PERCOLATION PROCESSES

I. CRYSTALS AND MAZES

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ABSTRACT. The paper studies, in a general way, how the random properties of a 'medium' influence the percolation of a 'fluid' through it. The treatment differs from conventional diffusion theory, in which it is the random properties of the fluid that matter. Fluid and medium bear general interpretations: for example, solute diffusing through solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, disease infecting a community, etc.

1. *Introduction.* There are many physical phenomena in which a *fluid* spreads randomly through a *medium*. Here fluid and medium bear general interpretations: we may be concerned with a solute diffusing through a solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, or disease infecting a community. Besides the random mechanism, external forces may govern the process, as with water percolating through limestone under gravity. According to the nature of the problem, it may be natural to ascribe the random mechanism either to the fluid or to the medium. Most mathematical analyses are confined to the former alternative, for which we retain the usual name of *diffusion process*: in contrast, there is (as far as we know) little published work on the latter alternative, which we shall call a *percolation process*. The present paper is a preliminary exploration of percolation processes; and, although our conclusions are somewhat scanty, we hope we may encourage others to investigate this terrain, which has both pure mathematical fascinations and many practical applications.

Some examples will clarify the distinction between diffusion and percolation processes.

Example 1. The simplest example of a diffusion process is the one-dimensional Pólya walk. In this, a particle (the fluid) takes steps of unit length along a straight line (the medium) starting from the origin. After any number of steps, the particle has, independently of its previous history, equal probabilities (each $\frac{1}{2}$) of taking its next step to the right or to the left. As is well known, the position of the particle after n steps is then a linear transformation of a binomial variate and has a distribution with zero mean and variance n. When n is infinite, the particle visits every point of the medium infinitely often with probability 1.

Example 2. In the percolation process which is analogous to Example 1, fluid and medium are the same as before; but the stochastic mechanism resides in the medium rather than in the particle. Specifically, each point of the medium has, independently of the other points, equal probabilities (each $\frac{1}{2}$) of being a 'right-sense' or a 'left-sense' point. The particle starts from the origin and takes steps of unit length, the direction of any step being that of the sense of the point from which that step starts. Thus the state of the medium entirely determines the motion of the particle, which

moves steadily in one direction until it encounters successive points of opposite sense, whereupon it oscillates between them. The distribution of terminal position is nothing like binomial, and it has zero mean and variance $\frac{1}{18}(81-(-1)^n-(3n+5)(\frac{1}{2})^{n-4})$ after n steps. When n is infinite, there is probability 1 that the particle will visit only finitely many points.

Some physical situations may be regarded either as diffusion or as percolation processes, as in Example 3. Other situations may demand one model to the exclusion of the other.

Example 3. Suppose each individual in a branching (or cascade) process has, independently of the other individuals, respective probabilities q^2, $2pq$, p^2 of giving birth to 0, 1, 2 descendants in the next generation. We may visualize this as a diffusion process by thinking of a branching fluid advancing from generation to generation: any one branch of the fluid at any generation carries with it a random mechanism that decides whether it provides 0, 1 or 2 branches of fluid in the next generation. But equally, we may think of a system of channels leading from the original ancestor such that each channel divides into precisely two channels at each generation. Each of these channels has, independently of the other channels, a probability q of being dammed. This random set of dams in the channels (the medium) will determine how fluid introduced at the ancestor will spread; this description is now a percolation process. It is a well known result that, if $p \leqslant \frac{1}{2}$, only finitely many channels will be wetted by the fluid with probability 1, and that the corresponding probability of ultimate extinction is q^2/p^2 when p exceeds the critical value $\frac{1}{2}$. Critical probabilities play similar roles in more general percolation processes, as we shall see later.

Example 4. Gas molecules, adsorbed on the surface of a porous solid, move by surface diffusion through all pores large enough to admit them; the problem is to determine the proportion of the interior of the solid reached by gas molecules. In the percolation model we represent the solid (the medium) by a regular structure of interconnecting paths equivalent to the pore system, and assign to each path independently a probability p of being wide enough to allow passage to gas molecules (the fluid). We shall discover that when p is less than some critical value, there is effectively no adsorption of gas in the interior.

Example 5. The trees in a large orchard are to be planted at the intersections of a square lattice, and the spacing between nearest neighbours is to be determined so that, if a single tree becomes blighted, there is a negligible probability of the blight spreading from tree to tree and infecting a large number of trees in the orchard. Thus we must choose the spacing so that the probability p of infection between neighbouring trees is less than the critical value of a square lattice.

Generally, the intrinsic and the random characteristics of the medium, together with any external laws which may operate, completely determine the progress of the fluid. The intrinsic characteristics of the medium consist in its interconnecting structure, for instance, the interconnecting structure of pores in Example 4 or the square lattice in Example 5. We shall formulate this structure in abstract terms, since we do not wish to limit the discussion to particular physical situations. We have in mind

a structure, to be called an *abstract crystal*, which is homogeneous in the large though it may possess local variations. Thus the structure might be that of an edge-centred cubic atomic lattice, which is homogeneous in the large in the sense that all cells are alike, although it has local variations inasmuch as atoms at the centre of an edge have two nearest neighbours whereas atoms at the corner of a cube have six. All physical crystalline lattices will be special cases of abstract crystals, and so will be many other configurations (such as the branching process of Example 3). Despite this degree of generality, there is in our definition of abstract crystals a preoccupation with a certain strict regularity, the homogeneity in the large, which may prove restrictive in some practical applications. Hammersley (2) gave definitions for an abstract crystal in connexion with a Markovian diffusion process. As we are now dealing with percolation processes, we employ somewhat different definitions, self contained and quite independent of (2). They are more general in being unbound by metric considerations, but less general in being subject to rather stricter interconnexion postulates.

The random characteristics of the medium are introduced by randomly damming some of its connexions. The resultant system will be called a *random maze*. Fluid supplied at various points flows along all the undammed paths. Generally, we are concerned with the spread of this fluid, which is determined by the *permeability* of the medium. These vague terms will be made more precise later.

The fluid will be able to flow from one point to another if and only if there is a connexion without dams between them, and this will be so if and only if there is an undammed *self-avoiding* walk connecting them (i.e. a walk which visits no intermediate point more than once). It is, therefore, appropriate to study the self-avoiding walks in crystals. Hammersley and Morton (3) discussed these walks on a particular crystal: more general results are stated in the next section.

In the third section the permeability of a *random maze* is considered from two aspects: the probability that liquid spreading from a single point will wet only a finite number of other points, and the probability that a given point in the interior of the medium will be wet when liquid is introduced at a boundary. These two aspects, and the self-avoiding walks in the medium, are all inter-related.

2. *The connective constant of a crystal.* We deal with abstract objects called *atoms* and *bonds* (which in other language are the nodes and loops of an oriented linear graph). A bond is a path between two atoms, and may either be two-way or may permit a walk in one direction only. An *n-stepped walk* is an ordered connected path along n bonds, each step being in a permitted direction along its bond and starting from the atom reached by the previous step. Two walks are *distinct* if with due regard to the order of their steps they do not traverse the same bonds. A walk is *self-avoiding* if it visits no atom more than once. Notice that distinctness is in terms of bonds and self-avoidance in terms of atoms. Let $S_n(A)$ denote an n-stepped self-avoiding walk starting from the atom A. Two atoms A and B are *outlike* if for each n the number of distinct $S_n(A)$ equals the number of distinct $S_n(B)$ ('out' emphasizes that the number depends primarily on the number of bonds whose direction is *away* from each atom). An *outlike class* is a (finite, countable or uncountable) set of pairwise outlike atoms.

A crystal is an infinite set of atoms and bonds satisfying the three postulates:

P 1: Each atom of the crystal belongs to just one of a *finite* number of outlike classes, denoted by $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$.

P 2: The number of bonds leading *from* (but not necessarily *to*) any atom of the crystal is finite.

P 3: If a subset of atoms *either* (*a*) contains only finitely many atoms, *or* (*b*) does not contain any atoms of at least one outlike class, then this subset contains an atom from which a bond leads to some atom not in the subset.

Of these postulates, P 3 most essentially characterizes a crystal; it ensures no outlike class is isolated and averts finite cul-de-sacs from which no walk can escape. P 1 and P 2 avoid theoretical complications unlikely to arise in practice; without them Theorem 1 would be false, as shown in Examples 8 and 9.

Theorem 1 below shows that a *connective constant*, κ, which gives information on the number of distinct long self-avoiding walks, can be meaningfully defined. In a certain sense κ measures the richness of the connexions in a crystal. The proof of Theorem 1 will be published separately (4), since the mathematics is not immediately relevant to the stochastic processes which are the subject of this paper.

Let $f_A(n, r)$ denote the number of distinct n-stepped walks starting from A, each of which can be broken into r or less self-avoiding walks. In the special case $r = 1$ these walks are themselves self-avoiding, and P 1 and the definitions of an outlike class permit the definition

$$f_i(n) = f_A(n, 1) \quad (A \in \Gamma_i,\ i = 1, 2, \ldots, k).$$

We define

$$\phi(n) = (1/n) \log \max_{1 \leqslant i \leqslant k} f_i(n).$$

Then for a given crystal there corresponds a connective constant κ defined by

$$\kappa = \inf_{n \geqslant 1} \phi(n).$$

THEOREM 1. *For any atom A of a crystal with connective constant κ,*

$$0 \leqslant \kappa = \inf_{n \geqslant 1} \phi(n) = \lim_{n \to \infty} (1/n) \log f_A(n, r(n)) < \infty,$$

provided that

$$\lim_{n \to \infty} r(n)/n = 0.$$

Hammersley and Morton (3) proved a special case of Theorem 1 with $r = k = 1$ and gave a Monte Carlo method of estimating κ. In the more general case, the same Monte Carlo method of estimating κ holds with only trivial modifications; known analytical bounds for κ are less informative than such Monte Carlo estimates.

The special case $r(n) = 1$ is sufficiently important to warrant rephrasing:

COROLLARY 1. *The number of distinct n-stepped self-avoiding walks starting from any given atom of a crystal with connective constant κ is*

$$e^{\kappa n + o(n)}$$

as $n \to \infty$.

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Example 6. Suppose a crystal consists of atoms numbered i ($i = \ldots, -1, 0, 1, \ldots$) and there are M ($1 \leqslant M < \infty$) one-way bonds from atom i to atom $(i+1)$. Then κ is clearly $\log M$.

Example 7. Suppose each atom in a branching process has M direct descendant atoms to each of which there is a one-way bond. Here κ is again $\log M$; the number of distinct $S_n(A)$ is the same as in Example 6 for all n and A.

These values of κ will be compared later with measures of the permeability of these simple crystals. Meanwhile, two further examples show that Theorem 1 is not true unless P 1 and P 2 are satisfied. Clearly P 3 must be satisfied if Theorem 1 is to hold; for example, we might otherwise call two separate crystals with different κ a single crystal.

Example 8. Suppose the atoms i ($i = 1, 2, \ldots$) are joined by one one-way bond from i to $(i+1)$ if the integer part of $\log_2 i$ is even, and by two one-way bonds if the integer part is odd. No two atoms are outlike and so P 1 does not hold. Further,

$$\liminf_{n\to\infty} (1/n)\log f_i(n) = \tfrac{1}{3}\log 2, \quad \limsup_{n\to\infty} (1/n)\log f_i(n) = \tfrac{2}{3}\log 2.$$

Hence, $\lim_{n\to\infty} (1/n)\log f_i(n)$ does not exist for any i.

Example 9. Suppose the atoms are the points $(2i, j)$ and $(2i-1, 0)$ in the Euclidean plane ($i, j = 0, 1, 2, \ldots$). There is a one-way bond from every atom on the x axis to its right-hand neighbour on the x axis; every atom not on the x axis is joined by a two-way bond to the atom on the x axis with the same x coordinate. There are two outlike classes: Γ_1 consists of all atoms of the form $(2i, 0)$ and Γ_2 of the remaining atoms. P 2 is violated at every atom of Γ_1.

$$f_i(n) = \infty \quad \text{if } n \equiv i \pmod 2,$$
$$f_i(n) = 1 \quad \text{otherwise.}$$

Hence, $\lim_{n\to\infty} (1/n)\log f_i(n)$ does not exist for any i.

3. *Random mazes.* 3·1. Suppose that in an infinite set of atoms joined by bonds some (or all) of the bonds are *dammed* in a random manner. Fluid is supplied to a (finite, countable or uncountable) subset of atoms called *source atoms*, and then percolates the set in the following way. An atom of the set is said to be *wet* by the fluid either if it is a source atom or if there exists a walk to the atom from a source atom, the walk traversing undammed bonds only and in the permitted directions. All atoms not wet are said to be *dry*. We are interested in the properties of the wet atoms, and these naturally depend on the structure and connexions of the given set, on the manner in which bonds are dammed, and on the source atoms.

In particular, there are two questions we discuss below. When there is only one source atom, is the number of wet atoms finite or infinite? When the source atoms form a boundary round a large subset of the maze (this idea is made precise below) what proportion of atoms in the subset are wet? The definition of proportion involves some limiting operation; in particular cases it will usually be clear what limiting operation is appropriate. These two questions we show to be related, and the

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permeability, which the answers to these questions define, is also related to the connective constant κ of § 2.

We shall restrict the very general medium described above. A *random maze* or, more briefly, a *maze* is such a medium which satisfies the two postulates P 4 and P 5 below. However, before enunciating these two postulates, we remark that some crystals have the property that, when the direction of each bond in the crystal is reversed, the

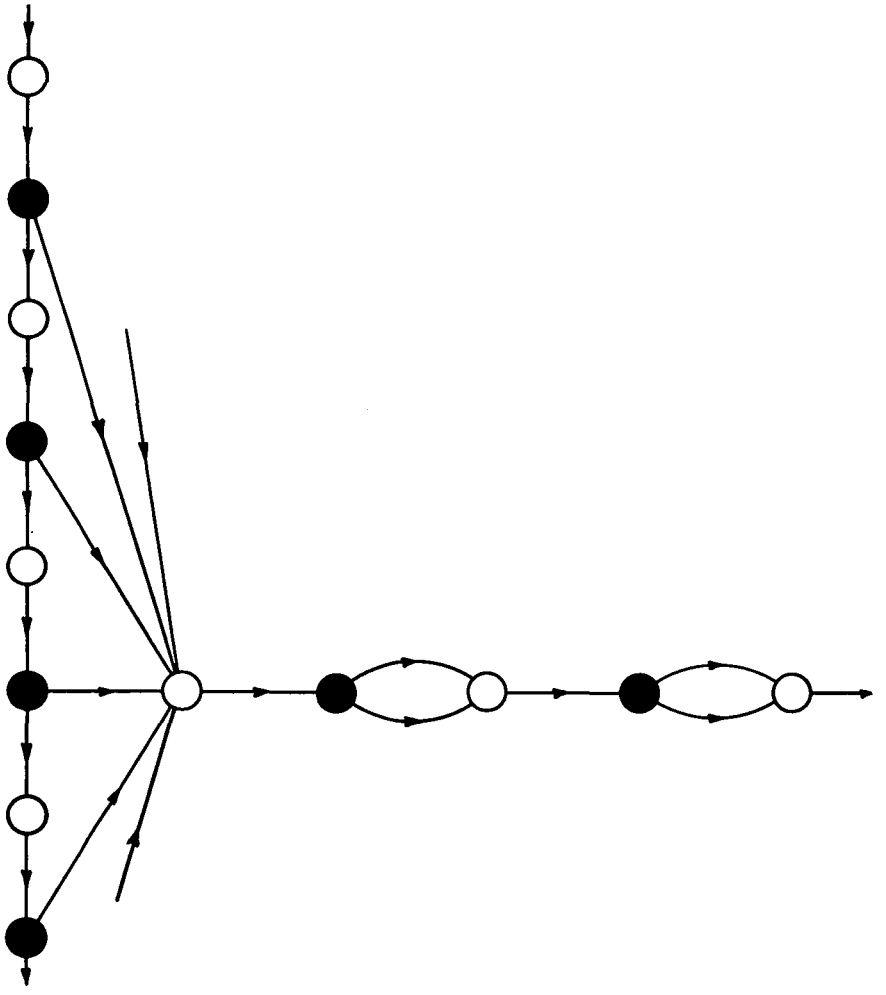

Fig. 1

resulting set of atoms and bonds is also a crystal. A crystal with this property is called *reversible*, and the resulting crystal is called the *reverse crystal*. Example 6 is a reversible crystal. The crystal in Example 10 is not reversible, since the reverse set of atoms and bonds violates all three postulates P 1, P 2 and P 3.

Example 10 (see Fig. 1). Atoms at the integral points of the x and y axes of the Euclidean plane are joined by one-way bonds as follows: $(2i-1, 0)$ is joined to $(2i, 0)$ by one bond, and $(2i, 0)$ to $(2i+1, 0)$ by two bonds ($i = 1, 2, \ldots$); $(0, 2i)$ is joined to $(0, 2i-1)$

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by one bond and to $(1, 0)$ by another ($i = 0, \pm 1, \ldots$); $(0, 2i+1)$ is joined to $(0, 2i)$ by one bond. This crystal has two outlike classes, atoms with even coordinates and atoms with an odd coordinate. It will also be noticed that each atom $(i, 0)$ ($i = 1, 2, \ldots$) has the following property, that there is no atom in the crystal from which the shortest walk to the atom $(i, 0)$ has $(i+2)$ or more steps.

Postulates P 4 and P 5 are as follows:

P 4. The set of atoms and bonds from which a maze is derived constitute a reversible crystal.

P 5. Each bond of a maze has, independently of all other bonds, a fixed probability $q = 1-p$ of being dammed.

A *reverse maze* is the maze obtained by reversing the directions of all bonds in the original maze, leaving dammed bonds dammed during reversal and undammed undammed. Atoms in the reverse crystal or maze are denoted by primes; e.g. an atom A in the original crystal is denoted by A' in the reverse. Thus there will be an undammed walk in the reverse maze from A' to B' if and only if there is an undammed walk from B to A in the original maze.

At first sight it may seem that P 5 is a severe limit on the use of a random maze in applications, since only one probability of damming a bond is allowed. However, suitable choice of the number of bonds between atoms and of p will always be possible in order to approximate as closely as desired to any finite number of distinct probabilities for percolation between pairs of atoms.

3·2. Let $d(A, p)$ denote the probability that only a finite number of atoms are wet when A is the only source atom in a random maze. This has the value 1 at $p = 0$ and is a monotone decreasing function of p; we define the *critical probability* $p_d(A)$ by

$$p_d(A) = \sup_{d(A,p)=1} p.$$

THEOREM 2. *If there is a finite walk from A to B and another (not necessarily distinct) from B to A in a crystal then in the corresponding maze $p_d(A) = p_d(B)$.*

Proof. Suppose a shortest walk from A to B has n steps. Being shortest, it is self avoiding. The probability that this walk is undammed in the maze is p^n, and so

$$1 - d(A, p) \geqslant p^n[1 - d(B, p)].$$

Hence $d(A, p) = 1$ implies $d(B, p) = 1$ and $p_d(A) \leqslant p_d(B)$. The reversed inequality may be proved similarly and the theorem follows.

In a large class of crystals, every atom is connected by at least one walk to every other atom, and such a crystal is called *interconnected*. We derive immediately

COROLLARY 2. *$p_d(A)$ is independent of A when the maze is derived from an interconnected crystal.*

In some crystals $d(A, p)$ can be calculated by elementary methods. The reader will find it easy to show in Example 6 that $d(A, p) = 1$ for all $p < 1$, and hence that $p_d(A) = 1$, where A is any atom of the crystal. In Example 7 the total number of descendants from any atom A in n generations is $(1 - M^{n+1})/(1 - M)$. When A is a source atom in the random maze the expected total of wet atoms in n generations is

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$[1-(Mp)^{n+1}]/[1-Mp]$. It is well known (see, for example, Good (1)) that $d(A,p)$ is 1 for $p \leqslant 1/M$, and for larger p is the smaller positive root x of $(q+px)^M = x$. It may be noted that $d(A,p)$ is 1 when $p = 1/M$ although the expected total of wet atoms is infinite; also that the expected *proportion* of the descendants wet in n generations tends to zero as $n \to \infty$, for all $p < 1$, even though the expected *number* wet may be infinite. Similar assertions can be made when there are N one-way bonds joining each atom to each of its M descendants.

Example 11. Consider the crystal whose atoms are the points with integer co-ordinates in the Euclidean plane. The atom (x,y) is joined by two-way bonds to its neighbours $(x-1,y)$ and $(x+1,y)$ in the same layer (i.e. with the same y coordinate). It is also joined by one-way bonds *from* one of the atoms with y coordinate $(y+1)$ and *to* one of the atoms with y coordinate $(y-1)$. The latter bonds form a one-one correspondence between the atoms of adjacent layers. Let this be a random correspondence in the sense recently defined by Rényi, i.e. such that, given any finite subset of one layer and any finite subset in an adjacent layer, every atom of the first subset is (conditionally) equally likely to correspond with any atom of the second. From such a crystal we obtain a maze called a *redistribution maze*, since the effect of the random correspondence is as if any wet atoms in a layer were distributed at random before the liquid percolates to the next layer. This ensures a certain independence between neighbours, and allows us to obtain specific solutions. These solutions, besides exemplifying our theorems, may be used to provide bounds for other plane Euclidean mazes.

Let there be a single source atom. The probability that the liquid wets just r further atoms in the same layer $(r = 0, 1, \ldots)$ and then s atoms in the following layer $(s = 0, 1, \ldots, r+1)$ is

$$(r+1)\,p^r q^2 \binom{r+1}{s} p^s q^{r+1-s};$$

and hence the probability generating function for the number wet in the second layer (before any spreading in this layer occurs) is

$$\begin{aligned} F(x) &= q \sum_{r=0}^{\infty} (r+1)\,p^r \sum_{s=0}^{r+1} \binom{r+1}{s} p^s q^{r+1-s} x^s \\ &= q^2(q+px)/(q+p^2-p^2x)^2. \end{aligned} \tag{1}$$

With probability one, the atoms wet in the second layer will be arbitrarily far apart (due to the random correspondence between layers) and so will behave like s isolated sources. Hence, with probability one the percolation process behaves like a branching process with probability generating function (1). Using Steffensen's theorem (Good (1)) we deduce from (1) that

$$d(A,p) = \begin{cases} \frac{1}{2} + qp^{-2} - (\frac{1}{4} + qp^{-3})^{\frac{1}{2}} & (p > \sqrt{2}-1), \\ 1 & (p \leqslant \sqrt{2}-1). \end{cases}$$

3·3. Let the *n-set to A* denote the set of all atoms in a crystal from which the shortest walk to the atom A has just n steps. We have already seen (Example 10) that an n-set to A may be an empty set. However, for a reversible crystal this set has at least one member for each n. In the corresponding maze let $w(A,n,p)$ denote the

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probability that A is wet when every atom in the n-set to A is a source atom and no other atom in the maze is a source atom. This is a monotone decreasing function of n and, since it is bounded, tends to a *limiting function* $w(A, p)$ as $n \to \infty$. This has the value 0 at $p = 0$ and is a monotone increasing function of p; we define

$$p_w(A) = \inf_{w(A,p)=0} p.$$

This quantity $p_w(A)$ is a critical probability, just as $p_d(A)$ was. The precise connexion between these two critical probabilities will emerge in Theorem 5.

Corresponding to Theorem 2, we state (deferring the proof until later)

THEOREM 3. *If there is a finite walk from A to B and another (not necessarily distinct) from B to A in a crystal, then in the corresponding maze $p_w(A) = p_w(B)$.*

We say that two atoms A and B of a crystal are *inlike* if the probability that A is wet when all atoms of the n-set to A are source atoms equals the corresponding probability for B, for each $n = 1, 2, \ldots$; the probability in this definition is to be taken as zero whenever an n-set to A or to B is empty. When the crystal is reversible, so that a maze may be derived from it, the definition of inlikeness amounts to

$$w(A, n, p) = w(B, n, p) \quad (n = 1, 2, \ldots).$$

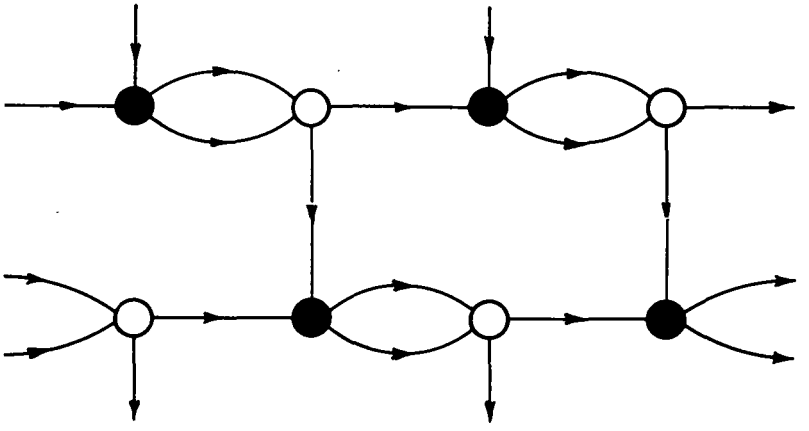

Fig. 2

An *inlike class* is a (finite, countable or uncountable) set of atoms every pair of which are inlike. In Example 10 each atom on the y axis belongs to a single inlike class (but to one of two outlike classes); no two atoms on the x axis are inlike. This example shows that inlike atoms are not necessarily outlike and vice versa. However, Example 10 is not a reversible crystal, and since outlikeness depends primarily on paths from atoms, and inlikeness on paths to atoms, it is tempting to conjecture that outlike atoms are inlike in the reverse (this is false, as Example 12 shows), or that inlike atoms are outlike in reverse. The latter we have been unable to prove or disprove.

Example 12. The integer points in the Euclidean plane are joined by one-way bonds as shown (Fig. 2). There is only one outlike class (from every atom there are two possible steps and every walk is self avoiding). But $w(A, 2, p)$ is larger when A is a black

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atom than when A is a white atom. In the reverse black and white change properties. A black and a white atom are therefore outlike, but not inlike in the reverse.

3·4. Let $\mathfrak{B}$ be a given set of atoms in a crystal. The atom A is called an *enclosed atom* of $\mathfrak{B}$ if there exists an integer N, perhaps depending on A and $\mathfrak{B}$, such that every $S_N(A)$ passes through at least one atom of $\mathfrak{B}$. The *interior of* $\mathfrak{B}$ is defined to be the (perhaps empty) set of all enclosed atoms of $\mathfrak{B}$ which do not belong to $\mathfrak{B}$. A set of atoms with a non-empty interior is called a *boundary* of that interior. An atom A is said to be n steps away from a set of atoms $\mathfrak{B}$ if some $S_n(A)$, but no $S_{n-1}(A)$, contains an atom of $\mathfrak{B}$.

THEOREM 4. *If A is an atom in the interior of a boundary $\mathfrak{B}$, and if A is at least n steps away from $\mathfrak{B}$, then the probability that A' becomes wet in the reverse maze having sources throughout $\mathfrak{B}'$ lies between $w(A,p)$ and $w(A,n,p)$, both inclusive.*

Proof. Since A and $\mathfrak{B}$ are fixed, there exists a fixed integer N such that the N-set to A' encloses $\mathfrak{B}'$, which in turn encloses the n-set to A'. Thus, if P is the probability that A' is wet from $\mathfrak{B}'$,

$$w(A,p) \leqslant w(A,N,p) \leqslant P \leqslant w(A,n,p).$$

Theorem 4 has the following practical significance. Suppose that a lump of porous material is placed in a bucket of water, and we wish to know how much of the interior becomes wet. Suppose that we can represent the pore system of the material as a maze, water (i.e. the fluid) being able to percolate along sufficiently large pores (i.e. undammed bonds) to the interstices (i.e. atoms) of the interior. The physical boundary of the lump becomes a boundary of source atoms. In ordinary circumstances, the pore system will be microscopic, and the boundary will not be pathologically irregular; so that we may say that there are a large number of atoms in the interior and that most of them lie a large number of steps away from the boundary. Such atoms will have probabilities of being wet almost equal to $w(A,p)$, by Theorem 4. If there are only a small number of inlike classes in the maze, there will only be the same small number of different values of $w(A,p)$; and (assuming a more or less equitable distribution of atoms in these classes) there will be a large number of atoms in each class, and hence (with probability effectively one), the proportion of wet atoms will be effectively the value of $w(A,p)$ for that class.

The foregoing paragraph is in vague terms because its detailed application will depend upon the physical circumstances of any particular problem. Had we attempted to frame a rigorous general description, we should have encountered difficulties: for instance, some boundary surfaces are so irregular that almost all interior atoms are near the boundary; and, worse still, certain crystals are so pathological that all their finite boundaries are irregular. Thus, Example 13 describes a crystal in which more than 99·99 % of the interior atoms of any finite boundary are only a single step from that boundary, no matter what the size and shape of that boundary may be.

Example 13. Let M be a fixed integer not less than 10^4. Consider the crystal which represents a branching process (infinite in both directions of time) such that every atom has just one parent and just M children. All bonds are one way from parent to child.

All atoms belong to a single inlike and a single outlike class. Let $\mathfrak{B}$ be any finite boundary in this crystal. A little reflexion shows that, for this crystal, the interior of $\mathfrak{B}$ must also be finite. Number the successive generations of the crystal $i = \ldots, -1, 0, 1, 2, \ldots$ such that the generations $i = 1$ and $i = m$ respectively, contain an earliest and a latest atom of the union of $\mathfrak{B}$ with its interior. In generation i, let there be just b_i members of $\mathfrak{B}$ and c_i members of the interior of $\mathfrak{B}$. We have

$$b_{i+1} + c_{i+1} \geqslant Mc_i \quad (i = 1, 2, \ldots, m-1); \qquad c_m = 0;$$

for if these conditions are not satisfied there will be a walk which escapes from some interior atom to the indefinite future without passing through an atom of $\mathfrak{B}$. Writing $B = b_1 + b_2 + \ldots + b_m$ and $C = c_1 + c_2 + \ldots + c_{m-1}$ for the total number of atoms in $\mathfrak{B}$ and the interior of $\mathfrak{B}$ respectively, we find, by adding all the above inequalities to the inequality $b_1 + c_1 > 0$, that $B > (M-1)C$. Now let D denote the total number of atoms in the interior of $\mathfrak{B}$ which are only one step from $\mathfrak{B}$. Since each atom has only M children, we have $MD \geqslant B > (M-1)C$. Thus $D/C > 1 - M^{-1} \geqslant 0{\cdot}9999$; and this is the required result.

3·5. We now return to Example 11. Suppose that in the redistribution maze a proportion $w_i > 0$ of the atoms with y coordinate i are source atoms, distributed within this layer in any manner. Consider any atom in layer $(i+1)$, say $(0, i+1)$. The probability this atom is wet directly from a source atom is $w_i p$. The probability that $(r, i+1)$ $(r = 1, 2, \ldots)$ is the first atom to the right of this atom to be wet directly from a source atom, and that liquid reaches this atom via $(r, i+1)$ is

$$w_i p (1 - w_i p)^{r-1} p^r.$$

Hence, the probability that $(0, i+1)$ is *not* wet via some atom on the right is

$$1 - w_i \sum_{r=1}^{\infty} (1 - w_i p)^{r-1} p^{r+1} = q/(q + w_i p^2).$$

Therefore, the probability that any atom in layer $(i+1)$ is wet (*either* directly from layer i *or* indirectly from the right *or* indirectly from the left *or* by more than one of these ways) is

$$1 - (1 - w_i p)\,[q/(q + w_i p^2)]^2 = w_i p (1 - p^2 + w_i p^3)/(q + w_i p^2)^2.$$

The last expression may be written w_{i+1}, the expected proportion wet in layer $(i+1)$, and we therefore have a recurrence relation describing the fall of liquid through successive layers of the redistribution maze. Since $w_i > 0$ it is easy to deduce that w_i tends monotonely to a limit as $i \to \infty$, say to $w(p)$ given by

$$w(p) = \begin{cases} \frac{1}{2} - qp^{-2} + (\frac{1}{4} + qp^{-3})^{\frac{1}{2}} & (p > \sqrt{2} - 1), \\ 0 & (p \leqslant \sqrt{2} - 1). \end{cases}$$

In particular, this is the probability that an atom is wet when every atom in the layer infinitely far above it is a source atom. It is, therefore, the w function previously defined, since clearly every atom in the maze is inlike.

It will be noted from § 3·2 that in the redistribution maze $d(A, p) = 1 - w(p)$. This result is generally true and will be proved in the next section.

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3·6. THEOREM 5. *$w(A,p)+d(A',p)=1$; and hence $d(A',p)$ is invariant for all A in the same inlike class.*

Proof. Let E'_n denote the event that A' in the reverse maze wets at least n atoms when A' is the only source. Similarly, let E'_∞ denote the event that A' wets infinitely many atoms. In the probability space concerned, E'_n is a non-increasing sequence of sets with E'_∞ as the limit set; so that

$$\lim_{n\to\infty} \operatorname{prob} E'_n = \operatorname{prob} E'_\infty = 1-d(A',p),$$

the last step following from the definition of $d(A',p)$.

Suppose now that A is wet when all atoms of the n-set to A are the only source atoms. By definition, this event (call it E_n) has probability $w(A,n,p)$. When E_n occurs there is at least one undammed self-avoiding walk of n steps from the n-set to A; and hence, in the reverse, A' wets all atoms on this path. Thus E_n implies E'_n. Consequently

$$w(A,n,p) \leqslant \operatorname{prob} E'_n.$$

Since $w(A,n,p)$ is a non-increasing function of n, we deduce

$$w(A,p) \leqslant \operatorname{prob} E'_n.$$

Letting $n\to\infty$ in this last relation, we deduce

$$w(A,p) \leqslant 1-d(A',p).$$

On the other hand, suppose that E'_∞ occurs. Then, in the original maze E_n occurs for every given n. Hence,

$$1-d(A',p) = \operatorname{prob} E_\infty \leqslant \operatorname{prob} E_n = w(A,n,p).$$

Letting $n\to\infty$, we have

$$1-d(A',p) \leqslant w(A,p);$$

and Theorem 5 follows.

THEOREM 6. *$p_w(A) = p_d(A') = p_0(A)$, say.*

This follows immediately from Theorem 5, and enables us to write the common critical probability $p_0(A)$. Theorem 3 now follows from Theorem 2.

Theorem 5 also enables us to think of $d(A,p)$ as the probability that, in the reverse, when the atoms of the n-set to A' are source atoms, A' is *dry* when $n\to\infty$, whereas $w(A',p)$ is the probability A' is *wet*.

3·7. Our discussion of permeability is now concluded. Theorem 5 is our main result, and shows how percolation *from* an atom is related to percolation *to* the atom in the reverse. In applications we have the choice of calculating $d(A,p)$ by studying the liquid spreading from a single atom, or $w(A,p)$ by choice of some suitable boundary system.

Finally, we relate our results to § 2 in which the connective constant of a crystal is defined. We first show that the constant gives a lower bound to the critical probability.

THEOREM 7. *$p_0(A') \geqslant e^{-\kappa}$.*

Proof. Write $p(j,n)$ for the probability that just j of the $S_n(A)$ are undammed. If infinitely many atoms are wet from the source atom A, then at least one of these walks is undammed for all n, and

$$0 \leqslant 1-d(A,p) \leqslant \sum_{j\geqslant 1} p(j,n) \leqslant \sum_{=0}^{\infty} jp(j,n).$$

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The last expression is the expected number of distinct undammed $S_n(A)$ and equals

$$p^n f_i(n), \quad (A \in \Gamma_i).$$

Hence
$$0 \leqslant 1 - d(A,p) \leqslant p^n f_i(n).$$

When $p < e^{-\kappa}$, and $n \to \infty$, the last expression tends to zero by Theorem 1 and it follows $p_d(A) \geqslant e^{-\kappa}$. Theorem 7 follows from Theorem 6.

We have been unable to decide whether $d(A, e^{-\kappa})$ is always 1.

Theorem 7 has been verified in Examples 6 and 7 (§§ 2, 3·2). In Example 6 we found $p_d(A) \approx 1$ and $e^{-\kappa} = 1/M$ $(1 \leqslant M < \infty)$. Thus for $M > 2$ we have strict inequality in Theorem 7. In Example 7 we found that $p_d(A)$ and $e^{-\kappa}$ were each $1/M$, and equality in Theorem 7 is required. These examples also serve to prove:

THEOREM 8. *The permeability of a crystal* (*i.e.* $d(A,p)$ *and* $w(A,n,p)$) *cannot be deduced from its connectivity* (*i.e. the number of* $S_n(A)$).

Proof. The number of $S_n(A)$ for all n and all A are the same in Examples 6 and 7, but for $M \geqslant 2$ the values of $p_d(A)$ are different. *A fortiori* $d(A,p)$ and $w(A,n,p)$ are different.

As a converse to Theorem 8, Example 6 has $d(A,p) = 1$ and Example 13 has $w(A,n,p) = p^n$, in each case independently of the value of M. Thus knowledge of either one of these two functions does not provide knowledge of the connectivity. We do not, however, possess a counter-example on the effect of knowing *both* functions.

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