INFINITE EULER GRAPHS

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1. Introduction, definitions. It is well known that for finite connected graphs the following are equivalent:

(i) *X is Euler* (i.e., every vertex of X has positive even degree);

(ii) *X is traceable* (i.e., the edges of X can be arranged in a sequence $e_1, \ldots, e_n$ such that $e_i \neq e_j$ if $i \neq j$, and e_i, e_{i+1} are adjacent, $i = 1, \ldots, n$, subscripts considered mod n);

(iii) *X is cyclically coverable* (i.e., X contains a family of non-overlapping circuits whose union is X).

The equivalence (i) $\Leftrightarrow$ (ii) was proved by Euler (**4**, 3.1.1); (i) $\Leftrightarrow$ (iii) is due to Veblen (**3**, Kapitel II, §5, Satz 11), and actually holds for arbitrary (not necessarily connected) locally finite graphs as well. We shall refer to this as "Veblen's theorem."

For arbitrary infinite Euler graphs both (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) fail to hold. A characterization of infinite traceable graphs was given by Erdös, Grünwald, and Vázsonyi (**2**). In the present note we propose to study extensions of Veblen's theorem. We show in particular that the "correct" generalization of that theorem is that every Euler graph is "traceably" coverable (Section 2).

If A is a set, $|A|$ is the cardinal of A. A function $f: A \to A$ is a *pairing function* for A if and only if f is one–one, onto, and $fa \neq a = f^2a$ for every $a \in A$. A pairing function exists if and only if $|A|$ is even or infinite.

A *graph* X is a set $V(X)$ (vertices) together with a set $E(X)$ (edges) of unordered pairs of distinct elements of $V(X)$. Unordered pairs will be indicated by brackets. We shall write $x \in X$ for $x \in V(X)$, and $e \in X$ for $e \in E(X)$. If $e = [x, y] \in E(X)$, (e) will denote the graph consisting of e and its two ends x, y.

The *order* of a graph X, denoted by $|X|$, is the cardinal of $V(X)$.

For $x \in X$ we define

$$V(X; x) = \{y \in X : [x, y] \in E(X)\},$$
$$E(X; x) = \{[x, y] : y \in V(X; x)\}.$$

$|V(X; x)|$ is called the *degree* of x in X, and is denoted by $d(X; x)$, or d_x when no confusion is likely. X is *Euler* if for any $x \in X$, d_x is positive and even

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or infinite. X is *locally finite*, if d_x is finite for every $x \in X$. If n is a cardinal, X is *n-regular* if $d_x = n$ for every $x \in X$. A non-empty connected 2-regular graph is called a *circuit*. Every non-empty Euler graph contains a circuit.

A graph X is *acyclic* if it contains no finite circuit. A connected acyclic graph is called a *tree*. A vertex x of an acyclic graph X is called an *end-vertex* if $d(X; x) = 1$, x is a *branch-vertex* if $d(X; x) \geqslant 3$. If X is acyclic, and x and y belong to the same component of X, there is a unique path in X which joins x and y. This path will be denoted by X_{xy}. If $x, y, z \in X$, then y *lies between* x and z if $y \in X_{xz}$ but $y \neq x, z$.

If Y is a subgraph of X we denote by $X \setminus Y$ the smallest subgraph of X with $E(X \setminus Y) = E(X) - E(Y)$. If X and Y are Euler, and if Y is locally finite, then $X \setminus Y$ is Euler.

If A is a subset of $V(X)$ denote by $X - A$ the maximal subgraph of X with $V(X - A) = V(X) - A$. If X is connected, A is a *cut set* of X if $X - A$ is disconnected. If a cut set consists of a single vertex x, then x is called a cut vertex.

Given graphs X, Y, a *homomorphism* $\phi : X \rightarrow Y$ is a function from $V(X)$ to $V(Y)$ such that $[x, y] \in E(X)$ implies $[\phi x, \phi y] \in E(Y)$. (In the terminology of (**4**, p. 85) our homomorphisms are *independent*.) A homomorphism ϕ induces a function $\tilde{\phi} : E(X) \rightarrow E(Y)$ by $\tilde{\phi}[x, y] = [\phi x, \phi y]$, $[x, y] \in E(X)$. ϕ is called *strong* if and only if $\tilde{\phi}$ is one–one; ϕ is an *epimorphism* if ϕ and $\tilde{\phi}$ are onto. Clearly, for graphs without isolated vertices, ϕ is an epimorphism if and only if $\tilde{\phi}$ is onto.

A graph X is called *traceable* if and only if there exists a strong epimorphism $\phi : C \rightarrow X$, where C is a (finite or infinite) circuit. It is easily verified that this definition of traceability coincides with the usual one (cf. **4**, p. 42).

The composition of two strong epimorphisms is again a strong epimorphism. Hence if X is traceable, and $\psi : X \rightarrow Y$ is a strong epimorphism, then Y is traceable.

By $\square$ we shall denote the empty set or the empty graph, depending on the context. If X and Y are graphs, we shall say that X and Y *do not overlap* if and only if $E(X) \cap E(Y) = \square$. X and Y may have common vertices.

Given a graph X, a *cover* of X is a family $\mathfrak{A}$ of subgraphs of X such that (i) $A \in \mathfrak{A}$ implies $E(A)$ is non-empty; (ii) $\bigcup \{A : A \in \mathfrak{A}\} = X$; (iii) if A and B are two distinct members of $\mathfrak{A}$, then $E(A)$ and $E(B)$ are disjoint (A and B do not overlap). A cover of X is called *cyclic* if and only if every $A \in \mathfrak{A}$ is a circuit. X is called *cyclically coverable* (c.c.) if it possesses a cyclic cover; it is called finitely *cyclically coverable* (f.c.c.) if it has a cover consisting of finite circuits.

It is easily verified that if $\phi : X \rightarrow Y$ is a strong epimorphism and $\mathfrak{A}$ is a cover of X, then $\phi\mathfrak{A} = \{\phi A : A \in \mathfrak{A}\}$ is a cover of Y.

2. Traceable covers.

(2.1) *Definition*. A cover $\mathfrak{A}$ of a graph X is called *traceable* if and only if every member of $\mathfrak{A}$ is traceable. X is called *traceably coverable* (t.c.) if it possesses a traceable cover.

By (**4**, 3.2.2, γ_2) every traceable graph is Euler; hence every t.c. graph is Euler.

It will be shown below (Theorem (2.2)) that every Euler graph is a strongly homomorphic image of a locally finite Euler graph. Thus the failure of Veblen's theorem to hold for arbitrary Euler graphs is due to the fact that the class of all c.c. graphs is not closed under strong homomorphisms (a strongly homomorphic image of an infinite circuit need not be c.c.). The class of all t.c. graphs, on the other hand, does have this closure property, i.e., if X is t.c. and $\phi : X \to Y$ is a strong epimophism, then Y is t.c. For if $\mathfrak{A}$ is a traceable cover of X, then $\phi\mathfrak{A}$ is a cover of Y and since the strongly homomorphic image of a traceable graph is traceable, each member of $\phi\mathfrak{A}$ is traceable. It follows in particular that the strongly homomorphic image of a c.c. graph is t.c.

(2.2) THEOREM. *Let X be an Euler graph. Then there exists a strong epimorphism $\phi : Y \to X$ such that Y is 2-regular.*

Proof. For each $x \in X$, let f_x be a pairing function for $E(X;x)$. Since $d(X;x)$ is even or infinite, such an f_x exists. For $x \in X$, $e \in E(X;x)$, denote by ex the two-element set $\{e, f_x e\}$. Suppose that $ex = e'x'$. Then either (i) $e = e'$ and $f_x e = f_{x'} e'$, or (ii) $e = f_{x'} e'$ and $e' = f_x e$. In case (i), e lies in the domain of both f_x and $f_{x'}$, i.e., $e \in E(X;x) \cap E(X;x')$. If $x \neq x'$ this means that $e = [x, x']$. But $f_x e \in E(X;x)$, $f_{x'} e' \in E(X;x')$; hence $f_x e \in E(X;x) \cap E(X;x')$, so that $f_x e = e$, a contradiction against the definition of f_x. Hence $x = x'$. In case (ii), $e = f_{x'} e'$ implies that $e, e' \in E(X;x')$; similarly, $e' = f_x e$ implies that $e, e' \in E(X;x)$. $e' = f_x e \neq e$; hence $x = x'$. It follows that $ex = e'x'$ if and only if $x = x'$ and $e = e'$ or $e' = f_x e$ (the sufficiency of this condition is obvious).

Now define Y by

$$V(Y) = \{ex : x \in X,\ e \in E(X;x)\},$$
$$E(Y) = \{[ex, e'x'] : ex \cap e'x' = [x, x']\}.$$

It is easily verified that for any $ex \in V(Y)$,

$$V(Y; ex) = \{ex', e'x''\},$$

where $[x, x'] = e$, and $e' = [x, x''] = f_x e$. Since $e \neq f_x e$, i.e., $x' \neq x''$, it follows that $ex' \neq e'x''$; hence $d(Y; ex) = 2$. Thus Y is 2-regular.

Define $\phi : Y \to X$ by $\phi(ex) = x$. Since $ex = e'x'$ implies $x = x'$, ϕ is well defined. It is immediate from the definition of Y that ϕ is an epimorphism. To see that ϕ is strong, let $\epsilon_i = [e_i x_i, e_i' x_i'] \in E(Y)$, $i = 0, 1$, and suppose that $\tilde{\phi}\epsilon_0 = \tilde{\phi}\epsilon_1$, i.e., $[x_0, x_0'] = [x_1, x_1']$. Hence (i) $x_0 = x_1, x_0' = x_1'$ or (ii) $x_0 = x_1'$, $x_1 = x_0'$. $[x_i, x_i'] = e_i x_i \cap e_i' x_i'$ implies $g_i e_i = g_i' e_i' = [x_i, x_i']$, where $g_i e_i$ denotes e_i or $f_{x_i} e_i$, as the case may be; similarly, $g_i' e_i'$, $i = 0, 1$. Hence, in case (i),

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$$\epsilon_1 = [e_1 x_1, e_1' x_1'] = [e_1 x_0, e_1' x_0'] = [(g_0 e_0)x_0, (g_0' e_0')x_0'] = [e_0 x_0, e_0' x_0'] = \epsilon_0,$$

and in case (ii)

$$\epsilon_1 = [e_1 x_1, e_1' x_1'] = [e_1 x_0', e_1' x_0] = [(g_0' e_0')x_0', (g_0 e_0)x_0] = [e_0' x_0', e_0 x_0] = \epsilon_0.$$

(2.3) THEOREM. *A necessary and sufficient condition that a graph X be t.c. is that X be Euler.*

Proof. Necessity: Cf. the remark following Definition (2.1). Sufficiency: Let $\phi : Y \to X$ be as in (2.2). Being 2-regular, Y is a union of disjoint circuits, and hence c.c. (2.3) then follows from the remarks preceding (2.2).

For acyclic graphs Theorem (2.3) can be carried a step further.

(2.4) THEOREM. *If X is an acyclic Euler graph, then X is cyclically coverable.*

We shall prove this by a sequence of simple lemmas. These are stated in somewhat greater generality than actually needed.

(2.5) LEMMA. *If $X \subset Y$, $Y \setminus X$ is finite, and X has only finitely many finite components, then X does not have fewer infinite components than Y.*

Proof. Suppose there is an infinite component K of Y which contains no infinite component of X. Then $K \cap X$ is either empty or consists only of finite components of X, and since these are finite in number, $K \cap X$ is finite. Hence $K \setminus (K \cap X)$ is infinite. But $Y \setminus X \supset K \setminus (K \cap X)$, and hence is infinite, a contradiction.

(2.6) LEMMA. *Let X be Euler, $x \in X$. Then either x belongs to a finite circuit, or X is infinite, and $X \setminus Y$ has at least $d(Y; x)$ infinite components, where Y is any finite subgraph of X containing x.*

Proof. Assume that x does not belong to a finite circuit. By Veblen's theorem this means that X is infinite. Let $Y \subset X$ be finite, $x \in Y$. Denote by Y_x the smallest subgraph of Y with $E(Y_x) = E(Y; x)$. Since x belongs to no finite circuit of X, x is a cut vertex of X, and no two distinct members of $V(X; x)$ belong to the same component of $X \setminus Y_x$. Each component of $X \setminus Y_x$ is infinite (it would otherwise contain exactly one vertex of odd degree, viz. the vertex belonging to $V(Y; x)$). Hence $X \setminus Y_x$ has at least $d(Y; x)$ infinite components. Now $X \setminus Y \subset X \setminus Y_x$, and $(X \setminus Y_x) \setminus (X \setminus Y) = Y \setminus Y_x$ is finite; moreover, $X \setminus Y$ has at most $|Y|$ components. Hence by Lemma (2.5), $X \setminus Y$ has at least $d(Y; x)$ infinite components.

If $V(Y; x)$ is a proper subset of $V(X; x)$, then $X \setminus Y$ has $d(Y; x) + 1$ infinite components. For, in that case $X \setminus Y_x$ consists of the components containing the vertices of $V(Y; x)$, as well as the component containing x.

(2.7) LEMMA. *An acyclic graph is traceable if and only if it is an infinite circuit.*

Proof. Sufficiency is obvious. Necessity: Suppose X is acyclic and traceable. Being traceable, X is connected and Euler; hence $d(X;x) \geqslant 2$ for every $x \in X$ (**4**, 3.2.2, γ_2). Suppose that $d(X;x) \geqslant 3$ for some $x \in X$. Let Y be any finite subgraph of X containing three edges incident with x. By Lemma (2.6) and the acyclicity of X, $X \setminus Y$ has at least three infinite components. But this is a contradiction against the traceability of X (**4**, 3.2.2, δ_2). It follows that $d(X;x) = 2$ for all $x \in X$; i.e., X is a circuit. By the acyclicity of X it must be an infinite circuit.

Proof of Theorem (2.4). By Theorem (2.3), X has a cover $\mathfrak{A}$ such that every $A \in \mathfrak{A}$ is traceable. But $A \in \mathfrak{A}$ implies $A \subset X$, and hence A is acyclic. By Lemma (2.7) this means that A is an infinite circuit. Thus X has a cover consisting of infinite circuits.

3. Finitely cyclically coverable graphs.

(3.1) LEMMA. *Let X be a countable graph. Then X is f.c.c. if and only if given any finite subgraph Y of X, there is a finite Euler subgraph Z of X such that $Y \subset Z$.*

Proof. Necessity: Let $\mathfrak{A}$ be a cover of X consisting of finite circuits. If Y is a finite subgraph of X, then $\mathfrak{A}_Y = \{A \in \mathfrak{A} : E(A) \cap E(Y) \neq \square\}$ is finite; in fact, $|\mathfrak{A}_Y| \leqslant |E(Y)|$. Hence $Z_Y = \bigcup \mathfrak{A}_Y$ is a finite Euler subgraph of X containing Y. Here countability of X is not used.

Sufficiency: Since X is countable, $E(X) = \{e_1, e_2, \ldots\}$. Take $Y_1 = (e_1)$. By hypothesis there is a finite Euler graph $Z_1 \subset X$ such that $e_1 \in Z_1$. Let $\mathfrak{A}_1$ be a cyclic cover of Z_1. By i_2 denote the smallest subscript such that $e_{i_2} \notin Z_1$. Clearly $i_2 \geqslant 2$. Take $Y_2 = Z_1 \cup (e_{i_2})$. Then there is a finite Euler graph $Z_2 \subset X$ such that $e_1, \ldots, e_{i_2} \in Z_2$. $Z_2 \setminus Z_1$ is a non-empty finite Euler graph, hence f.c.c. Let $\mathfrak{B}_2$ be a cyclic cover of $Z_2 \setminus Z_1$. Then $\mathfrak{A}_2 = \mathfrak{A}_1 \cup \mathfrak{B}_2$ is a cyclic cover of Z_2 such that $\mathfrak{A}_1 \subset \mathfrak{A}_2$. Continue in this manner obtaining finite Euler subgraphs Z_n of X, $n = 1, 2, \ldots$, such that $Z_1 \cup Z_2 \cup \ldots = X$, and covers $\mathfrak{A}_n$ consisting of finite circuits and such that $\mathfrak{A}_n \subset \mathfrak{A}_{n+1}$ for all n. Then

$$\mathfrak{A} = \bigcup_{n=1}^{\infty} \mathfrak{A}_n$$

is a cover of X consisting of finite circuits.

(3.2) COROLLARY. *Any countable graph X contains a maximal f.c.c. subgraph.*

Proof. If X is acyclic, the empty graph is the maximal f.c.c. subgraph of X. If X is not acyclic, the collection $\mathfrak{F}$ of all f.c.c. subgraphs of X is non-empty. Let $\mathfrak{F}_0$ be a chain in $\mathfrak{F}$, $F_0 = \bigcup \mathfrak{F}_0$. Since X is countable, so is F_0.

Let Y be a finite subgraph of F_0. Then $Y \subset F$ for some $F \in \mathfrak{F}_0$. By (3.1) there is a finite Euler subgraph Z of F such that $Y \subset Z$. But then $Y \subset Z \subset F_0$. Hence, by (3.1), F_0 is f.c.c. Thus $\mathfrak{F}$ has a maximal element.

We have been unable to determine whether (3.1) holds for uncountable graphs as well. If it does, then it would follow that the property of being f.c.c. is of finite character, and hence any graph would contain a maximal f.c.c. subgraph. Instead of (3.1) it is natural to try the following condition: For any $Y \subset X$ with $|Y| < |X|$ there exists an f.c.c. subgraph Z of X such that $Y \subset Z$ and $|Y| = |Z|$. Certainly this condition is necessary if X is to be f.c.c.; again we were unable to prove sufficiency. One might attempt to obtain a condition for finite cyclic coverability by first characterizing traceable f.c.c. graphs and then using Theorem (2.3). Traceable f.c.c. graphs are easy to describe; cf. (3.5). By the same arguments one can also obtain a relatively general condition for finite cyclic coverability; cf. Theorem (3.4).

(3.3) *Definition*. Let X be an infinite graph, $|X| = \aleph_\tau$. For $0 \leqslant \sigma \leqslant \tau$ let $I_\sigma = \{x \in X : d_x \leqslant \aleph_\sigma\}$. X will be called *stratified* if and only if $|I_\sigma| \leqslant \aleph_\sigma$ for each σ, $0 \leqslant \sigma \leqslant \tau$.

Note that if X is countable, then X is stratified.

(3.4) THEOREM. *Let X be a stratified Euler graph having the following property (P): if Y is a subgraph of X with $|Y| < |X|$, then $X \diagdown Y$ has exactly one infinite component. Then X is f.c.c.*

Proof. Suppose that $|X| = \aleph_\tau$. For $0 \leqslant \sigma \leqslant \tau$ put

$$E_\sigma = \bigcup_{x \in I_\sigma} E(X; x).$$

Since $x \in I_\sigma$ implies $d_x = |E(X; x)| \leqslant \aleph_\sigma$, and since $|I_\sigma| \leqslant \aleph_\sigma$ (stratification), it follows that $|E_\sigma| \leqslant \aleph_\sigma$. Hence, the edges of X can be indexed by the ordinals $\alpha < \omega_\tau$ in such a way that for any $\sigma \leqslant \tau$,

$$e_\alpha \in E_\sigma \quad \text{implies } \alpha < \omega_\sigma. \tag{1}$$

That is, we first index the members of E_0, then those in E_1 which do not belong to E_0, and so on.

We now define for each ordinal $\alpha < \omega_\tau$ two graphs X_α, A_α and a cover $\mathfrak{A}_\alpha$ of A_α with the following properties:

(i) X_α is an infinite Euler graph having property (P);

(ii) $X_\alpha = X \diagdown A_\alpha$;

(iii) $e_\beta \in E(A_\alpha)$ for all $\beta < \alpha$;

(iv) A_α is finite for $\alpha < \omega$, $|A_\alpha| \leqslant |\alpha|$ if α is infinite;

(v) $\mathfrak{A}_\alpha$ is a cover of A_α consisting of finite circuits;

(vi) $\mathfrak{A}_\beta \subset \mathfrak{A}_\alpha$ for all $\beta < \alpha$.

Put $X_0 = X$, $A_0 = \square$, $\mathfrak{A}_0 = \square$. For $1 \leqslant \alpha < \omega_\tau$ define X_α, A_α, $\mathfrak{A}_\alpha$ as follows. If α is not a limit ordinal and $e_{\alpha-1} \in A_{\alpha-1}$, take $X_\alpha = X_{\alpha-1}$, $A_\alpha = A_{\alpha-1}$, $\mathfrak{A}_\alpha = \mathfrak{A}_{\alpha-1}$. If $e_{\alpha-1} \notin A_{\alpha-1}$, i.e., $e_{\alpha-1} \in X_{\alpha-1}$, then by (2.6) (using that $X_{\alpha-1}$ has property (P)) there is a finite circuit C_α such that $e_{\alpha-1} \in C_\alpha \subset X_{\alpha-1}$. In this case put $A_\alpha = A_{\alpha-1} \cup C_\alpha$, $\mathfrak{A}_\alpha = \mathfrak{A}_{\alpha-1} \cup \{C_\alpha\}$, $X_\alpha = X \diagdown A_\alpha$. If α is a limit ordinal, put

$$A_\alpha = \bigcup_{\beta<\alpha} A_\beta, \qquad \mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta,$$

and $X_\alpha = X \diagdown A_\alpha$.

That X_α, A_α, and $\mathfrak{A}_\alpha$ so defined satisfy (ii), (iii), (v), and (vi) is obvious.

(iv): Since C_α is finite, A_α is clearly finite for finite α. If α is infinite and not a limit ordinal, then either $|A_\alpha| = |A_{\alpha-1}| \leqslant |\alpha - 1| = |\alpha|$ or

$$|A_\alpha| = |E(A_{\alpha-1})| + |E(C_\alpha)| \leqslant |\alpha - 1| = |\alpha|.$$

If α is a limit ordinal, then

$$|A_\alpha| \leqslant \sum_{\beta<\alpha} |A_\beta| = \sum_{\beta<\alpha} |\beta| = |\alpha|.$$

It remains to show (i). For finite α it is obvious that X_α is infinite Euler. Suppose that α is infinite, and let $x \in X_\alpha = X \diagdown A_\alpha$. If $x \notin A_\alpha$, then $d(X_\alpha; x) = d(X; x)$, and hence is even or infinite. If $x \in A_\alpha$, then $E(X_\alpha; x) = E(X; x) - E(A_\alpha; x)$. There is a unique $\sigma < \tau$ such that

$$\omega_\sigma \leqslant \alpha < \omega_{\sigma+1}. \tag{2}$$

By (iii), $e_\beta \in E(A_\alpha)$ for all $\beta < \omega_\sigma$. Together with (1) this implies that $E_\sigma \subset E(A_\alpha)$. Hence, if $d(X; x) \leqslant \aleph_\sigma$, then $x \in I_\sigma$. By the definition of E_σ, $E(X; x) \subset E_\sigma$, so that $E(X; x) \subset E(A_\alpha)$. But then $x \notin X \diagdown A_\alpha$, a contradiction. It follows that $d(X; x) = |E(X; x)| > \aleph_\sigma$. On the other hand, by (2),

$$|E(A_\alpha; x)| \leqslant |E(A_\alpha)| \leqslant |\alpha| = \aleph_\sigma.$$

Hence

$$d(X_\alpha; x) = |E(X_\alpha; x)| = |E(X; x) - E(A_\alpha; x)| = |E(X; x)| > \aleph_\sigma.$$

Thus X_α is infinite Euler. To see that it has property (P), take any $Z \subset X_\alpha$ with $|Z| < |X_\alpha|$. Then $X_\alpha \diagdown Z = X \diagdown (A_\alpha \cup Z)$, and

$$|A_\alpha \cup Z| \leqslant |E(A_\alpha)| + |E(Z)| \leqslant |\alpha| + |E(Z)| < |X|.$$

Since X has property (P), $X \diagdown (A_\alpha \cup Z)$ has exactly one infinite component.

By (iii),

$$\bigcup_{\alpha<\omega_\tau} A_\alpha = X.$$

Hence

$$\mathfrak{A} = \bigcup_{\alpha<\omega_\tau} \mathfrak{A}_\alpha$$

is a cover of X consisting of finite circuits.

(3.5) COROLLARY. *Let X be an infinite traceable graph. Then a necessary and sufficient condition that X be f.c.c. is that given any finite $Y \subset X$, $X \setminus Y$ have exactly one infinite component.*

Proof. Since X is a countable connected Euler graph, sufficiency follows from (3.4). Necessity: Let X be infinite, traceable, and f.c.c. If X does not have property (P), then there exists a finite $Y \subset X$ such that $X \setminus Y$ has at least two infinite components. By (3.1), there exists a finite Euler graph Z such that $Y \subset Z \subset X$. Since Z is finite, $X \setminus Z$ has at most finitely many components. $X \setminus Z \subset X \setminus Y$, and the two graphs differ only by finitely many edges. Hence by Lemma (2.5), $X \setminus Z$ has at least as many infinite components as $X \setminus Y$, i.e., at least two, a contradiction against **(4**, 3.2.2, δ_3**)**.

We now make an application of (3.2) to obtain a slight generalization of Veblen's theorem.

(3.6) THEOREM. *Let X be an Euler graph containing exactly one vertex x_0 of infinite degree. Then X is either cyclically coverable or has a cover consisting of circuits and a ray starting at x_0* (i.e. a graph R with $V(R) = \{x_0, x_1, \ldots\}$, $E(R) = \{[x_i, x_{i+1}] : i = 0, 1, \ldots\}$).

In order to prove (3.6) for uncountable X, we need the following lemma, which, in view of its close connection with Section 4, we state in greater generality than needed at this point.

(3.7) LEMMA. *Let X be a connected uncountable graph, $|X| = \aleph_\alpha$. Put $I = \{x \in X : d_x > \aleph_0\}$, $I_\alpha = \{x \in X : d_x = \aleph_\alpha\}$. Then*

(i) *I is non-empty, and $|I| = \aleph_\alpha$ or I is a cut set of X;*

(ii) *if $\aleph_\alpha$ is a regular cardinal, then I_α is non-empty, and $|I_\alpha| = \aleph_\alpha$ or I_α is a cut set of X.*

Proof. Consider case (ii) first. Suppose $d_x < \aleph_\alpha$ for every $x \in X$. Fix $x_0 \in X$, and for $n = 0, 1, \ldots$ let $A_n = \{x \in X : \rho(x, x_0) = n\}$, where ρ denotes distance. $A_0 = \{x_0\}$, and for $n \geqslant 1$,

$$A_n \subset \bigcup \{V(X; x) : x \in A_{n-1}\}.$$

Hence $|A_0| = 1$, and

$$|A_n| \leqslant \sum_{x \in A_{n-1}} d_x.$$

Thus, by induction, if $|A_{n-1}| < \aleph_\alpha$, then the regularity of $\aleph_\alpha$ implies $|A_n| < \aleph_\alpha$. But

$$|X| = \sum_{n=0}^{\infty} |A_n|, \qquad \text{and} \qquad |A_n| < \aleph_\alpha, \quad n = 0, 1, \ldots,$$

contradicts the regularity of $\aleph_\alpha$.

If $|I_\alpha| < \aleph_\alpha$ and I_α is not a cut set of X, then $X - I_\alpha$ is a connected graph of order $\aleph_\alpha$, and hence there is an $x \in X - I_\alpha$ with $d(X - I_\alpha; x) = \aleph_\alpha$. But then $x \in I_\alpha$, a contradiction.

The proof of (i) differs from that of (ii) only in that the assumption $d_x \leqslant \aleph_0$ for all $x \in X$ implies $|A_n| \leqslant \aleph_0$, $n = 0, 1, \ldots$, and hence $|X| \leqslant \aleph_0$.

That (3.7) (ii) fails to hold for every singular cardinal is shown by the following example. Let n be a singular cardinal, $\{n_\alpha : \alpha \in A\}$ a set of cardinals such that $n_\alpha < n$ for all $\alpha \in A$, $|A| < n$, and

$$\sum_{\alpha \in A} n_\alpha = n.$$

Let a be a vertex not in A, $\{S_\alpha : \alpha \in A\}$ a collection of mutually disjoint sets such that $|S_\alpha| = n_\alpha$, $\alpha \in A$, and $\bigcup_{\alpha \in A} S_\alpha$ is disjoint from $A \cup \{a\}$. Define a graph X by

$$V(X) = \{a\} \cup A \cup \bigcup_{\alpha \in A} S_\alpha,$$

$$E(X) = \{[a, \alpha] : \alpha \in A\} \cup \{[\alpha, s] : s \in S_\alpha, \alpha \in A\}.$$

X is a tree of finite diameter, $d_a = |A|$, $d_\alpha = n_\alpha$, $\alpha \in A$, $d_s = 1$ for $s \in \bigcup S_\alpha$. Hence $d_x < n$ for every $x \in X$. But

$$|X| = 1 + |A| + \sum_{\alpha \in A} |S_\alpha| = n.$$

Proof of Theorem (3.6). Let X_0 be the component of X which contains x_0. If K is any component of X distinct from X_0, then K is locally finite, and hence by Veblen's theorem cyclically coverable. Thus it will suffice to prove (3.6) for X_0, i.e., it is sufficient to assume that X is connected.

Case 1. X is countable. By (3.2), X contains a maximal f.c.c. subgraph Y. Y being maximal, $Z = X \setminus Y$ is acyclic. Let $z \in Z$. If $z \neq x_0$, then $d(X; z)$ is finite; hence $d(Z; z)$ is even. Thus Z has at most one vertex of odd degree, viz. x_0. If Z is Euler, then by (2.4), Z is c.c., and hence X is c.c. If Z is not Euler, then it is locally finite, and x_0 is the only vertex of odd degree of Z. Hence the component Z_0 of Z which contains x_0 is infinite. By **(4**, 2.4.2**)**, Z_0 contains a ray R starting at x_0. But then $Z \setminus R$ is an acyclic Euler graph and hence is c.c. In this case X has a cover consisting of R, a cyclic cover of Y, and a cyclic cover of $Z \setminus R$.

Case 2. X is uncountable. By Lemma (3.7), (i), x_0 is a cut vertex of X, and every component of $X - x_0$ is at most countable. For any component K of $X - x_0$ let $\bar{K}$ be the maximal subgraph of X with $V(\bar{K}) = V(K) \cup \{x_0\}$. $\bar{K}$ is finite or countable, and connected. Moreover, either (i) $\bar{K}$ is Euler, or (ii) $\bar{K}$ has exactly one vertex of odd degree, viz. x_0. In case (i), $\bar{K}$ has a cover $\mathfrak{A}_K$ consisting of circuits or of circuits together with a ray R_K starting at x_0 (by Veblen's theorem or case 1, according as $\bar{K}$ is locally finite or not). In case (ii), $\bar{K}$ is locally finite and infinite, and hence contains a ray R_K starting at x_0. Then $\bar{K} \setminus R_K$ is locally finite Euler; hence $\bar{K}$ has a cover $\mathfrak{A}_K$ consisting of R_K and circuits covering $\bar{K} \setminus R_K$.

Now take two distinct components K_1, K_2 of $X - x_0$ such that $\bar{K}_1$ and $\bar{K}_2$

are not c.c. Then $C = R_{K_1} \cup R_{K_2}$ is an infinite circuit which does not overlap with any member of any $\mathfrak{A}_K$, $K \neq K_1, K_2$; also, C does not overlap with any member of $\mathfrak{A}_{K_1} \cup \mathfrak{A}_{K_2}$ except R_{K_1} and R_{K_2}. Thus $\{C\} \cup (\mathfrak{A}_{K_1} - \{R_{K_1}\}) \cup (\mathfrak{A}_{K_2} - \{R_{K_2}\})$ is a cyclic cover of $\bar{K}_1 \cup \bar{K}_2$. Hence, it follows that if the number of components K of $X - x_0$ for which $\bar{K}$ is not c.c. is even or infinite, then X is c.c.; if the number of such components is odd, then X has a cover consisting of circuits together with a ray starting at x_0 (one of the R_K's).

4. A theorem of G. A. Dirac. Theorem A of **(1)** can essentially be rephrased as follows.

THEOREM A'. *If X is a connected graph, x not a cut vertex of X, and if d_x is even or infinite, then there exists an f.c.c. subgraph Y of X such that $x \in Y$ and $d(Y; x) = d_x$.*

Actually the information given in **(1)** is considerably more precise in that the structure of Y is described more or less explicitly depending on the size of d_x. Unfortunately Dirac's Theorem D (p. 224), on which the proof of Theorem A for uncountable d_x is based, does not hold for all uncountable cardinals but only for regular ones. This is due to the fact that Theorem D rests on Lemma (3.7), (ii), which is false for all singular cardinals. Nevertheless, Theorem A' is true; in fact, we propose to prove the following refinement.

(4.1) THEOREM. *Let X be a connected graph, $x \in X$, $E \subset E(X; x)$. If x is not a cut vertex of X then there is a finitely cyclically coverable subgraph Y of X such that $x \in Y$, $E(Y; x) \subset E$, and $|E - E(Y; x)| \leqslant 1$.*

That the case $|E - E(Y; x)| = 1$ can occur for infinite E is shown by the following example. Let T be the infinite tree of Fig. 1, x a vertex not belonging to T, X the graph obtained by joining x to every end-vertex of T. X is connected, x is not a cut-vertex of X, and it is the only vertex of infinite degree in X. Consider C_0, a finite circuit containing the edge $[x, a_0]$. If n is the largest

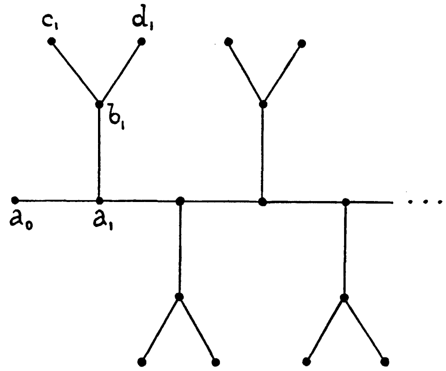

FIGURE 1

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subscript such that $a_n \in C_0$, then $n \geqslant 1$, and $b_n \in C_0$; also, exactly one of c_n, d_n belongs to C_0, say $c_n \in C_0$. But then any finite circuit containing $[x, d_n]$ overlaps C_0. Thus there is no f.c.c. $Y \subset X$ with $E(Y; x) = E(X; x)$.

We shall prove (4.1) using a suitable extension of the concept of "disentangling pairing" (**1**, Section 2). We shall make use of the following notation. If T is a tree and A a set of vertices of T we define

$$T_A = \bigcup_{a,\, b \in A} T_{ab}.$$

It is easily verified that T_A is connected, and is, in fact, the smallest subtree of T containing A. It will be called the subtree *generated* by A.

(4.2) *Definition.* Let T be an acyclic graph, A a set of vertices of T. A *disentangling pairing* of A is a set $\mathfrak{P}$ of non-overlapping proper paths of T such that (i) every $a \in A$ is an end-vertex of exactly one $P \in \mathfrak{P}$, and (ii) the end-vertices of every $P \in \mathfrak{P}$ belong to A. That is, a disentangling pairing is a pairing function $f : A \to A$ such that $\{T_{a,fa} : a \in A\}$ is a set of non-overlapping paths.

Note that this agrees with the definition given in (**1**) in that no two distinct members of $\mathfrak{P}$ have more than one vertex in common (in an acyclic graph they would otherwise have a common edge).

(4.3) Lemma. *If T is a tree without end-vertices, and if A is a set of generators of T, then A has a disjoint disentangling pairing. That is, there exists a pairing function f for A such that $\{T_{a,fa} : a \in A\}$ is a collection of disjoint paths.*

Proof. Let Φ be the collection of all pairs (B, f_B), where (i) B is a subset of A such that $A \cap V(T_B) \subset B$, and (ii) f is a pairing function for B such that $\{T_{b,f_B b} : b \in B\}$ is a collection of disjoint paths. Partially order Φ by setting $(B, f_B) \leqslant (C, f_C)$ if and only if $B \subset C$ and $f_B = f_C|B$. It is routine to verify that Φ is inductive; let (M, f_M) be a maximal element of Φ. We wish to show that $M = A$.

Fix $a \in A$ and suppose there is an $x \in A - M$. Since A generates T, there is a $y \in A$ with $x \in T_{ay}$. If $y \in M$, then by (i), $x \in M$. Hence $y \in A - M$ whenever $x \in T_{ay}$. Since x is not an end-vertex, there is a $y \in A$ such that x lies between a and y, and there is no vertex of A between x and y. Now put $N = M \cup \{x, y\}$, and define $f_N : N \to N$ by

$$f_N z = \begin{cases} f_M z & \text{if } z \in M, \\ y & \text{if } z = x, \\ x & \text{if } z = y. \end{cases}$$

Then $(N, f_N) \in \Phi$, contrary to the maximality of (M, f_M).

If x is a branch-vertex of a tree T, let M_x be the set of all end-vertices y of T such that there is no branch-vertex of T between x and y. Of course, M_x may be empty.

(4.4) COROLLARY. *Let T be a tree such that* (i) $|M_x| \leqslant 1$ *for every branch-vertex x of T, and* (ii) *the end-vertices of T generate T. Then the set of end-vertices of T has a disjoint disentangling pairing.*

Proof. Let A_0 be the set of all branch-vertices x of T for which $|M_x| = 1$, T_0 the subtree generated by A_0. By (i), there is a one–one correspondence $x \rightarrow y_x$ of A_0 with the set A of end-vertices of T, where $M_x = \{y_x\}$ for $x \in A_0$. By (i), T_0 has no end-vertices, and hence by (4.3) there exists a disjoint disentangling pairing $\mathfrak{P}_0$ of A_0. Now any member of A can be written as y_x for a unique $x \in A_0$. x belongs to a unique path $P_x \in \mathfrak{P}_0$; let z be the other end-vertex of P_x. Then $z \in A_0$. Put

$$N_x = T_{y_x x} \cup P_x \cup T_{z y_z}.$$

By the symmetry of the construction of $N_x, N_x = N_z$. Hence $\mathfrak{P} = \{N_x : x \in A_0\}$ is a disentangling pairing of A, and since $\mathfrak{P}_0$ is disjoint, so is $\mathfrak{P}$.

(4.5) LEMMA. *Let T be a tree, A the set of end-vertices of T. If A generates T, then A has a disentangling pairing or there exists an $a \in A$ such that $A - \{a\}$ has a disentangling pairing.*

Proof. For every ordinal α we define a subset $A^{(\alpha)}$ of A, a tree $T^{(\alpha)}$, and a disentangling pairing $\mathfrak{P}^{(\alpha)}$ of $A - A^{(\alpha)}$ with the property that if $\beta < \alpha$, then

$$A^{(\beta)} \supset A^{(\alpha)}, \qquad T^{(\beta)} \supset T^{(\alpha)}, \qquad \mathfrak{P}^{(\beta)} \subset \mathfrak{P}^{(\alpha)}.$$

Put $A^{(0)} = A$, $T^{(0)} = T$, $\mathfrak{P}^{(0)} = \square$. For $\alpha > 0$ proceed as follows. If α is a limit ordinal put

$$A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)} \qquad \mathfrak{P}^{(\alpha)} = \bigcup_{\beta<\alpha} \mathfrak{P}^{(\beta)},$$

and let $T^{(\alpha)}$ be the subtree of T generated by $A^{(\alpha)}$.

If α is not a limit ordinal, we distinguish two cases.

Case 1. $T^{(\alpha-1)}$ contains a branch-vertex. For a branch-vertex x of $T^{(\alpha-1)}$, let M_x be the set of all end-vertices $y \in T^{(\alpha-1)}$ such that there is no branch-vertex of $T^{(\alpha-1)}$ between x and y. Put $m_x = |M_x|$. By $B^{(\alpha-1)}$ denote the set of all branch-vertices $x \in T^{(\alpha-1)}$ for which $m_x > 0$. For each $x \in B^{(\alpha-1)}$ with m_x odd, choose a fixed $a_x \in M_x$, and let $A^{(\alpha)}$ be the set of all such a_x. Let $T^{(\alpha)}$ be the subtree of T generated by $A^{(\alpha)}$. Note that $T^{(\alpha)}$ will be a proper subgraph of $T^{(\alpha-1)}$ if and only if there exists an $x \in B^{(\alpha-1)}$ with $m_x \geqslant 2$. For, $T^{(\alpha)}$ does not contain any vertex of M_x (if m_x is even or infinite) or of $M_x - \{a_x\}$ (if m_x is odd).

We define $\mathfrak{P}^{(\alpha)}$ as follows. Consider any $x \in B^{(\alpha-1)}$. If m_x is even or infinite, let f be a pairing function for M_x and define a set of paths

$$\mathfrak{Q}_x^{(\alpha)} = \{T_{a,fa}^{(\alpha-1)} : a \in M_x\}.$$

If m_x is odd, let g be a pairing function for $M_x - \{a_x\}$. In this case define

$$\mathfrak{Q}_x^{(\alpha)} = \{T_{a,ga}^{(\alpha-1)} : a \in M_x - \{a_x\}\}.$$

Put

$$\mathfrak{Q}^{(\alpha)} = \bigcup \{\mathfrak{Q}_x^{(\alpha)} : x \in B^{(\alpha-1)}\}.$$

$\mathfrak{Q}^{(\alpha)}$ is a disentangling pairing for $A^{(\alpha-1)} - A^{(\alpha)}$. Finally put

$$\mathfrak{P}^{(\alpha)} = \mathfrak{P}^{(\alpha-1)} \cup \mathfrak{Q}^{(\alpha)}.$$

This is then a disentangling pairing for

$$(A - A^{(\alpha-1)}) \cup (A^{(\alpha-1)} - A^{(\alpha)}) = A - A^{(\alpha)}.$$

Case 2. $T^{(\alpha-1)}$ contains no branch-vertex. Since $A^{(\alpha-1)}$ is a set of end-vertices of T, and since $A^{(\alpha-1)}$ generates $T^{(\alpha-1)}$, $A^{(\alpha-1)}$ is the set of end-vertices of $T^{(\alpha-1)}$. Hence $|A^{(\alpha-1)}| \leqslant 2$ (otherwise $T^{(\alpha-1)}$ has a branch-vertex). If $|A^{(\alpha-1)}| = 0$, then $T^{(\alpha-1)}$ is empty and we define $A^{(\alpha)} = \square$, $T^{(\alpha)} = \square$, $\mathfrak{P}^{(\alpha)} = \mathfrak{P}^{(\alpha-1)}$. In this case $\mathfrak{P}^{(\alpha-1)}$ is a disentangling pairing for A. If $|A^{(\alpha-1)}| = 1$, then $T^{(\alpha-1)}$ is a single vertex a, and we define $A^{(\alpha)} = \{a\}$, $T^{(\alpha)} = a$, $\mathfrak{P}^{(\alpha)} = \mathfrak{P}^{(\alpha-1)}$. In this case $\mathfrak{P}^{(\alpha-1)}$ is a disentangling pairing for $A - \{a\}$. If $|A^{(\alpha-1)}| = 2$, then $T^{(\alpha-1)}$ is a proper path whose end-vertices are the two members of $A^{(\alpha-1)}$. In this case we put

$$A^{(\alpha)} = \square,\ T^{(\alpha)} = \square,\ \text{and}\ \mathfrak{P}^{(\alpha)} = \mathfrak{P}^{(\alpha-1)} \cup \{T^{(\alpha-1)}\}.$$

Then $\mathfrak{P}^{(\alpha)}$ is a disentangling pairing for A.

Now let σ be the smallest ordinal such that $T^{(\sigma)} = T^{(\sigma+1)}$. According to the construction of the trees $T^{(\alpha)}$, we must then have either (i) $T^{(\sigma)} = \square$ or (ii) $T^{(\sigma)} = a$((i) and (ii) arising from case 2), or (iii) $m_x = 1$ for all $x \in B^{(\sigma)}$ (arising from case 1). In case (i), $\mathfrak{P}^{(\sigma)}$ is a disentangling pairing for A; in case (ii), $\mathfrak{P}^{(\sigma)}$ is a disentangling pairing for $A - \{a\}$; finally, in case (iii), $T^{(\sigma)}$ satisfies the hypotheses of (4.4), and hence $A^{(\sigma)}$ has a disentangling pairing $\mathfrak{Q}$. Then

$$\mathfrak{P} = \mathfrak{Q} \cup [\bigcup \{\mathfrak{P}^{(\alpha)} : \alpha < \sigma\}]$$

is a disentangling pairing of A.

Proof of Theorem (4.1). For $e \in E(X;x)$, let x_e be that vertex of X for which $[x, x_e] = e$. We may assume that $d(X;x_e) = 2$ for every $e \in E(X;x)$. For, if this is not the case, subdivide every edge $e \in E(X;x)$ by a new vertex a_e. Denote the graph so obtained by X'. Clearly X' is connected, x is not a cut-vertex of X', $V(X';x) = \{a_e : e \in E(X;x)\}$, and $d(X';a_e) = 2$ for every $e \in E(X;x)$. Put $E' = \{[x, a_e] : e \in E\}$, and suppose there is an f.c.c. subgraph Y' of X' such that $x \in Y'$ and $E(Y';x) = E'$. If $a_e \in Y'$, then $[x, a_e]$, $[a_e, x_e] \in Y'$ (since $d(X';a_e) = d(Y';a_e) = 2$). Denote by Y the subgraph of X obtained from Y' by replacing each pair of edges $[x, a_e]$, $[a_e, x_e]$ by the single edge $[x, x_e]$, $e \in E$. In this process, non-overlapping circuits give rise to non-overlapping circuits; hence Y is an f.c.c. subgraph of X with $x \in Y$ and $E(Y;x) = E$.

Since x is not a cut-vertex of X, $X - x$ is connected. By **(4**, 6.5.1**)** there is a tree $S \subset X - x$ with $V(S) = V(X - x)$. Since $d(X; x_e) = 2$ for every $e \in E(X; x)$, $A = \{x_e : e \in E\}$ is a set of end-vertices of S. Let T be the subtree of S generated by A. A is the set of end-vertices of T. Hence, by Lemma (4.5), A has a disentangling pairing or there is an $e_0 \in E$ such that $A - \{x_{e0}\}$ has a disentangling pairing. Put $A_0 = A$ or $A - \{x_{e0}\}$, as the case may be, and let $\mathfrak{P}$ be a disentangling pairing of A_0 in T. Any $x_e \in A_0$ is an end-vertex of exactly one $P_e \in \mathfrak{P}$; denote the other end-vertex of P_e by y_e. Since P is a disentangling pairing, $y_e \in A_0$, i.e., $y_e = x_{e'}$ for a unique $e' \in E$. $C_e = (e) \cup P_e \cup (e')$, $e \in E$ (or $E - \{e_0\}$) is a finite circuit, and by the symmetry of the construction, $C_{e'} = C_e$. No two of the circuits C_e overlap (since the members of $\mathfrak{P}$ do not overlap). Hence $Y = \cup C_e$ is an f.c.c. subgraph of X with $E(Y; x) = E$ or $E - \{e_0\}$.

5. Almost cyclic covers. Theorem (4.1) permits us to prove a further extension of Veblen's theorem (Theorem (5.1)). Let us say that a cover $\mathfrak{A}$ of a graph is *almost cyclic* if and only if $\mathfrak{A}$ consists of circuits and at most a finite number of rays. A graph with such a cover will be called *almost cyclically coverable* (a.c.c.). Theorem (3.6) then says that every Euler graph with at most one vertex of infinite degree is a.c.c. We wish to investigate whether this statement holds for Euler graphs containing an arbitrary finite number of vertices of infinite degree.

For a graph X denote by I the set of vertices of infinite degree, by J the set of vertices whose degree is not even. Clearly $I \subset J$, and $I = J$ if and only if X is Euler.

If X is a.c.c., then among its almost cyclic covers there is one, $\mathfrak{A}_0$, which contains the smallest possible number of rays. $\mathfrak{A}_0$ has the following properties: (i) no two distinct rays $R, R' \in \mathfrak{A}_0$ start at the same vertex of X; (ii) every ray $R \in \mathfrak{A}_0$ starts at some vertex in J. For if R and R' both start at x, then $R \cup R'$ is an infinite circuit, and hence $\mathfrak{A}_1 = (\mathfrak{A}_0 - \{R, R'\}) \cup \{R \cup R'\}$ is an almost cyclic cover of X with fewer rays than A_0. If R starts at x, and d_x is even, then $d(X \diagdown R; x)$ is odd; hence $\mathfrak{A}_0 - \{R\}$ contains a ray starting at x. This contradicts (i). It follows from (i) and (ii) that if X is an a.c.c. graph, then it has an almost cyclic cover containing at most $|J|$ rays. The same argument shows that if X is a graph with a cover $\mathfrak{A}$ consisting of circuits and rays (not necessarily finitely many), then X has a cover $\mathfrak{A}_0$, which consists of circuits and rays, and satisfies (i) and (ii).

(5.1) THEOREM. *Let X be a graph. If J is finite, then there exists a finite acyclic subgraph F of X such that* (i) *if $d(F; x)$ is odd for some $x \in F$, then $x \in J$; and* (ii) *$X \diagdown F$ is a.c.c.*

In the proof of this theorem we make use of a simple property of the symmetric difference of two graphs.

(5.2) *Definition.* The *symmetric difference* of two graphs X and Y is the graph $X + Y = (X \cup Y) \setminus (X \cap Y)$.

(5.3) LEMMA. *Let X, Y be graphs such that* (i) *X is Euler*, (ii) *Y is locally finite, and* (iii) *every vertex $y \in Y$ for which $d(Y; y)$ is odd belongs to X and $d(X; y)$ is infinite. Then $X + Y$ is Euler.*

Proof. Let $x \in X + Y$. Then

$$E(X+Y;x) = \begin{cases} E(X;x) & \text{if } x \notin Y, \\ E(Y;x) & \text{if } x \notin X, \\ (E(X;x) \cup E(Y;x)) - (E(X;x) \cap E(Y;x)) & \\ & \text{if } x \in X \cap Y. \end{cases}$$

Accordingly,

$$d(X+Y;x) = \begin{cases} d(X;x) & \text{if } x \notin Y, \\ d(Y;x) & \text{if } x \notin X, \\ d(X;x) + d(Y;x) - 2|E(X;x) \cap E(Y;x)| & \\ & \text{if } x \in X \cap Y. \end{cases}$$

If $x \in Y$ and $d(Y; x)$ is odd, then by (iii) $x \in X \cap Y$ and $d(X; x) + d(Y; x)$ is infinite. On the other hand,

$$|E(X;x) \cap E(Y;x)| \leqslant |E(Y;x)| = d(Y;x) < \infty,$$

by (ii). Hence $d(X + Y; x)$ is infinite. In all other cases $d(X + Y; x)$ is clearly even or infinite.

(5.4) LEMMA. *If X is a connected graph containing exactly one vertex x_0 of infinite degree and if x_0 is not a cut vertex of X, then there exists an a.c.c. Euler subgraph Y of X such that $x_0 \in Y$ and $E(Y; x_0) = E(X; x_0)$.*

Proof. By Theorem (4.1) there is an f.c.c. $Y_0 \subset X$ such that $x_0 \in Y_0$ and $|E(X; x_0) - E(Y_0; x_0)| \leqslant 1$. If $E(X; x_0) = E(Y_0; x_0)$, put $Y = Y_0$. If $|E(X; x_0) - E(Y_0; x_0)| = 1$, there is a unique $y_0 \in X - x_0$ such that $e_0 = [x_0, y_0] \in X \setminus Y_0$. Put $X \setminus Y_0 = Z$. Z is locally finite, $d(Z; x_0) = 1$, and x_0 is the only vertex of odd degree of Z. Hence, the component Z_0 of Z which contains x_0 is infinite. By (**4**, 2.4.2), Z_0 contains a ray R starting at x_0, i.e., R contains e_0. Now consider $Y = Y_0 + R$. By (5.3), Y is Euler; x_0 is the only vertex of infinite degree in Y. Hence by (3.6), Y is a.c.c. Since $e_0 \notin E(Y_0)$,

$$E(Y;x_0) = E(Y_0;x_0) \cup \{e_0\} = E(X;x_0).$$

(5.5) LEMMA. *Let X be a graph, J a finite subset of $V(X)$, $\mathfrak{F}$ a cover of X consisting of finite acyclic graphs such that if $d(A; x)$ is odd for some $x \in A$, $A \in \mathfrak{F}$, then $x \in J$, and $V(A) \cap V(B) \subset J$ for any two distinct $A, B \in \mathfrak{F}$. Then X contains a finite acyclic subgraph F such that* (i) *if $d(F; x)$ is odd for some $x \in F$, then $x \in J$, and* (ii) *$X \setminus F$ is f.c.c.*

Remark. In this lemma, J may be any finite subset, not just the set of those vertices which do not have even degree.

Proof. Let $A \in \mathfrak{F}$. By N_A denote the set of all $x \in A$ for which $d(A;x)$ is odd. By hypothesis, $N_A \subset J$. Since A is finite, $|N_A|$ is even. Hence N_A has a disentangling pairing $\mathfrak{P}_A$. Put $\mathfrak{P} = \bigcup \{\mathfrak{P}_A : A \in \mathfrak{F}\}$.

Denote the members of J by $x_1, \ldots, x_n$, and for $1 \leqslant i < j \leqslant n$ let $\mathfrak{Q}_{ij}$ be the set of all $P \in \mathfrak{P}$ which join x_i and x_j. No two distinct members of $\mathfrak{Q}_{ij}$ belong to the same $\mathfrak{P}_A$ (otherwise A would contain a finite circuit); hence $\mathfrak{Q}_{ij}$ is a collection of paths no two of which have anything but their end-vertices in common. That is, the union of two distinct members of $\mathfrak{Q}_{ij}$ is a finite circuit. It follows that if $|\mathfrak{Q}_{ij}|$ is even or infinite, then $Q_{ij} = \bigcup \mathfrak{Q}_{ij}$ is f.c.c.; in this case put $P_{ij} = \square$. If $|\mathfrak{Q}_{ij}|$ is odd, let P_{ij} be an arbitrary member of $\mathfrak{Q}_{ij}$, and then $Q_{ij} = \bigcup (\mathfrak{Q}_{ij} - \{P_{ij}\})$ is f.c.c. Next note that if $(i,j) \neq (i',j')$, then no member of $\mathfrak{Q}_{ij}$ overlaps with any member of $\mathfrak{Q}_{i'j'}$. For if $P \in \mathfrak{Q}_{ij}$ and $P' \in \mathfrak{Q}_{i'j'}$, then $P \in \mathfrak{P}_A$, $P' \in \mathfrak{P}_{A'}$. Hence if $A \neq A'$, then $E(P) \cap E(P') = \square$ since $E(A) \cap E(A') = \square$; if $A = A'$, then P and P' are distinct members of $\mathfrak{P}_A$ (since $(i,j) \neq (i',j')$) and hence again do not overlap. It follows that

$$Q = \bigcup_{1 \leqslant i < j \leqslant n} Q_{ij}$$

is an f.c.c. subgraph of X and

$$F = X \setminus Q = \bigcup_{1 \leqslant i < j \leqslant n} P_{ij}$$

has properties (i), (ii).

Proof of Theorem (5.1). We use induction on $|J|$. For $|J| = 0$, X is locally finite Euler, and hence (5.1) holds trivially. Assume (5.1) true for all graphs X' with $|J'| < n$, and let X be a graph with $|J| = n$. We may suppose that X is infinite. For if X is finite, let X_0 be a maximal f.c.c. subgraph of X, and then $F = X \setminus X_0$ is a finite acyclic graph satisfying (i). Next consider the graph X_J consisting of all edges of X which join two vertices in J together with the end-vertices of these edges. Put $X' = X \setminus X_J$. $J' \subset J$ and no two members of J' are adjacent in X'. Assume that (5.1) holds for X'. Let F' be a finite acyclic subgraph of X' satisfying (i), (ii) with respect to X', and let Y be a maximal f.c.c. subgraph of the (finite) graph $X_J \cup F'$. Note that if $d(X_J \cup F'; z)$ is odd for some $z \in X_J \cup F'$, then $z \in J$. Hence $F = (X_J \cup F') \setminus Y$ is a finite acyclic subgraph of X satisfying (i). Also,

$$X \setminus (Y \cup F) = X \setminus (X_J \cup F') = X' \setminus F';$$

hence $X \setminus (Y \cup F)$ is a.c.c. Since Y is f.c.c., $X \setminus F$ is likewise a.c.c. It follows that it is sufficient to assume that no two vertices of J are adjacent in X. This is equivalent to

$$(1) \qquad V(X;x) \subset V(X - J) \qquad \text{for every } x \in J.$$

Finally, we may assume that X is connected. For if X is disconnected, and not all members of J belong to the same component of X, then the induction hypothesis is applicable to every component of X, and since at most n components of X are not c.c., (5.1) also holds for X.

Case 1. J is not a cut set of X. Choose an arbitrary x_0 in J and let X_0 be the graph consisting of $X - J$ and all edges of X joining x_0 with some vertex of $X - J$. Since $X - J$ is connected, it follows from (1) that X_0 is likewise connected, and that x_0 is not a cut vertex of X_0. Again by (1), $E(X_0; x_0) = E(X; x_0)$. There are now two possibilities: (i) d_{x_0} is odd, (ii) d_{x_0} is infinite. In case (i), X_0 is locally finite, and infinite; hence by **(4**, 2.4.2**)** there is a ray $R_0 \subset X_0$ starting at x_0. $X' = X \setminus R_0$ is then a graph with $J' = J - \{x_0\}$. Hence by induction hypothesis there is a finite acyclic $F' \subset X'$ satisfying (i) and (ii) with respect to X'. But F' also satisfies (i), (ii) with respect to X. In case (ii) it follows from (5.4) that there is an a.c.c. Euler subgraph Y_0 of X_0 such that $x_0 \in Y_0$ and $E(Y_0; x_0) = E(X_0; x_0)$, i.e., $E(Y_0; x_0) = E(X; x_0)$. Put $X' = X \setminus Y_0$. Then $J' = J - \{x_0\}$, and again the induction hypothesis is applicable.

Case 2. $X - J$ is disconnected. Let $\mathfrak{K}$ be the set of components of $X - J$. For $K \in \mathfrak{K}$ denote by $\bar{K}$ the graph consisting of K and all edges of X which join a vertex in K with a vertex in J. Put $L_K = J \cap V(\bar{K})$. Then $\bar{K} - L_K = K$. In view of (1) this means that no subset of L_K is a cut set of $\bar{K}$. Let $J_K = \{x \in \bar{K} : d(\bar{K}; x)$ is not even$\}$. Clearly $J_K \subset L_K$; hence J_K is not a cut set of $\bar{K}$. Hence by induction hypothesis or case 1 (according as $|J_K| < n$ or $= n$), there is a finite acyclic $F_K \subset \bar{K}$ satisfying (i), (ii) with respect to $\bar{K}$. Let $\mathfrak{A}_K$ be an almost cyclic cover of $\bar{K} \setminus F_K$. Now consider

$$Z = \bigcup_{K \in \mathfrak{K}} F_K.$$

Since no two distinct graphs $\bar{K}_1, \bar{K}_2$ overlap, the corresponding graphs F_{K_1}, F_{K_2} likewise do not overlap, i.e., $\mathfrak{F} = \{F_K : K \in \mathfrak{K}\}$ is a cover of Z satisfying the hypotheses of Lemma (5.5). Let F be the finite acyclic subgraph of Z which satisfies conditions (i), (ii) of Lemma (5.5). (i) is simply (i) of (5.1); by (ii), $Z \setminus F$ has a cyclic cover $\mathfrak{B}$.

$$X \setminus F = (X \setminus Z) \cup (Z \setminus F) = \bigcup_{K \in \mathfrak{K}} (\bar{K} \setminus F_K) \cup (Z \setminus F);$$

hence

$$\mathfrak{B} \cup \bigcup_{K \in \mathfrak{K}} \mathfrak{A}_K$$

is an almost cyclic cover of $X \setminus F$.

If $|J| = 1$, one can always assume the graph F of (5.1) to be empty. Simple examples show that without further restrictions this cannot even be done when $|I| = 2$.

However, we conjecture the following:

(5.6) CONJECTURE. *If X is a connected Euler graph such that I is finite and $X - I$ has only finitely many components, then X is a.c.c.*

We have only been able to prove this for $|I| \leqslant 2$. There are examples which show that our method of proof fails for $|I| \geqslant 3$.

The proof is based on the observation that if $|I| = 2$, then the F of (5.1) is either empty or a proper path joining x_1 and x_2, where $I = \{x_1, x_2\}$. If $F = \square$, there is nothing to prove. Suppose then that F is a proper path joining x_1 and x_2. Since $X - I$ has only finitely many components, one of these, K_0, is such that $x_1 \in \bar{K}_0$ and $d(\bar{K}_0; x_1)$ is infinite (notation as in the proof of (5.1)). But this means that K_0 is an infinite locally finite graph; hence by (**4**, 2.4.2), $\bar{K}_0$ contains a ray starting at x_1. Note that K_0 is actually a component of $(X \diagdown F) - I$. Consider $X' = (X \diagdown F) \diagdown R$. X' is Euler and $I' = I$. By (5.1) there is a finite acyclic graph $F' \subset X'$ whose end-vertices belong to I and such that $X' \diagdown F'$ has an almost cyclic cover $\mathfrak{A}'$. For F' there are two alternatives: $F' = \square$, or F' is a proper path joining x_1 and x_2. If $F' = \square$, put $\mathfrak{A} = \mathfrak{A}' \cup \{R \cup F\}$. Since $R \cup F$ is a ray starting at x_2, $\mathfrak{A}$ is an almost cyclic cover of X. If F' is a proper path, then $F \cup F'$ is a finite circuit. In this case $\mathfrak{A} = \mathfrak{A}' \cup \{R\} \cup \{F \cup F'\}$ is an almost cyclic cover of X.

Added in Proof. The proofs of Theorems (3.6) and (5.1) can be simplified considerably by a different kind of maximality argument; e.g., in (3.6) take Y to be the union of a maximal collection of non-overlapping finite circuits (rather than a maximal f.c.c. subgraph). These simpler proofs also make no use of (3.7) and (4.1), respectively.

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