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Conditional mean dimension

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Abstract. We introduce some notions of conditional mean dimension for a factor map between two topological dynamical systems and discuss their properties. With the help of these notions, we obtain an inequality to estimate the mean dimension of an extension system. The conditional mean dimension for G-extensions is computed. We also exhibit some applications in dynamical embedding problems.

Key words: amenable group, conditional mean dimension, G-extension, dynamical embedding
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1. *Introduction*

Let Γ be a countable amenable group. By a *dynamical system* $\Gamma \curvearrowright X$, we mean a compact metrizable space X associated with a continuous action of Γ. Suppose $\Gamma \curvearrowright Y$ is another dynamical system and $\pi : X \to Y$ is a continuous Γ-equivariant surjective map, that is, a *factor map* between X and Y. In such a setting, we call $\Gamma \curvearrowright X$ an *extension system* and $\Gamma \curvearrowright Y$ a *factor system*.

The mean (topological) dimension is a dynamical invariant newly introduced by Gromov [**7**], which measures the average dimension information of dynamical systems based on the covering dimension for compact Hausdorff spaces. It plays a crucial role in the embedding problem of dynamical systems [**8–11**, **13**, **19**].

Since each fiber $\pi^{-1}(y)$ is a closed subset of the ambient system $\Gamma \curvearrowright X$, taking advantage of the ambient action, we can also discuss the mean dimension $\mathrm{mdim}(\pi^{-1}(y), \Gamma)$ for the fiber $\pi^{-1}(y)$ (see Definition 2.14). When computing the mean dimension of moduli spaces of Brody curves, Tsukamoto established an inequality to estimate the mean dimension of the extension system in terms of the mean dimension of the factor system [**25**, Theorem 4.6]. Based on this inequality, he asked the following question [**25**, Problem 4.8].

QUESTION 1.1. *For a factor map* $\pi : X \to Y$*, is it true that* $\mathrm{mdim}(X) \le \mathrm{mdim}(Y) + \sup_{y\in Y} \mathrm{mdim}(\pi^{-1}(y), \Gamma)$*?*

Observe that as Γ is trivial, the above inequality recovers as the classical Hurewicz formula [14, Theorem VI 7] [6, Theorem 1.12.4, 3.3.10]:

$$\dim(X) \le \dim(Y) + \sup_{y\in Y} \dim(\pi^{-1}(y)).$$

So Question 1.1 can be regarded as a dynamical concern of the Hurewicz inequality.

We can also consider Question 1.1 in parallel with entropy theory. Historically, for $\Gamma = \mathbb{Z}$, in [3], Bowen estimated the topological entropy $h(X)$ of $\Gamma \curvearrowright X$ in terms of the entropy of fibers $h(\pi^{-1}(y), \Gamma)$ for $y \in Y$, that is,

$$h(X) \le h(Y) + \sup_{y\in Y} h(\pi^{-1}(y), \Gamma).$$

This verifies a conjecture of [1, Conjecture 5] concerning the entropy of a skew product system. Later, some versions of conditional entropy $h(X|Y)$ relative to a factor $\Gamma \curvearrowright Y$ were introduced and the related variational principles were established [5, 20, 27]. It is shown that

$$h(X|Y) = \sup_{y\in Y} h(\pi^{-1}(y), \Gamma)$$

in the case $\Gamma = \mathbb{Z}$ [5] and in the general case Γ is amenable [27]. This brings us the third motivation to study Question 1.1.

Motivated from these points of view, we introduce some conditional versions of mean dimension relative to a factor system and study their properties. When the factor is trivial, these conditional mean dimensions recover as the mean dimension.

In §2, we first define the conditional mean (topological) dimension. We study a class of extensions, called *G-extensions*, which generalize the (topological) principal group extensions (Definition 2.6). A G-extension is based on another dynamical system $\Gamma \curvearrowright G$ (typically G is a compact metrizable group and the action is by automorphisms). It turns out the notion of conditional mean dimension brings us a proper notion to strengthen some results on embedding problems of dynamical systems in terms of the Rokhlin dimension of the factor systems. We present such applications in Theorem 2.11 and Corollary 2.13. Under certain conditions, Question 1.1 is confirmed.

In §3, in terms of the notion of conditional mean dimension, we prove an inequality to estimate the mean dimension of an extension system.

THEOREM 1.2. *For any factor map* $\pi : X \to Y$*, we have*

$$\mathrm{mdim}(X) \le \mathrm{mdim}(Y) + \mathrm{mdim}(X|Y).$$

The key technique of the proof is to approximate a sufficiently large Følners set by smaller Følner sets as in the proof of [25, Theorem 4.6]. In fact, we can slightly adjust the proof to obtain a more general inequality for a composition of two factor maps, or even the setting of the inverse limit of factor maps.

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We remind the reader that $\mathrm{mdim}(X|Y)$ is an upper bound of $\sup_{y\in Y} \mathrm{mdim}(\pi^{-1}(y), \Gamma)$ (see Proposition 2.16) and in some cases they coincide with each other. Thus, the above estimation can be thought of as a weak version of the inequality in Question 1.1 (see Proposition 2.17).

Note that as X is a product system $Y \times Z$ for some dynamical system $\Gamma \curvearrowright Z$ associated with the diagonal action, π is the projection map, then we have $\mathrm{mdim}(X|Y) = \mathrm{mdim}(Z)$ (see Proposition 2.4). This recovers as the subadditivity formula for Cartesian products of dynamical systems [**19**, Proposition 2.8].

As a cousin of mean dimension, Lindenstrauss and Weiss introduced the metric mean dimension as an upper bound of mean dimension [**19**]. This notion is a dynamical analogue of lower box dimension. In §4, we define the conditional version of the metric mean dimension. It is natural to ask whether the conditional metric mean dimension is an upper bound of conditional mean dimension (Question 4.4).

Downarowicz and Serafin introduced the topological fiber entropy given a measure on the factor system [**5**, Definition 8]. Motivated by this approach, we introduce an analogue for mean dimension. It turns out that this mean dimension given a measure serves as a lower bound for the conditional mean dimension (See Propositions 5.4 and 5.5).

2. *Conditional mean topological dimension*

In this section, we define the notion of conditional mean topological dimension, discuss its properties, and compute some examples.

Let us first recall some machinery of amenable groups in the preparation of defining dynamical invariants. For a countable group Γ, denote by $\mathcal{F}(\Gamma)$ the set of all non-empty finite subsets of Γ.

2.1. *Amenable groups.*

For each $K \in \mathcal{F}(\Gamma)$ and $\delta > 0$, denote by $\mathcal{B}(K, \delta)$ the set of all $F \in \mathcal{F}(\Gamma)$ satisfying $|\{t \in F : Kt \subseteq F\}| \geq (1-\delta)|F|$. Here, Γ is called *amenable* if $\mathcal{B}(K, \delta)$ is not empty for each pair (K, δ).

The collection of pairs (K, δ) forms a net Λ in the sense that $(K', \delta') \succeq (K, \delta)$ if $K' \supseteq K$ and $\delta' \leq \delta$. For a real-valued function φ defined on $\mathcal{F}(\Gamma) \cup \{\emptyset\}$, we say that *$\varphi(F)$ converges to $c \in \mathbb{R}$ when $F \in \mathcal{F}(\Gamma)$ becomes more and more invariant*, denoted by $\lim_F \varphi(F) = c$, if for any $\varepsilon > 0$ there is some $(K, \delta) \in \Lambda$ such that $|\varphi(F) - c| < \varepsilon$ for all $F \in \mathcal{B}(K, \delta)$. In general, $\overline{\lim}_F \varphi(F)$ is defined as

$$\overline{\lim_{F}}\, \varphi(F) := \lim_{(K,\delta)\in\Lambda} \sup_{F\in\mathcal{B}(K,\delta)} \varphi(F).$$

In the rest of this paper, Γ will always denote a countable amenable group. The following fundamental lemma, due to Ornstein and Weiss, is crucial to define dynamical invariants for amenable group actions [**19**, Theorem 6.1].

LEMMA 2.1. *Let $\varphi: \mathcal{F}(\Gamma) \to [0, +\infty)$ be a map satisfying:*

(1) $\varphi(Fs) = \varphi(F)$ *for all* $F \in \mathcal{F}(\Gamma)$ *and* $s \in \Gamma$*;*
(2) $\varphi(E) \leq \varphi(F)$ *for all* $E \subseteq F \in \mathcal{F}(\Gamma)$*;*
(3) $\varphi(F_1 \cup F_2) \leq \varphi(F_1) + \varphi(F_2)$ *for all* $F_1, F_2 \in \mathcal{F}(\Gamma)$.

Then the limit $\lim_F \varphi(F)/|F|$ *exists.*

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2.2. *Conditional mean topological dimension.* Let X be a compact metrizable space. For two finite open covers $\mathcal{U}$ and $\mathcal{V}$ of X, the joining $\mathcal{U} \vee \mathcal{V}$ is defined as $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in V\}$. We say $\mathcal{U}$ *refines* $\mathcal{V}$, denoted by $\mathcal{U} \succeq \mathcal{V}$, if every element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Denote by ord $(\mathcal{U})$ the overlapping number of $\mathcal{U}$, that is,

$$\text{ord }(\mathcal{U}) = \max_{x \in X} \sum_{U \in \mathcal{U}} 1_U(x) - 1.$$

Now fix a factor map $\pi : X \to Y$ and a finite open cover $\mathcal{U}$ of X. Consider the number $\mathcal{D}(\mathcal{U}|Y) := \min(\text{ord }(\mathcal{W}))$ for $\mathcal{W}$ ranging over all finite open covers of X such that $\{\pi^{-1}(y)\}_{y \in Y} \vee \mathcal{W}$ *refines* $\mathcal{U}$. We put $\mathcal{D}(\mathcal{U}) := \mathcal{D}(\mathcal{U}|Y)$ for Y being a singleton. For any $F \in \mathcal{F}(\Gamma)$, denote by $\mathcal{U}^F$ the finite open cover $\vee_{s \in F} s^{-1}\mathcal{U}$. To see the function $\varphi : \mathcal{F}(\Gamma) \cup \{\emptyset\} \to \mathbb{R}$ sending F to $\mathcal{D}(\mathcal{U}^F|Y)$ satisfies the conditions of Lemma 2.1, we have a conditional version of [19, Proposition 2.4] to assist us.

LEMMA 2.2. *Suppose that $\pi : X \to Y$ is a continuous map and $\mathcal{U}$ is a finite open cover of X. Then $\mathcal{D}(\mathcal{U}|Y) \le k$ if and only if there exists a continuous map $f : X \to P$ for some polyhedron P with* $\dim(P) = k$ *such that $\{f^{-1}(p) \cap \pi^{-1}(y)\}_{(p,y) \in P \times Y}$ refines $\mathcal{U}$.*

Proof. First suppose that we have such a continuous map $f : X \to P$. Let $\varphi : X \to P \times Y$ be the map sending x to $(f(x), \pi(x))$. By [19, Proposition 2.4], there exists a finite open cover $\mathcal{V}$ of $P \times Y$ such that $\varphi^{-1}(\mathcal{V})$ refines $\mathcal{U}$. Without loss of generality, we may assume that $\mathcal{V}$ is of a form $\mathcal{W} \times \mathcal{V}_1$ for some finite open cover $\mathcal{W}$ of P and finite open cover $\mathcal{V}_1$ of Y. Then for every $W \in \mathcal{W}$ and $y \in Y$, we have $y \in V$ for some $V \in \mathcal{V}_1$ and $\varphi^{-1}(W \times V) \subseteq U$ for some $U \in \mathcal{U}$. Thus

$$f^{-1}(W) \cap \pi^{-1}(y) \subseteq f^{-1}(W) \cap \pi^{-1}(V) = \varphi^{-1}(W \times V) \subseteq U.$$

This concludes that $f^{-1}(\mathcal{W}) \vee \{\pi^{-1}(y)\}_y$ refines $\mathcal{U}$. Choose a finite open cover $\mathcal{V}_2$ of P refining $\mathcal{W}$ such that ord $(\mathcal{V}_2) \le \dim(P)$. Then ord $(f^{-1}(\mathcal{V}_2)) \le$ ord $(\mathcal{V}_2) \le \dim P$ and $f^{-1}(\mathcal{V}_2) \vee \{\pi^{-1}(y)\}_y$ refines $\mathcal{U}$. It follows that $\mathcal{D}(\mathcal{U}|Y) \le \dim P = k$.

Now suppose that $\mathcal{D}(\mathcal{U}|Y) \le k$. By definition, there exists a finite open cover $\mathcal{W}$ of X such that $\{\pi^{-1}(y)\}_y \vee \mathcal{W}$ refines $\mathcal{U}$ and ord $(\mathcal{W}) \le k$. Let $\{g_W\}_{W \in \mathcal{W}}$ be a partition of unity subordinate to $\mathcal{W}$ and $\Delta_{\mathcal{W}}$ be the polyhedron induced from the nerve complex of $\mathcal{W}$. Define the map $g : X \to \Delta_{\mathcal{W}}$ sending x to $\sum_{W \in \mathcal{W}} g_W(x) e_W$, where e_W stands for the vertex indexed with $W \in \mathcal{W}$. Then for every $q \in \Delta_{\mathcal{W}}$, $g^{-1}(q)$ is contained in an element W of $\mathcal{W}$ corresponding to a vertex of the least dimensional simplex of $\Delta_{\mathcal{W}}$ containing q. So for each $y \in Y$, $g^{-1}(q) \cap \pi^{-1}(y) \subseteq W \cap \pi^{-1}(y) \subseteq U$ for some U in $\mathcal{U}$. Choose a topological embedding $h : \Delta_{\mathcal{W}} \to P$ for some polyhedron P with $\dim P = k$. Then the map $f := h \circ g$ is what we need. □

From Lemma 2.2, we see that φ is sub-additive and so φ satisfies the conditions of Lemma 2.1. Thus the limit $\lim_F \mathcal{D}(\mathcal{U}^F|Y)/|F|$ exists.

Definition 2.3. We define the *conditional mean topological dimension of* $\Gamma \curvearrowright X$ *relative to* $\Gamma \curvearrowright Y$ as

$$\text{mdim}(X|Y) := \sup_{\mathcal{U}} \lim_F \frac{\mathcal{D}(\mathcal{U}^F|Y)}{|F|}$$

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for $\mathcal{U}$ running over all finite open covers of X. For simplicity, we may also say $\mathrm{mdim}(X|Y)$ is the conditional mean dimension of X relative to Y.

When Y is a singleton, $\mathrm{mdim}(X|Y)$ recovers the mean topological dimension of $\Gamma \curvearrowright X$, which we denote by $\mathrm{mdim}(X)$ (see [19, Definition 2.6]). Moreover, as $\Gamma = \{e_\Gamma\}$ is the trivial group, $\mathrm{mdim}(X)$ recovers the *(covering) dimension* of X, which we denote by $\dim(X)$.

PROPOSITION 2.4. *Let $\Gamma \curvearrowright Y$ and $\Gamma \curvearrowright Z$ be two dynamical systems. Let Γ act on $Y \times Z$ diagonally and $\pi : Y \times Z \to Y$ the projection map. Then* $\mathrm{mdim}(Y \times Z|Y) = \mathrm{mdim}(Z)$.

Proof of Theorem 1.2. Fix a finite open cover $\mathcal{U}$ of Z. To show $\mathrm{mdim}(Z) \leq \mathrm{mdim}(Y \times Z|Y)$, for any $F \in \mathcal{F}(\Gamma)$, it suffices to show $\mathcal{D}(\mathcal{U}^F) \leq \mathcal{D}(\mathcal{V}^F|Y)$ for $\mathcal{V} := \{Y \times U : U \in \mathcal{U}\}$.

Suppose that $\mathcal{D}(\mathcal{V}^F|Y) = \mathrm{ord}\ (\mathcal{W})$ for some finite open cover $\mathcal{W}$ of $Y \times Z$ such that $\mathcal{W} \vee \{\pi^{-1}(y)\}_{y\in Y}$ refines $\mathcal{V}^F$. Consider the topological embedding $\varphi : Z \to Y \times Z$ sending z to (y_0, z) for some fixed y_0 in Y. Then for every $W \in \mathcal{W}$, there exists $U \in \mathcal{U}^F$ such that $W \cap \pi^{-1}(y_0) \subseteq Y \times U$. It follows that $\varphi^{-1}(W) \subseteq U$ and so $\varphi^{-1}(\mathcal{W})$ refines $\mathcal{U}^F$. Thus

$$\mathcal{D}(\mathcal{U}^F) \leq \mathrm{ord}\ (\varphi^{-1}(\mathcal{W})) \leq \mathrm{ord}\ (\mathcal{W}) = \mathcal{D}(\mathcal{V}^F|Y).$$

To show the other direction, for any finite open covers $\mathcal{U}_0$ and $\mathcal{V}_0$ of Y and Z respectively, we need only to show $\mathcal{D}((\mathcal{U}_0 \times \mathcal{V}_0)^F|Y) \leq \mathcal{D}(\mathcal{V}_0^F)$ for all $F \in \mathcal{F}(\Gamma)$.

Let $\mathcal{D}(\mathcal{V}_0^F) = \mathrm{ord}\ (\mathcal{V})$ for some finite open cover $\mathcal{V}$ of Z refining $\mathcal{V}_0^F$. Denote by p_Z the projection map from $Y \times Z$ onto Z. Then for any $y \in Y$ and $V \in \mathcal{V}$, one has $p_Z^{-1}(V) \cap \pi^{-1}(y) = \{y\} \times V \subseteq U \times V$ for any $U \in \mathcal{U}_0^F$ containing y. Therefore, $p_Z^{-1}(\mathcal{V}) \cap \{\pi^{-1}(y)\}_{y\in Y}$ refines $(\mathcal{U}_0 \times \mathcal{V}_0)^F$. Thus

$$\mathcal{D}((\mathcal{U}_0 \times \mathcal{V}_0)^F|Y) \leq \mathrm{ord}\ (p_Z^{-1}(\mathcal{V})) = \mathrm{ord}\ (\mathcal{V}) = \mathcal{D}(\mathcal{V}_0^F). \qquad \square$$

Now we introduce a metric approach to the conditional mean dimension in line with [4, Theorem 6.5.4]. Let $\pi : X \to Y$ be a factor map and ρ a compatible metric on X. For any $\varepsilon > 0$, denote by $\mathrm{Wdim}_\varepsilon(X|Y, \rho)$ the minimal dimension of a polyhedron P which admits a continuous map $f : X \to P$ such that $\mathrm{diam}(f^{-1}(p) \cap \pi^{-1}(y), \rho) < \varepsilon$ for every $(p, y) \in P \times Y$. We call such a map a (ρ, ε) *-embedding relative to Y*. For every $F \in \mathcal{F}(\Gamma)$, denote by ρ_F the metric on X defined as

$$\rho_F(x, y) := \max_{s\in F} \rho(sx, sy).$$

Then it is easy to check that the function $\mathcal{F}(\Gamma) \cup \{\emptyset\} \to \mathbb{R}$ sending F to $\mathrm{Wdim}_\varepsilon(X|Y, \rho_F)$ satisfies the conditions of Lemma 2.1. Thus the limit $\lim_F(\mathrm{Wdim}_\varepsilon(X|Y, \rho_F)/|F|)$ exists.

PROPOSITION 2.5. *For a compatible metric ρ on X, we have*

$$\mathrm{mdim}(X|Y) = \sup_{\varepsilon>0} \lim_F \frac{\mathrm{Wdim}_\varepsilon(X|Y, \rho_F)}{|F|}.$$

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Proof. For the direction ' $\leq$', fix a finite open cover $\mathcal{U}$ of X. Picking a Lebesgue number λ of $\mathcal{U}$ with respect to ρ, it suffices to show

$$\mathcal{D}(\mathcal{U}^F|Y) \leq \mathrm{Wdim}_\lambda(X|Y, \rho_F)$$

for every $F \in \mathcal{F}(\Gamma)$. Let $f : X \to P$ be a continuous map with $\dim(P) = \mathrm{Wdim}_\lambda(X|Y, \rho_F)$ such that $\mathrm{diam}(f^{-1}(p) \cap \pi^{-1}(y), \rho_F) < \lambda$ for every $(p, y) \in P \times Y$. By choice of λ, we have that $\{f^{-1}(p) \cap \pi^{-1}(y)\}_{(p,y)}$ refines $\mathcal{U}^F$. Applying Lemma 2.2 to $\mathcal{U}^F$, it follows that $\mathcal{D}(\mathcal{U}^F|Y) \leq \dim(P) = \mathrm{Wdim}_\lambda(X|Y, \rho_F)$.

Now we show the converse direction of the equality. Fix $\varepsilon > 0$ and pick a finite open cover $\mathcal{U}$ of X consisting of some open sets of the diameter less than ε under the metric ρ. It reduces to show

$$\mathrm{Wdim}_\varepsilon(X|Y, \rho_F) \leq \mathcal{D}(\mathcal{U}^F|Y)$$

for each $F \in \mathcal{F}(\Gamma)$. Applying Lemma 2.2 to $\mathcal{U}^F$, we have a continuous map $f : X \to P$ with $\dim(P) = \mathcal{D}(\mathcal{U}^F|Y)$ such that $\{f^{-1}(p) \cap \pi^{-1}(y)\}_{(p,y)}$ refines $\mathcal{U}^F$. By choice of $\mathcal{U}$, we see that f is a (ρ_F, ε)-embedding relative to Y. Thus

$$\mathrm{Wdim}_\varepsilon(X|Y, \rho_F) \leq \dim(P) = \mathcal{D}(\mathcal{U}^F|Y). \qquad \square$$

2.3. *G-extensions.* Let us compute the conditional mean dimension of G-extensions.

Definition 2.6. [3, p. 411] Let $\pi : X \to Y$ be a factor map and $\Gamma \curvearrowright G$ be another dynamical system. Here, X is called a *G-extension of Y* if there exists a continuous map $X \times G \to X$ sending (x, g) to xg such that for any $x \in X$, $g, g' \in G$, and $t \in \Gamma$, we have:
(1) $\pi^{-1}(\pi(x)) = xG$;
(2) $xg = xg'$ exactly when $g = g'$;
(3) $t(xg) = (tx)(tg)$.

Note that when G is a group and the action $\Gamma \curvearrowright G$ is trivial, the factor map π recovers as a principal group extension.

Example 2.7. Let $\Gamma \curvearrowright Y$ and $\Gamma \curvearrowright G$ be two dynamical systems such that G is a compact group and Γ acts on G by continuous automorphisms. A (continuous) *cocycle* is a continuous map $\sigma : \Gamma \times Y \to G$ such that

$$\sigma(st, y) = \sigma(s, ty) \cdot s(\sigma(t, y))$$

for every $s, t \in \Gamma$, and $y \in Y$. It induces an action of Γ on $Y \times G$ by

$$s(y, g) := (sy, \sigma(s, y) \cdot (sg))$$

for all $s \in \Gamma$, $y \in Y$, and $g \in G$. Then $Y \times G$ is a G-extension of Y in light of the map $(Y \times G) \times G \to Y \times G$ sending $((y, g), h)$ to (y, gh). We denote by $Y \times_\sigma G$ the G-extension from such a cocycle σ.

Another source of G-extension arises when the underlying systems have group structure. Recall that a dynamical system $\Gamma \curvearrowright X$ is called an *algebraic action* if X is a compact metrizable group and the action of Γ on X is by continuous automorphisms. Let $\pi : X \to Y$

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be a factor map between algebraic actions such that π is a group homomorphism. Put $G = \ker(\pi)$. Then X is a G-extension given by sending $(x, g) \in X \times G$ to xg.

PROPOSITION 2.8. *Let X be a G-extension of Y for some compact metrizable space G. Then* $\mathrm{mdim}(X|Y) \geq \mathrm{mdim}(G)$. *If $\pi: X \to Y$ admits a continuous section $\tau: Y \to X$ in the sense that τ is continuous such that $\pi \circ \tau = \mathrm{id}_Y$, we have* $\mathrm{mdim}(X|Y) = \mathrm{mdim}(G)$.

Proof. Fix $F \in \mathcal{F}(\Gamma)$. Let ρ_X, ρ_G be two compatible metrics on X and G respectively.

Pick a point x_0 from X. By definition of G-extension, for any $x \in G$ and $g, g' \in G$, $xg = xg'$ exactly when $g = g'$. Thus by compactness of X and G, for any $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following property: for any $g, g' \in G$ such that $\rho_X(xg, xg') < \delta$ for some $x \in X$, we have

$$\rho_G(g, g') < \varepsilon. \tag{2.1}$$

Let $\psi: X \to P$ be a $(\rho_{X,F}, \delta)$-embedding relative to π. Then for any $g, g' \in G$ with $\psi(x_0 g) = \psi(x_0 g')$, since $\pi(x_0 g) = \pi(x_0) = \pi(x_0 g')$, we have $\rho_{X,F}(x_0 g, x_0 g') < \delta$. By inequality (2.1), we obtain $\rho_{G,F}(g, g') < \varepsilon$. Denote by φ the map $G \to X$ sending g to $x_0 g$. This concludes that the map $\psi \circ \varphi$ is a $(\rho_{G,F}, \varepsilon)$-embedding. The desired inequality then follows from a limit argument.

Now we assume that π admits a continuous cross-section $\tau: Y \to X$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\rho_X(xg, xg') < \varepsilon$$

for any $x \in X$ and $g, g' \in G$ such that $\rho_G(g, g') < \delta$.

Assume that $\varphi: G \to Q$ is a $(\rho_{G,F}, \delta)$-embedding for some polyhedron Q. For each $x \in X$, since $\pi(\tau(\pi(x))) = \pi(x)$, we have $x, \tau(\pi(x)) \in \pi^{-1}(\pi(x)) = xG$ and hence $x = \tau(\pi(x))g_x$ for a unique $g_x \in G$. Now define $\psi: X \to Q$ by sending x to $\varphi(g_x)$. Then the continuity of ψ is guaranteed by the continuity of τ. For any $x, x' \in X$ with the same image under π and ψ, since φ is a $(\rho_{G,F}, \delta)$-embedding, we have $\rho_G(sg_x, sg_{x'}) < \delta$ for any $s \in F$. By the design of δ, we have

$$\begin{aligned}\rho_X(sx, sx') &= \rho_X(s(\tau(\pi(x))g_x), s(\tau(\pi(x'))g_{x'})) \\ &= \rho_X((s(\tau(\pi(x))))(sg_x), (s(\tau(\pi(x))))(sg_{x'})) < \varepsilon.\end{aligned}$$

That implies that $\mathrm{Wdim}_\varepsilon(X|Y, \rho_{X,F}) \leq \dim(Q)$. The inequality then follows by running some limit argument. □

Example 2.9. Let $\mathbb{Z}\Gamma$ be the integral group ring of Γ and $f \in \mathbb{Z}\Gamma$ (see [22] for more details about group rings). Consider that Γ acts on $(\mathbb{R}/\mathbb{Z})^\Gamma$ by left shift. Let $R(f): X := (\mathbb{R}/\mathbb{Z})^\Gamma \to (\mathbb{R}/\mathbb{Z})^\Gamma$ be the group homomorphism sending x to xf. Set $G := \ker(R(f))$. Then the induced factor map $\pi_f: X \to Y := \mathrm{im}(R(f))$ shows that X is a G-extension of Y. Suppose that $fuf = f$ for some $u \in \mathbb{Z}\Gamma$. Then π_f admits a continuous section $Y \to X$ by sending y to yu. From Proposition 2.8, we have $\mathrm{mdim}(X|Y) = \mathrm{mdim}(G)$.

2.4. *Embedding problem.* By a *dynamical emmbedding* $\varphi: X \to Y$ between two dynamical systems $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$, we mean φ is a continuous injective map such

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that $\varphi(sx) = s\varphi(x)$ for every $s \in \Gamma$ and $x \in X$. With the help of the conditional mean dimension, we can strengthen some results regarding the dynamical embedding problem in terms of the Rokhlin dimension [12, Theorem 3.1].

The notion of Rokhlin dimension appears in the classification of C^*-algebras induced from topological dynamical systems. The definition in the setting of topological dynamical systems is due to Winter, explicitly formulated by Szabo for $\mathbb{Z}^k$-actions [23, Definition 2.1], and extended to the action of residually finite groups by Szabó *et al* [24]. The Rokhlin dimension of infinite finitely generated nilpotent group actions is estimated in [24, Corollary 8.5].

Now we can consider the definition for the action of amenable groups.

Definition 2.10. Let $\Gamma \curvearrowright X$ be a continuous action by a countable amenable group Γ on a compact metrizable space X. We say $\Gamma \curvearrowright X$ has *Rokhlin dimension d*, denoted as

$$\dim_{\mathrm{Rok}}(X, \Gamma) = d,$$

if d is the smallest non-negative integer such that for every finite subset K of Γ and every $\delta > 0$, there exists $(d+1)$ subsets $F_0, \ldots, F_d \in \mathcal{B}(K, \delta)$ and $(d+1)$ open sets $U_0, \ldots, U_d$ satisfying:

(i) the subsets $\{sU_i\}_{s \in F_i}$ are pairwise disjoint for every $i = 0, \ldots, d$;

(ii) the union $\bigcup_{i=0}^{d} \bigsqcup_{s \in F_i} sU_i$ covers the whole space X.

Observe that this definition allows the distinct towers $F_i U_i$ values to overlap and the Rokhlin dimension does not increase when passing to the extension systems.

With the help of the notion of conditional mean dimension, we can improve the statement of [12, Theorem 3.1] in the following. The proof of [12, Theorem 3.1] works here by applying Lemma 2.12 as a conditional version of [13, Lemma 2.1].

THEOREM 2.11. *Let D be a positive integer and L a non-negative integer. Suppose that $\pi : X \to Y$ is a factor map with $\dim_{Rok}(X, \Gamma) = D$ and $\mathrm{mdim}(X|Y) < L/2$. Then there exists a dynamical embedding from X to $(([0, 1]^{(D+1)L})^{\Gamma}) \times Y$ where the latter is endowed with the product action from the shift action on $([0, 1]^{(D+1)L})^{\Gamma}$ and the action $\Gamma \curvearrowright Y$.*

The following lemma is a conditional version of [13, Lemma 2.1], whose proof works here.

LEMMA 2.12. *Let $\pi : X \to Y$ be a continuous map between a compact metrizable space with a compatible metric ρ on X. Suppose that $f : X \to [0, 1]^L$ is a continuous map such that $\|f(x) - f(x')\|_\infty < \delta$ for every $x, x' \in X$ with $\rho(x, x') < \varepsilon$. Assume that $\mathrm{Wdim}_\varepsilon(X|Y, \rho) < L/2$. Then there exists a (ρ, ε)-embedding $g : X \to [0, 1]^L$ relative to Y satisfying that $\sup_{x \in X} \|f(x) - g(x)\|_\infty < \delta$.*

Combining Theorem 2.11 with Proposition 2.8, we obtain the following dynamical embedding result for G-extensions.

COROLLARY 2.13. *Let X be a G-extension of Y for some compact metrizable space G. Suppose that the factor map $\pi : X \to Y$ admits a continuous section. Assume that*

$\mathrm{mdim}(G) < L/2$ and $\dim_{Rok}(Y, \Gamma) = D$ for some positive integer L and non-negative integer D. Then there exists a dynamical embedding from X to $([0, 1]^{(D+1)L})^{\Gamma} \times Y$.

2.5. *Mean dimension of fibers.* Given a finite open cover $\mathcal{U}$ of X, for any closed subset K of X, denote by $\mathcal{U}|_K$ the finite open cover of K restricted from $\mathcal{U}$, that is, $\mathcal{U}|_K = \{U \cap K : U \in \mathcal{U}\}$. Taking advantage of Γ-invariance of X, we can similarly consider the mean dimension of K as [7, §1.5] and Tsukamoto [25, Remark 4.7].

Definition 2.14. Fix a *Følner sequence* $\mathcal{F} := \{F_n\}_{n\geq 1}$ of Γ, that is, for any $s \in \Gamma$, $|sF_n \Delta F_n|/|F_n|$ converges to 0 as n goes to the infinity. We define the *mean dimension of K* as

$$\mathrm{mdim}(K, \Gamma) := \sup_{\mathcal{U}} \varlimsup_{n\to\infty} \frac{\mathcal{D}(\mathcal{U}^{F_n}|_K)}{|F_n|},$$

where $\mathcal{U}$ ranges over all finite open covers of X.

By the same argument of Proposition 2.5, we have the following.

PROPOSITION 2.15. *Fix a compatible metric ρ on X. We have*

$$\mathrm{mdim}(K, \Gamma) = \sup_{\varepsilon>0} \varlimsup_{n\to\infty} \frac{\mathrm{Wdim}_{\varepsilon}(K, \rho_{F_n})}{|F_n|}.$$

Considering the fibers of a factor map $\pi : X \to Y$, owing to the metric approach formulas in Propositions 2.5 and 2.15, we have the following estimation.

PROPOSITION 2.16. *For every $y \in Y$, we have* $\mathrm{mdim}(\pi^{-1}(y), \Gamma) \leq \mathrm{mdim}(X|Y)$.

By a modified argument of Proposition 2.8, we have the following.

PROPOSITION 2.17. *Let X be a G-extension of Y. Then* $\mathrm{mdim}(G) = \mathrm{mdim}(\pi^{-1}(y), \Gamma)$ *for every $y \in Y$.*

We have a satisfactory answer to Question 1.1 in the following case.

COROLLARY 2.18. *Let $\pi : X \to Y$ be a factor map of algebraic actions such that π is a group homomorphism. Write $G := \ker(\pi)$. Then* $\mathrm{mdim}(G) = \mathrm{mdim}(\pi^{-1}(y), \Gamma)$ *for every $y \in Y$. In particular, we have*

$$\mathrm{mdim}(X) = \mathrm{mdim}(Y) + \sup_{y\in Y} \mathrm{mdim}(\pi^{-1}(y), \Gamma).$$

Proof. Clearly X is a G-extension of Y. Thus the first statement is true from Proposition 2.17. By the addition formula for mean dimension of algebraic actions [16, Corollary 6.1], we have

$$\begin{aligned}\mathrm{mdim}(X) &= \mathrm{mdim}(Y) + \mathrm{mdim}(G)\\ &= \mathrm{mdim}(Y) + \sup_{y\in Y} \mathrm{mdim}(\pi^{-1}(y), \Gamma).\end{aligned}$$

□

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3. *Proof of Theorem 1.2*

In this section, we give the proof of Theorem 1.2.

First, we recall the quasi-tiling lemma of amenable groups as follows [21, p. 24, Theorem 6] [15, Theorem 8.3]. In fact, one can require all quasi-tiles contain the identity element e_Γ of Γ. Let $\varepsilon > 0$ and $F_1, \ldots, F_m \in \mathcal{F}(\Gamma)$, we say $\{F_j\}_{j=1}^m$ are *ε-disjoint* if there exists $F'_j \subseteq F_j$ for every $j = 1, \ldots, m$ such that $\{F_j\}_{j=1}^m$ are pairwise disjoint and $|F'_j| \ge (1-\varepsilon)|F_j|$ for every $j = 1, \ldots, m$.

LEMMA 3.1. *Let $\varepsilon > 0$ and $K \in \mathcal{F}(\Gamma)$. Then there exists $\delta > 0$ and $K', F_1, \ldots, F_m \in \mathcal{F}(\Gamma)$ such that:*

(1) $e_\Gamma \in F_j \in \mathcal{B}(K, \varepsilon)$, *for all $j = 1, \ldots, m$;*

(2) *for each $A \in \mathcal{B}(K', \delta)$, there exist $D_1, \ldots, D_m \in \mathcal{F}(\Gamma)$ such that the family $\{F_j c : c \in D_j, j = 1, \ldots, m\}$ are ε-disjoint subsets of A, and $|A \setminus \bigcup_{j=1}^m F_j D_j| \le \varepsilon |A|$.*

We call those F_j values *quasitiles* of Γ and D_j values the *tiling centers* of A.

Proof of Theorem 1.2. Fix a finite open cover $\mathcal{U}$ of X. Let $0 < \varepsilon < 1$ and $K \in \mathcal{F}(\Gamma)$. By Lemma 3.1, there exist $\delta > 0$, $K' \in \mathcal{F}(\Gamma)$, and tiles $F_1, \ldots, F_m \in \mathcal{F}(\Gamma)$, such that each $A \in \mathcal{B}(K', \delta)$ admits tiling centers $D_1, \ldots, D_m \in \mathcal{F}(\Gamma)$ satisfying the conditions in Lemma 3.1.

For each $j = 1, \ldots, m$ choose a finite open cover $\mathcal{W}_j$ of X such that $\mathrm{ord}\,(\mathcal{W}_j) = \mathcal{D}(\mathcal{U}^{F_j}|Y)$ and $\mathcal{W}_j \vee \{\pi^{-1}(y)\}_{y\in Y}$ refines $\mathcal{U}^{F_j}$. Without loss of generality, we may assume $\mathcal{W}'_j \vee \{\pi^{-1}(y)\}_y$ still refines $\mathcal{U}^{F_j}$ for $\mathcal{W}'_j := \{\overline{W}\}_{W\in\mathcal{W}_j}$. Then for each $y \in Y$ and $W \in \mathcal{W}_j$, there exists $U \in \mathcal{U}^{F_j}$ and an open neighborhood $V_{y,W}$ of y such that

$$\overline{W} \cap \pi^{-1}(V_{y,W}) \subseteq U.$$

By compactness, there exists a subfamily $\mathcal{V}_j$ of $\{\bigcap_{W\in\mathcal{W}_j} V_{y,W} : y \in Y\}$ such that $\mathcal{V}_j$ still makes an open cover of Y. Clearly $\mathcal{W}_j \vee \pi^{-1}(\mathcal{V}_j)$ refines $\mathcal{U}^{F_j}$. Put $\mathcal{V} = \bigvee_{j=1}^m \mathcal{V}_j$ (depending only on $\mathcal{U}$ and K). It follows that $\mathcal{W}_j \vee \pi^{-1}(\mathcal{V})$ refines $\mathcal{U}^{F_j}$ for every $j = 1, \ldots, m$.

Now for $A \in \mathcal{B}(K', \delta)$, choose a finite open cover $\mathcal{W}_A$ of Y such that $\mathrm{ord}\,(\mathcal{W}_A) = \mathcal{D}(\mathcal{V}^A)$ and $\mathcal{W}_A$ refines $\mathcal{V}^A$. Since F_j contains the identity of Γ, we have $r\mathcal{W}_A$ refines $\mathcal{V}$ for every $r \in D_j$ and $j = 1, \ldots, m$. By construction of $\mathcal{V}$, we have $\mathcal{W}_j \vee \pi^{-1}(r\mathcal{W}_A)$ refines $\mathcal{U}^{F_j}$. Hence, $(\bigvee_{j=1}^m \vee_{r\in D_j} r^{-1}\mathcal{W}_j) \vee \pi^{-1}(\mathcal{W}_A) = \bigvee_{j=1}^m \vee_{r\in D_j} r^{-1}(\mathcal{W}_j \vee \pi^{-1}(r\mathcal{W}_A))$ refines $\mathcal{U}^{\bigcup_j F_j D_j}$. Thus

$$\begin{aligned}\mathcal{D}(\mathcal{U}^A) &\le \mathcal{D}(\mathcal{U}^{\cup_j F_j D_j}) + \mathcal{D}(\mathcal{U}^{A\setminus \cup_j F_j D_j}) \\ &\le \mathrm{ord}\,\Big(\Big(\bigvee_{j=1}^m \bigvee_{r\in D_j} r^{-1}\mathcal{W}_j\Big) \vee \pi^{-1}(\mathcal{W}_A)\Big) + \varepsilon|A|\mathcal{D}(\mathcal{U}) \\ &\le \mathcal{D}(\mathcal{V}^A) + \sum_j |D_j|\mathcal{D}(\mathcal{U}^{F_j}|Y) + \varepsilon|A|\mathcal{D}(\mathcal{U}).\end{aligned}$$

Since $\{F_j c\}_{j,c}$ are ε-disjoint subsets of A, we have

$$\sum_{j=1}^m |F_j||D_j| \le \frac{|A|}{1-\varepsilon}.$$

It follows that

$$\sum_{j=1}^{m} |D_j|\mathcal{D}(\mathcal{U}^{F_j}|Y) = \sum_{j=1}^{m} |F_j||D_j|\frac{\mathcal{D}(\mathcal{U}^{F_j}|Y)}{|F_j|} \le \frac{|A|}{1-\varepsilon} \sup_{F\in\mathcal{B}(K,\varepsilon)} \frac{\mathcal{D}(\mathcal{U}^{F}|Y)}{|F|}.$$

So

$$\frac{\mathcal{D}(\mathcal{U}^{A})}{|A|} \le \frac{\mathcal{D}(\mathcal{V}^{A})}{|A|} + \frac{1}{1-\varepsilon} \sup_{F\in\mathcal{B}(K,\varepsilon)} \frac{\mathcal{D}(\mathcal{U}^{F}|Y)}{|F|} + \varepsilon\mathcal{D}(\mathcal{U}).$$

Since $A \in \mathcal{B}(K', \delta)$ is arbitrary, we get

$$\mathrm{mdim}(\mathcal{U}) \le \mathrm{mdim}(Y) + \frac{1}{1-\varepsilon} \sup_{F\in\mathcal{B}(K,\varepsilon)} \frac{\mathcal{D}(\mathcal{U}^{F}|Y)}{|F|} + \varepsilon\mathcal{D}(\mathcal{U})$$

for $\mathrm{mdim}(\mathcal{U}) := \lim_F(\mathcal{D}(\mathcal{U}^F)/|F|)$. Taking the limit for K and ε, we have

$$\mathrm{mdim}(\mathcal{U}) \le \mathrm{mdim}(Y) + \mathrm{mdim}(X|Y).$$

Since $\mathcal{U}$ is arbitrary, this completes the proof. □

Remark 3.2. There is a number of reasons why the converse inequality in Theorem 1.2 can fail. It is well known that a Cantor set can continuously map onto any compact metrizable space. In particular, for the action of trivial group, we have that the converse inequality in Theorem 1.2 fails for such a surjective map. Moreover, Boltyanskiĭ constructed an example of a compact metrizable space X such that $\dim(X \times X) < 2\dim(X)$ (see [**2**]). As a consequence, we know that the converse of inequality in Theorem 1.2 can fail even for a projection map.

COROLLARY 3.3. *Let $\Gamma \curvearrowright Y$ be a dynamical system and $\Gamma \curvearrowright G$ an algebraic action. Suppose that $\sigma : \Gamma \times Y \to G$ is a continuous cocycle. Then for the induced G-extension $\Gamma \curvearrowright Y \times_\sigma G$, we have*

$$\mathrm{mdim}(Y \times_\sigma G) \le \mathrm{mdim}(Y) + \sup_{y\in Y} \mathrm{mdim}(\pi^{-1}(y), \Gamma).$$

Proof. Clearly $Y \times_\sigma G$ admits a continuous cross-section. Thus by Propositions 2.8 and 2.17, we have $\mathrm{mdim}(Y \times_\sigma G|Y) = \mathrm{mdim}(G) = \mathrm{mdim}(\pi^{-1}(y), \Gamma)$ for every $y \in Y$. Applying Theorem 1.2, we have

$$\begin{aligned}\mathrm{mdim}(Y \times_\sigma G) &\le \mathrm{mdim}(Y) + \mathrm{mdim}(Y \times_\sigma G|Y)\\ &= \mathrm{mdim}(Y) + \sup_{y\in Y} \mathrm{mdim}(\pi^{-1}(y), \Gamma).\end{aligned}$$ □

4. *Conditional metric mean dimension*

In contrast with the metric mean dimension, it is natural to consider its conditional version. For a metrizable space X with a compatible metric ρ and $\varepsilon > 0$, a subset $E \subseteq X$ is called (ρ, ε) *-separating* if $\rho(x, x') \ge \varepsilon$ for every distinct $x, x' \in E$. Denote by $N_\varepsilon(X, \rho)$ the maximal cardinality of (ρ, ε)-separating subsets of X.

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Definition 4.1. Let $\Gamma \curvearrowright X$ be a dynamical system. Fix a compatible metric ρ on X. Set

$$N_\varepsilon(X|Y,\rho) = \max_{y\in Y} N_\varepsilon(\pi^{-1}(y),\rho).$$

We define the *conditional metric mean dimension of* $\Gamma \curvearrowright (X,\rho)$ *relative to* $\Gamma \curvearrowright Y$ as

$$\mathrm{mdim}_{\mathrm{M}}(X|Y,\rho) := \varliminf_{\varepsilon\to 0} \varlimsup_{F} \frac{\log N_\varepsilon(X|Y,\rho_F)}{|\log\varepsilon||F|}.$$

Again, when Y is a singleton, $\mathrm{mdim}_{\mathrm{M}}(X|Y,\rho)$ recovers as the metric mean dimension of $\Gamma \curvearrowright (X,\rho)$, which we denote by $\mathrm{mdim}_{\mathrm{M}}(X,\rho)$ (see [**19**, Definition 4.1]).

Remark 4.2. Recall that the *mesh* of a finite open cover $\mathcal{U}$ for (X,ρ) is defined by

$$\mathrm{mesh}(\mathcal{U},\rho) := \max_{U\in\mathcal{U}} \mathrm{diam}(U,\rho).$$

In terms of this quantity, one can also give an equivalent definition of conditional metric mean dimension by considering the function $\mathcal{F}(\Gamma) \to \mathbb{R}$ sending F to

$$\log \max_{y\in Y} \min_{\mathrm{mesh}(\mathcal{U}_y,\rho_F)<\varepsilon} |\mathcal{U}_y|$$

for $\mathcal{U}_y$ ranging over all finite open covers of $\pi^{-1}(y)$. It is easy to check that this function satisfies the conditions of Lemma 2.1.

PROPOSITION 4.3. *Let X be a G-extension of Y for some compact metrizable space G. Suppose that* ρ_X *and* ρ_G *are two compatible metrics on X and G respectively such that*

$$\rho_X(xg, xg') = \rho_G(g, g')$$

for all $x \in X$ *and* $g, g' \in G$. *Then*

$$\mathrm{mdim}_M(X|Y,\rho_X) = \mathrm{mdim}_M(G,\rho_G).$$

Proof. By definition, a subset E of $\pi^{-1}(y)$ is of a form x_0G_0 for some $x_0 \in X$ and $G_0 \subseteq G$. Then E is a (ρ_F,ε)-separating subset of $\pi^{-1}(y)$ if and only if G_0 is a (ρ_F,ε)-separating subset of G. By a limit argument, we have $\mathrm{mdim}_{\mathrm{M}}(X|Y,\rho_X) \le \mathrm{mdim}_M(G,\rho_G)$.

To see the converse of equality, it suffices to notice that if a subset G_0 of G is (ρ_F,ε)-separating, then for every $x \in X$, xE is a (ρ_F,ε)-separating subset of $\pi^{-1}(\pi(x))$. □

Question 4.4. For a factor map $\pi : X \to Y$, is it true that $\mathrm{mdim}(X|Y) \le \mathrm{mdim}_{\mathrm{M}}(X|Y,\rho)$ for every compatible metric ρ on X?

5. *Mean dimension given a measure*

In this section, we define the mean dimension given a measure on the factor system and discuss its properties. We start with a key lemma.

LEMMA 5.1. *Suppose that* $\varphi : X \to Y$ *is a continuous map between compact metrizable spaces and* $\mathcal{U}$ *is a finite open cover of X. Then the map* $Y \to \mathbb{R}$ *sending* y *to* $\mathcal{D}(\mathcal{U}|_{\varphi^{-1}(y)})$ *is upper semicontinuous.*

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Proof. Fix $y \in Y$ and put $\mathcal{D}(\mathcal{U}|_{\varphi^{-1}(y)}) = d$. By definition, there exists a finite open cover $\mathcal{V}$ of X such that $\mathcal{V}|_{\varphi^{-1}(y)} \succ \mathcal{U}|_{\varphi^{-1}(y)}$ and $\mathrm{ord}(\mathcal{V}|_{\varphi^{-1}(y)}) = d$. Write $\mathcal{V}$ as $\mathcal{V} = \{V_i\}_{i \in I}$. Then for any $J \subseteq I$ such that $|J| > d + 1$, one has

$$(\cap_{j \in J} V_j) \cap \varphi^{-1}(y) = \emptyset.$$

Since X is normal, there exists a finite open cover $\mathcal{V}' = \{V_i'\}_{i \in I}$ such that $\overline{V_i'} \subseteq V_i$ for all $i \in I$ (see [4, Corollary 1.6.4]). In particular, $\varphi^{-1}(y)$ has the empty intersection with $\cap_{j \in J} \overline{V_j'}$ for all $J \subseteq I$ such that $|J| > d + 1$. Thus we conclude that y sits inside the open set

$$Y \setminus \varphi(\cap_{j \in J} \overline{V_j'}) = \{z \in Y : \varphi^{-1}(z) \subseteq (\cap_{j \in J} \overline{V_j'})^c\}$$

for each $J \subseteq I$ with $|J| > d + 1$. That means, as z approaches to y, $\varphi^{-1}(z)$ has the empty intersection with $\cap_{j \in J} V_j'$ for every $J \subseteq I$ with $|J| > d + 1$. So by definition, $\mathcal{D}(\mathcal{U}|_{\varphi^{-1}(z)}) \leq \mathrm{ord}(\mathcal{V}'|_{\varphi^{-1}(z)}) \leq d$. This finishes the proof. □

Based on this lemma, we are safe to define the measure-theoretic conditional mean dimension.

Definition 5.2. Denote by $M_\Gamma(Y)$ the collection of Γ-invariant Borel probability measures on Y. For any $\nu \in M_\Gamma(Y)$, set

$$\mathcal{D}(\mathcal{U}|\nu) := \int_Y \mathcal{D}(\mathcal{U}|_{\pi^{-1}(y)}) d\nu(y).$$

Note that $\mathcal{D}(\mathcal{U}^{Fs}|_{\pi^{-1}(y)}) = \mathcal{D}(\mathcal{U}^F|_{\pi^{-1}(sy)})$ for any $s \in \Gamma$. It follows that the function $\mathcal{F}(\Gamma) \cup \{\emptyset\} \to \mathbb{R}$ sending F to $\mathcal{D}(\mathcal{U}^F|\nu)$ satisfies the conditions of Lemma 2.1. We define the *mean dimension of* $\Gamma \curvearrowright X$ *given* ν as

$$\mathrm{mdim}(X|\nu) := \sup_{\mathcal{U}} \lim_F \frac{\mathcal{D}(\mathcal{U}^F|\nu)}{|F|},$$

for $\mathcal{U}$ ranging over all finite open covers of X.

Example 5.3. In the same setting of Proposition 2.4, it is easy to see that $\mathrm{mdim}(Y \times Z|\nu) = \mathrm{mdim}(Z)$ for any $\nu \in M_\Gamma(Y)$.

Following the similar argument as in the proof of [15, Lemma 6.8] by taking liminf instead, we have the following.

PROPOSITION 5.4. *Let* $\pi : X \to Y$ *be a factor map. Then*

$$\sup_{\nu \in M_\Gamma(Y)} \mathrm{mdim}(X|\nu) \leq \sup_{y \in Y} \mathrm{mdim}(\pi^{-1}(y), \Gamma).$$

Combining Proposition 2.16 with 5.4, we see that the conditional mean dimension given a measure serves as a lower bound of the conditional mean dimension.

Recall that a finite subset T of Γ is called a *tile* if there exists a subset C of Γ such that $\{Tc\}_{c \in C}$ makes a partition of Γ. A Følner sequence F_n is called a *tiling Følner sequence*

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if each F_n is a tile. It is well known that all elementary amenable groups including abelian groups admit a tiling Følner sequence [26].

PROPOSITION 5.5. *When* Γ *is an abelian group, we have*

$$\sup_{\nu\in M_\Gamma(Y)} \mathrm{mdim}(X|\nu) = \sup_{y\in Y} \mathrm{mdim}(\pi^{-1}(y),\Gamma),$$

where $\mathrm{mdim}(\pi^{-1}(y),\Gamma)$ *is defined along a tiling Følner sequence of* Γ.

Proof. From Proposition 5.4, we need only to prove that for every $y \in Y$, there exists a $\mu \in M_\Gamma(Y)$ such that $\mathrm{mdim}(\pi^{-1}(y),\Gamma) \le \mathrm{mdim}(X|\mu)$.

Fix a finite open cover $\mathcal{U}$ of X. For every $F \in \mathcal{F}(\Gamma)$ and $z \in Y$, set $f_F(z) = \mathcal{D}(\mathcal{U}^F|_{\pi^{-1}(z)})$. By Lemma 5.1, f_F is upper semicontinuous. Note that [18, Lemma 3.6] holds when every f_F is upper semicontinuous. Pick a cluster point of the measures $(1/|F_n|)\sum_{s\in F_n}\delta_{sy}$ under the weak*-topology, we have $\mu \in M_\Gamma(Y)$. Applying [18, Lemma 3.6] to the measures $\nu_n = \delta_y$, it follows that

$$\varlimsup_{n\to\infty} \frac{f_{F_n}(y)}{|F_n|} = \varlimsup_{n\to\infty} \int_Y \frac{f_{F_n}}{|F_n|}d\nu_n \le \varlimsup_{n\to\infty} \int_Y \frac{f_{F_n}}{|F_n|}d\mu.$$

Therefore,

$$\begin{aligned}\mathrm{mdim}(\pi^{-1}(y),\Gamma) &= \sup_{\mathcal{U}} \varlimsup_{n\to\infty} \frac{f_{F_n}(y)}{|F_n|} \\ &\le \sup_{\mathcal{U}} \varlimsup_{n\to\infty} \int_Y \frac{f_{F_n}}{|F_n|}d\mu = \mathrm{mdim}(X|\mu).\end{aligned}$$

Since y is arbitrary, this finishes the proof. □

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