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Radial symmetry of large solutions of semilinear elliptic equations with convection

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We study the radial symmetry of large solutions of the semilinear elliptic problem $\Delta u + \nabla h \cdot \nabla u = f(|x|, u)$, and we provide sharp conditions under which the problem has a radial solution. The result is independent of the rate of growth of the solution at infinity.

1. Introduction

The radial symmetry of the solutions of $\Delta u = f(|x|, u)$ on $\mathbb{R}^n$ is a well-studied problem and various conditions on the rate of growth and monotonicity of $f(|x|, u)$, as well as the behaviour of $u(x)$ at infinity, have been presented to guarantee radial symmetry of the solutions. In this paper we study the radial symmetry of large solutions of the semilinear elliptic problem

$$\left.\begin{aligned} \Delta u(x) + \nabla h(x) \cdot \nabla u(x) = f(|x|, u(x)), &\quad x \in \mathbb{R}^n \ (n \geqslant 2), \\ u(x) \to \infty, &\quad x \to \infty. \end{aligned}\right\} \tag{1.1}$$

We assume that for large values of $|x|$ and u the function $f(|x|, u)$ is positive and superlinear, and that $\lim_{|x|\to\infty} u(x) = \infty$, but we do not assume a particular rate of growth at infinity for the solution. Our main focus is the effect of the convection term on the radial symmetry of the solutions. The case $h \equiv 0$ with a similar setting has been studied in [6, 7] and, in contrast to the large boundary condition, symmetry of the small solutions $\lim_{|x|\to\infty} u(x) = 0$ of the same problem has been studied in [1–4].

If $u(x)$ is radial, then all of the terms in (1.1), except perhaps $h(x)$, will be radial, which automatically implies radial symmetry of $h(x)$, at least whenever u is not constant. Thus, it is natural to assume that h is radial and, whenever clear, we abuse the notation $h(x) = h(|x|)$. We require that the convection term h satisfy a

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particular integrability condition given by

$$\int_1^\infty \mathrm{e}^{-h(r)} r^{1-n}\,\mathrm{d}r < \infty. \tag{1.2}$$

This condition is shown to be sharp in the sense that if violated, while all the other conditions hold, there are examples with no radial solution. Having this condition on h, a change of variable is proved to be well defined, which converts the radial solutions of the partial differential equation (PDE) into the solutions of a corresponding ordinary differential equation (ODE). The available ODE theory developed in [6] combined with comparison arguments can then be used to prove the existence and symmetry of the solutions.

2. Statements and proofs

To set the appropriate conditions on $f(|x|, u)$, we compare it with a function $g(r, s)$ that satisfies the following conditions.

(c1) $g(r, s)$ and $g_s(r, s)$ are continuous and positive on

$$\Omega = \{(r, s) \mid r > r_0,\ s > s_0\},$$

where r_0, s_0 are positive constants.

(c2) $g(r, s)$ is superlinear in s on Ω in the sense that there exists $\lambda > 1$ such that $g(r, vs) \geqslant v^\lambda g(r, s)$ for all $v > 1$ and $(r, s) \in \Omega$.

(c3) $p(r)\mathrm{e}^{h(r)} g(r, s)$ is monotone in r on Ω, where the function $p(r)$ is given by

$$p(r) := -\int_r^\infty \mathrm{e}^{-h(z)} z^{1-n}\,\mathrm{d}z. \tag{2.1}$$

The following theorem is the main result of this paper.

Theorem 2.1. *Let $h(r)$ be continuous and satisfy (1.2). Let $f(r, s)$ and $f_s(r, s)$ be continuous and positive. Assume that there exists a function $g(r, s)$ such that*

$$\lim_{(r,s)\to(\infty,\infty)} \frac{f(r, s)}{g(r, s)} = 1,$$

where $g(r, s)$ satisfies (c1)–(c3). Assume also that $f(r, s)$ is superlinear in s on Ω. Then, the following hold.

(i) *All C^2-solutions of (1.1) are radial.*

(ii) *If (1.1) has a C^2-solution, then there exist $R \geqslant 0$, $\hat{u} > 0$ such that*

$$-\int_{|x|>R} p(|x|)\mathrm{e}^{h(|x|)} f(x, s)\,\mathrm{d}x < \infty \quad \forall s > \hat{u}, \tag{2.2}$$

where $p(r)$ is given by (2.1).

(iii) *If, in addition, $f(|x|, u)$ satisfies (c3), (2.2) is also a sufficient condition for the existence of a solution to (1.1).*

In [6] and [7], Taliaferro studies the relevance of the conditions on $f(r,s)$ for the problem without the convection term. For example, it is shown that the superlinearity of $f(r,s)$ is a sharp condition for the radial symmetry of the solutions of (2.1). Indeed, there are non-radial solutions of (2.1) when this condition fails.

Condition (1.2) on h is a sharp condition in the sense that if it does not hold, then there are cases with no radial solution to (1.1). To see this, let $h(x) = \beta \log(|x|)$. Note that (1.2) holds for $\beta > 2-n$ and it is violated if $\beta \leqslant 2-n$. Consider the critical case when $\beta = 2-n$ and let $f(r,s) = f(s)$ be a superlinear function. We claim that there is no radial solution to (1.1). Assume, on the contrary, that there exists a radial solution $u(x) = u(|x|)$. We have that

$$\begin{aligned} f(u) &= \Delta u(x) + \nabla \log(|x|^{(2-n)}) \cdot \nabla u(|x|) \\ &= \Delta u(x) + (2-n)\frac{x}{|x|^2} \cdot \frac{x}{|x|} u'(|x|). \end{aligned}$$

Hence,

$$\begin{aligned} f(u) &= \left\{ u''(r) + \frac{n-1}{r} u'(r) \right\} + \frac{2-n}{r} u'(r) \\ &= u''(r) + \frac{1}{r} u'(r), \end{aligned}$$

where $r = |x|$. Now, define the radial function $v \colon \mathbb{R}^2 \to \mathbb{R}$ by $v(y) := u(|y|)$. Then, v is a radial solution of $\Delta v = f(v)$ in $\mathbb{R}^2$. This is a contradiction because Osserman showed in [5] that for a superlinear function $f(v)$ the problem $\Delta v(x) = f(v)$ has no large solution in $\mathbb{R}^2$. Therefore, (1.1) has no radial solution or it has no solution at all, which both indicate necessity of the condition (1.2).

Based on (1.2) on h, we can use the following change of variables to transform radial solutions of (1.1) into the corresponding ODE solutions.

LEMMA 2.2. *Let $h(r)$ satisfy (1.2) and let $f(|x|,u)$ satisfy the conditions of theorem 2.1. Then, $u(x)$ is a radial solution of (1.1) if and only if $z(t) := u(p^{-1}(t))$ solves*

$$\left.\begin{aligned} z''(t) &= F(t, z(t)), \\ \lim_{t \to 0^-} z(t) &= \infty, \end{aligned}\right\} \tag{2.3}$$

where $p(r)$ is given by (2.1) and $F(t,z)$ is given by

$$F(t,z) := (p^{-1}(t))^{2n-2} \exp(2h(p^{-1}(t))) f(p^{-1}(t), z). \tag{2.4}$$

Proof. Let $r = |x|$, $t = p(r)$ and $z(t) = u(p^{-1}(t))$. This is a valid change of variable because, by definition, $p(r)$ is continuous and strictly increasing. We have that

$$\begin{aligned} f(r, u(r)) &= \left[u''(r) + \frac{n-1}{r} u'(r) \right] + h'(r) u'(r) \\ &= p'(r)^2 z''(p(r)) + \left(p''(r) + \frac{n-1}{r} p'(r) + h'(r) p'(r) \right) z'(p(r)) \\ &= p'(r)^2 z''(p(r)). \end{aligned}$$

Therefore,

$$\begin{aligned} z''(p(r)) &= \frac{1}{p'(r)^2} f(r, z(p(r))) \\ &= \mathrm{e}^{2h(r)} r^{2n-2} f(r, z(p(r))) \\ &= F(p(r), z(p(r))). \end{aligned}$$

Also, the boundary condition $\lim_{|x|\to\infty} u(x) = \infty$ is equivalent to $\lim_{t\to 0^-} z(t) = \infty$, because $\lim_{r\to\infty} p^{-1}(r) = 0$. □

REMARK 2.3. Note that the definition of $F(t, z)$ implies that for large values of t and z both $F(t, z)$ and $F_z(t, z)$ are continuous and non-negative, and that $F(t, z)$ is superlinear in z. This fact is useful when we study the ODE that corresponds to (1.1). Lemma 2.2 plays an important role in our arguments. In particular, in the proof of theorem 2.1 we need to construct two sequences of radial functions for the comparison arguments. The sequences can be constructed, with the help of lemma 2.2, from the ODE counterparts described in lemma A.2, as follows. Assuming that the conditions of lemma 2.2 hold, for each $M, m > s_0$ and $r_1 > r_0$ there exist an increasing sequence $\{\rho_k\}_{k=1}^{\infty} \subseteq (r_1, \infty)$, with $\lim_{k\to\infty} \rho_k = \infty$, and two sequences of C^2-radial functions $\{u_k(x)\}$ and $\{U_k(x)\}$ such that

(i) $U_0(x)$ and $u_0(x)$, $u_1(x)$, ... are radial solutions of

$$\begin{aligned} \Delta u(x) + \nabla h(x) \cdot \nabla u(x) &= f(|x|, u(x)), && |x| \geqslant r_1, \\ u(x) &= m, && |x| = r_1, \end{aligned}$$

(ii) $U_1(x), U_2(x), \ldots$ are solutions of

$$\begin{aligned} \Delta U_k(x) + \nabla h(x) \cdot \nabla U_k(x) &= f(|x|, U_k(x)), && r_1 \leqslant |x| \leqslant \rho_k, \\ U_k &\to \infty, && |x| \to \rho_k^-, \\ U_k(x) &= M, && |x| = r_1, \end{aligned}$$

(iii) $\lim_{|x|\to\infty} u_0(x) = \lim_{|x|\to\infty} U_0(x) = \infty$,

(iv) $u_1(x)$, $u_2(x)$, ... are all bounded as $|x| \to \infty$ and

(v) for each $|x| > r_1$ we have $\lim_{k\to\infty} u_k(x) = u_0(x)$ and $\lim_{k\to\infty} U_k(x) = U_0(x)$.

We are now ready to prove theorem 2.1.

Proof of theorem 2.1(iii). We start by proving part (iii) where we have the additional monotonicity condition (c3) on $f(|x|, u)$. In fact, we prove that, having (c3), (2.2) is both necessary and sufficient for the existence of a solution to (1.1). This fact will be useful in the proof of other parts. Let $t = p(|x|)$, where $p(r)$ is given by (2.1). We have that

$$-\int_{\Omega} p(|x|)\mathrm{e}^{h(|x|)} f(|x|, s)\,\mathrm{d}x = -\sigma_n \int_R^{\infty} r^{n-1} p(r) \mathrm{e}^{h(r)} f(r, s)\,\mathrm{d}r,$$

where σ_n is the perimeter of the unit ball in $\mathbb{R}^n$. Therefore,

$$
\begin{aligned}
-\int_{\Omega} p(|x|)\mathrm{e}^{h(|x|)} f(x,s)\,\mathrm{d}x &= -\sigma_n \int_R^{\infty} r^{2n-2} p(r)\mathrm{e}^{2h(r)} f(r,s)(\mathrm{e}^{-h(r)} r^{(1-n)})\,\mathrm{d}r \\
&= -\sigma_n \int_R^{\infty} p(r) F(p(r),s) p'(r)\,\mathrm{d}r \\
&= -\sigma_n \int_{t_0}^{0} tF(t,s)\,\mathrm{d}t, \qquad (2.5)
\end{aligned}
$$

where in the second equality we used the fact that $p'(r) = \mathrm{e}^{-h(r)} r^{(1-n)}$. Assuming that (2.2) holds, (2.5) implies that

$$
-\int_{t_0}^{0} tF(t,s)\,\mathrm{d}t < \infty \quad \forall s > \hat{u}. \qquad (2.6)
$$

By lemma A.1, (2.6) is a necessary and sufficient condition for the existence of a solution $z(t)$ of $z''(t) = F(t, z(t))$. By lemma 2.2, the solution $z(t)$ of the ODE $z''(t) = F(t, z(t))$ can be transformed into a radial solution $u(x) = z(p(|x|))$ of (1.1). Conversely, if there is a radial solution to (1.1), using lemma 2.2, we can transform it into a solution of $z'' = F(t, z)$. This implies that (2.6) holds. Therefore, by (2.5) we have that condition (2.2) is true. □

Proof of theorem 2.1(ii). Let $g(r, s)$ be as in the statement of the theorem. Because

$$
\lim_{(|x|,s)\to(\infty,\infty)} \frac{f(x,s)}{g(x,s)} = 1,
$$

without loss of generality we can assume that r_0 and s_0 are large enough such that $g(r, s) < 2f(r, s)$ on Ω. Define $l(r, s) := \frac{1}{2}g(r, s)$. We work with $l(r, s)$ because we want to use the monotonicity condition (c3), which is not available for $f(|x|, u)$. We claim that if (1.1) has a solution, then the problem

$$
\left.
\begin{aligned}
\Delta y(x) + \nabla h(x)\cdot\nabla y(x) &= l(|x|, u(x)), \quad && x \in \mathbb{R}^n \ (n \geqslant 2), \\
y(x) &\to \infty, && x \to \infty,
\end{aligned}
\right\} \qquad (2.7)
$$

has a radial solution. Assume, on the contrary, that (1.1) has a solution, while there is no radial solution to (2.7). To reach a contradiction, we study another related PDE. Consider the constants $s_1 > s_0$ and $r_1 > r_0$ such that

$$
\max_{|x|=r_0} u(x) < s_1 < \max_{|x|=r_1} u(x). \qquad (2.8)
$$

These constants exist because $\lim_{|x|\to\infty} u(x) = \infty$. We want to prove that there exists $r_2 > r_1$ such that there exists a radial solution to the PDE

$$
\left.
\begin{aligned}
\Delta v(x) + \nabla h(x)\cdot\nabla v(x) &= l(x, v), \quad && r_0 < |x| < r_2, \\
v(x) &= s_1, && |x| = r_0, \\
v(x) &= s_1, && |x| = r_1, \\
v(x) &\to \infty, && |x| \to r_2^{-}.
\end{aligned}
\right\} \qquad (2.9)
$$

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Setting

$$t = p(r), \quad z(t) = v(p^{-1}(t)) \quad \text{and} \quad F(t,z) = (p^{-1}(t))^{2n-2} \mathrm{e}^{2a(p^{-1}(t))} l(p^{-1}(t), z),$$

the problem of finding r_2 is equivalent to finding $t_2 \in (t_1, 0)$ such that there exists a solution to

$$\left.\begin{aligned} z''(t) &= F(t,z), \\ z(t_0) &= z(t_1) = s_1, \\ \lim_{t \to t_2^-} z(t) &= \infty. \end{aligned}\right\} \tag{2.10}$$

Note that because we assumed that (2.7) has no solution, proof of part (iii) implies that

$$\int_{|x|>r_0} -p(|x|)\mathrm{e}^{h(|x|)} l(x,s)\,\mathrm{d}x = \infty, \tag{2.11}$$

which, again using part (iii), results in

$$-\int_{t_0}^{0} tF(t,s)\,\mathrm{d}t = \infty. \tag{2.12}$$

On the bounded interval $[t_0, t_1]$, with bounded boundary values $z(t_0) = z(t_1) = s_1$, we can use the Green function of $-\mathrm{d}^2/\mathrm{d}t^2$ to find a solution to $z'' = F(t,z)$ on this domain. Let $t_2 > t_1$ be the maximal time where $z(t)$ continuously solves $z'' = F(t,z)$. Since $z''(t) = F(t,z) \geqslant 0$ and $z(t_0) = z(t_1)$, we have that $z'(t) \geqslant 0$. There are only three possibilities. The first case is when $t_2 = 0$ and $\lim_{t\to 0^-} z(t) = \infty$. This possibility is ruled out because (2.12) implies that (2.3) has no solution. The second possibility is that $t_2 = 0$ and $\lim_{t\to 0^-} z(t) < \infty$. In this case, by integrating $z'' = F(t,z)$ twice we have that

$$-\int_{t_1}^{0} tF(t,z)\,\mathrm{d}t = z(0^-) - z(t_1) + t_1 z'(t_1) < \infty,$$

which is a contradiction by (2.12). The only remaining possibility is that $t_2 \in (t_1, 0)$ and $\lim_{t\to t_2^-} z(t) = \infty$. Therefore, we find t_2 with the required conditions. By converting (2.10) back into the corresponding PDE, there exists $r_2 = p^{-1}(t_2) \in (r_1, \infty)$ such that there is a radial solution to (2.9). The set $\Sigma = \{x \in (r_0, r_2) \mid u(x) > v(x)\}$ is open and non-empty because of the definition of s_1. Since $f(r,s) \geqslant h(r,s)$ on $\Sigma \subset \Omega$, we have that

$$\Delta(u-v)(x) + \nabla h(x)\cdot\nabla(u-v)(x) = f(x,u) - l(x,v) > 0 \quad \forall x \in \Sigma.$$

But $u(x) - v(x) = 0$ on $\partial\Sigma$. This is a contradiction by the maximum principle. Hence, assumption (2.11) is not true. Therefore, if (1.1) has a solution, then

$$\int_{\Omega} -p(|x|)\mathrm{e}^{h(|x|)} l(x,s)\,\mathrm{d}x < \infty.$$

Because

$$\lim_{(r,s)\to(\infty,\infty)} \frac{f(r,s)}{l(r,s)} = \frac{1}{2},$$

we have that

$$\exists \hat{s} \geqslant s_0 \int_{|x|>r_0} -p(|x|)\mathrm{e}^{h(|x|)} f(x,s)\,\mathrm{d}x < \infty \quad \forall s > \hat{s}.$$

□

Proof of theorem 2.1(i). We start by showing that the difference of any two C^2-solutions $u_a(x)$ and $u_b(x)$ of PDE (1.1) goes to zero at infinity. First, assume that $y_a(x)$ and $y_b(x)$ are two radial solutions of the PDE. By setting

$$t = p(r), \qquad z(t) = y(p^{-1}(t)),$$
$$F(t,z) := p^{-1}(t)^{2n-2}\exp(2h(p^{-1}(t)))f(p^{-1}(t),z),$$

lemma 2.2 implies that we can find two solutions $z_a(p(|x|)) = y_a(x)$ and $z_b(p(|x|)) = y_b(x)$ of the corresponding ODE. By lemma A.1, the difference of any two large solutions of the ODE $z'' = F(t,z)$ goes to zero as $t \to 0^-$. This implies that $\lim_{|x|\to\infty}|y_a(x) - y_b(x)| = 0$.

Let $r > 0$ be large enough such that $u(x) > s_0$ for $|x| > r$. Let $m = \min_{|x|=r} u_a(x)$ and $M = \max_{|x|=r} u_a(x)$. Now, consider the sequences $u_k(x)$ and $U_k(x)$ described in the remark of lemma 2.2. By the construction, $u_a(x) - u_k(x) > 0$ on $|x| = r$ and $\lim_{|x|\to\infty} u_a(x) - u_k(x) = \infty$. Since $f_s(r,s) \geqslant 0$, we have that

$$\Delta(u_a(x) - u_k(x)) + \nabla h(x)\cdot\nabla(u_a(x) - u_k(x)) = f(x,u_a(x)) - f(x,u_k(x)) \geqslant 0.$$

Therefore, the maximum principle implies that $u_a(x) \geqslant u_k(x)$ for all k and all $|x| > r$. Hence,

$$u_a(x) \geqslant u_0(x) = \lim_{k\to\infty} u_k(x) \quad \text{for } |x| > r.$$

Similarly, $u_a(x) \leqslant U_k(x)$ for $r < |x| < p_k$. Since $\lim_{k\to\infty} p_k = \infty$, we have that $u_a(x) \leqslant U_0(x)$ on $|x| > r$. Furthermore, $u_0(x)$ and $U_0(x)$ are two radial solutions of (1.1). By the discussion at the beginning of this step, $\lim_{|x|\to\infty}|U_0(x) - u_0(x)| = 0$. Since $u_0(x) \leqslant u_a(x) \leqslant U_0(x)$, we have that $\lim_{|x|\to\infty}|u_a(x) - u_0(x)| = 0$. By a similar argument for $u_b(x)$, we have that $\lim_{|x|\to\infty}|u_b(x) - u_0(x)| = 0$.

Now, assume that R is an orthonormal transformation on $\mathbb{R}^n$. We have that

$$\begin{aligned}[\nabla(h(Rx))]\cdot[\nabla(u(Rx))] &= [\nabla(h(Rx))]^{\mathrm{T}}[\nabla(u(Rx))] \\ &= [(\nabla h)(Rx)]RR^{\mathrm{T}}[(\nabla u)(Rx)] \\ &= \nabla h\cdot\nabla u(R(x)).\end{aligned}$$

Furthermore, the Laplace operator is interchangeable with orthonormal operators in the sense that for $u_R(x) = u(R(x))$ we have that $\Delta u(R(x)) = \Delta u_R(x)$. Therefore, for a given solution $u(x)$ of (1.1) we have that

$$\Delta(u_R - u)(x) + \nabla h\cdot\nabla(u_R - u)(x) = f(x,u_R) - f(x,u).$$

By the argument at the beginning of the proof, we know that $\lim_{|x|\to\infty}|u_R - u| = 0$. Because $f_s(r,s) \geqslant 0$, the maximum principle implies that $u_R \equiv u$. Therefore, $u(x)$ is radial. □

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Appendix A.

In this appendix we gather the statements of the ODE lemmas required for our arguments. See [6] for the proofs of the lemmas.

LEMMA A.1. *Let $\Gamma = \{(t,z) \mid \hat{t} \leqslant t < 0,\ 0 < \hat{z} < z\}$ be given. Assume that $F(t,z)$ and $F_z(t,z)$ are C^0 and non-negative on Γ. Assume also that $F_z(t,z)$ is superlinear in z and that $F(t,z)$ is monotone in t on Γ. Then, the problem*

$$\left.\begin{aligned} z''(t) &= F(t,z(t)),\\ \lim_{t\to 0^-} z(t) &= \infty \end{aligned}\right\} \tag{A 1}$$

has a C^2-solution if and only if there exists $c \in (\hat{t}, 0)$ such that

$$-\int_c^0 tF(t,z)\,\mathrm{d}t < \infty \quad \forall z > \hat{z}. \tag{A 2}$$

Furthermore, for any pair of solutions $z_1(t)$, $z_2(t)$ to (A 1) we have that

$$\lim_{t\to 0^-} |z_2(t) - z_1(t)| = 0.$$

LEMMA A.2. *Let $\Gamma = \{(t,z) \mid \hat{t} \leqslant t < 0,\ 0 < \hat{z} < z\}$ be given. Assume that $F(t,z)$ and $F_z(t,z)$ are C^0 and non-negative on Γ. Assume also that $F(t,z)$ is superlinear in z on Γ. Then, for each $\bar{z} > \hat{z}$ and $\bar{t} > \hat{t}$ there exist a sequence $\{\rho_k\}_{k=0}^{\infty} \subseteq (\bar{t}, 0)$, with $\lim_{k\to\infty}\rho_k = 0$, and two sequences of C^2-functions $\{z(t)\}_{k=0}^{\infty}$ and $\{Z(t)\}_{k=0}^{\infty}$ such that*

(i) *$Z_0(t)$ and $z_0(t)$, $z_1(t)$, … are solutions of*

$$\begin{aligned} z''(t) &= F(t,z(t)), \quad t \geqslant t_1,\\ z(\bar{t}) &= \bar{z}, \end{aligned}$$

(ii) *for all $k \geqslant 1$, $Z_k(t)$ is a solution of*

$$\begin{aligned} Z_k''(t) &= F(t,Z_k(t)), \quad t_1 \leqslant t < \rho_k,\\ \lim_{t\to\rho_k^-} Z_k(t) &= \infty,\\ Z_k(\bar{t}) &= \bar{z}, \end{aligned}$$

(iii) *$\lim_{t\to 0^-} z_0(t) = \lim_{t\to 0^-} Z_0(t) = \infty$,*

(iv) *for all $k \geqslant 1$, $z_k(t)$ is finite as $t \to 0^-$ and*

(v) *for each $t \in (\bar{t}, 0)$ we have that*

$$\lim_{k\to\infty} z_k(t) = z_0(t) \quad \text{and} \quad \lim_{k\to\infty} Z_k(t) = Z_0(t).$$

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