$

https://doi.org/10.1017/S1446181108000187 Published online by Cambridge University Press

where $\Gamma(\cdot)$ is the Gamma function. We may view this series as a formal power series (in powers of $1/z^{\alpha}$).

The generating function of the partial Bell polynomials $B_{n,k}(g_1, \ldots, g_{n-k+1})$ of the variables g_i, $i = 1, \ldots, n-k+1$, is given by

$$\Phi(t, u) = \exp\left(u \sum_{m=1}^{\infty} \frac{g_m t^m}{m!}\right) \tag{2.4}$$

$$= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} B_{n,k}(g_1, \ldots, g_{n-k+1}) \frac{u^k t^n}{n!} \tag{2.5}$$

(this can be found in the section on Bell polynomials in [4], for example). The Bell polynomials are well-suited for writing a function of a power series as a power series; this is true even if the power series is only a formal one.

In particular, let $g(t) = \sum_{m=0}^{\infty} (g_m t^m / m!)$. Then the function $h(g)$ may be written as

$$h(g(t)) = h\left(\sum_{m=0}^{\infty} \frac{g_m t^m}{m!}\right) \tag{2.6}$$

$$= h(0) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{d^k h}{dg^k}\bigg|_{t=0} B_{n,k}(g_1, \ldots, g_{n-k+1}) \frac{t^n}{n!}. \tag{2.7}$$

In what follows we will obtain the Laplace transform $L_S(z)$ of the total claim amount S as a function of $L_X(z)$. In other words,

$$L_S(z) = h[L_X(z)]. \tag{2.8}$$

Comparing (2.8) with (2.3) and (2.6), we see that in all cases we can set

$$g(z) = L_X(x) \quad \text{and} \quad g_m = (-1)^{m+1} \Gamma[m\alpha + 1] \lambda^m, \tag{2.9}$$

unless we particularize the Weibull distribution to the exponential distribution. This would happen when $\alpha = 1$, in which case (2.1) would become $f_X(x) = \lambda e^{-\lambda x}$ with $x > 0$, $\lambda > 0$, and (2.9) would turn into

$$g_m = (-1)^{m+1} m! \lambda^m. \tag{2.10}$$

3. $N \sim$ Poisson(θ)

The probability function of N is

$$P(N = n) = e^{-\theta} \frac{\theta^n}{n!}, \quad n = 0, 1, \ldots. \tag{3.1}$$

https://doi.org/10.1017/S1446181108000187 Published online by Cambridge University Press

According to (1.1), (1.2) and (3.1), we may write the Laplace transform of the total claim amount as

$$L_S(z) = e^{-\theta} \sum_{n=0}^{\infty} \frac{[\theta L_X(z)]^n}{n!} = e^{-\theta} \exp(\theta L_X(z)), \tag{3.2}$$

where $L_X(z)$ is given by (2.3).

Since $L_S(z) = e^{-\theta} \exp(\theta L_X(z))$, we see, by comparing with (2.4), that save for a constant factor $e^{-\theta}$, $L_S(z)$ is the generating function for the partial Bell polynomials (with $u = \theta$ and g_m as in (2.9)).

From (2.5) we get

$$L_S(z) = e^{-\theta}\left(1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_{n,k}(g_1, \ldots, g_{n-k+1}) \frac{\theta^k}{n! z^{n\alpha}}\right).$$

Given that $\delta(x) \doteqdot 1$ where $\delta(x)$ is the Dirac delta function, applying the inversion theorem (see [1]) to the sum above gives us the density of the total claim amount,

$$f_S(x) = e^{-\theta}\left(\delta(x) + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_{n,k}(g_1, \ldots, g_{n-k+1}) \frac{\theta^k x^{n\alpha-1}}{n!\Gamma(n\alpha)}\right). \tag{3.3}$$

Setting $\alpha = 1$ in this expression and using (2.10) yields the density of the aggregate claim amount for the exponential–Poisson combination,

$$f_S(x) = e^{-\theta}\left(\delta(x) + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_{n,k}(g_1, \ldots, g_{n-k+1}) \frac{\theta^k x^{n-1}}{n!\Gamma(n)}\right). \tag{3.4}$$

4. $N \sim$ NegativeBinomial(r, p)

Here $P(N = n) = C_n^{n+r-1} p^r q^n = \big((r)_n/(1)_n\big) p^r q^n$, $n = 0, 1, \ldots,$ where $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)$ is the Pochhammer symbol. Also, $p + q = 1$.

Using this in conjunction with (1.1) and (1.2), we get

$$L_S(z) = p^r \sum_{n=0}^{\infty} \frac{(r)_n}{(1)_n} [q L_X(z)]^n = p^r \sum_{n=0}^{\infty} (r)_n \frac{[q L_X(z)]^n}{n!} \tag{4.1}$$

$$= p^r \, {}_1F_0(r; q L_X(z)), \tag{4.2}$$

where ${}_1F_0(\alpha, \beta; z)$ is a hypergeometric function.

By virtue of (2.9), we can write $L_S(z)/p^r = h(g) = {}_1F_0(r; qg)$; this is (2.7). The kth derivative of this function is easily found to be $h^{(k)}(g) = q^k (r)_k \, {}_1F_0(r + k; qg)$. Therefore $h^{(k)}(g(0)) = q^k (r)_k$ for $k \geq 1$, and finally

$$L_S(z) = p^r \left(1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} q^k (r)_k B_{n,k}(g_1, \ldots, g_{n-k+1}) \frac{1}{n! z^{n\alpha}}\right).$$

https://doi.org/10.1017/S1446181108000187 Published online by Cambridge University Press

Proceeding as before, we use the inversion theorem to obtain

$$f_S(x) = p^r\left(\delta(x) + \sum_{n=1}^{\infty}\sum_{k=1}^{n} q^k (r)_k B_{n,k}(g_1, \ldots, g_{n-k+1})\frac{x^{n\alpha-1}}{n!\Gamma(n\alpha)}\right). \tag{4.3}$$

Setting $\alpha = 1$ in this expression and using (2.10), we find the density of the total claim amount for the exponential–negative binomial combination,

$$f_S(x) = p^r\left(\delta(x) + \sum_{n=1}^{\infty}\sum_{k=1}^{n} q^k (r)_k B_{n,k}(g_1, \ldots, g_{n-k+1})\frac{x^{n-1}}{n!\Gamma(n)}\right). \tag{4.4}$$

For $r = 1$ in (4.3), we obtain additionally the density of the total claim amount for the Weibull–geometric combination. In this case,

$$f_S(x) = p\left(\delta(x) + \sum_{n=1}^{\infty}\sum_{k=1}^{n} q^k k! B_{n,k}(g_1, \ldots, g_{n-k+1})\frac{x^{n\alpha-1}}{n!\Gamma(n\alpha)}\right). \tag{4.5}$$

Putting $\alpha = 1$ in (4.5) and using (2.10) should yield the well-known exponential–geometric result, $f_S(x) = p\delta(x) + \lambda pq \exp(-\lambda px)$ (see [5]). This is indeed the case: we obtain

$$f_S(x) = p\left(\delta(x) + \sum_{n=1}^{\infty}\sum_{k=1}^{n} q^k k! B_{n,k}(g_1, \ldots, g_{n-k+1})\frac{x^{n-1}}{n!\Gamma(n)}\right), \tag{4.6}$$

which is in excellent agreement with the result quoted above.

5. $N \sim$ Logarithmic(p)

The probability function of N is $P(N = n) = -q^n/(n \ln p)$, $n = 1, 2, \ldots$. In the same manner as before,

$$L_S(z) = -\frac{1}{\ln p}\sum_{n=1}^{\infty}\frac{[qL_X(z)]^n}{n} = \frac{1}{\ln p}\ln(1 - qL_X(z)).$$

This time, using (2.9), we write $L_S(z)\ln p = h(g) = \ln(1 - qg)$. By comparing this result with (2.6) and (2.7), we find the derivatives to be $h^{(k)}(g) = (-1)^k q^k (k-1)!/(qg-1)^k$. Therefore $h^{(k)}(g(0)) = -q^k(k-1)!$ for $k \geq 1$, and $h(g(0)) = 0$. Consequently,

$$L_S(z) = -\frac{1}{\ln p}\sum_{n=1}^{\infty}\sum_{k=1}^{n} q^k (k-1)! B_{n,k}(g_1, \ldots, g_{n-k+1})\frac{1}{n! z^{n\alpha}}.$$

Finally, by the inversion theorem,

$$f_S(x) = -\frac{1}{\ln p}\sum_{n=1}^{\infty}\sum_{k=1}^{n} q^k (k-1)! B_{n,k}(g_1, \ldots, g_{n-k+1})\frac{x^{n\alpha-1}}{n!\Gamma(n\alpha)}. \tag{5.1}$$

https://doi.org/10.1017/S1446181108000187 Published online by Cambridge University Press

Setting $\alpha = 1$ in (5.1) and using (2.10), we obtain the corresponding exponential–logarithmic case, namely

$$f_S(x) = -\frac{1}{\ln p}\sum_{n=1}^{\infty}\sum_{k=1}^{n} q^k (k-1)! B_{n,k}(g_1, \ldots, g_{n-k+1}) \frac{x^{n-1}}{n!\Gamma(n)}.$$

References

[1] G. Doetsch, *Introduction to the theory and applications of the Laplace transformation* (Springer, Berlin, 1974).

[2] R. Grübel and R. Hermesmeier, "Computation of compound distributions I: aliasing errors and exponential tilting", *Astin Bull.* **29** (1999) 197–214.

[3] R. Grübel and R. Hermesmeier, "Computation of compound distributions II: discretization errors and Richardson extrapolation", *Astin Bull.* **30** (2000) 309–331.

[4] M. Hazewinkel (ed.), *Encyclopaedia of mathematics* (Kluwer, 1990).

[5] T. Mikosch, *Non-life insurance mathematics. An introduction with stochastic processes* (Springer, Berlin, 2004).

[6] H. H. Panjer, "Recursive evaluation of a family of compound distributions", *Astin Bull.* **11** (1981) 22–26.