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NODAL SOLUTIONS FOR NONLINEAR NON-HOMOGENEOUS ROBIN PROBLEMS WITH AN INDEFINITE POTENTIAL

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Abstract We consider a nonlinear Robin problem driven by a non-homogeneous differential operator plus an indefinite potential term. The reaction function is Carathéodory with arbitrary growth near $\pm\infty$. We assume that it is odd and exhibits a concave term near zero. Using a variant of the symmetric mountain pass theorem, we establish the existence of a sequence of distinct nodal solutions which converge to zero.

Keywords: Robin boundary condition; nonlinear non-homogeneous differential operator; nonlinear regularity theory; nodal solutions

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear non-homogeneous Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)|u(z)|^{p-2}u(z) = f(z,u(z)) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega, \\ 1 < p < +\infty. \end{cases} \tag{1.1}$$

In this problem, the map $a\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ involved in the definition of the differential operator is continuous and monotone (and hence also maximal monotone) and satisfies certain other regularity and growth conditions listed in hypotheses $H(a)$ below. These hypotheses are general enough to incorporate in our framework many differential operators of interest, such as the p-Laplacian ($1 < p < +\infty$) and the $(p.q)$-Laplacian (that

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is, the sum of a p-Laplacian and a q-Laplacian, with $1 < q < p < +\infty$). The potential function $\xi \in L^\infty(\Omega)$ is indefinite (that is, sign changing). The reaction term f is a Carathéodory function (that is, for all $\zeta \in \mathbb{R}$, the map $z \longmapsto f(z,\zeta)$ is measurable, and for almost all (a.a.) $z \in \Omega$, the map $\zeta \longmapsto f(z,\zeta)$ is continuous). We do not impose any growth restriction on $f(z,\cdot)$ near $\pm\infty$. All the conditions on $f(z,\cdot)$ concern its behaviour near zero. So, we assume that near zero $f(z,\cdot)$ is odd and exhibits a concave term (that is, a $(p-1)$-superlinear term).

In the boundary condition, $(\partial u/\partial n_a)$ is the generalized normal derivative (conormal derivative), defined by extension of the map

$$u \longmapsto (a(Du), n)_{\mathbb{R}^N}, \quad \forall u \in C^1(\overline{\Omega}),$$

with n being the outward unit normal on $\partial\Omega$. This kind of directional derivative is dictated by the nonlinear Green's identity (see, for example, Gasiński-Papageorgiou [3]) and is also used by Lieberman [18], whose nonlinear regularity theory is employed in this work. The boundary coefficient $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$ and $\beta \geqslant 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$, we recover the Neumann problem.

We are looking for nodal (that is, sign changing) solutions. Using an abstract multiplicity result due to Heinz [16], Wang [31] and Kajikiya [17], together with suitable truncation and perturbation techniques, we establish the existence of a whole sequence $\{u_n\}_{n\geqslant 1} \subseteq C^1(\overline{\Omega})$ of distinct nodal solutions such that

$$u_n \longrightarrow 0 \quad \text{in } C^1(\overline{\Omega}).$$

Recently, nodal solutions for nonlinear Robin problems were obtained by Papageorgiou and Rădulescu [24, 28]. However, they do not prove the existence of a sequence of nodal solutions. Very recently, Papageorgiou and Rădulescu [26] proved the existence of a sequence of nodal solutions when $\xi \equiv 0$ and under stronger conditions on the reaction term f. Finally, we mention also the works of He *et al.* [15], who studied the Neumann problem (that is, $\beta \equiv 0$) driven by the p-Laplacian (that is, $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$ with $1 < p < +\infty$); Gasiński *et al.* [14], where the existence of positive solutions was obtained; and Gasiński and Papageorgiou [6–8, 11, 13], where some other types of boundary value problems with non-homogeneous operators were considered.

2. Mathematical background

Let X be a Banach space and let X^* denote its topological dual. By $\langle\cdot,\cdot\rangle$, denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X;\mathbb{R})$, we say that φ satisfies the *Palais–Smale condition* if the following property holds:

Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(u_n) \longrightarrow 0 \quad \text{in } X^*,$$

admits a strongly convergent subsequence.

The next result is a variant of the so-called 'symmetric mountain pass theorem' and is due to Heinz [16], Wang [31] and Kajikiya [17] (the most general version of the result is that of Kajikiya [17]).

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Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X;\mathbb{R})$ satisfies the Palais–Smale condition and is even and bounded below, $\varphi(0) = 0$, and for every $n \geqslant 1$ there exist an n-dimensional subspace $V_n \subseteq X$ and $\varrho_n > 0$ such that*

$$\sup\{\varphi(u) : \ u \in V_n, \|u\| = \varrho_n\} < 0,$$

then there exists a sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ of critical points of φ such that

$$u_n \neq 0, \quad \forall n \geqslant 1 \quad \text{and} \quad u_n \longrightarrow 0 \quad \text{in } X.$$

Let $\vartheta \in C^1(0,+\infty)$ be such that $\vartheta(t) > 0$ for all $t > 0$ and assume that

$$0 < \widehat{c}_0 \leqslant \frac{\vartheta'(t)t}{\vartheta(t)} \leqslant \widehat{c}_1 \quad \text{and} \quad c_1 t^{p-1} \leqslant \vartheta(t) \leqslant c_2(t^{\tau-1} + t^{p-1}) \quad 1 \leqslant \tau < p, \tag{2.1}$$

for some $c_1, c_2 > 0$. Then the conditions on the map $y \longmapsto a(y)$ are the following:

<u>$H(a)$:</u> $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,+\infty)$, $t \longmapsto a_0(t)t$ is strictly increasing on $(0,+\infty)$, $a_0(t)t \longrightarrow 0^+$ as $t \to 0^+$ and

$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) there exists $c_3 > 0$ such that

$$|\nabla a(y)| \leqslant c_3 \frac{\vartheta(|y|)}{|y|}, \quad \forall y \in \mathbb{R}^N \setminus \{0\};$$

(iii) we have

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant \frac{\vartheta(|y|)}{|y|}|\xi|^2, \quad \forall y \in \mathbb{R}^N \setminus \{0\}, \quad \xi \in \mathbb{R}^N;$$

(iv) if $G_0(t) = \int_0^t a_0(s)s\,\mathrm{d}s$ for $t > 0$, then there exists $q \in (1,p)$ such that the map $t \longmapsto G_0(t^{\frac{1}{q}})$ is convex and

$$\limsup_{t\to 0^+} \frac{qG_0(t)}{t^q} \leqslant c_3,$$

for some $c_3 > 0$.

Remark 2.2. Hypotheses $H(a)$(i),(ii),(iii) are motivated by the nonlinear regularity theory of Lieberman [18] and the nonlinear maximum principle of Pucci and Serrin [29]. Hypothesis $H(a)$(iv) serves the needs of our problem, but it is not restrictive and it is satisfied in most cases of interest, as the examples below illustrate.

Hypotheses $H(a)$ imply that the map $t \longmapsto G_0(t)$ is strictly increasing and strictly convex.

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We set

$$G(y) = G_0(|y|), \quad \forall y \in \mathbb{R}^N.$$

Then the map $y \longmapsto G(y)$ is convex and $G(0) = 0$.

Also, we have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y), \quad \forall y \in \mathbb{R}^N \setminus \{0\}, \qquad \nabla G(0) = 0.$$

Hence, G is the primitive of the map a.

The above properties lead to the following inequality

$$G(y) \leqslant (a(y), y)_{\mathbb{R}^N}, \quad \forall y \in \mathbb{R}^N. \tag{2.2}$$

The next lemma summarizes the main properties of the map a. It is an easy consequence of hypotheses $H(a)$(i),(ii) and (iii) and of (2.1) (see also Papageorgiou and Rădulescu [**25**]).

Lemma 2.3. *If hypotheses $H(a)$(i),(ii) and (iii) hold, then:*

(a) *the map $y \longmapsto a(y)$ is continuous, strictly monotone (and hence also maximal monotone);*

(b) $|a(y)| \leqslant c_4(|y|^{\tau-1} + |y|^{p-1})$ *for all* $y \in \mathbb{R}^N$ *and some* $c_4 > 0$;

(c) $(a(y), y)_{\mathbb{R}^N} \geqslant (c_1/p-1)|y|^p$ *for all* $y \in \mathbb{R}^N$.

This lemma and (2.2) lead to the following growth estimate for the primitive G.

Corollary 2.4. *If hypotheses $H(a)$(i),(ii) and (iii) hold, then*

$$\frac{c_1}{p(p-1)}|y|^p \leqslant G(y) \leqslant c_5(1 + |y|^p), \quad \forall y \in \mathbb{R}^N,$$

for some $c_5 > 0$.

Next, we present some examples of maps a which satisfy hypotheses $H(a)$ above. These examples illustrate the generality of our conditions on a.

Example 2.5. The following maps $y \longmapsto a(y)$ satisfy hypotheses $H(a)$.

(a) $a(y) = |y|^{p-2}y$ with $1 < p < +\infty$. This map corresponds to the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du), \quad \forall u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < +\infty$. This map corresponds to the (p, q)-Laplace differential operator defined by

$$\Delta_p u + \Delta_q u, \quad \forall u \in W^{1,p}(\Omega).$$

Such operators arise in problems of mathematical physics. Recently, there have been some multiplicity results for equations driven by such operators. We mention the works of Aizicovici *et al.* [1]; Gasiński and Papageorgiou [5, 9, 10, 12]; Mugnai and Papageorgiou [20]; Papageorgiou and Rădulescu [21, 23]; Sun *et al.* [30]; and Yang and Bai [32].

(c) $a(y) = (1 + |y|^2)^{(p-2/2)}y$ with $1 < p < +\infty$. This map corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}((1 + |Du|^2)^{(p-2/2)}Du), \quad \forall u \in W^{1,p}(\Omega).$$

(d) $a(y) = |y|^{p-2}y + ((|y|^{p-2}y)/(1 + |y|^p))$ with $1 < p < +\infty$. This map corresponds to the following differential operator

$$\Delta_p u + \operatorname{div}\left(\frac{|Du|^{p-2}Du}{1 + |Du|^p}\right), \quad \forall u \in W^{1,p}(\Omega),$$

which arises in problem of plasticity.

We will use the following function spaces in the study of problem (1.1):

- the Sobolev space $W^{1,p}(\Omega)$, for $1 < p < +\infty$;
- the Banach space $C^1(\overline{\Omega})$;
- the 'boundary' Lebesgue space $L^r(\partial\Omega)$, for $1 \leqslant r \leqslant +\infty$.

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$, defined by

$$\|u\| = (\|u\|_p^p + \|Du\|_p^p)^{1/p}, \quad \forall u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C^1(\overline{\Omega}) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a non-empty interior containing the set

$$D_+ = \{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$, we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ. Using this measure, we can define in the usual way the 'boundary' Lebesgue spaces $L^r(\partial\Omega)$ for $1 \leqslant r \leqslant +\infty$. We know that there exists a unique continuous, linear map $\gamma_0 : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$, known as the 'trace operator', such that

$$\gamma_0(u) = u|_{\partial\Omega}, \quad \forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map assigns boundary values to the Sobolev functions.

The trace map is compact into $L^q(\partial\Omega)$ for all $q \in [1, ((Np-p)/(N-p)))$ if $1 < p < N$ and into $L^q(\partial\Omega)$ for all $q \geqslant 1$ if $p \geqslant N$. Also, we have

$$\operatorname{im}\gamma_0 = W^{(1/p'),p}(\partial\Omega) \quad \text{and} \quad \ker\gamma_0 = W_0^{1,p}(\Omega).$$

In what follows, for the sake of notional simplicity, we drop the use of the map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

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Let $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\,\mathrm{d}z, \quad \forall u, h \in W^{1,p}(\Omega).$$

We know that this map is continuous, monotone and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$; see Gasiński and Papageorgiou [4]).

Finally, let us conclude this section with some basic notation, which will be used in the sequel.

If $\varphi \in C^1(X;\mathbb{R})$, then by K_φ we denote the critical set of φ defined by

$$K_\varphi = \{u \in X: \ \varphi'(u) = 0\}.$$

Also, let $\zeta \in \mathbb{R}$, let $\zeta^\pm = \max\{\pm\zeta, 0\}$. Then for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

3. Infinitely many nodal solutions

Our hypotheses on the other data of problem (1.1) are the following:

$\underline{H(\xi)}$: $\xi \in L^\infty(\Omega)$;

$\underline{H(\beta)}$: $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Remark 3.1. When $\beta \equiv 0$, we recover the Neumann problem.

$\underline{H(f)}$: $f\colon \Omega \times [-c, c] \longrightarrow \mathbb{R}$ (with $c > 0$) is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z,\cdot)$ is odd on $[-c, c]$ and

(i) there exists $a_c \in L^\infty(\Omega)_+$ such that

$$|f(z,\zeta)| \leqslant a_c(z) \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \leqslant c;$$

(ii) if $q \in (1, p)$ is as in hypothesis $H(a)$(iv), then

$$\lim_{\zeta\to 0} \frac{f(z,\zeta)}{|\zeta|^{q-2}\zeta} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

Remark 3.2. Hypothesis $H(f)$(ii) implies the presence of a concave term near zero.

In what follows,

$$F(z,\zeta) = \int_0^\zeta f(z,s)\,\mathrm{d}s$$

(the primitive of the reaction term $f(z,\zeta)$). Hypotheses $H(f)$(i) and (ii) imply that, given any $\eta > 0$ and $r > p$, we can find $c_4 = c_4(\eta, r) > 0$ such that

$$f(z,\zeta)\zeta \ \geqslant\ \eta|\zeta|^q - c_4|\zeta|^r \quad \text{for a.a. } z \in \Omega \text{ all } |\zeta| \leqslant c. \tag{3.1}$$

Also, consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_s u(z) + \xi(z)|u(z)|^{s-2}u(z) = \widetilde{\lambda}|u(z)|^{s-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_s} + \beta(z)|u|^{s-2}u = 0 & \text{on } \partial\Omega, \\ 1 < s < +\infty. \end{cases} \tag{3.2}$$

In this case, $\partial u/\partial n_s = |Du|^{s-2}(Du, n)_{\mathbb{R}^N}$.

Let $\widetilde{\gamma}\colon W^{1,s}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\widetilde{\gamma}(u) = \|Du\|_s^s + \int_\Omega \xi(z)|u|^s \,\mathrm{d}z + \int_{\partial\Omega} \beta(z)|u|^s \,\mathrm{d}\sigma, \quad \forall u \in W^{1,s}(\Omega).$$

From Mugnai and Papageorgiou [**19**] and Papageorgiou and Rădulescu [**22**], we know that problem (3.2) has smallest eigenvalue $\widetilde{\lambda}_1(s) \in \mathbb{R}$ (note that if $\xi = 0$ and $\beta \equiv 0$ (Neumann case), then $\widetilde{\lambda}_1(s) = 0$). This eigenvalue is simple and isolated, and the corresponding eigenfunctions are of constant sign. Moreover, we have

$$\widetilde{\lambda}_1(s) = \inf\left\{ \frac{\gamma(u)}{\|u\|_s^s} : \ u \in W^{1,s}(\Omega), \ u \neq 0 \right\} \tag{3.3}$$

and the infimum is realized on the corresponding one-dimensional eigenspace. Let $\widetilde{u}_1(s) \in W^{1,s}(\Omega)$ be the positive L^s-normalized (that is, $\|\widetilde{u}_1(s)\|_s = 1$) eigenfunction corresponding to the eigenvalue $\widetilde{\lambda}_1(s)$. From the nonlinear regularity theory (Lieberman [**18**]) and the nonlinear maximum principle (Pucci and Serrin [**29**, pp. 111, 120]), we have that

$$\widetilde{u}_1(s) \in D_+.$$

Motivated by the unilateral growth estimate (3.1) and with $\mu > \|\xi\|_\infty$ (see hypothesis $H(\xi)$), we consider the following auxiliary Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi^+(z)|u(z)|^{p-2}u(z) \\ \qquad = \eta|u(z)|^{q-2}u(z) - c_4|u(z)|^{r-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Proposition 3.3. *If hypotheses $H(a)$, $H(\xi)$ and $H(\beta)$ hold, then problem (3.4) admits a unique positive solution*

$$\underline{u} \in D_+$$

and, since the equation is odd, it follows that

$$\underline{v} = -\underline{u} \in D_+$$

is the unique negative solution of (3.4).

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Proof. Let $\psi_+ : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\psi_+(u) = \int_\Omega G(Du)\,\mathrm{d}z + \frac{1}{p}\int_\Omega \xi^+(z)|u|^p\,\mathrm{d}z + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u|^p\,\mathrm{d}\sigma$$
$$- \frac{\eta}{q}\|u^+\|_q^q + \frac{c_4}{r}\|u^+\|_r^r, \quad \forall u \in W^{1,p}(\Omega).$$

From Corollary 2.4 and since $q < p < r$, we see that ψ_+ is coercive.

Also, using the Sobolev embedding theorem and the compactness of the trace map, we infer that ψ_+ is sequentially weakly lower semicontinuous.

So, by the Weierstrass theorem, we can find $\underline{u} \in W^{1,p}(\Omega)$ such that

$$\psi_+(\underline{u}) = \inf_{u \in W^{1,p}(\Omega)} \psi_+(u). \tag{3.5}$$

Hypothesis $H(a)$(iv) implies that, given $\varepsilon \in (0, c_2)$, we can find $\delta \in (0,1)$ such that

$$G(y) \leqslant \frac{1}{q}(c_2+\varepsilon)|y|^q, \quad \forall |y| \leqslant \delta. \tag{3.6}$$

Recall that $\widetilde{u}_1(q) \in D_+$. So, we can find $t \in (0,1)$ small such that

$$t\widetilde{u}_1(q)(z) \leqslant \delta \quad \text{and} \quad |D(t\widetilde{u}_1(q))(z)| \leqslant \delta, \quad \forall z \in \overline{\Omega}. \tag{3.7}$$

We can always assume that $c_2 \geqslant 1$ (see hypothesis $H(a)$(iv)). Then we have

$$\begin{aligned}\psi_+(t\widetilde{u}_1(q)) &\leqslant \frac{1}{q}(c_2+\varepsilon)\bigg(\|D(t\widetilde{u}_1(q))\|_q^q + \int_\Omega \xi^+(z)(t\widetilde{u}_1(q))^q\,\mathrm{d}z \\ &\quad + \int_{\partial\Omega}\beta(z)(t\widetilde{u}_1(q))^q\,\mathrm{d}\sigma\bigg) + \frac{c_4}{r}\|t\widetilde{u}_1(q)\|_r^r - \frac{\eta}{q}t^q \\ &\leqslant \frac{t^q}{q}(2c_2|\widetilde{\lambda}_1(q)| - \eta) + \frac{t^r}{r}c_4\|\widetilde{u}_1(q)\|_r^r\end{aligned} \tag{3.8}$$

(since $c_2+\varepsilon > 1$, $\delta \in (0,1]$, $q < p$, $\|\widetilde{u}_1(q)\|_q = 1$ and $\varepsilon \in (0,c_2)$). Recall that $\eta > 0$ is arbitrary. So, if we choose $\eta > 2c_2\widetilde{\lambda}_1(q)$, then from (3.8) and since $q < p < r$, by choosing $t \in (0,1)$ small, we have

$$\psi_+(t\widetilde{u}_1(q)) < 0,$$

so

$$\psi_+(\underline{u}) < 0 = \psi_+(0)$$

(see (3.5)) and hence $\underline{u} \neq 0$.

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From (3.5) we have

$$\psi'_+(\underline{u}) = 0,$$

so

$$\langle A(\underline{u}), h\rangle + \int_\Omega \xi^+(z)|\underline{u}|^{p-2}\underline{u}h\,\mathrm{d}z + \int_{\partial\Omega} \beta(z)|\underline{u}|^{p-2}\underline{u}h\,\mathrm{d}\sigma$$
$$= \int_\Omega (\eta(\underline{u}^+)^{q-1} - c_4(\underline{u}^+)^{r-1})h\,\mathrm{d}z, \quad \forall h \in W^{1,p}(\Omega). \tag{3.9}$$

In (3.9) we choose $h = -\underline{u}^- \in W^{1,p}(\Omega)$. Then

$$\frac{c_1}{p-1}\|D\underline{u}^-\|_p^p + \int_\Omega \xi^+(z)(\underline{u}^-)^p\,\mathrm{d}z + \int_{\partial\Omega} \beta(z)(\underline{u}^-)^p\,\mathrm{d}\sigma \leqslant 0$$

(see Lemma 2.3), so

$$c_5\|\underline{u}^-\|^p \leqslant 0$$

for some $c_5 > 0$ (see hypothesis $H(\beta)$), and thus

$$\underline{u} \geqslant 0, \qquad \underline{u} \neq 0.$$

Then from (3.9) it follows that

$$\begin{cases} -\operatorname{div} a(D\underline{u}(z)) + \xi^+(z)\underline{u}(z)^{p-1} = \eta\underline{u}(z)^{q-1} - c_4\underline{u}(z)^{r-1} & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial \underline{u}}{\partial n_a} + \beta(z)\underline{u}^{p-1} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.10}$$

(see Papageorgiou and Rădulescu [**22**]).

From Papageorgiou and Rădulescu [**27**], we have

$$\underline{u} \in L^\infty(\Omega).$$

Then the regularity theory of Lieberman [**18**] implies that

$$\underline{u} \in C_+ \setminus \{0\}.$$

From (3.10) we have

$$\operatorname{div} a(D\underline{u}(z)) \leqslant (\|\xi\|_\infty + c_4\|\underline{u}\|_\infty^{r-p})\underline{u}(z)^{p-1} \quad \text{for a.a. } z \in \Omega,$$

so $\underline{u} \in D_+$ (see Pucci and Serrin [**29**, pp. 111, 120]).

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Next, we show the uniqueness of this positive solution. To this end, we introduce the integral functional $j\colon L^1(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega G(Du^{1/q})\,\mathrm{d}z + \dfrac{1}{p}\int_\Omega \xi^+(z)u^{p/q}\,\mathrm{d}z + \dfrac{1}{p}\int_{\partial\Omega}\beta(z)u^{p/q}\,\mathrm{d}\sigma & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \geqslant 0,\ u^{1/q} \in W^{1,p}(\Omega), & \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} & \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom} j = \{u \in L^1(\Omega) :\ j(u) < +\infty\}$ (the effective domain of j) and set

$$u = ((1-t)u_1 + tu_2)^{1/q} \quad \text{with } t \in [0,1].$$

Using Lemma 1 of Diaz and Saa [2], we have

$$|Du(z)| \leqslant ((1-t)|Du_1(z)^{1/q}|^q + t|Du_2(z)^{1/q}|^q)^{1/q} \quad \text{for a.a. } z \in \Omega.$$

Hence, we have

$$\begin{aligned} G_0(|Du|) &\leqslant G_0(((1-t)|Du_1^{1/q}|^q + t|Du_2^{1/q}|^q)^{1/q}) \\ &\leqslant (1-t)G_0(|Du_1^{1/q}|) + tG_0(|Du_2^{1/q}|) \quad \text{for a.a. } z \in \Omega \end{aligned}$$

(since G_0 is increasing and using hypothesis $H(a)$(iv)), so

$$G(Du) \leqslant (1-t)G(|Du_1^{1/q}|) + tG(|Du_2^{1/q}|) \quad \text{for a.a. } z \in \Omega,$$

and thus j is convex (recall that $q < p$ and see hypothesis $H(\beta)$).

Also, by Fatou's lemma, j is lower semicontinuous.

Let $\widetilde{\underline{u}}$ be another positive solution of the auxiliary problem (3.4). As for $\underline{u}$, we show that

$$\widetilde{\underline{u}} \in D_+.$$

Then $\underline{u}, \widetilde{\underline{u}} \in \operatorname{dom} j$, and for all $h \in C^1(\overline{\Omega})$ and for $|t| < 1$ small we have

$$\underline{u}^q + th \in \operatorname{dom} j, \qquad \widetilde{\underline{u}}^q + th \in \operatorname{dom} j.$$

We can easily see that j is Gâteaux differentiable at $\underline{u}^q$ and at $\widetilde{\underline{u}}^q$ in the direction $h \in C^1(\overline{\Omega})$. Moreover, by the chain rule and the nonlinear Green's theorem (Gasiński and Papageorgiou [3, p. 210]), we have

$$j'(\underline{u}^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(D\underline{u}) + \xi^+(z)\underline{u}^{p-1}}{\underline{u}^{q-1}} h\,\mathrm{d}z, \quad \forall h \in W^{1,p}(\Omega)$$

and

$$j'(\widetilde{\underline{u}}^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(D\widetilde{\underline{u}}) + \xi^+(z)\widetilde{\underline{u}}^{p-1}}{\widetilde{\underline{u}}^{q-1}} h\,\mathrm{d}z, \quad \forall h \in W^{1,p}(\Omega).$$

The convexity of j implies the monotonicity of j'. Hence

$$\begin{aligned} 0 &\leqslant \int_\Omega \left(\frac{-\operatorname{div} a(D\widetilde{\underline{u}})}{\widetilde{\underline{u}}^{q-1}} - \frac{-\operatorname{div} a(D\underline{u})}{\underline{u}^{q-1}} \right) (\underline{u}^q - \widetilde{\underline{u}}^q) \, \mathrm{d}z \\ &= \int_\Omega (\xi^+(z)(\widetilde{\underline{u}}^{p-q} - \underline{u}^{p-q}) + c_4(\widetilde{\underline{u}}^{r-q} - \underline{u}^{r-q}))(\underline{u}^q - \widetilde{\underline{u}}^q) \, \mathrm{d}z, \end{aligned}$$

so

$$\underline{u} = \widetilde{\underline{u}}$$

(since $q < p < r$ and $\mu > \|\xi\|_\infty$).

So, $\underline{u} \in D_+$ is the unique positive solution of (3.4).

Since problem (1.1) is odd, it follows that

$$\underline{v} = -\underline{u} \in -D_+$$

is the unique negative solution of (3.4). □

Let S_+ (respectively S_-) be the set of positive (respectively negative) solutions u (respectively solutions v) of problem (1.1) such that

$$u(z) \in [0, c] \quad (\text{respectively } v(z) \in [-c, 0]) \quad \text{for a.a. } z \in \Omega.$$

The nonlinear regularity theory and the nonlinear maximum principle imply that

$$S_+ \subseteq (D_+ \cap [0, c]) \cup \{0\} \quad \text{and} \quad S_- \subseteq ((-D_+) \cap [-c, 0]) \cup \{0\}.$$

Proposition 3.4. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then*

(a) $\underline{u} \leqslant u$ *for all* $u \in S_+$;

(b) $v \leqslant \underline{v}$ *for all* $v \in S_+$.

Proof. (a) Let $u \in S_+$ and consider the Carathéodory function $e_+ \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$e_+(z, \zeta) = \begin{cases} 0 & \text{if} \quad \zeta < 0, \\ \eta \zeta^{q-1} - c_4 \zeta^{r-1} + \mu \zeta^{p-1} & \text{if} \quad 0 \leqslant \zeta \leqslant u(z), \\ \eta u(z)^{q-1} - c_4 u(z)^{r-1} + \mu u(z)^{p-1} & \text{if} \quad u(z) < \zeta. \end{cases} \tag{3.11}$$

Let $E_+(z, \zeta) = \int_0^\zeta e_+(z, s) \, \mathrm{d}s$ and consider the C^1-functional $\widehat{\varphi}_+ \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \widehat{\varphi}_+(u) &= \int_\Omega G(Du) \, \mathrm{d}z + \frac{1}{p} \int_\Omega (\xi^+(z) + \mu) |u|^p \, \mathrm{d}z \\ &\quad + \frac{1}{p} \int_{\partial\Omega} \beta(z) |u|^p \, \mathrm{d}\sigma - \int_\Omega E_+(z, u) \, \mathrm{d}z, \quad \forall u \in W^{1,p}(\Omega). \end{aligned}$$

Corollary 2.4, together with (3.11) and the fact that $\mu > \|\xi\|_\infty$, imply that $\widehat{\varphi}_+$ is coercive. Also, $\widehat{\varphi}_+$ is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_* \in W^{1,p}(\Omega)$ such

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that

$$\widehat{\varphi}_+(\widetilde{u}_*) = \inf_{u \in W^{1,p}(\Omega)} \widehat{\varphi}_+(u). \tag{3.12}$$

As in the proof of Proposition 3.3, using (3.6), we see that for $t \in (0,1)$ small (at least such that $t\widetilde{u}_1(q) \leqslant u$, recall that $u \in D_+$), we have

$$\widehat{\varphi}_+(t\widetilde{u}_1(q)) < 0,$$

so

$$\widehat{\varphi}_+(\widetilde{u}_*) < 0 = \widehat{\varphi}_+(0)$$

(see (3.12)), and thus

$$\widetilde{u}_* \neq 0.$$

From (3.12) we have

$$\widehat{\varphi}'_+(\widetilde{u}_*) = 0,$$

so

$$\begin{aligned}\langle A(\widetilde{u}_*), h\rangle + \int_\Omega (\xi^+(z) + \mu)|\widetilde{u}_*|^{p-2}\widetilde{u}_* h\,\mathrm{d}z + \int_{\partial\Omega} \beta(z)|\widetilde{u}_*|^{p-2}\widetilde{u}_* h\,\mathrm{d}\sigma \\ = \int_\Omega e_+(z, \widetilde{u}_*)h\,\mathrm{d}z, \quad \forall h \in W^{1,p}(\Omega).\end{aligned} \tag{3.13}$$

In (3.13), first we choose $h = -\widetilde{u}_*^- \in W^{1,p}(\Omega)$. Then

$$\frac{c_1}{p-1}\|D\widetilde{u}_*^-\|_p^p + \int_\Omega (\xi^+(z) + \mu)(\widetilde{u}_*^-)^p\,\mathrm{d}z + \int_{\partial\Omega} \beta(z)(\widetilde{u}_*^-)^p\,\mathrm{d}\sigma \leqslant 0$$

(see Lemmas 2.3 and (3.11)), so

$$c_6\|\widetilde{u}_*^-\|^p \leqslant 0,$$

for some $c_6 > 0$ (recall that $\mu > \|\xi\|_\infty$), and thus

$$\widetilde{u}_* \geqslant 0, \qquad \widetilde{u}_* \neq 0.$$

Also, in (3.13) we choose $h = (\widetilde{u}_* - u)^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}&\langle A(\widetilde{u}_*), (\widetilde{u}_* - u)^+\rangle + \int_\Omega (\xi^+(z) + \mu)\widetilde{u}_*^{p-1}(\widehat{u}_* - u)^+\mathrm{d}z \\ &\quad + \int_{\partial\Omega} \beta(z)\widetilde{u}_*^{p-1}(\widetilde{u}_* - u)^+\mathrm{d}\sigma \\ &= \int_\Omega (\eta u^{q-1} - c_4 u^{r-1} + \mu u^{p-1})(\widetilde{u}_* - u)^+\mathrm{d}z \\ &\leqslant \int_\Omega (f(z,u) + \mu u^{p-1})(\widetilde{u}_* - u)^+\mathrm{d}z \\ &\leqslant \langle A(u), (\widetilde{u}_* - u)^+\rangle + \int_\Omega (\xi^+(z) + \mu)u^{p-1}(\widetilde{u}_* - u)^+\mathrm{d}z \\ &\quad + \int_{\partial\Omega} \beta(z)u^{p-1}(\widetilde{u}_* - u)^+\mathrm{d}\sigma\end{aligned}$$

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(see (3.11), (3.1) and recall that $u \in S_+$), so

$$\widetilde{u}_* \leqslant u.$$

Thus, we have proved that

$$\widetilde{u}_* \in [0, u], \qquad \widetilde{u}_* \neq 0, \tag{3.14}$$

where $[0, u] = \{y \in W^{1,p}(\Omega) : \ 0 \leqslant y(z) \leqslant u(z) \text{ for a.a. } z \in \Omega\}$.

From (3.11) and (3.14), it follows that equation (3.13) becomes

$$\begin{aligned} \langle A(\widetilde{u}_*), h\rangle + \int_\Omega \xi^+(z)\widetilde{u}_*^{p-1} h \,\mathrm{d}z + \int_{\partial\Omega} \beta(z)\widetilde{u}_*^{p-1} h \,\mathrm{d}\sigma \\ = \int_\Omega (\eta\widetilde{u}_*^{q-1} - c_4\widetilde{u}_*^{r-1}) h \,\mathrm{d}z, \quad \forall h \in W^{1,p}(\Omega), \end{aligned}$$

so

$$\begin{cases} -\mathrm{div}\, a(D\widetilde{u}_*(z)) + \xi^+(z)\widetilde{u}_*(z)^{p-1} = \eta\widetilde{u}_*(z)^{q-1} - c_4\widetilde{u}_*(z)^{r-1} & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial \widetilde{u}_*}{\partial n_a} + \beta(z)\widetilde{u}_*^{p-1} = 0 \quad \text{on } \partial\Omega \end{cases}$$

(see Papageorgiou and Rădulescu [**22**]), and then

$$\widetilde{u}_* = \underline{u} \in D_+$$

(see Proposition 3.3), thus

$$\underline{u} \leqslant u, \quad \forall u \in S_+.$$

(b) Similarly, we show that $v \leqslant \underline{v}$ for all $v \in S_-$. □

Consider the Carathéodoty function $\widehat{f} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\widehat{f}(z, \zeta) = \begin{cases} f(z, -c) - \mu c^{p-1} & \text{if } \zeta < -c, \\ f(z, \zeta) + \mu|\zeta|^{p-2}\zeta & \text{if } -c \leqslant \zeta \leqslant c, \\ f(z, c) + \mu c^{p-1} & \text{if } c < \zeta. \end{cases} \tag{3.15}$$

Let $\widehat{F}(z, \zeta) = \int_0^\zeta \widehat{f}(z, s) \,\mathrm{d}s$ and consider the C^1-functional $\widehat{\varphi}_+ : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \widehat{\varphi}_+(u) = \int_\Omega G(Du) \,\mathrm{d}z + \frac{1}{p}\int_\Omega (\xi(z) + \mu)|u|^p \,\mathrm{d}z \\ + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p \,\mathrm{d}\sigma - \int_\Omega \widehat{F}(z, u) \,\mathrm{d}z, \quad \forall u \in W^{1,p}(\Omega). \end{aligned}$$

It is clear that the functional $\widehat{\varphi}$ has the following properties:

- $\widehat{\varphi}$ is even and $\widehat{\varphi}(0) = 0$;
- $\widehat{\varphi}$ is coercive (see (3.15) and recall that $\mu > \|\xi\|_\infty$).

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So, $\widehat{\varphi}$ is bounded below and satisfies the Palais–Smale condition. Moreover, the nonlinear regularity theory implies that

$$K_{\widehat{\varphi}} \subseteq C^1(\overline{\Omega}).$$

Proposition 3.5. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then there exists $\widehat{M} > 0$ such that*

$$\|u\|_\infty < \widehat{M}, \quad \forall u \in K_{\widehat{\varphi}}.$$

Proof. From hypothesis $H(f)(i)$, (3.15) and since $\mu > \|\xi\|_\infty$, we see that we can find $\widehat{M} > 0$ big such that

$$|\widehat{f}(z,\zeta)| \leqslant (\xi(z)+\mu)\widehat{M}^{p-1} \quad \text{for a.a. } z \in \Omega \text{ all } \zeta \in \mathbb{R}. \tag{3.16}$$

Let $u \in K_{\widehat{\varphi}}$. We have

$$|\widehat{\varphi}'(u)| = 0,$$

so

$$\begin{aligned}
&\left|\langle A(u),h\rangle + \int_\Omega (\xi(z)+\mu)|u|^{p-2}uh\,\mathrm{d}z + \int_{\partial\Omega}\beta(z)|u|^{p-2}uh\,\mathrm{d}\sigma\right| \\
&\quad = \left|\int_\Omega \widehat{f}(z,u)h\,\mathrm{d}z\right| \leqslant \int_\Omega |\widehat{f}(z,u)||h|\,\mathrm{d}z \leqslant \int_\Omega (\xi(z)+\mu)\widehat{M}^{p-1}|h|\,\mathrm{d}z \\
&\quad \leqslant \int_\Omega (\xi(z)+\mu)\widehat{M}^{p-1}|h|\,\mathrm{d}z + \int_{\partial\Omega}\beta(z)\widehat{M}^{p-1}|h|\,\mathrm{d}\sigma
\end{aligned} \tag{3.17}$$

(see hypothesis $H(\beta)$).

Let $h = (u-\widehat{M})^+ \in W^{1,p}(\Omega)$. Then $|h| = h$, and note that $A(\widehat{M}) = 0$. From (3.17), we have

$$\begin{aligned}
&\langle A(u) - A(\widehat{M}), (u-\widehat{M})^+\rangle \\
&\quad + \int_\Omega (\xi(z)+\mu)(|u|^{p-2}u - \widehat{M}^{p-1})(u-\widehat{M})^+\,\mathrm{d}z \\
&\quad + \int_{\partial\Omega}\beta(z)(|u|^{p-2}u - \widehat{M}^{p-1})(u-\widehat{M})^+\,\mathrm{d}\sigma \leqslant 0,
\end{aligned}$$

so $u \leqslant \widehat{M}$.

Similarly, if we choose $h = (-\widehat{M}-u)^+ \in W^{1,p}(\Omega)$, then $|h| = h$ and we have

$$\begin{aligned}
0 \geqslant &\langle A(-\widehat{M}) - A(u), (-\widehat{M}-u)^+\rangle \\
&+ \int_\Omega (\xi(z)+\mu)(|-\widehat{M}|^{p-2}(-\widehat{M}) - |u|^{p-2}u)(-\widehat{M}-u)^+\,\mathrm{d}z \\
&+ \int_{\partial\Omega}\beta(z)(|-\widehat{M}|^{p-2}(-\widehat{M}) - |u|^{p-2}u)(-\widehat{M}-u)^+\,\mathrm{d}\sigma,
\end{aligned}$$

so $u \geqslant -\widehat{M}$.

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So, we conclude that

$$\|u\|_\infty \leqslant \widehat{M}, \quad \forall u \in K_{\widehat{\varphi}}. \qquad \square$$

Proposition 3.6. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$ and $H(f)$ hold, and $V_n \in W^{1,p}(\Omega)$ is an n-dimensional subspace, then we can find $\varrho_n > 0$ such that*

$$\widehat{\varphi}(u) < 0, \quad \forall u \in V_n, \ \|u\| = \varrho_n.$$

Proof. Hypothesis $H(a)$(iv) and Corollary 2.4 imply that we can find $c_7 > 0$ such that

$$G(y) \leqslant c_7(|y|^q + |y|^p), \quad \forall y \in \mathbb{R}^N. \tag{3.18}$$

Also, from hypothesis $H(f)$(ii), we see that given any $\widehat{\eta} > 0$, we can find $\delta = \delta(\widehat{\eta}) > 0$ such that

$$F(z,\zeta) \ \geqslant \ \widehat{\eta}|\zeta|^p \quad \text{for a.a. } z \in \Omega \text{ all } |\zeta| \leqslant \delta. \tag{3.19}$$

Since V_n is finite dimensional, all norms are equivalent. So, we can find $\varrho_n \in (0,1)$ small such that

$$\text{if } u_n \in V, \ \|u\| \leqslant \varrho \text{ then } |u_n(z)| \leqslant \delta \quad \text{for a.a. } z \in \Omega. \tag{3.20}$$

Then for $u \in V_n$ with $\|u\| = \varrho_n$, we have

$$\begin{aligned}\widehat{\varphi}(u) &\leqslant c_7(\|Du\|_q^q + \|Du\|_p^p) + c_8\|u\|^p - \widehat{\eta}c_9\|u\|^q \\ &\leqslant (c_7 - \widehat{\eta}c_9)\|u\|^q + c_{10}\|u\|^p,\end{aligned}$$

for some $c_8, c_9, c_{10} > 0$ (see (3.18)–(3.20) and recall that $q < p < r$, $\varrho_n \in (0,1)$ and all norms on V_n are equivalent). Since $\widehat{\eta} > 0$ is arbitrary, we choose $\widehat{\eta} > c_7/c_9$. Then

$$\widehat{\varphi}(u) \leqslant -c_{11}\|u\|^q + c_{10}\|u\|^p, \quad \forall u \in V_n, \ \|u\| = \varrho_n,$$

for some $c_{11} > 0$. Because $q < p$, by choosing $\varrho_n \in (0,1)$ even smaller if necessary, we have

$$\widehat{\varphi}(u) < 0, \quad \forall u \in V_n, \ \|u\| = \varrho_n. \qquad \square$$

Now we are ready for the multiplicity result concerning nodal solutions for problem (1.1).

Theorem 3.7. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then problem (1.1) admits a whole sequence of distinct nodal solutions $\{u_n\}_{n\geqslant 1}$ such that*

$$u_n \in C^1(\overline{\Omega}), \quad \forall n \geqslant 1 \quad \textit{and} \quad u_n \longrightarrow 0 \quad \textit{in } C^1(\overline{\Omega}).$$

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Proof. According to Theorem 2.1, we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ such that

$$u_n \in K_{\widehat{\varphi}}, \quad \forall n \geqslant 1 \quad \text{and} \quad u_n \longrightarrow 0 \quad \text{in } W^{1,p}(\Omega). \tag{3.21}$$

From Proposition 3.5 and the nonlinear regularity theory of Lieberman [**18**], we know that we can find $\alpha \in (0,1)$ and $\widetilde{M} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant \widetilde{M}, \quad \forall n \geqslant 1. \tag{3.22}$$

Exploiting the compactness of the embedding $C^{1,\alpha}(\overline{\Omega}) \subseteq C^1(\overline{\Omega})$, from (3.21) and (3.22) it follows that

$$u_n \longrightarrow 0 \quad \text{in } C^1(\overline{\Omega}). \tag{3.23}$$

Let $m^* < \min\{\min_{\overline{\Omega}} \underline{u}, -\min_{\overline{\Omega}} \underline{v}\}$ (recall that $\underline{u} \in D_+$ and $\underline{v} \in -D_+$; see Proposition 3.3). Then, from (3.23), we see that

$$u_n \in [-m^*, m^*], \quad \forall n \geqslant n_0,$$

so $\{u_n\}_{n\geqslant n_0}$ is the sequence of nodal solutions of (1.1) (see Proposition 3.4). □

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