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GENERATION OF RELATIVE COMMUTATOR SUBGROUPS IN CHEVALLEY GROUPS. II

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Abstract In the present paper, which is a direct sequel of our paper [**14**] joint with Roozbeh Hazrat, we prove an unrelativized version of the standard commutator formula in the setting of Chevalley groups. Namely, let Φ be a reduced irreducible root system of rank ≥ 2, let R be a commutative ring and let I, J be two ideals of R. We consider subgroups of the Chevalley group $G(\Phi, R)$ of type Φ over R. The unrelativized elementary subgroup $E(\Phi, I)$ of level I is generated (as a group) by the elementary unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in I$, of level I. Obviously, in general, $E(\Phi, I)$ has no chance to be normal in $E(\Phi, R)$; its normal closure in the absolute elementary subgroup $E(\Phi, R)$ is denoted by $E(\Phi, R, I)$. The main results of [**14**] implied that the commutator $[E(\Phi, I), E(\Phi, J)]$ is in fact normal in $E(\Phi, R)$. In the present paper we prove an unexpected result, that in fact $[E(\Phi, I), E(\Phi, J)] = [E(\Phi, R, I), E(\Phi, R, J)]$. It follows that the standard commutator formula also holds in the unrelativized form, namely $[E(\Phi, I), C(\Phi, R, J)] = [E(\Phi, I), E(\Phi, J)]$, where $C(\Phi, R, I)$ is the full congruence subgroup of level I. In particular, $E(\Phi, I)$ is normal in $C(\Phi, R, I)$.

Keywords: Chevalley groups; elementary subgroups; generation of mixed commutator subgroups; standard commutator formula

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In the present paper, which is a direct sequel of our paper [**14**] joint with Roozbeh Hazrat, we continue the study of mutual commutator subgroups of relative elementary subgroups $E(\Phi, R, I)$, unrelative elementary subgroups $E(\Phi, I)$ and congruence subgroups $G(\Phi, R, I)$ of level $I \trianglelefteq R$ in a Chevalley group $G(\Phi, R)$, $\mathrm{rk}(\Phi) \geq 2$. For $\mathrm{GL}(n, R)$ at the stable level such commutator formulas first occurred in the groundbreaking work of Hyman Bass [**2**]. Soon thereafter, they were extended, still at the stable level, to other classical groups by Anthony Bak, and to Chevalley groups by Michael Stein [**22**]. At about the same time, Alec Mason and Wilson Stothers obtained such birelative formulas for $\mathrm{GL}(n, R)$, see [**20**].

The next important breakthrough came with the work of Andrei Suslin [**29**], who, again for $\mathrm{GL}(n, R)$, $n \geq 3$, established one of Bass's key results, normality of the relative elementary subgroup $E(n, R, I)$, not at the stable level, but for *arbitrary* commutative rings.

This work was then immediately extended, by Suslin himself, Vyacheslav Kopeiko, Leonid Vaserstein, Zenon Borewicz, the first author and many others to other classical groups, and to other Bass commutator formulas. Eventually, both the normality of $E(\Phi,R,I)$ and the second Bass commutator formula were proven in the context of Chevalley groups by Giovanni Taddei and Vaserstein [**30**, **32**].

One extremely pregnant intermediary result asserts that the relative elementary subgroup $E(\Phi,R,I)$ is generated by the Stein–Tits–Vaserstein generators $z_\alpha(\xi,\eta)=x_{-\alpha}(\eta)x_\alpha(\xi)x_{-\alpha}(-\eta)$, where $\xi\in I$ for $\alpha\in\Phi$, while $\eta\in R$.

Later, Hong You, Alexei Stepanov, the authors and Roozbeh Hazrat also initiated the generalization of birelative commutator formulas to arbitrary commutative rings and to other types of groups, including Chevalley groups, see, for instance, [**12**, **25**, **38**] and references there. In particular, in the process of this work we obtained a generalization of the above generation result, proving that the birelative commutator subgroup $[E(\Phi,R,I),E(\Phi,R,J)]$ is generated *as a group* by the following three types of generator: (i) $z_\alpha(\xi\zeta,\eta)$, (ii) $[x_\alpha(\xi),x_{-\alpha}(\zeta)]$ and (iii) $[x_\alpha(\xi),z_\alpha(\zeta,\eta)]$, where $\alpha\in\Phi$, $\xi\in I$, $\zeta\in J$, $\eta\in R$, see [**14**].

In the second part of this work, we perform another unexpected feat in this direction. Namely, in the Main Lemma we establish that the third type of the above generators are redundant. In particular, both remaining types of generators already belong to the mutual commutator of *unrelative* commutator subgroups $[E(\Phi,I),E(\Phi,J)]$. This, in particular, implies the following amazing commutator formula: $[E(\Phi,R,I),E(\Phi,R,J)]=[E(\Phi,I),E(\Phi,J)]$. As we describe in the introduction below, this formula generalizes and explains a great number of preceding results.

1. Introduction

Let Φ be a reduced irreducible root system of rank ≥ 2, let R be a commutative ring with 1 and let $G(\Phi,R)$ be a Chevalley group of type Φ over R. For background on Chevalley groups over rings see [**33**] or [**35**], where one can find many further references. We fix a split maximal torus $T(\Phi,R)$ in $G(\Phi,R)$ and consider root unipotents $x_\alpha(\xi)$ elementary with respect to $T(\Phi,R)$. The subgroup $E(\Phi,R)$ generated by all $x_\alpha(\xi)$, where $\alpha\in\Phi$, $\xi\in R$, is called the *absolute* elementary subgroup of $G(\Phi,R)$.

Now, let $I\trianglelefteq R$ be an ideal of R. Then the *unrelativized elementary subgroup* $E(\Phi,I)$ of level I is defined as the subgroup of $E(\Phi,R)$, generated by all elementary root unipotents $x_\alpha(\xi)$ of level I,

$$E(\Phi,I)=\langle x_\alpha(\xi)\mid \alpha\in\Phi,\ \xi\in I\rangle.$$

In general, this subgroup has no chance to be normal in $E(\Phi,R)$. Its normal closure $E(\Phi,R,I)=E(\Phi,I)^{E(\Phi,R)}$ is called the *relative elementary subgroup* of level I.

The starting points of the present paper are the following (Theorems A–C below) three observations contained in [**14**]. The first one is the leftmost (nontrivial) inclusion in Theorem 3.1, whereas the other two are Corollary 5.2 and Corollary 5.1 of Theorem 1.3, respectively. In these results some additional assumptions are necessary in the cases $\Phi=\mathrm{C}_l,\mathrm{G}_2$. The first of these results relies on a calculation that is immediate for simply laced systems, but rather nontrivial in the exceptional cases $\Phi=\mathrm{C}_2,\mathrm{G}_2$. The other two are easy corollaries of this result and the main result of [**14**], describing generators of the mixed commutator subgroup $[E(\Phi, R, I), E(\Phi, R, J)]$.

In the rest of this paper we impose the following umbrella assumption.

(*) In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$, assume that R does not have residue fields $\mathbb{F}_2$ of two elements, and in the case $\Phi = \mathrm{C}_l$, $l \geq 2$, assume additionally that any $\theta \in R$ is contained in the ideal $\theta^2 R + 2\theta R$.

This condition arises in the computation of the lower level of $[E(\Phi, I), E(\Phi, J)]$ in [**12**, Lemma 17, **14**, Theorem 3.1]; see also further related results and discussion of this condition in [**24**, **25**].

Theorem A. *Let Φ be a reduced irreducible root system of rank ≥ 2 and let I, J be two ideals of a commutative ring R. Then one has the following inclusion:*

$$E(\Phi, R, IJ) \leq [E(\Phi, I), E(\Phi, J)].$$

Theorem B. *Let Φ be a reduced irreducible root system of rank ≥ 2 and let I, J be two ideals of a commutative ring R. Then the mixed commutator subgroup $[E(\Phi, I), E(\Phi, J)]$ is normal in $E(\Phi, R)$.*

Theorem C. *Let Φ be a reduced irreducible root system of rank ≥ 2 and let I, J be two ideals of a commutative ring R. Then*

$$[E(\Phi, I), E(\Phi, R, J)] = [E(\Phi, R, I), E(\Phi, R, J)].$$

What we have not considered when writing [**14**] is that modulo some further elementary calculations involving our generators of $[E(\Phi, R, I), E(\Phi, R, J)]$, Theorems A–C admit the following common generalization.

Theorem 1.1. *Let Φ be a reduced irreducible root system of rank ≥ 2 and let I, J be two ideals of a commutative ring R. Then*

$$[E(\Phi, I), E(\Phi, J)] = [E(\Phi, R, I), E(\Phi, R, J)].$$

As a matter of fact, Theorem 1.1 can be derived from the main result of [**14**, Theorem 1.3]. That theorem, which we recall as Theorem F in § 2, lists three types of generators of $[E(\Phi, R, I), E(\Phi, R, J)]$. Of those three types, the last two are contained already in $[E(\Phi, I), E(\Phi, J)]$, the second one by the definition itself and the last one by Theorem A above. It remains to be shown that the first type of generators, those of the form $[x_\alpha(\xi), z_\alpha(\zeta, \eta)]$ (see § 2 for precise definitions) are already in $[E(\Phi, I), E(\Phi, J)]$. Let us record the result for future reference.

Theorem 1.2. *Let Φ be a reduced irreducible root system of rank ≥ 2. In the cases $\Phi = \mathrm{B}_2, \mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements, and in the case $\Phi = \mathrm{B}_2$ assume additionally that any $c \in R$ is contained in the ideal $c^2 R + 2cR$.*

Further, let I and J be two ideals of a commutative ring R. Then the mixed commutator subgroup $[E(\Phi, R, I), E(\Phi, R, J)]$ is generated as a group by the elements of the form

- $z_\alpha(\xi\zeta, \eta)$,
- $[x_\alpha(\xi), x_{-\alpha}(\zeta)]$,

where in both cases $\alpha \in \Phi$, $\xi \in I$, $\zeta \in J$, $\eta \in R$.

This generalization of Theorem F is exactly the main new calculation in the present paper; the rest was mostly either known before, or is contained in [**14**] or easily follows.

Actually, our Theorem 1.1 also allows us to unrelativize the birelative standard commutator formula, established in this context by You Hong [**38**] via level calculations, and by ourselves [**12**] via a version of relative localization. Namely, let $\rho_I : R \longrightarrow R/I$ be the reduction modulo I. By functoriality, it defines the group homomorphism $\rho_I : G(\Phi, R) \longrightarrow G(\Phi, R/I)$. The kernel of ρ_I is denoted by $G(\Phi, R, I)$ and is called the *principal congruence subgroup* of $G(\Phi, R)$ of level I. In turn, the full pre-image of the centre of $G(\Phi, R/I)$ with respect to the reduction homomorphism ρ_I is called the *full congruence subgroup* of level I and is denoted by $C(\Phi, R, I)$. Now, the *birelative standard commutator formula*, see [**12**, Theorem 1], can be stated as follows.

Theorem D. *Let Φ be a reduced irreducible root system of rank ≥ 2. Further, let R be a commutative ring, and let $I, J \trianglelefteq R$ be two ideals of R. Then*

$$[E(\Phi, R, I), C(\Phi, R, J)] = [E(\Phi, R, I), E(\Phi, R, J)].$$

Now, Theorems 1.1 and D immediately imply the following result.

Theorem 1.3. *Let Φ be a reduced irreducible root system of rank ≥ 2 and let I, J be two ideals of a commutative ring R. Then*

$$[E(\Phi, I), C(\Phi, R, J)] = [E(\Phi, I), E(\Phi, J)].$$

Proof. Indeed, one has

$$\begin{aligned}[E(\Phi, I), E(\Phi, J)] &\leq [E(\Phi, I), C(\Phi, R, J)]\\ &\leq [E(\Phi, R, I), C(\Phi, R, J)] = [E(\Phi, R, I), E(\Phi, R, J)],\end{aligned}$$

where the first two inclusions are obvious, whereas the last equality is Theorem D. On the other hand, the left-hand side equals the right-hand side by Theorem 1.1. □

Setting $I = J$ in Theorem 1.3, we get the following freakish corollary.

Theorem 1.4. *Let Φ be a reduced irreducible root system of rank ≥ 2 and let I be an ideal of a commutative ring R. Then $E(\Phi, I)$ is normal in $C(\Phi, R, I)$.*

For the special case of $G = \mathrm{GL}(n, R)$, Theorems 1.1 and 1.3 were first verified by the first-named author in [**34**], while Theorem 1.4 in that case was proven already in [**21**]. However, in [**34**] the proof proceeded differently. First, Theorem 1.3 was derived from Theorems A and B by the same birelative version of decomposition of unipotents [**27**] that was already used in [**36**] to establish the respective special case of Theorem D. Then, Theorem 1.1 was derived as a corollary of Theorems 1.3 and D. Thereupon, the second author immediately suggested that per case one could achieve the same directly, by looking at the elementary generators in [**14**, Theorem 1.3]. This is exactly what we accomplish in the present paper. Technically, the proofs are not ticklish; the main difficulty was to convince ourselves that Theorems 1.1–1.4 could be true as stated!

This paper is organized as follows. In § 2 we recall notation and some background facts that will be used in our proofs. Also, we recall Theorem 1.3 of [**14**] and reduce the proof of Theorem 1.1 to a calculation in groups of rank 2. The technical core of the paper is § 3, where we consecutively verify our Main Lemma for types A_2, C_2 (which is the most difficult case) and G_2. After that, in § 4, we establish another related result, generation of $E(\Phi, R, I)$ by long root elements. Finally, in § 5 we mention some further related results and applications.

2. Notation and preliminary facts

To make this paper independent of [**14**], here we recall basic notation and the requisite facts which will be used in our proofs. For more background information on Chevalley groups over rings, see [**4**, **33**, **35**] and references therein.

2.1. Notation

Let G be a group. For any $x, y \in G$, ${}^x y = xyx^{-1}$ denotes the left x-conjugate of y. As usual, $[x, y] = xyx^{-1}y^{-1}$ denotes the (left normed) commutator of x and y. We shall make constant use of the obvious commutator identities, such as $[x, yz] = [x, y] \cdot {}^y[x, z]$ or $[xy, z] = {}^x[y, z] \cdot [x, z]$, usually without any specific reference.

As in the introduction, we denote by $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, the elementary generators of the (absolute) elementary Chevalley subgroup $E(\Phi, R)$. For a root $\alpha \in \Phi$ we denote by X_α the corresponding (elementary) root subgroup $X_\alpha = \{x_\alpha(\xi) \mid \xi \in R\}$. Recall that any conjugate ${}^g x_\alpha(\xi)$ of an elementary root unipotent, where $g \in G(\Phi, R)$, is called a *root element* or *root unipotent*; it is called *long* or *short*, depending on whether the root α itself is long or short.

As in the introduction, let I be an ideal of R. We denote by $X_\alpha(I)$ the intersection of X_α with the principal congruence subgroup $G(\Phi, R, I)$. Clearly, $X_\alpha(I)$ consists of all elementary root elements $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in I$, of level I:

$$X_\alpha(I) = \{x_\alpha(\xi) \mid \xi \in I\}.$$

By definition, $E(\Phi, I)$ is generated by $X_\alpha(I)$, for all roots $\alpha \in \Phi$. The same subgroups generate $E(\Phi, R, I)$ as a normal subgroup of the absolute elementary group $E(\Phi, R)$. Generators of $E(\Phi, R, I)$ *as a group* are recalled in the next subsection.

All results of the present paper are based on the Steinberg relations among the elementary generators, which will be repeatedly used without any specific reference. Especially important for us is the Chevalley commutator formula:

$$[x_\alpha(\xi), x_\beta(\zeta)] = \prod_{i\alpha + j\beta \in \Phi} x_{i\alpha + j\beta}(N_{\alpha\beta ij}\xi^i\zeta^j),$$

where $\alpha \neq -\beta$ and $N_{\alpha\beta ij}$ are the structure constants which do not depend on ξ and ζ. However, for $\Phi = G_2$, they may depend on the order of the roots in the product on the right-hand side. See [**3**, **22**, **23**, **35**] for more details regarding the structure constants $N_{\alpha\beta ij}$.

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2.2. Generation of mixed commutator subgroups

We shall extensively use the two following generation theorems. The first one is a classical result by Michael Stein [**22**], Jacques Tits [**31**] and Leonid Vaserstein [**32**]. The second one is the main result of [**15**, Theorem 1.3].

Theorem E. *Let Φ be a reduced irreducible root system of rank ≥ 2 and let I be an ideal of a commutative ring R. Then, as a group, $E(\Phi, R, I)$ is generated by elements of the form:*

$$z_\alpha(\xi, \eta) = x_{-\alpha}(\eta) x_\alpha(\xi) x_{-\alpha}(-\eta),$$

where $\xi \in I$, $\eta \in R$ and $\alpha \in \Phi$.

Theorem F. *Let Φ be a reduced irreducible root system of rank ≥ 2. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$, assume that R does not have residue fields $\mathbb{F}_2$ of two elements, and in the case $\Phi = \mathrm{C}_l$, $l \geq 2$, assume additionally that any $\theta \in R$ is contained in the ideal $\theta^2 R + 2\theta R$.*

Further, let I and J be two ideals of a commutative ring R. Then the mixed commutator subgroup $[E(\Phi, R, I), E(\Phi, R, J)]$ is generated as a group by elements of the form:

- $[x_\alpha(\xi), z_\alpha(\zeta, \eta)]$,
- $[x_\alpha(\xi), x_{-\alpha}(\zeta)]$,
- $z_\alpha(\xi\zeta, \eta)$,

where in all cases $\alpha \in \Phi$, $\xi \in I$, $\zeta \in J$, $\eta \in R$.

Now, the generators of the second type belong to $[E(\Phi, I), E(\Phi, J)]$ by definition. Generators of the third type belong to $[E(\Phi, I), E(\Phi, J)]$ by Theorem A. Thus, Theorem F implies that to prove Theorem 1.1 it suffices to establish the following result.

Main Lemma. *Let Φ be a reduced irreducible root system of rank ≥ 2 and let I, J be two ideals of a commutative ring R. Then*

$$[x_\alpha(\xi), z_\alpha(\zeta, \eta)] \in [E(\Phi, I), E(\Phi, J)],$$

for all $\alpha \in \Phi$, $\xi \in I$, $\zeta \in J$, $\eta \in R$.

Obviously, the proof of the Main Lemma immediately reduces to rank 2 systems. Thus, we only have to verify it for groups of types A_2, B_2 and G_2. For $\Phi = \mathrm{A}_2$ we reproduce an authentic calculation at the level of individual elementary generators, with actual signs (which in this case is an adaptation of an argument from [**34**]). We could do the same for $\Phi = \mathrm{C}_2, \mathrm{G}_2$, and this was, as a matter of fact, how we originally verified it. However, to make the text more readable, we prefer the following short-cut. Since we already know Theorems A and B, we can perform calculations modulo the subgroups $E(n, R, IJ)$ and $[E(n, I), E(n, J)]$. In turn, in many cases the easiest way to verify that some commutators fall into these subgroups is Levi decomposition, which we now recall in a slightly more precise form than the one used in [**14**].

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2.3. Parabolic subgroups

An important part in the proof of the Main Lemma for $\Phi = \mathrm{C}_2, \mathrm{G}_2$ is played by Levi decomposition for (elementary) parabolic subgroups. Classically, it asserts that any parabolic subgroup P of $G(\Phi, R)$ can be expressed as the semi-direct product $P = L_P \leftthreetimes U_P$ of its unipotent radical $U_P \trianglelefteq P$ and a Levi subgroup $L_P \leq P$. However, as in [14], we do not have to recall the general case.

- Since we calculate inside $E(n, R)$, we can limit ourselves to the *elementary* parabolic subgroups, spanned by some root subgroups X_α.
- Since we can choose the order on Φ arbitrarily, we can always assume that α is fundamental and, thus, limit ourselves to *standard* parabolic subgroups.
- Since the proof of the Main Lemma reduces to groups of rank 2, we could only consider rank 1 parabolic subgroups, which *in this case* are maximal parabolic subgroups.

Thus, we consider only *elementary* rank 1 parabolics, which are defined as follows. Namely, we fix an order on Φ, and let Φ^+ and Φ^- be the corresponding sets of positive and negative roots, respectively. Further, let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the corresponding fundamental system. For any r, $1 \leq r \leq l$, and we define the rth rank 1 *elementary* parabolic subgroup as:

$$P_{\alpha_r} = \langle U, X_{-\alpha_r} \rangle \leq E(\Phi, R).$$

Here $U = \prod X_\alpha$, $\alpha \in \Phi^+$, is the unipotent radical of the standard Borel subgroup B. Then the unipotent radical of P_{α_r} has the form:

$$U_{\alpha_r} = \prod X_\alpha, \quad \alpha \in \Phi^+, \quad \alpha \neq \alpha_r,$$

whereas $L_{\alpha_r} = \langle X_{\alpha_r}, X_{-\alpha_r} \rangle$ is the (standard) Levi subgroup of P_r. Clearly, L_{α_r} is isomorphic to the elementary subgroup $E(2, R)$ in $\mathrm{SL}(2, R)$, or to its projectivized version $\mathrm{PE}(2, R)$ in $\mathrm{PGL}(2, R)$. In the sequel we usually (but not always) abbreviate $P_{\alpha_r}, U_{\alpha_r}, L_{\alpha_r}$, etc., to P_r, U_r, L_r, etc.

Levi decomposition (which in the case of elementary parabolics immediately follows from the Chevalley commutator formula) asserts that the group P_r is the semi-direct product $P_r = L_r \leftthreetimes U_r$ of $U_r \trianglelefteq P_r$ and $L_r \leq P_r$. The most important part is the (obvious) claim is that U_r is normal in P_r.

Simultaneously with P_r, one considers the opposite parabolic subgroup P_r^- defined as:

$$P_r^- = \langle U^-, X_{\alpha_r} \rangle \leq E(\Phi, R).$$

Here $U^- = \prod X_\alpha$, $\alpha \in \Phi^-$, is the unipotent radical of the Borel subgroup B^- opposite to the standard one. Clearly, P_r and P_r^- share the common (standard) Levi subgroup L_r, whereas the unipotent radical U_r^- of P_r^- is opposite to that of P_r and has the form:

$$U_r^- = \prod X_\alpha, \quad \alpha \in \Phi^-, \quad \alpha \neq -\alpha_r.$$

Now, Levi decomposition takes the form $P_r^- = L_r \leftthreetimes U_r^-$ with $U_r^- \trianglelefteq P_r^-$. In other words, U_r and U_r^- are both normalized by L_r.

Actually, we need a slightly more precise form of this last statement. Namely, let I be an ideal of R. Denote by $L_r(I)$ the principal congruence subgroup of level I in L_r and by $U_r(I)$ and $U_r^-(I)$ the respective intersections of U_r and U_r^- with $G(\Phi, R, I)$, or, equivalently, with $E(\Phi, R, I)$:

$$U_r(I) = U_r \cap E(\Phi, R, I), \quad U_r^-(I) = U_r^- \cap E(\Phi, R, I).$$

Obviously, $U_r(I), U_r^-(I) \leq E(\Phi, I)$ are normalized by L_r. The following fact will be repeatedly used in the proof of the Main Lemma.

Lemma. *Let I and J be two ideals of R. Then*

$$[L_r(I), U_r(J)] \leq U_r(IJ), \quad [L_r(I), U_r^-(J)] \leq U_r^-(IJ).$$

In particular, both commutators are contained in $E(\Phi, IJ) \leq E(\Phi, R, IJ)$.

Proof. This is classically known. For a recent reference, see, for instance, [**2**, Lemma 3.1]. □

3. Proof of Main Lemma

In this section we prove the Main Lemma, and thus also Theorems 1.1–1.4. As above, let $x = [x_\alpha(\xi), z_\alpha(\zeta, \eta)]$, where $\xi \in I$, $\zeta \in J$, $\eta \in R$. We divide the proof into four cases.

(i) α can be embedded in a root subsystem of type A_2. This proves the Main Lemma for simply laced Chevalley groups and for the Chevalley group of type F_4. It also proves the inclusion in the Main Lemma for *short* roots in Chevalley groups of type C_l, $l \geq 3$, for *long* roots in Chevalley groups of type B_l, $l \geq 3$, and for *long* roots in the Chevalley group of type G_2.

(ii) α can be embedded in a root subsystem of type C_2 as a *long* root. This proves the Main Lemma for Chevalley groups of type C_l, $l \geq 3$.

(iii) α can be embedded in a root subsystem of type C_2 as a *short* root. This proves the Main Lemma for Chevalley groups of type B_l, $l \geq 3$, and finishes the proof for the case C_2.

(iv) α can be embedded in a root subsystem of type G_2 as a *short* root. This proves the Main Lemma for the last remaining case, the group of type G_2.

For the first case, we reproduce an actual computation at the level of root elements that could ultimately be refined to an explicit formula expressing $x = [x_\alpha(\xi), z_\alpha(\zeta, \eta)]$ as a product of conjugates of commutators of the form $[x_\gamma(\epsilon), x_\delta(\theta)]$, for some roots $\gamma, \delta \in \Phi$ and some $\epsilon \in I$, $\theta \in J$. This argument is a transcript of the initial argument from [**34**], which corresponds to the first (and difficult) item in the proof of [**34**, Theorem 1].

• First, assume that α can be embedded in a root system of type A_2. We wish to prove that $[x_\alpha(\xi), z_\alpha(\zeta, \eta)] \in [E(A_2, I), E(A_2, J)]$. Indeed, in this case there exist roots

$\beta, \gamma \in \Phi$, of the same length as α such that $\alpha = \beta + \gamma$ and $N_{\beta\gamma 11} = 1$. Then

$$x = [x_\alpha(\xi), z_\alpha(\zeta, \eta)] = x_\alpha(\xi) \cdot {}^{z_\alpha(\zeta,\eta)}x_\alpha(-\xi) = x_\alpha(\xi) \cdot {}^{z_\alpha(\zeta,\eta)}[x_\beta(1), x_\gamma(-\xi)].$$

Thus,

$$\begin{aligned} x &= x_\alpha(\xi) \cdot [{}^{z_\alpha(\zeta,\eta)}x_\beta(1), {}^{z_\alpha(\zeta,\eta)}x_\gamma(-\xi)] \\ &= x_\alpha(\xi) \cdot [x_\beta(1 - \zeta\eta)x_{-\gamma}(-\eta\zeta\eta), x_{-\beta}(-\xi\eta\zeta\eta)x_\gamma(-\xi(1 - \eta\zeta))] \\ &= x_\alpha(\xi) \cdot [x_\beta(1)y, x_\gamma(-\xi)z], \end{aligned}$$

where

$$y = x_\beta(-\zeta\eta)x_\gamma(-\eta\zeta\eta) \in E(\mathrm{A}_2, J), \quad z = x_{-\beta}(-\xi\eta\zeta\eta)x_\gamma(\xi\eta\zeta) \in E(\mathrm{A}_2, IJ).$$

Since $x_\gamma(\xi) \in E(\mathrm{A}_2, I)$, the second factor of the above commutator belongs to $E(\mathrm{A}_2, I)$. Thus,

$$[x_\beta(1)y, x_\gamma(-\xi)z] = {}^{x_\beta(1)}[y, x_\gamma(-\xi)z] \cdot [x_\beta(1), x_\gamma(-\xi)z].$$

Now the first commutator on the right-hand side belongs to $[E(\mathrm{A}_2, I), E(\mathrm{A}_2, J)]$, which is normal in $E(\mathrm{A}_2, R)$, so that the conjugation by $x_\beta(1)$ still leaves us there.

On the other hand, the second commutator equals

$$[x_\beta(1), x_\gamma(-\xi)] \cdot {}^{x_\gamma(-\xi)}[x_\beta(1), z].$$

The second commutator in the last expression belongs to $E(\mathrm{A}_2, R, IJ)$ and remains there after elementary conjugations, whereas the first commutator equals $x_\alpha(-\xi)$.

Summarizing the above, we see that

$$x \in x_\alpha(\xi)[E(\mathrm{A}_2, I), E(\mathrm{A}_2, J)]x_\alpha(-\xi) \cdot E(\mathrm{A}_2, R, IJ) \le [E(\mathrm{A}_2, I), E(\mathrm{A}_2, J)],$$

as claimed.

For the three remaining cases, where $\Phi = \mathrm{C}_2$ or $\Phi = \mathrm{G}_2$, the idea of the proof is similar, but its implementation requires more care because of the more complicated form of the Chevalley commutator formula. In these cases too we *could* come up with explicit formulas, but to restrain the length, we prefer to repeatedly invoke the above Lemma on unipotent radicals, and Theorems A, B. In other words, all calculations are performed modulo $[E(\Phi, I), E(\Phi, J)]$, which is already normal in $E(\Phi, R)$. At the moment we discover that a certain factor falls into $E(\Phi, R, IJ)$ or into $[E(\Phi, I), E(\Phi, J)]$ itself, we immediately lose interest in the explicit form of this factor.

Remark. The referee suggested that the proof for these three cases could be written uniformly by arguing not in terms of individual elements but rather in terms of the subgroups $E(\Phi, R, IJ)$, $[E(\Phi, I), E(\Phi, J)]$, etc. This is indeed the case, and this is how this proof was organized in the first draft of the present paper. However, later, in view of prospective applications to width problems, we decided to add some more careful analysis, with explicit calculations in each case, since we will need these details in our next paper.

The argument proceeds as follows. When α is *short*, we express it in the form $\alpha = \beta + \gamma$, where β is long and γ is short. Similarly, when α is *long*, we express it in the form

$\alpha = \beta + 2\gamma$, with the same β, γ as above. Since we are only looking at *one* instance of the Chevalley commutator formula, the parametrization of the corresponding root subgroups can be chosen in such a way that all the resulting structure constants are positive, so that the formula takes the form

$$[x_\beta(\xi), x_\gamma(\theta)] = x_{\beta+\gamma}(\xi\theta)x_{\beta+2\gamma}(\xi\theta^2)$$

in the case of $\Phi = \mathrm{C}_2$, and the form

$$[x_\beta(\xi), x_\gamma(\theta)] = x_{\beta+\gamma}(\xi\theta)x_{\beta+2\gamma}(\xi\theta^2)x_{\beta+3\gamma}(\xi\theta^3)x_{2\beta+3\gamma}(2\xi^2\theta^3)$$

in the case of $\Phi = \mathrm{G}_2$; see [**3**, **23**] or [**35**] and references there.

As above, we rewrite x as $x = x_\alpha(\xi) \cdot {}^{z_\alpha(\zeta,\eta)}x_\alpha(-\xi)$ and plug in the expression of $x_\alpha(-\xi)$ as the commutator $[x_\beta(\xi), x_\gamma(1)]^{-1} = [x_\gamma(1), x_\beta(\xi)]$ times the tail consisting of the remaining factors $x_\delta(\eta)$ from the above instances of the Chevalley commutator formula, which up to sign are equal to ξ or $2\xi^2$ and in any case belong to $E(\Phi, I)$. By the above Lemma, the conjugates ${}^{z_\alpha(\zeta,\eta)}x_\delta(\eta)$ of the remaining factors are congruent to these factors themselves, modulo $E(\Phi, R, IJ)$. As in the first case, this leaves us with analysis of the commutator ${}^{z_\alpha(\zeta,\eta)}[x_\gamma(1), x_\beta(\xi)]$, slightly different between cases, owing to disparate configurations of roots. Anyway, in each case the result will be that modulo elementary conjugations and factors that cancel with $x_\alpha(\xi)$, or with the outstanding factors coming from the Chevalley commutator formula, the relevant part of the commutator falls into $[E(\Phi, I), E(\Phi, J)]$.

Now we pass to the case-by-case analysis.

• First, assume that α can be embedded into C_2 as a *long* root. In this case there exist a long root β and a short root γ such that $\alpha = \beta + 2\gamma$. Choosing the signs in the Chevalley commutator formula as above, we can write $x_\alpha(-\xi) = [x_\gamma(1), x_\beta(\xi)]x_{\beta+\gamma}(\xi)$. Plugging this into the expression for x, we get

$$x = x_\alpha(\xi) \cdot {}^{z_\alpha(\zeta,\eta)}[x_\gamma(1), x_\beta(\xi)] \cdot {}^{z_\alpha(\zeta,\eta)}x_{\beta+\gamma}(\xi).$$

As we know from the Lemma,

$${}^{z_\alpha(\zeta,\eta)}x_{\beta+\gamma}(\xi) \equiv x_{\beta+\gamma}(\xi) \pmod{E(\mathrm{C}_2, IJ)},$$

so that ${}^{z_\alpha(\zeta,\eta)}x_{\beta+\gamma}(\xi)$ can be rewritten in the form $x_{\beta+\gamma}(\xi)z$, for some $E(\mathrm{C}_2, IJ)$.

Next, we look at the second factor. Clearly,

$$y = {}^{z_\alpha(\zeta,\eta)}[x_\gamma(1), x_\beta(\xi)] = [{}^{z_\alpha(\zeta,\eta)}x_\gamma(1), {}^{z_\alpha(\zeta,\eta)}x_\beta(\xi)] = [{}^{z_\alpha(\zeta,\eta)}x_\gamma(1), x_\beta(\xi)].$$

As we know from the Lemma, ${}^{z_\alpha(\zeta,\eta)}x_\gamma(1) \equiv x_\gamma(1) \pmod{E(\mathrm{C}_2, J)}$. Rewriting ${}^{z_\alpha(\zeta,\eta)}x_\gamma(1)$ in the form ${}^{z_\alpha(\zeta,\eta)}x_\gamma(1) = x_\gamma(1)w$, for some $w \in E(\mathrm{C}_2, J)$, we get

$$y = [x_\gamma(1)w, x_\beta(\xi)] = {}^{x_\gamma(1)}[w, x_\beta(\xi)] \cdot [x_\gamma(1), x_\beta(\xi)],$$

where the first commutator belongs to $[E(\mathrm{C}_2, I), E(\mathrm{C}_2, J)]$ and stays there after elementary conjugation.

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Combining the above, and expanding $[x_\gamma(1), x_\beta(\xi)]$ by the Chevalley commutator formula, we see that

$$x = x_\alpha(\xi) \cdot {}^{x_\gamma(1)}[x_\beta(\xi), w] \cdot x_\alpha(-\xi)x_{\beta+\gamma}(-\xi) \cdot x_{\beta+\gamma}(\xi)z \in [E(\mathrm{C}_2, I), E(\mathrm{C}_2, J)],$$

as claimed.

• Next, assume that α can be embedded in C_2 as a *short* root. Choose β and γ such that $\alpha = \beta + \gamma$, while $N_{\beta\gamma 11} = N_{\beta\gamma 12} = 1$. Then, clearly, $x_\alpha(-\xi)$ can be expressed as $x_\alpha(-\xi) = [x_\gamma(1), x_\beta(\xi)]x_{\beta+2\gamma}(\xi)$. Thus,

$$x = x_\alpha(\xi) \cdot {}^{z_\alpha(\zeta,\eta)}[x_\gamma(1), x_\beta(\xi)] \cdot {}^{z_\alpha(\zeta,\eta)}x_{\beta+2\gamma}(\xi).$$

Again by the Lemma ${}^{z_\alpha(\zeta,\eta)}x_{\beta+2\gamma}(\xi) = x_{\beta+2\gamma}(\xi)z$ for some $z \in E(\mathrm{C}_2, IJ)$.

Looking at the second factor, we see that

$$y = {}^{z_\alpha(\zeta,\eta)}[x_\gamma(1), x_\beta(\xi)] = [{}^{z_\alpha(\zeta,\eta)}x_\gamma(1), {}^{z_\alpha(\zeta,\eta)}x_\beta(\xi)].$$

The tail consisting of the remaining factors $x_\delta(\eta)$ from the above instances of the Chevalley commutator formula, which in any case belong to $E(\Phi, IJ)$.

• This leaves us with the analysis of the case when α is a *short* root of $\Phi = \mathrm{G}_2$. Choose a long root β and a short root γ such that $\alpha = \beta + \gamma$, and the structure constants are as above, $N_{\beta\gamma 11} = N_{\beta\gamma 12} = N_{\beta\gamma 13} = 1$, $N_{\beta\gamma 23} = 2$. Then $x_\alpha(-\xi)$ can be expressed as:

$$x_\alpha(-\xi) = [x_\gamma(1), x_\beta(\xi)] \cdot x_{\beta+2\gamma}(\xi)x_{\beta+3\gamma}(\xi)x_{2\beta+3\gamma}(2\xi^2).$$

Plugging this into the expression for x, we get

$$x = x_\alpha(\xi) \cdot {}^{z_\alpha(\zeta,\eta)}[x_\gamma(1), x_\beta(\xi)] \cdot {}^{z_\alpha(\zeta,\eta)}(x_{\beta+2\gamma}(\xi)x_{\beta+3\gamma}(\xi)x_{2\beta+3\gamma}(2\xi^2)).$$

By the same token, we see that the last factor belongs to the unipotent radical of the parabolic subgroup P_α and, thus, by the Lemma can be rewritten as:

$${}^{z_\alpha(\zeta,\eta)}(x_{\beta+2\gamma}(\xi)x_{\beta+3\gamma}(\xi)x_{2\beta+3\gamma}(2\xi^2)) = x_{\beta+2\gamma}(\xi)x_{\beta+3\gamma}(\xi)x_{2\beta+3\gamma}(2\xi^2) \cdot z,$$

for some $z \in E(\mathrm{G}_2, IJ)$. Now, repeating exactly the same calculation as in the previous case, we see that the second factor in the above expression for x has the form

$${}^{x_\gamma(1)}[w, {}^{z_\alpha(\zeta,\eta)}x_\beta(\xi)] \cdot [x_\gamma(1), x_\beta(\xi)] \cdot {}^{x_\beta(\xi)}[x_\gamma(1), v],$$

for some $w \in E(\mathrm{G}_2, J)$ and $v \in E(\mathrm{G}_2, IJ)$.

Combining the above, and once more expanding $[x_\gamma(1), x_\beta(\xi)]$ by the Chevalley commutator formula, we see that

$$\begin{aligned} x = x_\alpha(\xi) \cdot {}^{x_\gamma(1)}[w, {}^{z_\alpha(\zeta,\eta)}x_\beta(\xi)] \cdot x_\alpha(-\xi)x_{\beta+2\gamma}(-\xi)x_{\beta+3\gamma}(-\xi)x_{2\beta+3\gamma}(-2\xi^2) \\ \cdot {}^{x_\beta(\xi)}[x_\gamma(1), v] \cdot x_{\beta+2\gamma}(\xi)x_{\beta+3\gamma}(\xi)x_{2\beta+3\gamma}(2\xi^2) \cdot z \end{aligned}$$

(recall that for the above choice of structure constants, $[x_a(\xi), x_{\beta+2\gamma}(\eta)] = x_{2\beta+3\gamma}(3\xi\eta)$, whereas root elements corresponding to the roots $\beta+2\gamma, \beta+3\gamma, 2\beta+3\gamma$ commute). Here, the first commutator belongs to $[E(\mathrm{G}_2, I), E(\mathrm{G}_2, J)]$ and stays there after elementary conjugation, and the second commutator belongs to $E(\mathrm{G}_2, R, IJ)$ and stays there after elementary conjugation, while the outstanding factor z already belongs to $E(\mathrm{G}_2, IJ)$, as claimed.

This completes the proof of the Main Lemma, and thus also of all other new results stated in the Introduction.

4. Generation of elementary subgroups by long root unipotents

In this section we prove another result pertaining to generation of relative elementary subgroups, closely related to the contents of [**14**] and the present paper. Namely, we prove that $E(\Phi, R, I)$ is generated by *long* root unipotents. There is no doubt that this result has been known for several decades and is immediately obvious to experts. However, we are not aware of any explicit source.

The purpose of including this result here is twofold. First, we need it for future reference in work by the first-named author on the width of root type unipotents in $\mathrm{Sp}(2l, R)$ and in $G(\mathrm{F}_4, R)$ with respect to the elementaries. Second, it would be very interesting to understand what this result means for the generation of relative commutator subgroups $[E(\Phi, R, I), E(\Phi, R, J)]$ and whether one could accordingly reduce their sets of generators obtained in [**14**, Theorem 1.3].

Theorem 4.1. *Let* $\mathrm{rk}(\Phi) \geq 2$*; for* $\Phi = \mathrm{G}_2$ *assume additionally that* R *does not have residue field* $\mathbb{F}_2$ *of two elements. Then for any ideal* $I \trianglelefteq R$ *the relative elementary group* $E(\Phi, R, I)$ *is generated by long root elements.*

Proof. For $\Phi = \mathrm{A}_l, \mathrm{D}_l, E_l$ there is nothing to prove. Thus, let $\Phi = \mathrm{B}_l, \mathrm{C}_l, F_4$ or G_2. It suffices to prove that any elementary short root element $x_\beta(\xi)$, where $\beta \in \Phi_s$ and $\xi \in I$, is a product of long root elements $x_1, \dots, x_m \in E(\Phi, R, I)$. If this is the case, then for any $g \in E(\Phi, R)$, its conjugate ${}^g x_\beta(\xi) = {}^g x_1 \cdot \ldots \cdot {}^g x_m$ is also a product of long root elements from $E(\Phi, R, I)$.

First, let $\Phi \neq \mathrm{G}_2$. Then there exists a long root α and a short root γ such that $\beta = \alpha + \gamma$. Then the root $\alpha + 2\gamma = \beta + \gamma$ is long, and, carrying the corresponding factor to the left-hand side in the commutator formula

$$[x_\alpha(\xi), x_\gamma(1)] = x_\beta(\pm\xi) x_{\beta+\gamma}(\pm\xi),$$

we express $x_\beta(\pm\xi)$ as the product of three long root unipotents

$$x_\beta(\pm\xi) = x_\alpha(\xi)(x_\gamma(1) x_\alpha(-\xi) x_\gamma(-1)) x_{\beta+\gamma}(\mp\xi),$$

sitting in $E(\Phi, R, I)$.

On the other hand, for the case $\Phi = \mathrm{G}_2$ there exists a long root α and a short root γ such that $\beta = \alpha + 2\gamma$. Then the root $\alpha + \gamma = \beta - \gamma$ is short, whereas the roots $\alpha + 3\gamma = \beta + \gamma$ and $2\alpha + 3\gamma = 2\beta - \gamma$ are both long. Plugging in the Chevalley commutator formula,

$$[x_\alpha(\eta), x_\gamma(\zeta)] = x_{\alpha+\gamma}(\pm\eta\zeta) x_\beta(\pm\eta\zeta^2) x_{\beta+\gamma}(\pm\eta\zeta^3) x_{2\beta-\gamma}(\pm\eta^2\zeta^3)$$

first with $\eta = \xi \in I$ and $\zeta = \theta \in R$ and then with $\eta = \xi\theta$ and $\zeta = 1$, for the same $\xi \in I$ and $\theta \in R$, and carrying over the factors corresponding to the long roots $\beta + \gamma$ and $2\beta - \gamma$ to the left-hand side of the resulting commutator formulas, we get the following expressions. First,

$$y = x_{\alpha+\gamma}(\pm\xi\theta)x_\beta(\pm\xi\theta^2) = x_\alpha(\xi)(x_\gamma(\theta)x_\alpha(-\xi)x_\gamma(-\theta))x_{\beta+\gamma}(\mp\xi\theta^3)x_{2\beta-\gamma}(\mp\xi^2\theta^3)$$

is a product of four long root unipotents sitting in $E(\Phi, R, I)$. Similarly,

$$z = x_{\alpha+\gamma}(\pm\xi\theta)x_\beta(\pm\xi\theta) = x_\alpha(\xi\theta)(x_\gamma(1)x_\alpha(-\xi\theta)x_\gamma(-1))x_{\beta+\gamma}(\mp\xi\theta)x_{2\beta-\gamma}(\mp\xi^2\theta^2)$$

is a product of four such long root unipotents. Comparing these equalities, we get an expression

$$x_\beta(\pm\xi(\theta^2 - \theta)) = yz^{-1}$$

as a product of not more than six long root elements from $E(\Phi, R, I)$. Since R does not have a residue field of two elements, the ideal generated by $\theta^2 - \theta$, where $\theta \in R$, is not contained in any maximal ideal and thus coincides with R. This means that $x_\beta(\xi)$ is a product of finitely many long root elements from $E(\Phi, R, I)$. □

5. Final remarks

The main results presented here were completely unexpected to us and to several other experts in the structure theory of algebraic groups over rings, with whom we discussed the subject of the present paper. Once more, these results highlight the relative commutator subgroups $[E(\Phi, R, I), E(\Phi, R, J)]$ as an ubiquitous class of subgroups that occur *surprisingly* often.

For the general linear group $\mathrm{GL}(n, R)$, these and other concomitant birelative groups were first considered in the seminal work of Hyman Bass [**2**] and then systematically studied by Alec Mason and Wilson Stothers [**17**–**20**]. At that stage, the standing premise was that $n \geq \mathrm{sr}(R) + 1$.

In [**36**, **37**] the first author and Alexei Stepanov observed that the standard commutator formula holds for arbitrary commutative rings, and in [**6**, **7**] Roozbeh Hazrat and the second author proposed an approach based on localization. As part of that approach, in the linear case they found generators of relative commutator groups, which was a starting point for the present work.

Later, we together with Roozbeh Hazrat generalized the relative and birelative versions of localization, the commutator formulas themselves, and results on generation of relative commutator groups to unitary groups [**9**, **15**] and to Chevalley groups [**12**, **14**]. Luckily, at that time we were not aware of the pioneering work by Hong You [**38**]; see the footnote on page 265 of [**12**].

These results were instrumental in the work by Alexei Stepanov on the commutator width of Chevalley groups, see [**25**, **26**, **28**]. See also [**1**, **8**] for other interesting occurrences of the above commutator subgroups in the theory of Chevalley groups. One can find many further related results, applications and open problems in our surveys and conference papers [**10**, **11**, **13**, **16**].

So far, we have not even mentioned another extremely important line of research, which initially was our main motivation to focus on relative commutator subgroups. Namely, the study of subgroups normalized by a relative elementary subgroup; see [**5**, **10**, **11**, **13**, **16**]

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for an outline and further references. We plan to return to this problem in the context of Chevalley groups in our next publication.

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