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LOOK, KNAVE

THOMAS MORRILL

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Dedicated to the life and works of John Horton Conway

Abstract

We examine a recursive sequence in which s_n is a literal description of what the binary expansion of the previous term s_{n-1} is not. By adapting a technique of Conway, we determine the limiting behaviour of $\{s_n\}$ and dynamics of a related self-map of $2^{\mathbb{N}}$. Our main result is the existence and uniqueness of a pair of binary sequences, each the complement-description of the other. We also take every opportunity to make puns.

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1. Introduction

The Look-Say sequence is defined as follows. Let $s_1 = 1$. Given s_n, the next term of the sequence is a literal description of the digits of the previous one [4]. The first few terms are

$$1, 11, 21, 1211, 111221, \ldots .$$

We use $|s|$ to denote the length of a finite string s.

THEOREM 1.1 (Conway [1, 5]). *Let s_n be the nth term of the Look-Say sequence. Then*

$$\lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} = \lambda,$$

where

$$\lambda = 1.3035\ldots .$$

Shockingly, λ is an algebraic integer of degree 71 [5]. Theorem 1.1 follows from Conway's cosmological theorem [1]. In short, the terms of any Look-Say-type sequence (not necessarily starting at $s_1 = 1$) will eventually decompose into a concatenation of certain fundamental substrings identified by Conway as 'elements'.

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This problem has also been considered in terms of binary strings. Given a binary string s_n, the next term of the binary Look-Say sequence is a literal description of the bits of the previous term, where the counts are expressed in base two [6]. The first few terms are

$$1, 11, 101, 111011, \ldots .$$

THEOREM 1.2 (Johnston [2]). *Let s_n be the nth term of the binary Look-Say sequence. Then*

$$\lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} = \lambda,$$

where

$$\lambda = 1.465571 \ldots .$$

We shake this up by introducing a new player, a Knave in the style of Smullyan. As opposed to the previous recursions, our s_n is instead the literal description of what the bits of s_{n-1} are not. Our main result concerns the limiting behaviour of the *Look-Knave sequence*.

THEOREM 1.3. *There is a unique pair of binary sequences S_{even} and S_{odd} such that S_{even} is a literal description of the bitwise complement of S_{odd} and vice versa.*

The rest of the paper is organised as follows. In Section 2 we define the Look-Knave sequence and pose our problem. Then, in Section 3, we simplify the problem and prove Theorem 1.3. Finally, in Section 4, we offer avenues for future work.

2. The Knave

Recall Smullyan's game of Knights and Knaves, a logic puzzle in which Knights always tell the truth and Knaves are always compelled to lie [3]. Our Knave is a very idiosyncratic liar. When looking at a string of n 0s, the Knave correctly tells us they see n bits of the same parity, but they will lie by saying that there are n 1s. Likewise, while looking at k 1s, the Knave will happily tell us there are k `0`s instead.

The Knave understands how to express natural numbers in base two and will write down their observations for us as such. Thus, when the Knave looks at the string

`110`,

they write down

`10 0 1 1`

for the two `0`s and one `1` they claim to have seen. Here, we have inserted white space to enhance the Knave's handwriting.

Now, our Knave has not yet realised that they could have lied about their count by inverting the bits representing n and k above. I won't tell them if you won't.

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TABLE 1. The first 10 entries of the Look-Knave sequence.

s_{2n+1}	s_{2n+2}
1	10
1011	1011100
1011110101	1011100011101110
1011110111101111011	1011100011101011100011100
1011110111101110111101111110101	1011100011101011110111000111010111101110

Thus begins our new game. We will supply a binary string and command 'Look, Knave'. Dutifully, the Knave will read the string and then record the observations on a fresh piece of paper for us. We return this paper to the Knave, who reads their own report and transcribes it in the only way a Knave can. The game continues.

Let us begin with the string $s_1 = 1$ and take s_n to be the Knave's description of s_{n-1}. This defines the Look-Knave sequence. For example, $s_3 = \mathtt{1011}$. We see that there is one bit which is not $\mathtt{0}$, followed by one bit which is not $\mathtt{1}$ and then two bits which are not $\mathtt{0}$. Thus, s_4 must be the string $\mathtt{1011100}$. In short, s_n is a a binary string describing precisely what s_{n-1} is not.

Looking at Table 1, it is tempting to conjecture that the subsequences $\{s_{2n+1}\}$ and $\{s_{2n+2}\}$ are approaching some bitwise limits. So, do there exist binary sequences S_{even} and S_{odd} such that S_{odd} is the Knave's description of S_{even} and *vice versa*?

A binary sequence S can be described by the Knave, so long as the tail end of S is not all $\mathtt{0}$s or all $\mathtt{1}$s. Let $\mathcal{S} \subset 2^{\mathbb{N}}$ be the set of all such sequences. Then the Knave imposes a map $k : \mathcal{S} \to \mathcal{S}$.

It will be convenient to view finite strings as belonging to $2^{\mathbb{N}}$. We say that a string whose final bit is $\mathtt{0}$ is followed by a tail of all $\mathtt{1}$s and *vice versa*. For example,

$$\mathtt{101} \leftrightarrow \mathtt{101000}\ldots,$$
$$\mathtt{100} \leftrightarrow \mathtt{100111}\ldots.$$

Our Knave does not have the patience for these infinite matters, so when we do compel them to act on $2^{\mathbb{N}}$, the Knave will report

$$\mathtt{000}\ldots$$

as

$$\mathtt{111}\ldots$$

and *vice versa*. Thus, these tails will never interfere with the preceding string. We will (somewhat abusively) treat these either as sequences or strings, depending on which is more convenient.

Note that k is not invertible; already

$$k(\mathtt{10}) = k(\mathtt{00000}) = 1011.$$

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3. Metamorphosis

For a natural number n, let $[n]$ denote the string which represents n in base two. We will call any string of n $\mathtt{0}$s or k $\mathtt{1}$s a *ribbit*, short for repeated bit. If we need to clarify what bit is repeated, we can say that $\mathtt{111}$ is a *ribbit of three* $\mathtt{1}$*s* or an *odd ribbit*. Likewise, $\mathtt{000}$ is a *ribbit of three* $\mathtt{0}$*s* and an *even ribbit*. Thus, any binary sequence $S \in \mathcal{S}$ decomposes into a sequence of ribbits of alternating parity.

Let $S \in \mathcal{S}$. Since the Knave must begin the report with a $\mathtt{1}$, we assume that S begins with an odd ribbit. Then S decomposes into ribbits as

$$S = r_1\ r_2\ r_3\ \ldots .$$

Happily, this means that odd ribbits are indexed by odd subscripts and *vice versa*.

We may write

$$k(S) = [|r_1|]\ \mathtt{0}\ [|r_2|]\ \mathtt{1}\ [|r_3|]\ \mathtt{0}\ldots .$$

It is unfortunate here that the leftmost $\mathtt{1}$ arising from $r_{2\ell+1}$ will always form a ribbit with $[|r_{2\ell+2}|]$, as in

$$k(101) = 101110.$$

However, the decomposition of s_n into even and odd ribbits allows us to get the Knave's reports piecemeal; keeping

$$S = r_1\ r_2\ r_3\ \ldots$$

with r_1 odd, then

$$k(S) = k(r_1)\ k(r_2\ r_3)\ k(r_4\ r_5)\ldots .$$

Thus, we can determine the behaviour of k by examining all possible pairs of ribbits occurring in the decomposition of all s_n. Fortunately, there are not many to check. We will call a ribbit r belonging to S *maximal* if it is not contained in a ribbit of larger size.

LEMMA 3.1. *Let $\{s_n\}$ be the Look-Knave sequence. A maximal ribbit occurring in s_n cannot have length greater than five.*

PROOF. Suppose that n is the smallest index such that s_n contains a ribbit r of length six or greater, either

$$s_n = \ldots \mathtt{1}\overbrace{\mathtt{0}\ldots\mathtt{0}}^{\geq 6}\mathtt{1}\ldots$$

or

$$s_n = \ldots \mathtt{0}\overbrace{\mathtt{1}\ldots\mathtt{1}}^{\geq 6}\mathtt{0}\ldots .$$

What is s_n describing? Or, rather, what *isn't* s_n describing? If r is even, then s_{n-1} contains a ribbit of length at least 64; this ribbit can only occur if s_{n-1} has a ribbit r' such that the binary representation of $|r'|$ has at least six $\mathtt{0}$s. This is a contradiction.

The case where r is odd is more complicated. We already see that such an r could arise from an r' in s_{n-1}, where the binary representation of $|r'|$ has at least five 1s, which is again impossible.

However, r could represent the concatenation of two separate descriptions of ribbits; the first even and the second odd. In this case,

$$s_n = \ldots \overbrace{\ldots \mathtt{01} \ldots \mathtt{1}}\ \mathtt{1}\ \overbrace{\mathtt{1} \ldots \mathtt{10} \ldots}\ \mathtt{0}\ \ldots,$$

where the first overbrace indicates the binary expansion of the length of an odd ribbit in s_{n-1}, and the second overbrace indicates the binary expansion of the length of an even ribbit in s_{n-1}. From our assumption on n, we see that the only acceptable arrangement is

$$s_n = \ldots \overbrace{\ldots \mathtt{111}}\ \mathtt{1}\ \overbrace{\mathtt{11}}\ \mathtt{0}\ \ldots.$$

Unfortunately,

$$\overbrace{\ldots \mathtt{111}}$$

is the binary expansion of some $n \geq 7$ and we croak. □

In fact, once we know the bound for maximal ribbits in general, we can tighten up the proof for some edge cases.

COROLLARY 3.2. *A maximal even ribbit occurring in s_n cannot have length greater than three.*

COROLLARY 3.3. *If 11111 occurs in s_n, it is not preceded by* $\mathtt{000}$.

PROOF. If we consider s_n as

$$s_n = \ldots \overbrace{\mathtt{00}}\ \mathtt{0}\ \overbrace{\mathtt{11}}\ \mathtt{1}\ \overbrace{\mathtt{11}}\ \mathtt{0}\ \ldots,$$

then s_{n-1} contains at least four 1s. Otherwise, if we consider s_n as

$$s_n = \ldots \overbrace{\ldots \mathtt{011}}\ \mathtt{1}\ \overbrace{\mathtt{11}}\ \mathtt{0}\ \ldots,$$

then s_{n-1} contains at least 11 1s. Both possibilities contradict our previous results. □

We may now examine the Knave's behaviour on all possible ribbit pairs (r, r') occurring in some s_n, with r' possibly empty. This is shown in Table 2. Note that in all cases, $k(r\ r')$ is no shorter than rr'.

From our observation in Table 1, we want to determine if the sequences $\{s_{2n+1}\}$ and $\{s_{2n+2}\}$ converge in $\mathcal{S}$. To this end, we will endow $2^{\mathbb{N}}$ with a simple metric. Two distinct binary sequences S, S' which first differ at the nth bit satisfy $d(S, S') = 2^{-n}$. As expected, we set $d(S, S) = 0$. Note that $\mathcal{S}$ is not complete under this metric, but $2^{\mathbb{N}}$ is.

For $\ell \geq 1$, let β_ℓ be the string given by the first ℓ bits in s_ℓ, which are then truncated to the last maximal ribbit. Here β stands for $\beta\rho\epsilon\kappa\epsilon\kappa\epsilon\kappa\epsilon\kappa$, of course. For example, β_3 is the string $\mathtt{10}$, taken from $s_3 = \mathtt{1011}$.

LEMMA 3.4. *For $\ell \geq 1$, the strings $s_{\ell+1}$ and $s_{\ell+3}$ agree up to the $(|\beta_\ell| + 1)$th bit.*

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TABLE 2. Elements of the Knave map.

$r\,r'$	$k(r\,r')$
0	1
00	101
000	111
1	10
01	1110
001	10110
0001	11110
011	11100
0011	101100
00011	111100
0111	11110
00111	101110
000111	111110
01111	111000
001111	1011000
0001111	1111000
011111	111010
0011111	1011010

PROOF. We induct on ℓ. According to Table 2, we have $|k(r\,r')| \geq |r\,r'|$ for all elements of the Knave map. Because β_ℓ begins with 10, we see that $|k(\beta_\ell)| > |r_\ell|$. In the induction, we see that the first $|\beta_\ell|$ bits of s_ℓ and $s_{\ell+2}$ determine at least the first $|\beta_\ell| + 1$ bits of $s_{\ell+1}$ and $s_{\ell+3}$. □

Note that $\ell - 4 \leq |r_\ell| \leq \ell$.

COROLLARY 3.5. *The sequences* $\{k^{2n}(1)\}$ *and* $\{k^{2n}(10)\}$ *converge in* $\mathcal{S}$.

Thus, we can take $S_{even} = \lim_{n\to\infty} k^{2n}(10)$ and $S_{odd} = \lim_{n\to\infty} k^{2n}(1)$. It turns out that not only are S_{even} and S_{odd} fixed points of k^2, they attract all other orbits under k in $\mathcal{S}$.

THEOREM 3.6. *Let* $S \in \mathcal{S}$ *be a binary sequence. Then either*

$$\lim_{n\to\infty} d(k^n(S), k^n(1)) = 0$$

or

$$\lim_{n\to\infty} d(k^n(S), k^n(10)) = 0.$$

PROOF. We see that $k(S)$ must begin with an odd ribbit. If $k(S)$ begins with an odd ribbit of length $\ell \geq 2$, then $k^2(S)$ begins with an odd ribbit of length strictly less than ℓ.

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Otherwise, $k(S)$ begins with $\mathtt{10}$ and so does $k^2(S)$. Thus, some iterate $k^n(S)$ begins with $\mathtt{10}$.

Then $k^{n+1}(S)$ begins with $\mathtt{101}$, and $k^{n+2}(S') = \mathtt{10}r\ldots$, where r is a maximal odd ribbit of length at least three. If $|r| \geq 5$, then $k^{n+3}(S) = \mathtt{10}r'\ldots$, where r' is a maximal odd ribbit of length at most $2 + \log_2(r)$. Further iteration of the Knave map reduces to the case $|r| = 3, 4$.

If $|r| = 3, 4$, using the argument in Lemma 3.4, we see that the prefix of $k^{n+2}(S)$ determines a longer prefix of $k^{n+4}(S')$ and so on. Then

$$\lim_{n\to\infty} d(k^n(S), k^n(1)) = 0$$

or

$$\lim_{n\to\infty} d(k^n(S), k^n(\mathtt{10})) = 0,$$

depending on the parities of $|r|$ and n. □

Corollary 3.7. *Let S be any binary sequence in $\mathcal{S}$. Then $\lim_{n\to\infty} k^{2n}(S)$ exists and is equal to one of S_{even} or S_{odd}.*

Corollary 3.8. *The only fixed points of k^2 in $\mathcal{S}$ are S_{even} and S_{odd}.*

Corollary 3.9. *The only fixed points of k^2 in $2^{\mathbb{N}}$ are S_{even} and S_{odd}, which are attracting, and $\mathtt{000}\ldots$ and $\mathtt{111}\ldots$, which are repelling.*

4. Future study

We have left open the question of the asymptotic growth of $|s_n|$. Experimentally, we expect that

$$\lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} = 1.12\ldots.$$

Adapting Johnston's argument to this problem would be an appropriate problem for a student.

Further, we conjecture that the binary strings S are in fact the sections of a larger dynamical system via the diagonal entries of certain Kermitian matrices.

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References

[1] J. H. Conway, 'The weird and wonderful chemistry of audioactive decay', *Open Problems in Communication and Computation* (eds. T. M. Cover and B. Gopinath) (Springer, New York, NY, 1987), 173–188.

[2] N. Johnston, 'The binary 'look-and-say' sequence', http://www.njohnston.ca/2010/11/the-binary-look-and-say-sequence/, 2010.

[3] R. M. Smullyan, *What is the Name of This Book? The Riddle of Dracula and Other Logical Puzzles* (Prentice-Hall, Upper Saddle River, NJ, 1978).
[4] *The On-Line Encyclopedia of Integer Sequences* (2020), https://oeis.org/A005150.
[5] *The On-Line Encyclopedia of Integer Sequences* (2020), https://oeis.org/A014715.
[6] *The On-Line Encyclopedia of Integer Sequences* (2020), https://oeis.org/A001387.

THOMAS MORRILL, School of Science,
The University of New South Wales Canberra, Australia
e-mail: t.morrill@adfa.edu.au; morrillt@trine.edu

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