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Abstract

Let $f\colon Y \to X$ be a proper flat morphism of locally noetherian schemes. Then the locus in $X$ over which $f$ is smooth is stable under generization. We prove that, under suitable assumptions on the formal fibers of $X$, the same property holds for other local properties of morphisms, even if $f$ is only closed and flat. Our proof of this statement reduces to a purely local question known as Grothendieck's localization problem. To answer Grothendieck's problem, we provide a general framework that gives a uniform treatment of previously known cases of this problem, and also solves this problem in new cases, namely for weak normality, seminormality, $F$-rationality, and the 'Cohen–Macaulay and $F$-injective' property. For the weak normality statement, we prove that weak normality always lifts from Cartier divisors. We also solve Grothendieck's localization problem for terminal, canonical, and rational singularities in equal characteristic zero.

## 1. Introduction

Let $f\colon Y \to X$ be a proper flat morphism of locally noetherian schemes. By [EGAIV$_3$, Théorème 12.2.4(iii)], the locus of points $x \in X$ such that $f^{-1}(x)$ is smooth over $\kappa(x)$ is open, and, in particular, is stable under generization. In [EGAIV$_3$, (12.0.2)], Grothendieck and Dieudonné asked whether similar statements hold for other local properties of morphisms, in the following sense.

*Question* 1.1. Let $\mathbf{R}$ be a property of noetherian local rings, and consider a proper flat morphism $f\colon Y \to X$ of locally noetherian schemes. Is the locus

$$U_{\mathbf{R}}(f) \coloneqq \{x \in X \mid f^{-1}(x) \text{ is geometrically } \mathbf{R} \text{ over } \kappa(x)\} \subseteq X$$

stable under generization?

Question 1.1 was answered for many properties $\mathbf{R}$ in [EGAIV$_3$, § 12], and is a global version of Problem 1.2 below, which is known as Grothendieck's localization problem (see, for example, [AF94]). Our goal is to provide a general framework with which to answer Question 1.1, assuming that $\mathbf{R}$ is well behaved in the sense that it satisfies the following four permanence conditions.

($\mathrm{R}'_{\mathrm{I}}$) (Ascent via geometrically $\mathbf{R}$ homomorphisms) For every flat local homomorphism $A \to B$ of noetherian local rings with geometrically $\mathbf{R}$ fibers, if $A$ satisfies $\mathbf{R}$, then $B$ satisfies $\mathbf{R}$.

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($\mathrm{R_{II}}$) (Descent) For every flat local homomorphism $A \to B$ of noetherian local rings, if $B$ satisfies $\mathbf{R}$, then $A$ satisfies $\mathbf{R}$.

($\mathrm{R_{IV}}$) (Lifting from Cartier divisors) For every noetherian local ring $A$ and for every nonzerodivisor $t$ in its maximal ideal, if $A/tA$ satisfies $\mathbf{R}$, then $A$ satisfies $\mathbf{R}$.

($\mathrm{R_V}$) (Localization) If a noetherian local ring $A$ satisfies $\mathbf{R}$, then $A_{\mathfrak{p}}$ satisfies $\mathbf{R}$ for every prime ideal $\mathfrak{p} \subseteq A$.

These conditions on $\mathbf{R}$ are studied in [EGAIV$_2$, § 7] (see also Conditions 3.1), and are satisfied by many common properties $\mathbf{R}$ (see Table 2). The notation ($\mathrm{R_I'}$) is used instead of ($\mathrm{R_I}$) because the latter condition in [EGAIV$_2$, (7.3.10)] asserts that $\mathbf{R}$ ascends via geometrically *regular* homomorphisms. The condition ($\mathrm{R_{IV}}$) is called 'deformation' in commutative algebra, is related to inversion of adjunction-type results in birational geometry, and can also be thought of as an inverse to local Bertini-type theorems. The terminology 'lifts from Cartier divisors' was suggested to us by János Kollár.

Our main result says that under an additional assumption on the formal fibers of the local rings of $X$, a more general version of Question 1.1 holds.

Theorem A. *Let $\mathbf{R}$ be a property of noetherian local rings satisfying ($\mathrm{R_I'}$), ($\mathrm{R_{II}}$), ($\mathrm{R_{IV}}$), and ($\mathrm{R_V}$), such that regular local rings satisfy $\mathbf{R}$. Consider a flat morphism $f\colon Y \to X$ of locally noetherian schemes.*

(i) *Suppose that $f$ maps closed to closed points, and that the local rings of $X$ at closed points have geometrically $\mathbf{R}$ formal fibers. If every closed fiber of $f$ is geometrically $\mathbf{R}$, then all fibers of $f$ are geometrically $\mathbf{R}$.*

(ii) *Suppose that $f$ is closed, and that the local rings of $X$ have geometrically $\mathbf{R}$ formal fibers. Then the locus*

$$U_{\mathbf{R}}(f) \coloneqq \{x \in X \mid f^{-1}(x) \text{ is geometrically } \mathbf{R} \text{ over } \kappa(x)\} \subseteq X$$

*is stable under generization.*

In the statement above, a locally noetherian scheme $X$ over a field $k$ is *geometrically* $\mathbf{R}$ over $k$ if, for all finite field extensions $k \subseteq k'$, every local ring of $X \otimes_k k'$ satisfies $\mathbf{R}$. We consider the fiber $f^{-1}(x)$ of a morphism $f$ as a scheme over the residue field $\kappa(x)$ at $x \in X$. We will use similar terminology for noetherian algebras over a field $k$ and homomorphisms of noetherian rings. A semi-local noetherian ring $A$ has *geometrically* $\mathbf{R}$ *formal fibers* if the $\mathfrak{m}$-adic completion homomorphism $A \to \hat{A}$ has geometrically $\mathbf{R}$ fibers, where $\mathfrak{m}$ is the product of the maximal ideals in $A$.

Theorem A(ii) answers Question 1.1 since proper morphisms are closed. Combined with the constructibility results proved in [EGAIV$_3$, § 9] and [BF93, § 7], Theorem A(ii) implies that many properties of fibers are open on the target for closed flat morphisms of finite type between locally noetherian schemes with nice formal fibers. In Theorem A(i), the condition that a morphism maps closed points to closed points is much weaker than properness or even closedness, since it is satisfied by all morphisms locally of finite type between algebraic varieties, or more generally between Jacobson schemes [EGAI$_{\mathrm{new}}$, Corollaire 6.4.7 and Proposition 6.5.2].

Theorem A does not need to assume that $f$ is proper or even of finite type, and therefore answers a question of Shimomoto [Shi17, p. 1058]. Shimomoto proved versions of (i) and (ii) for morphisms of finite type between excellent noetherian schemes [Shi17, Main Theorem 1 and Corollary 3.8], and asked whether similar results hold without finite type hypotheses.



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The main ingredient in the proof of Theorem A is a purely commutative-algebraic statement, which is of independent interest. In [EGAIV$_2$], Grothendieck and Dieudonné asked whether the following local version of Question 1.1 holds.

*Problem* 1.2 (Grothendieck's localization problem; see [EGAIV$_2$, Remarque 7.5.4(i)]). Let $\mathbf{R}$ be a property of noetherian local rings, and consider a flat local homomorphism $\varphi\colon A \to B$ of noetherian local rings. If $A$ has geometrically $\mathbf{R}$ formal fibers and the closed fiber of $\varphi$ is geometrically $\mathbf{R}$, then are all fibers of $\varphi$ geometrically $\mathbf{R}$?

In other words, Problem 1.2 asks whether the property of having geometrically $\mathbf{R}$ fibers localizes for flat local homomorphisms of noetherian local rings. We call Problem 1.2 'Grothendieck's localization problem' following Avramov and Foxby [AF94], who proved many cases of this problem; see Table 1. We resolve Problem 1.2 for well-behaved properties $\mathbf{R}$, extending [EGAIV$_2$, Proposition 7.9.8] and [Mar84, Theorem 2.1] to non-zero residue characteristic.

Theorem B. *Problem 1.2 holds for properties* $\mathbf{R}$ *of noetherian local rings satisfying* $(\mathrm{R}'_{\mathrm{I}})$, $(\mathrm{R}_{\mathrm{II}})$, $(\mathrm{R}_{\mathrm{IV}})$, *and* $(\mathrm{R}_{\mathrm{V}})$, *such that regular local rings satisfy* $\mathbf{R}$.

The proof of Theorems A and B now proceeds as follows.

(I) We reduce Theorem A to Theorem B by replacing $X$ and $Y$ with $\operatorname{Spec}(\mathcal{O}_{X,x})$ and $\operatorname{Spec}(\mathcal{O}_{Y,y})$ for suitable points $x \in X$ and $y \in Y$. This step uses the assumption either that $f$ maps closed points to closed points, or that $f$ is closed.

(II) We reduce to the case when $A$ is quasi-excellent by replacing $A$ with its completion $\hat{A}$. This step uses $(\mathrm{R}'_{\mathrm{I}})$, $(\mathrm{R}_{\mathrm{II}})$, and the condition on the formal fibers of $A$.

(III) We reduce to the case when $A$ is a regular local ring by applying Gabber's weak local uniformization theorem [ILO14, Exposé VII, Théorème 1.1]. This step uses $(\mathrm{R}'_{\mathrm{I}})$, $(\mathrm{R}_{\mathrm{II}})$, $(\mathrm{R}_{\mathrm{IV}})$, $(\mathrm{R}_{\mathrm{V}})$, and the quasi-excellence of $A$ obtained in (II).

(IV) Finally, we are in a situation where we can apply [EGAIV$_2$, Lemme 7.5.1.1], which is a statement similar to Problem 1.2 when $A$ is regular. This step uses $(\mathrm{R}_{\mathrm{IV}})$.

The main innovation in our approach to Theorems A and B is the use of Gabber's theorem in (III). Grothendieck and Dieudonné [EGAIV$_2$, Proposition 7.9.8] and Marot [Mar84, Theorem 2.1] separately proved versions of Theorem B under the assumption that every reduced module-finite $A$-algebra has a resolution of singularities. This assumption on resolutions of singularities holds when $A$ is a quasi-excellent $\mathbf{Q}$-algebra [Hir64, Chapter I, § 3, Main Theorem I($n$)], or when $A$ is quasi-excellent of dimension at most three [Lip78, Theorem; CP19, Theorem 1.1], but is not known to hold in general.

On the other hand, Gabber's weak local uniformization theorem says that a variant of resolutions of singularities exists for arbitrary quasi-excellent noetherian schemes [ILO14, Exposé VII, Théorème 1.1]. Gabber's theorem is a version of de Jong's alteration theorem [dJ96, Theorem 4.1] for quasi-excellent noetherian schemes that are not necessarily of finite type over a field or a discrete rank one valuation ring (DVR). Both of their theorems say that variants of resolutions of singularities exist after possibly passing to a finite extension of the function field. As presented by Kurano and Shimomoto [KS21, Main Theorem 2], Gabber previously used this theorem to prove that quasi-excellence is preserved under ideal-adic completion.

We note that one can also ask about generization on the source space $Y$ in the context of Theorem A. Such a statement follows from Theorem B for arbitrary flat morphisms $f\colon Y \to X$ of locally noetherian schemes, with appropriate assumptions on the formal fibers of the local rings of $X$; cf. [EGAIV$_2$, Remarque 7.9.10(ii)].



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In the second half of this paper, we answer Question 1.1 and Problem 1.2 in specific cases. Starting with André's theorem on the localization of formal smoothness [And74, Théorème on p. 297], previous cases of Problem 1.2 were proved using a variety of methods, including André–Quillen homology [And67, Qui70], Grothendieck duality [Har66], and the Cohen factorizations of Avramov, Foxby, and Herzog [AFH94]. See Table 1 for known cases of Problem 1.2. By checking the conditions $(R_I')$, $(R_{II})$, $(R_{IV})$, and $(R_V)$ (see Table 2), we give a uniform treatment of most of the results in Table 1 using Theorem B. Additionally, Theorem B resolves Problem 1.2 for weak normality, seminormality, $F$-rationality, and the 'Cohen–Macaulay and $F$-injective' property, the latter three of which were previously known only under additional finiteness assumptions [Has01, Theorem 5.8 and Remark 6.7; Shi17, Corollaries 3.4 and 3.10; PSZ18, Theorem 5.13]. The result for $F$-rationality completely answers a question of Hashimoto [Has01, Remark 6.7].

For weak normality, we prove that weak normality lifts from Cartier divisors for all noetherian local rings (Proposition 4.10), extending a result of Bingener and Flenner [BF93, Corollary 4.1] to non-excellent rings. We also solve Grothendieck's localization problem for terminal, canonical, and rational singularities in equal characteristic zero (Corollary 4.18) using [$\mathrm{EGAIV}_2$, Proposition 7.9.8].

As an application of Theorem B, we consider the following version of Grothendieck's lifting problem for semi-local rings.

*Problem* 1.3 (Local lifting problem; cf. [$\mathrm{EGAIV}_2$, Remarque 7.4.8A]). Let $A$ be a noetherian semi-local ring that is $I$-adically complete with respect to an ideal $I \subseteq A$. If $A/I$ has geometrically $\mathbf{R}$ formal fibers, then does $A$ have geometrically $\mathbf{R}$ formal fibers?

We call Problem 1.3 the 'local lifting problem' following Nishimura and Nishimura [NN88], in order to distinguish Problem 1.3 from Grothendieck's original lifting problem asked in [$\mathrm{EGAIV}_2$, Remarque 7.4.8A], which does not restrict to semi-local rings. Using Theorem B and the axiomatic approach to Problem 1.3 due to Brezuleanu and Ionescu [BI84, Theorem 2.3], we give a solution to Problem 1.3 under the additional assumption that $A/I$ is Nagata. This result extends [Mar84, Theorem 5.2] to non-zero residue characteristic.

Theorem C. *Let* $\mathbf{R}$ *be a property of noetherian local rings that satisfies the hypotheses in Theorem B. Suppose, moreover, that the locus*

$$U_{\mathbf{R}}\big(\mathrm{Spec}(C)\big) \coloneqq \{\mathfrak{p} \in \mathrm{Spec}(C) \mid C_{\mathfrak{p}} \textit{ satisfies } \mathbf{R}\} \subseteq \mathrm{Spec}(C)$$

*is open for every noetherian complete local ring* $C$*. If* $A$ *is a noetherian semi-local ring that is* $I$*-adically complete with respect to an ideal* $I \subseteq A$*, and if* $A/I$ *is Nagata and has geometrically* $\mathbf{R}$ *formal fibers, then* $A$ *is Nagata and has geometrically* $\mathbf{R}$ *formal fibers.*

The condition on $U_{\mathbf{R}}(\mathrm{Spec}(C))$ is condition $(R_{III})$ in Conditions 3.1. Recall that a noetherian semi-local ring is Nagata if and only if it has geometrically reduced formal fibers; see Remark 2.5. Theorem C therefore answers Problem 1.3 for properties $\mathbf{R}$ such that regular $\Rightarrow \mathbf{R} \Rightarrow$ reduced, and gives a uniform treatment of most known cases of Problem 1.3; see Table 1. Theorem C also implies that for semi-local Nagata rings, the property of having geometrically $\mathbf{R}$ formal fibers is preserved under ideal-adic completion; see Corollary 3.11.

We end this introduction with one question that is still open. Nishimura showed that the non-local version of Problem 1.3 is false for all properties $\mathbf{R}$ such that regular $\Rightarrow \mathbf{R} \Rightarrow$ reduced [Nis81, Example 5.3], and Greco showed that similar questions for the properties 'universally catenary' and 'excellent' are also false [Gre82, Proposition 1.1]. Instead, inspired by the axiomatic



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approach of Valabrega [Val78, Theorem 3], Imbesi conjectured that the following formulation of Problem 1.3 for non-semi-local rings may hold.

*Problem* 1.4 [Imb95, p. 54]. Let $A$ be a noetherian ring that is $I$-adically complete with respect to an ideal $I \subseteq A$. If $A/I$ satisfies $\mathbf{R}$-2 and every local ring of $A/I$ has geometrically $\mathbf{R}$ formal fibers, then is it true that $A$ satisfies $\mathbf{R}$-2 and that every local ring of $A$ has geometrically $\mathbf{R}$ formal fibers?

Here, a ring $A$ *satisfies* $\mathbf{R}$-2 if for every $A$-algebra $B$ of finite type, the locus $U_{\mathbf{R}}(\operatorname{Spec}(B))$ is open in $\operatorname{Spec}(B)$ [Val78, Definition 1]. Nishimura's aforementioned example [Nis81, Example 5.3] shows that Problem 1.4 does not hold when $\mathbf{R}$ = 'reduced'. On the other hand, Problem 1.4 for $\mathbf{R}$ = 'normal' follows from a result proved by Brezuleanu and Rotthaus [BR82, Satz 1] and by Chiriacescu [Chi82, Theorem 1.5] around the same time, and a result due to Nishimura and Nishimura [NN88, Theorem A]. Moreover, Gabber proved Problem 1.4 for $\mathbf{R}$ = 'regular' [KS21, Main Theorem 1].

**Outline**

This paper is structured as follows.

In the first half of the paper, we set up the general framework with which we prove Theorems A, B, and C. To so do, we define geometrically $\mathbf{R}$ morphisms and formal fibers, and review the necessary background on (quasi-)excellent rings and on Gabber's weak local uniformization theorem in § 2. In § 3 we state the various conditions we put on local properties $\mathbf{R}$ of noetherian rings, and then prove Theorem B, first in the quasi-excellent case (Theorem 3.4), and then for rings with geometrically $\mathbf{R}$ formal fibers (Corollary 3.6). We obtain Theorems A and C as consequences.

The second half of the paper consists of § 4, where we verify the necessary conditions to apply Theorems A, B, and C to specific properties $\mathbf{R}$, in particular proving that weak normality lifts from Cartier divisors (Proposition 4.10). We then solve Problem 1.2 for terminal, canonical, and rational singularities in equal characteristic zero (Corollary 4.18) using [EGAIV$_2$, Proposition 7.9.8]. Finally, we conclude this paper with two tables: Table 1 contains references for special cases of Problems 1.2 and 1.3, and Table 2 contains references for the conditions used in our theorems.

**Notation**

All rings are commutative with identity, and all ring homomorphisms are unital. We follow the notation in Definition 2.1 for properties $\mathbf{R}$ of noetherian local rings and their associated properties of morphisms and of formal fibers. We also follow the notation in Conditions 3.1 for the conditions on $\mathbf{R}$ appearing in our results. In addition, if $\mathbf{R}$ is a property of noetherian local rings, then, following [EGAIV$_2$, Proposition 7.3.12], the $\mathbf{R}$ *locus* in a locally noetherian scheme $X$ is the locus

$$U_{\mathbf{R}}(X) \coloneqq \{x \in X \mid \mathcal{O}_{X,x} \text{ satisfies } \mathbf{R}\} \subseteq X$$

at which $X$ satisfies $\mathbf{R}$.

## 2. Preliminaries

### 2.1 Geometrically R morphisms and formal fibers

We begin by defining geometrically $\mathbf{R}$ morphisms and rings with geometrically $\mathbf{R}$ formal fibers.



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Definition 2.1 [EGAIV$_2$, (7.3.1), (7.5.0), and (7.3.13)]. Let $\mathbf{R}$ be a property of noetherian local rings, and let $k$ be a field. A locally noetherian $k$-scheme $X$ is *geometrically* $\mathbf{R}$ over $k$ if $X$ satisfies the following property.

$$\text{For all finite field extensions } k \subseteq k', \text{ every local ring of } X \otimes_k k' \text{ satisfies } \mathbf{R}. \tag{1}$$

A noetherian $k$-algebra $A$ is *geometrically* $\mathbf{R}$ over $k$ if $\operatorname{Spec}(A)$ is geometrically $\mathbf{R}$ over $k$.

A morphism $f\colon Y \to X$ of locally noetherian schemes is *geometrically* $\mathbf{R}$ if it is flat and if the scheme $f^{-1}(x)$ is geometrically $\mathbf{R}$ for every $x \in X$. We consider the fiber $f^{-1}(x)$ of a morphism $f$ as a scheme over the residue field $\kappa(x)$ at $x \in X$. A ring homomorphism $\varphi\colon A \to B$ is *geometrically* $\mathbf{R}$ if $\operatorname{Spec}(\varphi)$ is geometrically $\mathbf{R}$. The *geometrically* $\mathbf{R}$ *locus* of a flat morphism $f\colon Y \to X$ of locally noetherian schemes is the locus

$$U_{\mathbf{R}}(f) \coloneqq \{x \in X \mid f^{-1}(x) \text{ is geometrically } \mathbf{R} \text{ over } \kappa(x)\} \subseteq X$$

of points in $X$ over which $f$ is geometrically $\mathbf{R}$.

A noetherian semi-local ring $A$ has *geometrically* $\mathbf{R}$ *formal fibers* if the $\mathfrak{m}$-adic completion homomorphism $A \to \hat{A}$ is geometrically $\mathbf{R}$, where $\mathfrak{m}$ is the product of the maximal ideals in $A$.

*Remark* 2.2. We note that the property in (1) is called $\mathbf{P}$ in [EGAIV$_2$, (7.5.0)]. Following this terminology, geometrically $\mathbf{R}$ morphisms are called $\mathbf{P}$-morphisms in [EGAIV$_2$, (7.3.1)], and semi-local rings with geometrically $\mathbf{R}$ formal fibers are called $\mathbf{P}$-rings in [EGAIV$_2$, (7.3.13)].

### 2.2 (Quasi-)excellent rings and schemes

We next define (quasi-)excellent rings and schemes. In the definition below, we recall that a noetherian ring $A$ is a *G-ring* if $A_{\mathfrak{p}}$ has geometrically regular formal fibers for every prime ideal $\mathfrak{p} \subseteq A$ [Mat89, p. 256].

Definition 2.3 [EGAIV$_2$, Définition 7.8.2 and (7.8.5)] (cf. [Mat89, Definition on p. 260]). A noetherian ring $A$ is *quasi-excellent* if $A$ is a $G$-ring and if $A$ is $J$-2, that is, if for every $A$-algebra $B$ of finite type the regular locus in $\operatorname{Spec}(B)$ is open. A quasi-excellent ring $A$ is *excellent* if $A$ is universally catenary.

A locally noetherian scheme $X$ is *quasi-excellent* (respectively, *excellent*) if it admits an open affine covering $X = \bigcup_i \operatorname{Spec}(A_i)$, such that every $A_i$ is quasi-excellent (respectively, excellent).

The condition $J$-2 is the condition $\mathbf{R}$-2 for $\mathbf{R}$ = 'regular' in the sense mentioned in § 1. We also define the following closely related notion.

Definition 2.4 [EGAIV$_1$, Chapitre 0, Définition 23.1.1]. A noetherian domain $A$ is *Japanese* if, for every finite extension $L$ of the fraction field of $A$, the integral closure of $A$ inside $L$ is module-finite over $A$. A noetherian ring $A$ is *Nagata* or *universally Japanese* if every domain $B$ of finite type over $A$ is Japanese.

Every quasi-excellent noetherian ring is Nagata [EGAIV$_2$, Corollaire 7.7.3].

*Remark* 2.5. By theorems of Zariski and Nagata [EGAIV$_2$, Théorèmes 7.6.4 and 7.7.2] (see also [Mat89, p. 264]), a noetherian ring $A$ is Nagata if and only if:

(a) $A_{\mathfrak{p}}$ has geometrically reduced formal fibers for every prime ideal $\mathfrak{p} \subseteq A$; and
(b) for every domain $B$ that is module-finite over $A$, the normal locus is open in $\operatorname{Spec}(B)$.

For semi-local rings $A$, (a) implies (b) by [EGAIV$_2$, Théorème 7.6.4 and Corollaire 7.6.5], and (a) holds if and only if $A$ has geometrically reduced formal fibers [EGAIV$_2$, Proposition 7.3.14



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and Corollaire 7.4.5]. Thus, a semi-local ring is Nagata if and only if it has geometrically reduced formal fibers.

### 2.3 Gabber's weak local uniformization theorem

We now recall Gabber's weak local uniformization theorem, which is a variant of resolutions of singularities for arbitrary quasi-excellent noetherian schemes. Gabber's result is a version of de Jong's alteration theorem [dJ96, Theorem 4.1] for quasi-excellent noetherian schemes that are not necessarily of finite type over a field or a DVR.

To state Gabber's result, we first need to define maximally dominating morphisms. We recall that a point $x$ on a scheme $X$ is *maximal* if it is the generic point of an irreducible component of $X$ [EGAI$_{\text{new}}$, Chapitre 0, (2.1.1)]. We then have the following definition.

Definition 2.6 [ILO14, Exposé II, Définition 1.1.2]. A morphism $f\colon Y \to X$ of schemes is *maximally dominating* if every maximal point of $Y$ maps to a maximal point of $X$.

We now define the alteration topology on a noetherian scheme.

Definition 2.7 [ILO14, Exposé II, Définition 1.2.2 and (2.3.1)]. Let $X$ be a noetherian scheme. The category $\mathrm{alt}/X$ is the category whose objects are reduced schemes that are maximally dominating, generically finite, and of finite type over $X$, and whose morphisms are morphisms as schemes over $X$. All morphisms in $\mathrm{alt}/X$ are maximally dominating, generically finite, and of finite type [ILO14, Exposé II, Définition 1.2.2 and (2.3.1)].

The *alteration topology* on $X$ is the Grothendieck topology on $\mathrm{alt}/X$ associated to the pretopology generated by:

(i) étale coverings; and
(ii) proper surjective morphisms that are maximally dominating and generically finite.

We will use the following alternative characterization for coverings in the alteration topology when $X$ is irreducible. This result implies that coverings in the alteration topology on $\mathrm{alt}/X$ are coverings in Voevodsky's $h$-topology [Voe96, Definition 3.1.2]; see [ILO14, p. 263].

Theorem 2.8 [ILO14, Exposé II, Théorème 3.2.1]. *Let $X$ be an irreducible noetherian scheme. Then, for every finite covering $\{Y_i \to X\}_{i=1}^m$ in the alteration topology on $X$, there exist a proper surjective morphism $\pi\colon V \to X$ in $\mathrm{alt}/X$ such that $V$ is integral, and a Zariski open covering $V = \bigcup_{i=1}^m V_i$, together with a collection $\{h_i\colon V_i \to Y_i\}_{i=1}^m$ of morphisms, such that the diagram*

$$\begin{array}{ccc} V_i & \hookrightarrow & V \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y_i & \xrightarrow{f_i} & X \end{array}$$

*commutes for every $i \in \{1, 2, \ldots, m\}$, where the morphisms $V_i \hookrightarrow V$ are the natural open immersions.*

We now state a special case of Gabber's weak local uniformization result, which is the main technical ingredient in the proof of Theorem B.

Theorem 2.9 (Gabber [ILO14, Exposé VII, Théorème 1.1]). *Let $X$ be a quasi-excellent noetherian scheme. Then there exists a finite covering $\{Y_i \to X\}_{i=1}^m$ in the alteration topology on $X$, such that $Y_i$ is regular and integral for every $i \in \{1, 2, \ldots, m\}$.*



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## 3. Grothendieck's localization problem and the local lifting problem

In this section we solve Grothendieck's localization problem (Problem 1.2) by proving Theorem B in a sequence of steps. We first fix notation for our conditions on local properties $\mathbf{R}$ in § 3.1. In § 3.2 we prove Theorem B under the additional assumption that $A$ is quasi-excellent (Theorem 3.4), in which case Gabber's weak local uniformization theorem (Theorem 2.9) applies. We then prove Theorem B in § 3.3 by taking a completion to reduce to the quasi-excellent case, using $(\mathrm{R}'_{\mathrm{I}})$ and the fact that $A$ has geometrically $\mathbf{R}$ formal fibers. Finally, we obtain Theorems A and C as consequences in §§ 3.4 and 3.5, respectively.

### 3.1 Conditions on R

We fix the following notational conventions for permanence conditions on the local properties $\mathbf{R}$.

Condition 3.1. Fix a full subcategory $\mathscr{C}$ of the category of noetherian rings, and let $\mathbf{R}$ be a property of noetherian local rings. We consider the following conditions on the property $\mathbf{R}$.

- $(\mathrm{R}_0)$ The property $\mathbf{R}$ holds for every field $k$.
- $(\mathrm{R}_{\mathrm{I}}^{\mathscr{C}})$ (Ascent via geometrically regular homomorphisms) For every geometrically regular local homomorphism $\varphi\colon A \to B$ of noetherian local rings in $\mathscr{C}$, if $A$ satisfies $\mathbf{R}$, then $B$ satisfies $\mathbf{R}$.
- $(\mathrm{R}_{\mathrm{II}}^{\mathscr{C}})$ (Descent) For every flat local homomorphism $\varphi\colon A \to B$ of noetherian local rings such that $B$ is in $\mathscr{C}$, if $B$ satisfies $\mathbf{R}$, then $A$ satisfies $\mathbf{R}$.
- $(\mathrm{R}_{\mathrm{III}})$ (Openness) For every noetherian complete local ring $C$, the locus $U_{\mathbf{R}}(\mathrm{Spec}(C))$ is open.
- $(\mathrm{R}_{\mathrm{IV}}^{\mathscr{C}})$ (Lifting from Cartier divisors) For every noetherian local ring $A$ in $\mathscr{C}$ and for every nonzerodivisor $t$ in its maximal ideal, if $A/tA$ satisfies $\mathbf{R}$, then $A$ satisfies $\mathbf{R}$.
- $(\mathrm{R}_{\mathrm{V}}^{\mathscr{C}})$ (Localization) If a noetherian local ring $A$ in $\mathscr{C}$ satisfies $\mathbf{R}$, then $A_{\mathfrak{p}}$ satisfies $\mathbf{R}$ for every prime ideal $\mathfrak{p} \subseteq A$.

In addition, we consider the following variant of $(\mathrm{R}_{\mathrm{I}}^{\mathscr{C}})$.

- $(\mathrm{R}_{\mathrm{I}}^{\mathscr{C}\prime})$ (Ascent via geometrically $\mathbf{R}$ homomorphisms) For every local geometrically $\mathbf{R}$ homomorphism $\varphi\colon A \to B$ of noetherian local rings in $\mathscr{C}$, if $A$ satisfies $\mathbf{R}$, then $B$ satisfies $\mathbf{R}$.

We also consider the following conditions for $\mathbf{R}$ that affect geometrically $\mathbf{R}$ homomorphisms.

- $(\mathrm{P}_{\mathrm{I}}^{\mathscr{C}})$ If $\varphi\colon A \to B$ and $\psi\colon B \to C$ are a geometrically $\mathbf{R}$ homomorphism and a geometrically regular homomorphism of noetherian rings, respectively, and $B$ and $C$ are in $\mathscr{C}$, then $\psi \circ \varphi$ is a geometrically $\mathbf{R}$ homomorphism.
- $(\mathrm{P}_{\mathrm{II}}^{\mathscr{C}})$ (Descent) If $\varphi\colon A \to B$ and $\psi\colon B \to C$ are two homomorphisms of noetherian rings such that $\psi \circ \varphi$ is geometrically $\mathbf{R}$, $\psi$ is faithfully flat, and $C$ is in $\mathscr{C}$, then $\varphi$ is geometrically $\mathbf{R}$.
- $(\mathrm{P}_{\mathrm{III}})$ Every field $k$ is geometrically $\mathbf{R}$ over itself.
- $(\mathrm{P}_{\mathrm{IV}}^{\mathscr{C}})$ (Stability under finitely generated ground field extensions) If a noetherian ring $A$ in $\mathscr{C}$ is geometrically $\mathbf{R}$ over a field $k$, then, for all finitely generated field extensions $k \subseteq k'$, the ring $A \otimes_k k'$ is geometrically $\mathbf{R}$ over $k'$.

In addition, we consider the following variant of $(\mathrm{P}_{\mathrm{I}}^{\mathscr{C}})$.

- $(\mathrm{P}_{\mathrm{I}}^{\mathscr{C}})$ If $\varphi\colon A \to B$ and $\psi\colon B \to C$ are geometrically $\mathbf{R}$ homomorphisms of noetherian rings such that $B$ and $C$ are in $\mathscr{C}$, then $\psi \circ \varphi$ is also geometrically $\mathbf{R}$.

We drop $\mathscr{C}$ from our notation if $\mathscr{C}$ is the entire category of noetherian rings.



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*Remark* 3.2. The list in Conditions 3.1 is a subset of that in [Mar84, Conditions 1.1 and 1.2], although our naming convention mostly follows [EGAIV$_2$]. Specifically,

- $(R_0)$, $(R_I)$, and $(R_{II})$ appear in [EGAIV$_2$, (7.3.10)], although $(R_0)$ is not named;
- $(R_{III})$ appears in [EGAIV$_2$, Proposition 7.3.18];
- $(R_{IV}^{\mathscr{C}})$ specializes to the condition $(R_{IV})$ in [EGAIV$_2$, Théorème 7.5.1] when $\mathscr{C}$ is the category of noetherian complete local rings;
- $(R_V^{\mathscr{C}})$ is unrelated to the condition $(R_V)$ in [EGAIV$_2$, Corollaire 7.5.2];
- $(R_I')$ appears in [EGAIV$_2$, Remarque 7.3.11];
- $(P_I)$, $(P_{II})$, and $(P_{III})$ appear in [EGAIV$_2$, (7.3.4)];
- $(P_{IV})$ appears in [EGAIV$_2$, (7.3.6)]; and
- $(P_I')$ appears in [EGAIV$_2$, Remarque 7.3.5(iii)].

See also [EGAIV$_2$, (7.9.7)], [Val78, p. 201], and [BI84, (2.1) and (2.4)].

We will use the following relationships between different conditions on $\mathbf{R}$.

Lemma 3.3 (cf. [EGAIV$_2$, (7.3.10), Remarque 7.3.11, and Lemme 7.3.7; DM20, Proposition 4.10]). *Fix a full subcategory $\mathscr{C}$ of the category of noetherian rings that is stable under homomorphisms essentially of finite type, and let $\mathbf{R}$ be a property of noetherian local rings.*

(i) *If $\mathbf{R}$ satisfies $(R_0)$, then $\mathbf{R}$ satisfies $(P_{III})$.*

(ii) *Let $\mathbf{R}'$ be another property of noetherian local rings. Suppose that, for every local geometrically $\mathbf{R}'$ homomorphism $B \to C$ of noetherian local rings in $\mathscr{C}$, if $B$ satisfies $\mathbf{R}$, then $C$ satisfies $\mathbf{R}$. If $\varphi\colon A \to B$ and $\psi\colon B \to C$ are geometrically $\mathbf{R}$ and geometrically $\mathbf{R}'$ homomorphisms of noetherian rings, respectively, and $B$ and $C$ are in $\mathscr{C}$, then $\psi \circ \varphi$ is geometrically $\mathbf{R}$.*

*In particular, if $\mathbf{R}$ satisfies $(R_I^{\mathscr{C}})$, then $\mathbf{R}$ satisfies $(P_I^{\mathscr{C}})$, and if $\mathbf{R}$ satisfies $(R_I^{\mathscr{C}'})$, then $\mathbf{R}$ satisfies $(P_I^{\mathscr{C}'})$.*

(iii) *If $\mathbf{R}$ satisfies $(R_{II}^{\mathscr{C}})$, then $\mathbf{R}$ satisfies $(P_{II}^{\mathscr{C}})$.*

(iv) *If $\mathbf{R}$ satisfies the special cases of $(R_I^{\mathscr{C}})$ and $(R_{II}^{\mathscr{C}})$ when the homomorphisms $\varphi$ are essentially of finite type, then $\mathbf{R}$ satisfies $(P_{IV}^{\mathscr{C}})$.*

(v) *If $\mathbf{R}$ satisfies $(P_{IV}^{\mathscr{C}})$, and if $\varphi\colon A \to B$ is a geometrically $\mathbf{R}$ homomorphism of noetherian rings such that $B$ is in $\mathscr{C}$, then for every $A$-algebra $C$ essentially of finite type, the base change $\varphi \otimes_A \mathrm{id}_C\colon C \to B \otimes_A C$ is geometrically $\mathbf{R}$.*

*Proof.* (i) is clear from definition of being geometrically $\mathbf{R}$.

To show (ii) and (iii), it suffices to consider the case when $A$ is a field $k$ by transitivity of fibers; see [EGAIV$_2$, Remarque 7.3.5(ii)]. Note that for (ii) (respectively, (iii)), we use the hypothesis on $\mathscr{C}$ to guarantee that after this reduction, $B$ and $C$ (respectively, $C$) are still in $\mathscr{C}$. Consider the composition

$$k \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

of homomorphisms of noetherian rings, and consider the base change

$$k' \xrightarrow{\varphi \otimes_k \mathrm{id}_{k'}} B \otimes_k k' \xrightarrow{\psi \otimes_k \mathrm{id}_{k'}} C \otimes_k k' \tag{2}$$

for a finite field extension $k \subseteq k'$. For (ii), we first note that since $B \to B \otimes_k k'$ is a module-finite homomorphism, it induces finite field extensions on residue fields. Thus, the base change $\psi \otimes_k \mathrm{id}_{k'}$ of $\psi$ is geometrically $\mathbf{R}'$. Since the local rings of $B \otimes_k k'$ satisfy $\mathbf{R}$ by assumption, we see that the local rings of $C \otimes_k k'$ also satisfy $\mathbf{R}$. For (iii), we note that in (2), the local rings of



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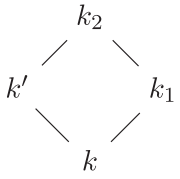
$C \otimes_k k'$ satisfy $\mathbf{R}$ by assumption. Thus, the local rings of $B \otimes_k k'$ also satisfy $\mathbf{R}$ by $(\mathrm{R}_{\mathrm{II}}^{\mathscr{C}})$, since $\psi \otimes_k \mathrm{id}_{k'}$ is faithfully flat by base change.

For (iv), let $k \subseteq k'$ be a finitely generated field extension. By [DM20, Lemma 4.9], there exist a finite extension $k \subseteq k_1$ and a diagram




of finitely generated field extensions, where $k_1 \subseteq k_2 \coloneqq (k' \otimes_k k_1)_{\mathrm{red}}$ is a separable field extension. Since $A$ is geometrically $\mathbf{R}$ over $k$, the local rings of $A \otimes_k k_1$ satisfy $\mathbf{R}$. We therefore see that the local rings of $A \otimes_k k_2$ also satisfy $\mathbf{R}$, since $k_1 \to k_2$ is regular and both $A \otimes_k k_1$ and $A \otimes_k k_2$ are in $\mathscr{C}$. Finally, the local rings of $A \otimes_k k'$ satisfy $\mathbf{R}$ by $(\mathrm{R}_{\mathrm{II}}^{\mathscr{C}})$.

For (v), we note that $\varphi \otimes_A \mathrm{id}_C$ is flat by base change, and hence it suffices to show that, for every prime ideal $\mathfrak{q} \subseteq C$, the fiber $B \otimes_A C \otimes_C \kappa(\mathfrak{q})$ is geometrically $\mathbf{R}$. Letting $\mathfrak{p} = \mathfrak{q} \cap A$, the field extension $\kappa(\mathfrak{p}) \subseteq \kappa(\mathfrak{q})$ is finitely generated since $A \to C$ is essentially of finite type [$\mathrm{EGAI}_{\mathrm{new}}$, Proposition 6.5.10]. Since

$$B \otimes_A C \otimes_C \kappa(\mathfrak{q}) \simeq B \otimes_A \kappa(\mathfrak{q}) \simeq B \otimes_A \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}),$$

we see that $B \otimes_A C \otimes_C \kappa(\mathfrak{q})$ is geometrically $\mathbf{R}$ by $(\mathrm{P}_{\mathrm{IV}}^{\mathscr{C}})$. □

### 3.2 Problem 1.2 for quasi-excellent bases

The following result solves Grothendieck's localization problem when the ring $A$ is quasi-excellent. This step corresponds to (III) in § 1, and forms the technical core of the proof of Theorem B.

Theorem 3.4 (cf. [$\mathrm{EGAIV}_2$, Proposition 7.9.8; Mar84, Theorem 2.1]). *Fix a full subcategory $\mathscr{C}$ of the category of noetherian rings that is stable under homomorphisms essentially of finite type. Let $\mathbf{R}$ be a property of noetherian local rings, and consider a flat local homomorphism $\varphi\colon (A, \mathfrak{m}, k) \to (B, \mathfrak{n}, l)$ of noetherian local rings. Assume the following:*

(i) *the ring $A$ is quasi-excellent;*
(ii) *the ring $B$ appears in $\mathscr{C}$; and*
(iii) *the property $\mathbf{R}$ satisfies $(\mathrm{R}_{\mathrm{II}}^{\mathscr{C}})$, $(\mathrm{R}_{\mathrm{IV}}^{\mathscr{C}})$, $(\mathrm{R}_{\mathrm{V}}^{\mathscr{C}})$, and $(\mathrm{P}_{\mathrm{IV}}^{\mathscr{C}})$.*

*If the closed fiber of $\varphi$ is geometrically $\mathbf{R}$, then all fibers of $\varphi$ are geometrically $\mathbf{R}$.*

The proof in [Mar84] relies on the existence of resolutions of singularities, which we avoid by using Gabber's weak local uniformization theorem. As far as we are aware, the idea of using alterations instead of resolutions of singularities first appeared in [Has01, Remark 6.7], where Hashimoto proves Grothendieck's localization problem for $F$-rationality when the base ring $A$ is essentially of finite type over a field of positive characteristic using de Jong's alteration theorem [dJ96, Theorem 4.1]. When $\mathbf{R}$ = Cohen–Macaulay or $(S_n)$, one can use Kawasaki's Macaulayfication theorem [Kaw02, Theorem 1.1] to prove Theorem 3.4; see [BI84, Proposition 3.1; Ion08, Theorem 4.1 and Remark 4.2].

Our strategy will be to ultimately reduce to the following version of Problem 1.2 for regular bases. This statement corresponds to (IV) in § 1.

Lemma 3.5 [$\mathrm{EGAIV}_2$, Lemme 7.5.1.1]. *Fix a full subcategory $\mathscr{C}$ of the category of noetherian local rings that is stable under quotients. Let $\mathbf{R}$ be a property of noetherian local rings satisfying*



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$(\mathrm{R}_{\mathrm{IV}}^{\mathscr{C}})$, *and consider a flat local homomorphism* $\varphi\colon (C,\mathfrak{m},k)\to(D,\mathfrak{n},l)$ *of noetherian local rings in* $\mathscr{C}$*, where* $C$ *is regular. If* $D\otimes_C k$ *satisfies* $\mathbf{R}$*, then* $D$ *satisfies* $\mathbf{R}$*.*

We now prove Theorem 3.4.

*Proof of Theorem 3.4.* We want to show that, for every prime ideal $\mathfrak{p}\subseteq A$, the $\kappa(\mathfrak{p})$-algebra $B\otimes_A\kappa(\mathfrak{p})$ is geometrically $\mathbf{R}$ over $\kappa(\mathfrak{p})$. By noetherian induction, it suffices to show that if $\mathfrak{p}_0\subseteq A$ is a prime ideal and $B\otimes_A\kappa(\mathfrak{p})$ is geometrically $\mathbf{R}$ over $\kappa(\mathfrak{p})$ for every prime ideal $\mathfrak{p}\supsetneq\mathfrak{p}_0$, then $B\otimes_A\kappa(\mathfrak{p}_0)$ is geometrically $\mathbf{R}$ over $\kappa(\mathfrak{p}_0)$. Replacing $A$ by $A/\mathfrak{p}_0$ and $B$ by $B/\mathfrak{p}_0B$, we may therefore assume that $A$ is a domain and that $B\otimes_A\kappa(\mathfrak{p})$ is geometrically $\mathbf{R}$ over $\kappa(\mathfrak{p})$ for every non-zero prime ideal $\mathfrak{p}\subseteq A$. We note that $A/\mathfrak{p}_0$ is quasi-excellent by [EGAIV$_2$, Proposition 7.3.15(i)], and that $B/\mathfrak{p}_0B$ appears in $\mathscr{C}$ by (ii).

We now reduce to showing that the local rings of $B\otimes_A K$ satisfy $\mathbf{R}$, where $K\coloneqq\mathrm{Frac}(A)$. We want to show that, for every finite field extension $K\subseteq K'$, the local rings of $B\otimes_A K'$ satisfy $\mathbf{R}$. Let $A\subseteq A'$ be a module-finite extension such that $A'$ is a domain and such that $K'=\mathrm{Frac}(A')$. Then $A'$ is a semi-local ring, and setting $B'\coloneqq B\otimes_A A'$, we have $B\otimes_A K'\simeq B'\otimes_{A'}K'$. Note that every local ring of $B\otimes_A K'$ is a local ring of $B'_{\mathfrak{q}}\otimes_{A'_{\mathfrak{p}}}K'$ for some prime ideals $\mathfrak{p}\subseteq A'$ and $\mathfrak{q}\subseteq B'$. Moreover, denoting by $k'$ the residue field of $A'_{\mathfrak{p}}$, we have

$$B\otimes_A k'\simeq B\otimes_A A'\otimes_{A'}k'\simeq B'\otimes_{A'}k'.$$

Thus, $B'_{\mathfrak{q}}\otimes_{A'_{\mathfrak{p}}}k'$ is a localization of $B\otimes_A k'$, and hence $B'_{\mathfrak{q}}\otimes_{A'_{\mathfrak{p}}}k'$ is geometrically $\mathbf{R}$ over $k'$. We may therefore replace $A$ by $A'_{\mathfrak{p}}$, $B$ by $B'_{\mathfrak{q}}$, and $K$ by $K'$, in which case it suffices to show that the local rings of $B\otimes_A K$ satisfy $\mathbf{R}$. We note that $A'_{\mathfrak{p}}$ is quasi-excellent by [EGAIV$_2$, Proposition 7.3.15(i) and Théorème 7.7.2], and that $B'_{\mathfrak{q}}$ appears in $\mathscr{C}$ by (ii).

We now apply Gabber's weak local uniformization theorem. Since $X\coloneqq\mathrm{Spec}(A)$ is quasi-excellent by (i), Theorem 2.9 implies there exists a finite covering $\{f_i\colon Y_i\to X\}_{i=1}^m$ of $X$ in the alteration topology, where $Y_i$ is regular and integral for every $i$. Note that $X$ is irreducible by our reduction in the first paragraph and by construction of $A'$ in the previous paragraph. Thus, Theorem 2.8 implies there exist a proper surjective morphism $\pi\colon V\to X$ and a Zariski open covering $V=\bigcup_{i=1}^m V_i$ fitting into a commutative diagram

$$\begin{array}{ccc} V_i & \hookrightarrow & V \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y_i & \xrightarrow{f_i} & X \end{array}$$

for every $i$. By base change along the morphism $\mathrm{Spec}(\varphi)\colon X'\to X$, where $X'\coloneqq\mathrm{Spec}(B)$, we obtain the commutative diagram

$$\begin{array}{ccc} V'_i & \hookrightarrow & V' \\ {\scriptstyle h'_i}\downarrow & & \downarrow{\scriptstyle \pi'} \\ Y'_i & \xrightarrow{f'_i} & X' \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle \mathrm{Spec}(\varphi)} \\ Y_i & \xrightarrow{f_i} & X \end{array} \tag{3}$$

with cartesian squares for every $i$. We now consider a point $\eta'\in X'$ lying over the generic point of $X$. We want to show that $\mathcal{O}_{X',\eta'}$ satisfies $\mathbf{R}$. Since $\pi'$ is surjective, there exists a point $\xi'\in V'$ such that $\pi'(\xi')=\eta'$. Since $\pi'$ is closed, the specialization $\eta'\rightsquigarrow\mathfrak{n}$ in $X'=\mathrm{Spec}(B)$ then lifts to a specialization $\xi'\rightsquigarrow v'$ in $V'$ [Sta20, Tag 0066]. Since $V'=\bigcup_{i=1}^m V'_i$ is a Zariski open covering,



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there exists an index $i_0 \in \{1, 2, \dots, m\}$ such that $v' \in V'_{i_0}$, and since open sets are stable under generization, we have $\xi' \in V'_{i_0}$ as well. We claim that it suffices to show that $\mathcal{O}_{Y'_{i_0}, h'_{i_0}(\xi')}$ satisfies $\mathbf{R}$. Since the morphism $f_{i_0}$ is maximally dominating, base-changing the bottom square in (3) for $i = i_0$ along the morphism $\operatorname{Spec}(K) \to X = \operatorname{Spec}(A)$, localizing at the generic point of $Y_{i_0}$, and taking global sections yields the cocartesian square

$$\begin{array}{ccc} B \otimes_A L_{i_0} & \longleftarrow & B \otimes_A K \\ \uparrow & & \uparrow \\ L_{i_0} & \longleftarrow & K \end{array}$$

of rings, where $L_{i_0}$ is the function field of $Y_{i_0}$. The bottom horizontal arrow is faithfully flat; thus, the top horizontal arrow is also faithfully flat by base change. After localizing, we therefore obtain a faithfully flat homomorphism

$$\mathcal{O}_{X',\eta'} \simeq (B \otimes_A K)_{\eta'} \longrightarrow (B \otimes_A L_{i_0})_{h'_{i_0}(\xi')} \simeq \mathcal{O}_{Y'_{i_0}, h'_{i_0}(\xi')},$$

and $(\mathrm{R}^{\mathscr{C}}_{\mathrm{II}})$ implies that if $\mathcal{O}_{Y'_{i_0}, h'_{i_0}(\xi')}$ satisfies $\mathbf{R}$, then $\mathcal{O}_{X',\eta'}$ satisfies $\mathbf{R}$. Here we use the fact that $(B \otimes_A L_{i_0})_{h'_{i_0}(\xi')}$ is in $\mathscr{C}$ by (ii).

It remains to show that $\mathcal{O}_{Y'_{i_0}, h'_{i_0}(\xi')}$ satisfies $\mathbf{R}$. Setting $y' \coloneqq h'_{i_0}(v')$ and $y \coloneqq g_{i_0}(y')$, the residue field extension $k \subseteq \kappa(y)$ is finitely generated since $f_{i_0}$ is of finite type. Thus, the closed fiber of the flat homomorphism $\psi$ in the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{Y'_{i_0}, y'} & \longleftarrow & B \\ {\scriptstyle\psi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ \mathcal{O}_{Y_{i_0}, y} & \longleftarrow & A \end{array}$$

is geometrically $\mathbf{R}$ by $(\mathrm{P}^{\mathscr{C}}_{\mathrm{IV}})$, since $B \otimes_A k$ is in $\mathscr{C}$ by (ii). Since $\mathcal{O}_{Y_{i_0}, y}$ is regular by construction and $\mathcal{O}_{Y'_{i_0}, y'}$ appears in $\mathscr{C}$ by (ii), we can apply Lemma 3.5 (which uses $(\mathrm{R}^{\mathscr{C}}_{\mathrm{IV}})$) to deduce that $\mathcal{O}_{Y'_{i_0}, y'}$ satisfies $\mathbf{R}$. Finally, $(\mathrm{R}^{\mathscr{C}}_{\mathrm{V}})$ implies that $\mathcal{O}_{Y'_{i_0}, h'_{i_0}(\xi')}$ satisfies $\mathbf{R}$, since the specialization $\xi' \rightsquigarrow v'$ in $V'$ maps to the specialization $h'_{i_0}(\xi') \rightsquigarrow h'_{i_0}(v') = y'$ by continuity. □

### 3.3 Problem 1.2 in general and the proof of Theorem B

We now prove Theorem B by reducing to the complete (hence quasi-excellent) case proved in Theorem 3.4. This step corresponds to (II) in § 1.

We first show the following stronger statement that is more specific about what conditions from Conditions 3.1 are needed.

Corollary 3.6 (cf. [Mar84, Theorem 2.2; BI84, Proposition 1.2]). *Fix a full subcategory $\mathscr{C}$ of the category of noetherian rings that is stable under homomorphisms essentially of finite type. Let $\mathbf{R}$ be a property of noetherian local rings, and consider a flat local homomorphism $\varphi\colon (A, \mathfrak{m}, k) \to (B, \mathfrak{n}, l)$ of noetherian local rings. Assume the following:*

(i) *the ring $A$ has geometrically $\mathbf{R}$ formal fibers;*
(ii) *the rings $\hat{A}$ and $B^*$ appear in $\mathscr{C}$, where $\hat{A}$ and $B^*$ denote the $\mathfrak{m}$-adic completions of $A$ and $B$, respectively; and*
(iii) *the property $\mathbf{R}$ satisfies $(\mathrm{R}^{\mathscr{C}}_{\mathrm{II}})$, $(\mathrm{R}^{\mathscr{C}}_{\mathrm{IV}})$, $(\mathrm{R}^{\mathscr{C}}_{\mathrm{V}})$, $(\mathrm{P}^{\mathscr{C}'}_{\mathrm{I}})$, and $(\mathrm{P}^{\mathscr{C}}_{\mathrm{IV}})$.*

*If the closed fiber of $\varphi$ is geometrically $\mathbf{R}$, then all fibers of $\varphi$ are geometrically $\mathbf{R}$.*



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*Proof.* We have the commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ \widehat{A} & \xrightarrow{\varphi^*} & B^* \end{array}$$

where $\sigma$ and $\tau$ are the canonical $\mathfrak{m}$-adic completion homomorphisms. By (i), the homomorphism $\sigma$ is geometrically $\mathbf{R}$. We claim that $\varphi^*$ is geometrically $\mathbf{R}$. By [EGAI$_{\text{new}}$, Chapitre 0, Lemme 6.8.3.1], the ring $B^*$ is a noetherian local ring, and we have

$$B^* \otimes_{\hat{A}} k \simeq B \otimes_A k.$$

Theorem 3.4 therefore implies that $\varphi^*$ is geometrically $\mathbf{R}$, where we use the fact that the complete local ring $\hat{A}$ is excellent by [EGAIV$_2$, Scholie 7.8.3(iii)]. The composition $\tau \circ \varphi = \varphi^* \circ \sigma$ is geometrically $\mathbf{R}$ by $(\mathrm{P}_{\mathrm{I}}^{\mathscr{C}'})$, and therefore $\varphi$ is also by $(\mathrm{P}_{\mathrm{II}}^{\mathscr{C}})$ (which holds by $(\mathrm{R}_{\mathrm{II}}^{\mathscr{C}})$ and Lemma 3.3(iii)) since $\tau$ is faithfully flat [Mat89, Theorem 8.14]. □

We now deduce Theorem B as a consequence.

Theorem B. *Problem 1.2 holds for properties* $\mathbf{R}$ *of noetherian local rings satisfying* $(\mathrm{R}'_{\mathrm{I}})$, $(\mathrm{R}_{\mathrm{II}})$, $(\mathrm{R}_{\mathrm{IV}})$, *and* $(\mathrm{R}_{\mathrm{V}})$, *such that regular local rings satisfy* $\mathbf{R}$.

*Proof.* It suffices to show that the hypotheses in Theorem B imply those in Corollary 3.6 when $\mathscr{C}$ is the entire category of noetherian rings. Note that (i) is already a hypothesis in Theorem B and that (ii) is vacuously true. It therefore suffices to note that $(\mathrm{R}'_{\mathrm{I}})$ implies $(\mathrm{P}'_{\mathrm{I}})$ by Lemma 3.3(ii), and that $(\mathrm{R}_{\mathrm{I}})$ and $(\mathrm{R}_{\mathrm{II}})$ imply $(\mathrm{P}_{\mathrm{IV}})$ by Lemma 3.3(iv). Here, $(\mathrm{R}_{\mathrm{I}})$ holds by $(\mathrm{R}'_{\mathrm{I}})$ and the assumption that regular local rings satisfy $\mathbf{R}$. □

We can also prove a version of Corollary 3.6 for a specific choice of the category $\mathscr{C}$, as long as we put an extra condition on $B \otimes_A k$.

Corollary 3.7 (cf. [EGAIV$_2$, Corollaire 7.5.2; BI84, Proposition 1.2]). *Denote by* $\mathscr{C}$ *the smallest full subcategory of the category of noetherian rings containing noetherian complete local rings that is stable under homomorphisms essentially of finite type. Let* $\mathbf{R}$ *be a property of noetherian local rings, and consider a flat local homomorphism* $\varphi\colon (A, \mathfrak{m}, k) \to (B, \mathfrak{n}, l)$ *of noetherian local rings. Assume the following:*

(i) *the rings* $A$ *and* $B \otimes_A k$ *have geometrically* $\mathbf{R}$ *formal fibers; and*
(ii) *the property* $\mathbf{R}$ *satisfies* $(\mathrm{R}'_{\mathrm{I}})$, $(\mathrm{R}_{\mathrm{II}}^{\mathscr{C}})$, $(\mathrm{R}_{\mathrm{IV}}^{\mathscr{C}})$, $(\mathrm{R}_{\mathrm{V}}^{\mathscr{C}})$, *and* $(\mathrm{P}_{\mathrm{IV}}^{\mathscr{C}})$.

*If the closed fiber of* $\varphi$ *is geometrically* $\mathbf{R}$, *then all fibers of* $\varphi$ *are geometrically* $\mathbf{R}$.

*Proof.* We have the commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ \widehat{A} & \xrightarrow{\hat{\varphi}} & \widehat{B} \end{array}$$

where $\sigma$ and $\tau$ are the canonical $\mathfrak{m}$-adic and $\mathfrak{n}$-adic completion homomorphisms, respectively. By (i), the homomorphism $\sigma$ is geometrically $\mathbf{R}$. We claim that $\hat{\varphi}$ is geometrically $\mathbf{R}$. Note that $\hat{\varphi}$ is flat by [Mat89, Theorem 22.4(i)], and that $B \otimes_A k$ has geometrically $\mathbf{R}$ formal fibers by (i). Thus, the composition

$$k \longrightarrow B \otimes_A k \longrightarrow \hat{B} \otimes_{\hat{A}} k$$



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is geometrically $\mathbf{R}$ by applying $(\mathrm{P}'_{\mathrm{I}})$ (which holds by $(\mathrm{R}'_{\mathrm{I}})$ and Lemma 3.3(ii)). Since this composition is equal to $\hat{\varphi} \otimes_{\hat{A}} \mathrm{id}_k$, Theorem 3.4 implies that $\hat{\varphi}$ is geometrically $\mathbf{R}$, where we use the fact that the complete local ring $\hat{A}$ is excellent by [EGAIV$_2$, Scholie 7.8.3(iii)].

The composition $\tau \circ \varphi = \varphi^* \circ \sigma$ is geometrically $\mathbf{R}$ by $(\mathrm{P}'_{\mathrm{I}})$, and therefore $\varphi$ is also by $(\mathrm{P}'_{\mathrm{I}})$ (which holds by $(\mathrm{R}^{\mathscr{C}}_{\mathrm{II}})$ and Lemma 3.3(iii)) since $\tau$ is faithfully flat [Mat89, Theorem 8.14]. □

### 3.4 Global applications and the proof of Theorem A

We now prove Theorem A by reducing to the local statements proved above. This step corresponds to (I) in § 1.

We first give global versions of Theorem 3.4 and Corollaries 3.6 and 3.7. Theorem A(i) will be deduced from (ii) below. These results are related to a theorem of Shimomoto [Shi17, Main Theorem 1], which applies to morphisms of finite type between excellent noetherian schemes.

Proposition 3.8 (cf. [Shi17, Main Theorem 1]). *Fix a full subcategory $\mathscr{C}$ of the category of noetherian rings that is stable under homomorphisms essentially of finite type. Let $\mathbf{R}$ be a property of noetherian local rings, and consider a flat morphism $f\colon Y \to X$ of locally noetherian schemes mapping closed points to closed points. Assume one of the following:*

(i) *$\mathbf{R}$ satisfies the hypotheses of Theorem 3.4 for the category $\mathscr{C}$, the local rings of $X$ at closed points are quasi-excellent, and the local rings of $Y$ at closed points appear in $\mathscr{C}$;*
(ii) *$\mathbf{R}$ satisfies the hypotheses of Corollary 3.6 for the category $\mathscr{C}$, the local rings of $X$ at closed points have geometrically $\mathbf{R}$ formal fibers, and the rings $\hat{\mathcal{O}}_{X,f(y)}$ and $\mathcal{O}^*_{Y,y}$ appear in $\mathscr{C}$ for every closed point $y \in Y$, where $\mathcal{O}^*_{Y,y}$ denotes the $\mathfrak{m}_{f(y)}$-adic completion of $\mathcal{O}_{Y,y}$; or*
(iii) *$\mathbf{R}$ satisfies the hypotheses of Corollary 3.7 for the specific choice of $\mathscr{C}$ therein, the local rings of $X$ at closed points have geometrically $\mathbf{R}$ formal fibers, and the local rings $\mathcal{O}_{f^{-1}(x),y}$ have geometrically $\mathbf{R}$ formal fibers for every closed point $x \in X$ and every closed point $y \in f^{-1}(x)$.*

*If every closed fiber of $f$ is geometrically $\mathbf{R}$, then all fibers of $f$ are geometrically $\mathbf{R}$.*

*Proof.* Let $\xi \in Y$ be an arbitrary point, and let $\eta = f(\xi)$. We want to show that $\mathcal{O}_{Y,\xi}$ is geometrically $\mathbf{R}$ over $\kappa(\eta)$. By [Sta20, Tag 02IL], the point $\xi$ specializes to a closed point $y \in Y$. Since $f$ maps closed points to closed points, the point $x = f(y)$ is closed in $X$. After localization, we then obtain a flat local homomorphism

$$\varphi\colon \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y}$$

whose closed fiber is geometrically $\mathbf{R}$ by assumption. Finally, we can apply Theorem 3.4, Corollary 3.6, and Corollary 3.7 to the homomorphism $\varphi$ under the assumptions in (i), (ii), and (iii), respectively, to conclude that $\mathcal{O}_{Y,\xi}$ is geometrically $\mathbf{R}$ over $\kappa(\eta)$. □

We also prove that the locus over which $f$ has geometrically $\mathbf{R}$ fibers is stable under generization for closed flat morphisms. Theorem A(i) will be deduced from (ii) below.

Proposition 3.9. *Fix a full subcategory $\mathscr{C}$ of the category of noetherian rings that is stable under homomorphisms essentially of finite type. Let $\mathbf{R}$ be a property of noetherian local rings, and consider a closed flat morphism $f\colon Y \to X$ of locally noetherian schemes. Assume one*



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*of the following:*

(i) $\mathbf{R}$ *satisfies the hypotheses of Theorem 3.4 for the category* $\mathscr{C}$*, the local rings of* $X$ *are quasi-excellent, and the local rings of* $Y$ *appear in* $\mathscr{C}$*;*

(ii) $\mathbf{R}$ *satisfies the hypotheses of Corollary 3.6 for the category* $\mathscr{C}$*, the local rings of* $X$ *have geometrically* $\mathbf{R}$ *formal fibers, and the rings* $\hat{\mathcal{O}}_{X,f(y)}$ *and* $\mathcal{O}^*_{Y,y}$ *appear in* $\mathscr{C}$ *for every* $y \in Y$*, where* $\mathcal{O}^*_{Y,y}$ *denotes the* $\mathfrak{m}_{f(y)}$*-adic completion of* $\mathcal{O}_{Y,y}$*; or*

(iii) $\mathbf{R}$ *satisfies the hypotheses of Corollary 3.7 for the specific choice of* $\mathscr{C}$ *therein, the local rings of* $X$ *have geometrically* $\mathbf{R}$ *formal fibers, and the local rings* $\mathcal{O}_{f^{-1}(x),y}$ *have geometrically* $\mathbf{R}$ *formal fibers for every* $x \in X$ *and every* $y \in f^{-1}(x)$*.*

*Then the locus*

$$U_{\mathbf{R}}(f) \coloneqq \{x \in X \mid f^{-1}(x) \text{ is geometrically } \mathbf{R} \text{ over } \kappa(x)\} \subseteq X$$

*is stable under generization.*

*Proof.* Consider a specialization $\eta \rightsquigarrow x$ in $X$, and suppose that $f^{-1}(x)$ is geometrically $\mathbf{R}$ over $\kappa(x)$. We want to show that $f^{-1}(\eta)$ is geometrically $\mathbf{R}$ over $\kappa(\eta)$. It suffices to show that, for every $\xi \in f^{-1}(\eta)$, the local ring $\mathcal{O}_{Y,\xi}$ is geometrically $\mathbf{R}$ over $\kappa(\eta)$. Since $f$ is closed, the specialization $\eta \rightsquigarrow x$ lifts to a specialization $\xi \rightsquigarrow y$ [Sta20, Tag 0066]. After localization, we then obtain a flat local homomorphism

$$\varphi\colon \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y}$$

whose closed fiber is geometrically $\mathbf{R}$ by assumption. Finally, we can apply Theorem 3.4, Corollary 3.6, and Corollary 3.7 to the homomorphism $\varphi$ under the assumptions in (i), (ii), and (iii), respectively, to conclude that $\mathcal{O}_{Y,\xi}$ is geometrically $\mathbf{R}$ over $\eta$. □

We now deduce Theorem A as a consequence of Propositions 3.8 and 3.9.

Theorem A. *Let* $\mathbf{R}$ *be a property of noetherian local rings satisfying* $(\mathrm{R}'_{\mathrm{I}})$*,* $(\mathrm{R}_{\mathrm{II}})$*,* $(\mathrm{R}_{\mathrm{IV}})$*, and* $(\mathrm{R}_{\mathrm{V}})$*, such that regular local rings satisfy* $\mathbf{R}$*. Consider a flat morphism* $f\colon Y \to X$ *of locally noetherian schemes.*

(i) *Suppose that* $f$ *maps closed to closed points, and that the local rings of* $X$ *at closed points have geometrically* $\mathbf{R}$ *formal fibers. If every closed fiber of* $f$ *is geometrically* $\mathbf{R}$*, then all fibers of* $f$ *are geometrically* $\mathbf{R}$*.*

(ii) *Suppose that* $f$ *is closed, and that the local rings of* $X$ *have geometrically* $\mathbf{R}$ *formal fibers. Then the locus*

$$U_{\mathbf{R}}(f) \coloneqq \{x \in X \mid f^{-1}(x) \text{ is geometrically } \mathbf{R} \text{ over } \kappa(x)\} \subseteq X$$

*is stable under generization.*

*Proof.* As in the proof of Theorem B, it suffices to note that the hypotheses in (i) and (ii) imply those in Propositions 3.8(ii) and 3.9(ii), respectively, when $\mathscr{C}$ is the entire category of noetherian rings. □

### 3.5 The local lifting problem

To solve the local lifting problem, we use the following theorem of Brezuleanu and Ionescu. We state their theorem using the notation in Conditions 3.1 instead of that in [BI84, (2.1)].

Theorem 3.10 [BI84, Theorem 2.3]. *Let* $\mathbf{R}'$ *be a property of noetherian local rings. Assume the following:*

(i) *we have the sequence of implications regular* $\Rightarrow$ $\mathbf{R}'$ $\Rightarrow$ *reduced;*



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(ii) *the property* $\mathbf{R}'$ *satisfies* $(\mathrm{R}'_{\mathrm{I}})$, $(\mathrm{R}_{\mathrm{II}})$, $(\mathrm{R}_{\mathrm{III}})$, *and* $(\mathrm{R}_{\mathrm{V}})$;
(iii) *for every flat local homomorphism* $\varphi\colon A \to B$ *of noetherian complete local rings, if the closed fiber of* $\varphi$ *is geometrically* $\mathbf{R}'$, *then all fibers of* $\varphi$ *are geometrically* $\mathbf{R}'$.

*Let* $A$ *be a noetherian semi-local ring that is* $I$*-adically complete with respect to an ideal* $I \subseteq A$. *If* $A/I$ *is has geometrically* $\mathbf{R}'$ *formal fibers, then* $A$ *has geometrically* $\mathbf{R}'$ *formal fibers.*

We obtain Theorem C and an important corollary as immediate consequences.

Theorem C. *Let* $\mathbf{R}$ *be a property of noetherian local rings that satisfies the hypotheses in Theorem B. Suppose, moreover, that the locus*

$$U_{\mathbf{R}}\big(\mathrm{Spec}(C)\big) \coloneqq \{\mathfrak{p} \in \mathrm{Spec}(C) \mid C_{\mathfrak{p}} \text{ satisfies } \mathbf{R}\} \subseteq \mathrm{Spec}(C)$$

*is open for every noetherian complete local ring* $C$. *If* $A$ *is a noetherian semi-local ring that is* $I$*-adically complete with respect to an ideal* $I \subseteq A$, *and if* $A/I$ *is Nagata and has geometrically* $\mathbf{R}$ *formal fibers, then* $A$ *is Nagata and has geometrically* $\mathbf{R}$ *formal fibers.*

*Proof.* Since a noetherian semi-local ring is Nagata if and only if it has geometrically reduced formal fibers (Remark 2.5), it suffices to verify the hypotheses in Theorem 3.10 for $\mathbf{R}' = \mathbf{R}+$'reduced'. Both (i) and (ii) hold by assumption. Theorem B implies (iii) holds, since complete local rings have geometrically $\mathbf{R}$ formal fibers by the assumption that regular local rings satisfy $\mathbf{R}$. □

Corollary 3.11. *With assumptions as in Theorem C, if* $B$ *is a noetherian semi-local Nagata ring that has geometrically* $\mathbf{R}$ *formal fibers, then for every ideal* $I \subseteq B$ *the* $I$*-adic completion of* $B$ *is Nagata and has geometrically* $\mathbf{R}$ *formal fibers.*

*Proof.* The $I$-adic completion $\hat{B}$ of $B$ is a noetherian semi-local ring that is $I\hat{B}$-adically complete by [Bou98, Chapter III, § 3, n° 4, Proposition 8]. Thus, by Theorem C, it suffices to show that $\hat{B}/I\hat{B} \simeq B/I$ is a Nagata ring with geometrically $\mathbf{R}$ formal fibers. But $B/I$ has geometrically $\mathbf{R}$ formal fibers by [EGAIV$_2$, Proposition 7.3.15(i)], and is also Nagata by Definition 2.4. □

## 4. Specific properties R

We now explicitly consider our new cases of Grothendieck's localization problem and the local lifting problem. We have listed known cases of conditions $(\mathrm{R}'_{\mathrm{I}})$, $(\mathrm{R}_{\mathrm{II}})$, $(\mathrm{R}_{\mathrm{III}})$, $(\mathrm{R}_{\mathrm{IV}})$, and $(\mathrm{R}_{\mathrm{V}})$ in Table 2. While Problem 1.2 follows readily from these results when $\mathbf{R} =$'domain', 'Cohen–Macaulay and $F$-injective', and '$F$-rational', we will have to verify $(\mathrm{R}_{\mathrm{IV}})$ for weak normality (Proposition 4.10). We will deduce our results for terminal, canonical, and rational singularities from [EGAIV$_2$, Proposition 7.9.8] instead of Theorems A and B.

We will not explicitly formulate versions of Theorem A below, except for terminal, canonical, and rational singularities (Corollary 4.19).

### 4.1 Domain

We first note that a version of Problem 1.2 holds for $\mathbf{R} =$'domain'. This property satisfies the hypotheses in Theorem 3.4 (see Table 2). This extends a result of Marot [Mar84], which holds in residue characteristic zero. We use the terminology 'geometrically punctually integral' following [EGAIV$_2$, Définition 4.6.9] for the property obtained by applying Definition 2.1 to $\mathbf{R} =$ 'domain'.

Corollary 4.1 (cf. [Mar84, Theorem 2.1]). *Let* $\varphi\colon A \to B$ *be a flat local homomorphism of noetherian local rings, and assume that* $A$ *is quasi-excellent. If the closed fiber of* $\varphi$ *is geometrically punctually integral, then all fibers of* $\varphi$ *are geometrically punctually integral.*



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*Remark* 4.2. Since $(\mathrm{R}_{\mathrm{I}}^{\mathscr{C}'})$ is false for the property $\mathbf{R}$ = 'domain' [EGAIV$_2$, Remarques 6.5.5(ii) and 6.15.11(ii)], we only know that global versions of Corollary 4.1 hold under quasi-excellence assumptions by applying Propositions 3.8(i) and 3.9(i). See also [EGAIV$_2$, Corollaire 7.9.9 and Remarque 7.9.10(i)].

### 4.2 *F*-singularities

We now solve Problems 1.2 and 1.3 for 'Cohen–Macaulay and $F$-injective' and for $F$-rationality. See [Fed83, Definition on p. 473] and [HH94, Definition 4.1] for the definitions of $F$-injectivity and $F$-rationality, respectively. We recall (see [DM20, Remark A.4]) that we have the following sequence of implications:

$$\text{regular} \implies \text{F-rational} \implies \text{F-injective} \implies \text{reduced}. \tag{4}$$

Since the 'Cohen–Macaulay and $F$-injective' property satisfies the hypotheses in Theorems B and C (see Table 2 and (4)), we can solve Problems 1.2 and 1.3 for this property. The result for Problem 1.2 extends a result of Hashimoto [Has01] to the non-$F$-finite case.

Corollary 4.3 (cf. [Has01, Theorem 5.8]). *Grothendieck's localization problem and the local lifting problem hold for 'Cohen–Macaulay and $F$-injective', where $A$ is assumed to be of prime characteristic $p > 0$.*

*Remark* 4.4. Shimomoto and Zhang proved Grothendieck's localization problem for the 'Gorenstein and $F$-pure' property when $A$ and $B$ are $F$-finite [SZ09, Theorem 3.10]. Their result is a special case of Hashimoto's result, since $F$-purity and $F$-injectivity coincide for Gorenstein rings [Fed83, Lemma 3.3], and since Problem 1.2 holds for Gorensteinness [Mar84, Theorem 3.2]. Corollary 4.3 also extends Shimomoto and Zhang's result to the non-$F$-finite case. See also Remark 4.6.

Next, we consider Grothendieck's localization problem for $F$-rationality. The following extends a result of Hashimoto [Has01] to rings $A$ not necessarily essentially of finite type over a field, and a result of Shimomoto [Shi17] to homomorphisms not necessarily of finite type, giving a complete answer to a question of Hashimoto [Has01, Remark 6.7].

Corollary 4.5 (cf. [Has01, Remark 6.7; Shi17, Corollary 3.10]). *Let $\varphi\colon A \to B$ be a flat local homomorphism of noetherian local rings of prime characteristic $p > 0$. Assume that $A$ is quasi-excellent and that $B$ is excellent. If the closed fiber of $\varphi$ is geometrically $F$-rational, then all fibers of $\varphi$ are geometrically $F$-rational.*

*Proof.* We apply Theorem 3.4 when $\mathscr{C}$ is the category of excellent rings, which is stable under homomorphisms essentially of finite type by [EGAIV$_2$, Scholie 7.8.3(ii)]. To verify the hypotheses of Theorem 3.4, it suffices to note that excellent local rings are homomorphic images of Cohen–Macaulay rings [Kaw02, Corollary 1.2], and hence $(\mathrm{R}_{\mathrm{IV}}^{\mathscr{C}})$, $(\mathrm{R}_{\mathrm{V}}^{\mathscr{C}})$, and $(\mathrm{P}_{\mathrm{IV}}^{\mathscr{C}})$ hold by Table 2. □

*Remark* 4.6. Patakfalvi, Schwede, and Zhang also obtained a version of Problem 1.2 for proper flat morphisms $f\colon Y \to X$, showing that the locus $U_{\mathbf{R}}(f)$ defined in Definition 2.1 is open when $\mathbf{R}$ = 'Cohen–Macaulay and $F$-injective' (respectively, '$F$-rational') under the assumption that $X$ is an excellent integral scheme with a dualizing complex [PSZ18, Theorem 5.13]. Theorem A(ii) (respectively, Proposition 3.9(i)) implies that the locus $U_{\mathbf{R}}(f)$ is stable under generization, even if $f$ is only closed and flat (respectively, $f$ is closed and flat, $X$ is quasi-excellent, and $Y$ is excellent).



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For $F$-rationality, we note that Theorem A(i) does not apply, but one can apply Proposition 3.8(i) instead.

### 4.3 Weak normality and seminormality

In this subsection we prove that weak normality lifts from Cartier divisors. We then solve Problems 1.2 and 1.3 for both weak normality and seminormality.

To fix notation, we first define weak normality and seminormality.

Definition 4.7 (see [Kol16, Definition 50]). Let $X$ be a noetherian scheme. A morphism $g\colon X' \to X$ is a *partial normalization* if $X'$ is reduced, $g$ is integral, and $X' \to X_{\mathrm{red}}$ is birational. A partial normalization is a *partial weak normalization* if $g$ is a universal homeomorphism. A partial weak normalization is a *partial seminormalization* if the induced extensions of residue fields

$$\kappa(x) \subseteq \kappa\big(g^{-1}(x)_{\mathrm{red}}\big) \tag{5}$$

are bijective for all $x \in X$.

We say that $X$ is *weakly normal* (respectively, *seminormal*) if every finite partial weak normalization (respectively, finite partial seminormalization) $g\colon X' \to X$ is an isomorphism. A noetherian ring $A$ is *weakly normal* (respectively, seminormal) if $\operatorname{Spec}(A)$ is weakly normal (respectively, seminormal).

We have the sequence of implications

$$\text{normal} \Longrightarrow \text{weakly normal} \Longrightarrow \text{seminormal} \Longrightarrow \text{reduced} \tag{6}$$

where for the last implication, we use that $X_{\mathrm{red}} \to X$ is a finite partial seminormalization; see [Kol16, (4.4)]. We also note that a partial normalization is a partial weak normalization if and only if the extensions (5) are purely inseparable for all $x \in X$ by [EGAIV$_4$, Corollaire 18.12.11].

To show that weak normality lifts from Cartier divisors, we follow the proof in [BF93], with suitable modifications to avoid excellence hypotheses.

Lemma 4.8 (cf. [BF93, Proposition 4.7]). *Let $(A, \mathfrak{m}_A)$ be a noetherian local ring with $\operatorname{depth}(A_{\mathrm{red}}) \geqslant 2$, and let $B$ be a module-finite $A$-algebra such that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a partial weak normalization. Set $U \coloneqq \operatorname{Spec}(A) - \{\mathfrak{m}_A\}$. Then the local hull*

$$C \coloneqq \Gamma\big(U, \tilde{B}|_U\big)$$

*is a module-finite local $B$-algebra with $\operatorname{depth}(C) \geqslant 2$, such that $\operatorname{Spec}(C) \to \operatorname{Spec}(B)$ is a partial weak normalization.*

Here, $\tilde{\cdot}$ denotes the sheaf associated to a module on an affine scheme.

*Proof.* Throughout this proof, if $R$ is a local ring, then we denote by $\mathfrak{m}_R$ and $k_R$ the maximal ideal and residue field of $R$, respectively. By replacing $A$ with its reduction $A_{\mathrm{red}}$, it suffices to consider the case when $A$ is reduced.

We first show that $B \to C$ is module-finite, for which it suffices to show that $A \to C$ is module-finite. By Kollár's finiteness theorem [Kol17, Theorem 2], (5) $\Rightarrow$ (1), it suffices to show that for every $\mathfrak{p} \in \operatorname{Ass}_A(B)$, the local hull

$$\Gamma\big(U, (A/\mathfrak{p})^{\sim}|_U\big)$$

is a finitely generated $A$-module. Since $A$ and $B$ are reduced and $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a homeomorphism, we see that $\operatorname{Ass}_A(A) = \operatorname{Ass}_A(B)$. Thus, applying Kollár's finiteness theorem [Kol17, Theorem 2], (1) $\Rightarrow$ (5) to $F = \mathcal{O}_{\operatorname{Spec}(A)}$, it suffices to show that the local hull $\Gamma(U, \tilde{A}|_U)$ is a finitely



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generated $A$-module. But this is automatic since the natural homomorphism $A \to \Gamma(U, \tilde{A}|_U)$ is an isomorphism by [Kol17, Lemma 14], using the condition $\operatorname{depth}(A) \geqslant 2$.

We now show that $\operatorname{Spec}(C) \to \operatorname{Spec}(B)$ is a universal homeomorphism. Let $A' \coloneqq A^{\mathrm{sh}}$ be a strict Henselization of $A$, and set $U' \coloneqq \operatorname{Spec}(A') - \{\mathfrak{m}_{A'}\}$. Denoting $B' \coloneqq B \otimes_A A^{\mathrm{sh}}$ and $C' \coloneqq C \otimes_A A^{\mathrm{sh}}$, it suffices to show that the morphism

$$\operatorname{Spec}(C') \longrightarrow \operatorname{Spec}(B') \tag{7}$$

is a universal homeomorphism by [EGAIV$_2$, Corollaire 2.6.4(iv)] since $A \to A'$ is faithfully flat. Setting

$$U' \coloneqq \operatorname{Spec}(A') - \{\mathfrak{m}_{A'}\} \quad \text{and} \quad V' \coloneqq \operatorname{Spec}(B') - \{\mathfrak{m}_{B'}\},$$

we see that (7) is an isomorphism over $V'$, and hence it suffices to show that there is only one prime ideal in $C'$ lying over $\mathfrak{m}_{B'}$, and that the residue field extension $k_{B'} \to k_{C'}$ is purely inseparable [EGAIV$_4$, Corollaire 18.12.11]. First, since $\operatorname{depth}(A') \geqslant 2$ by [BH98, Proposition 1.2.16(a)], Hartshorne's connectedness theorem [EGAIV$_2$, Théorème 5.10.7] implies that the punctured spectrum $U'$ is connected, and thus, $V'$ is also connected by the fact that $\operatorname{Spec}(B') \to \operatorname{Spec}(A')$ is a homeomorphism. By the Henselian property of $A' = A^{\mathrm{sh}}$, the semi-local ring $C'$ is a direct product of local rings [EGAIV$_4$, Proposition 18.5.9(ii)]. But the fact that $V'$ is connected forces there to be only one factor in this decomposition, since the morphism (7) is an isomorphism over $V'$, and the support of

$$C' \simeq \Gamma\big(U', \widetilde{B'}|_{U'}\big) \tag{8}$$

is the closure of the inverse image of $V'$ under (7) by [Kol17, Definition 1, (3′)]. Thus, $C'$ is a local ring, and there is therefore only one prime ideal in $C'$ lying over $\mathfrak{m}_{B'}$. Note that the isomorphism (8) follows from flat base change for local hulls [Kol17, (13)] since the preimage of $\{\mathfrak{m}_B\}$ in $\operatorname{Spec}(B')$ is $\{\mathfrak{m}_{B'}\}$. Finally, since $k_{A'}$ is separably closed, the residue field extension $k_{B'} \to k_{C'}$ is purely inseparable, and hence (7) is a universal homeomorphism.

We now note that

$$\operatorname{depth}_{\mathfrak{m}_C}(C) = \operatorname{depth}_{\mathfrak{m}_A}(C) \geqslant 2, \tag{9}$$

where the first equality is [EGAIV$_1$, Chapitre 0, Proposition 16.4.8], and the second inequality follows from [Kol17, Lemma 14] since $C$ is module-finite over $A$. Moreover, the morphism $\operatorname{Spec}(C) \to \operatorname{Spec}(B)$ is birational since it induces an isomorphism between $\operatorname{Spec}(B) - \{\mathfrak{m}_B\}$ and $\operatorname{Spec}(C) - \{\mathfrak{m}_C\}$, which are dense in $\operatorname{Spec}(B)$ and $\operatorname{Spec}(C)$, respectively. It remains to show that $C$ is reduced. Since $\operatorname{Spec}(C) \to \operatorname{Spec}(B)$ is birational, we see that $C$ satisfies $(R_0)$, and satisfies $(S_1)$ everywhere except possibly at $\mathfrak{m}_C$. But (9) implies

$$\operatorname{depth}_{\mathfrak{m}_C}(C) \geqslant 2 \geqslant 1 = \min\big\{1, \dim(C)\big\}.$$

Thus, $C$ satisfies $(S_1)$ at $\mathfrak{m}_C$, and $C$ is therefore reduced. □

The following result allows us to restrict a partial weak normalization to a reduced Cartier divisor.

Lemma 4.9 (cf. [BF93, Proposition 4.8]). *Let $(A, \mathfrak{m}_A)$ be a noetherian local ring, and let $t \in \mathfrak{m}$ be a nonzerodivisor such that $A/tA$ is reduced of dimension $\geqslant 1$. Let $B$ be a module-finite $A$-algebra such that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a partial weak normalization that is an isomorphism*



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*along* $V(t) - \{\mathfrak{m}_A\}$. *Then the local hull*

$$C \coloneqq \Gamma\big(U, \tilde{B}|_U\big)$$

*is a module-finite local* $B$*-algebra such that the composition*

$$\operatorname{Spec}(C/tC) \xrightarrow{h} \operatorname{Spec}(B/tB) \xrightarrow{g} \operatorname{Spec}(A/tA)$$

*is a partial weak normalization.*

*Proof.* Throughout this proof, if $R$ is a local ring, then we denote by $\mathfrak{m}_R$ the maximal ideal of $R$. We note that $A$ is reduced since reducedness lifts from Cartier divisors [EGAIV$_2$, Proposition 3.4.6].

Set $\bar{A} \coloneqq A/tA$ and $\bar{C} \coloneqq C/tC$. We have

$$\operatorname{depth}(\bar{A}) \geqslant \min\big\{1, \dim(\bar{A})\big\} \geqslant 1$$

since $A/tA$ satisfies $(S_1)$ [EGAIV$_2$, Proposition 5.8.5]. Thus, we have $\operatorname{depth}(A) \geqslant 2$ by [BH98, Proposition 1.2.10(d)], and hence $\operatorname{Spec}(C) \to \operatorname{Spec}(B)$ is a partial weak normalization by Lemma 4.8. Since $g \circ h$ is an integral universal homeomorphism by base change, to show that $g \circ h$ is a partial weak normalization, it suffices to show that $g \circ h$ is birational and that $\bar{C}$ is reduced.

We start by showing that $g \circ h$ is birational. Since $g \circ h$ is a universal homeomorphism and $\bar{A}$ is reduced, to show that $g \circ h$ is birational, it suffices to show that $\bar{A} \to \bar{C}$ localizes to an isomorphism $\bar{A}_{\bar{\mathfrak{p}}} \xrightarrow{\sim} \bar{C}_{\bar{\mathfrak{p}}}$ at every minimal prime $\bar{\mathfrak{p}} \subseteq \bar{A}$. Let $\mathfrak{p} \subseteq A$ be the prime in $A$ corresponding to such a minimal prime $\bar{\mathfrak{p}}$. Since regularity lifts from Cartier divisors [EGAIV$_1$, Chapitre 0, Corollaire 17.1.8] and $\bar{A}$ is reduced, we see that $A_{\mathfrak{p}}$ is regular, in which case the homomorphism $A \to C$ induces an isomorphism $A_{\mathfrak{p}} \xrightarrow{\sim} C_{\mathfrak{p}}$ by the fact that regular rings are weakly normal; see (6). Thus, $g \circ h$ is birational.

We now show that $\bar{C}$ is reduced. By our assumption that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is an isomorphism along $V(t) - \{\mathfrak{m}_A\}$, we know that $\operatorname{Spec}(C) \to \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is an isomorphism along $V(t) - \{\mathfrak{m}_A\}$ by [Kol17, Definition 1, $(4')$]. Since $\bar{A}$ is reduced and $g \circ h$ is birational, we see that $\bar{A} \to \bar{C}$ is injective by [EGAI$_{\text{new}}$, Corollaire 1.2.6]. Thus, the homomorphism $\bar{A} \to \bar{C}$ restricts to an isomorphism

$$\tilde{\bar{A}}|_{\bar{U}} \xrightarrow{\sim} \tilde{\bar{C}}|_{\bar{U}}$$

over $\bar{U} \coloneqq \operatorname{Spec}(\bar{A}) - \{\mathfrak{m}_{\bar{A}}\}$. Since $\bar{A}$ is reduced, this shows that $\bar{C}$ satisfies $(R_0)$, and satisfies $(S_1)$ everywhere except possibly at $\mathfrak{m}_{\bar{C}}$. It suffices to show $\operatorname{depth}(\bar{C}) \geqslant 1$, since this would imply

$$\operatorname{depth}(\bar{C}) \geqslant \min\big\{1, \dim(\bar{C})\big\} = 1$$

by the assumption that $\dim(\bar{A}) = \dim(\bar{C}) \geqslant 1$. Note that $\operatorname{depth}(C) \geqslant 2$ by Lemma 4.8, and hence it suffices to show that $t$ is a nonzerodivisor on $C$ by [BH98, Proposition 1.2.10(d)]. Consider the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \\ \operatorname{Frac}(A) & \xrightarrow{\sim} & \operatorname{Frac}(C) \end{array}$$

where the isomorphism of total rings of fractions on the bottom holds by the birationality of $g \circ h$ and the two vertical homomorphisms are injective since $A$ and $C$ are reduced. Since $A \to \operatorname{Frac}(A)$ is flat, it maps $t$ to a nonzerodivisor on $\operatorname{Frac}(A)$. By the commutativity of the diagram, $t$ is therefore a nonzerodivisor on $C$. □



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We can now show that weak normality satisfies ($R_{IV}$). This statement is due to Bingener and Flenner in the excellent case [BF93].

Proposition 4.10 (cf. [BF93, Corollary 4.1]). *Let $(A, \mathfrak{m})$ be a noetherian local ring, and let $t \in \mathfrak{m}$ be a nonzerodivisor. If $A/tA$ is weakly normal, then $A$ is weakly normal.*

*Proof.* We want to show that, for every module-finite $A$-algebra $B$ such that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a partial weak normalization, the homomorphism $A \to B$ is an isomorphism. We will show that $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is an isomorphism for every prime ideal $\mathfrak{p} \subseteq A$ containing $t$, which would suffice since $\mathfrak{m}$ is a prime ideal containing $t$.

Set $\bar{A} \coloneqq A/tA$ and $\bar{B} \coloneqq B/tB$. We induce on the height of $\bar{\mathfrak{p}} \coloneqq \mathfrak{p}\bar{A}$. If $\operatorname{ht}(\bar{\mathfrak{p}}) = 0$, then $\bar{A}_{\bar{\mathfrak{p}}} \simeq A_{\mathfrak{p}}/tA_{\mathfrak{p}}$ is regular since $\bar{A}$ is reduced by (6), and hence $A_{\mathfrak{p}}$ is regular since regularity lifts from Cartier divisors [$EGAIV_1$, Chapitre 0, Corollaire 17.1.8]. Thus, $A_{\mathfrak{p}}$ is weakly normal by (6), and $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is an isomorphism.

Now suppose that $\operatorname{ht}(\bar{\mathfrak{p}}) > 0$. By the inductive hypothesis, we know that $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is an isomorphism along $V(t) - \{\mathfrak{p}A_{\mathfrak{p}}\} \subseteq \operatorname{Spec}(A_{\mathfrak{p}})$. Applying Lemmas 4.8 and 4.9 on $A_{\mathfrak{p}}$, the local hull

$$C \coloneqq \Gamma\big(\operatorname{Spec}(A_{\mathfrak{p}}) - \{\mathfrak{p}A_{\mathfrak{p}}\}, \tilde{B}_{\mathfrak{p}}|_{\operatorname{Spec}(A_{\mathfrak{p}})-\{\mathfrak{p}A_{\mathfrak{p}}\}}\big)$$

is module-finite over $A$ and yields a partial weak normalization $\operatorname{Spec}(C) \to \operatorname{Spec}(B)$ such that the composition

$$\operatorname{Spec}(C/tC) \longrightarrow \operatorname{Spec}(\bar{B}_{\bar{\mathfrak{p}}}) \longrightarrow \operatorname{Spec}(\bar{A}_{\bar{\mathfrak{p}}})$$

is a partial weak normalization. Since $\bar{A}_{\bar{\mathfrak{p}}} \simeq A_{\mathfrak{p}}/tA_{\mathfrak{p}}$ is weakly normal by [Man80, Corollary IV.2], we see that this composition is an isomorphism. The Nakayama–Azumaya–Krull lemma [Mat89, Theorem 2.2] then implies that $A_{\mathfrak{p}} \to C$ is an isomorphism. Since $B_{\mathfrak{p}} \to C$ is injective by [$EGAI_{new}$, Corollaire 1.2.6], we see that $B_{\mathfrak{p}} \to C$ is an isomorphism as well, and hence $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is also an isomorphism. □

Since we have seen that both weak normality and seminormality satisfy the hypotheses in Theorems B and C (see Table 2 and (6)), we can solve Problems 1.2 and 1.3 for both weak normality and seminormality. The proof of Problem 1.2 for seminormality extends a result of Shimomoto [Shi17] in the finite type case.

Corollary 4.11 (cf. [Shi17, Corollary 3.4]). *Grothendieck's localization problem and the local lifting problem hold for weak normality and seminormality.*

### 4.4 Terminal, canonical, and rational singularities

We now consider Problem 1.2 for terminal, canonical, and rational singularities in equal characteristic zero, using the version of Theorem B in [$EGAIV_2$, Proposition 7.9.8].

See [Kol13, Definition 2.8] for the definition of terminal and canonical singularities, which apply to excellent noetherian schemes with dualizing complexes. For rational singularities, we will work with the following definition.

Definition 4.12 [KKMS73, p. 51; Mur21, Definition 7.2]. Let $A$ be a quasi-excellent local $\mathbf{Q}$-algebra. We say that $A$ has *rational singularities* if $A$ is normal and if, for every resolution of singularities $f\colon W \to \operatorname{Spec}(A)$, we have $R^i f_* \mathcal{O}_W = 0$ for all $i > 0$. If $X$ is a locally noetherian $\mathbf{Q}$-scheme whose local rings $\mathcal{O}_{X,x}$ are quasi-excellent, we say that $X$ *has rational singularities* if $\mathcal{O}_{X,x}$ has rational singularities for every $x \in X$. By [Mur21, Lemma 7.3], a quasi-excellent local $\mathbf{Q}$-algebra $R$ has rational singularities if and only if $\operatorname{Spec}(R)$ has rational singularities.



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*Remark* 4.13 (cf. [KKMS73, p. 51; Kol13, Corollary 2.86)]. The object $\mathbf{R}f_*\mathcal{O}_W$ does not depend on the choice of resolution $f\colon W \to \mathrm{Spec}(A)$, since higher direct images of structure sheaves vanish for morphisms of noetherian schemes of finite type over a quasi-excellent local $\mathbf{Q}$-algebra (one combines the strategy of [Hir64, (2) on pp. 144–145] with the version of elimination of indeterminacies in [Hir64, Chapter I, § 3, Main Theorem II(N)]). Thus, to show that a quasi-excellent local $\mathbf{Q}$-algebra has rational singularities, it suffices to show that $A$ is normal and that there exists a resolution of singularities $f\colon W \to \mathrm{Spec}(A)$ such that $R^i f_*\mathcal{O}_W = 0$ for all $i > 0$.

In particular, this shows that if $X$ is a quasi-excellent $\mathbf{Q}$-scheme of finite Krull dimension, then the rational locus is open in $X$: since resolutions of singularities exist in this context [Tem08, Theorem 1.1], the rational locus is the normal locus intersected with the complement of $\bigcup_{i>0}\mathrm{Supp}(R^i f_*\mathcal{O}_W)$, where $f\colon W \to X$ is a resolution of singularities.

We now show the following strong version of $(\mathrm{P_{IV}})$.

Lemma 4.14. *Let $R$ be a quasi-excellent local $\mathbf{Q}$-algebra, and let $k \to R$ be a homomorphism from a field of characteristic zero.*

(i) *If $R$ has a dualizing complex, then $\mathrm{Spec}(R)$ has terminal (respectively, canonical) singularities if and only if $\mathrm{Spec}(R \otimes_k k')$ has terminal (respectively, canonical) singularities for every finitely generated field extension $k \subseteq k'$.*
(ii) *$\mathrm{Spec}(R)$ has rational singularities if and only if $\mathrm{Spec}(R \otimes_k k')$ has rational singularities for every finitely generated field extension $k \subseteq k'$.*

*Proof.* $\Leftarrow$ holds by setting $k = k'$. It therefore suffices to show the converse. The terminal and canonical cases follow from [Kol13, Proposition 2.15], since $R \to R \otimes_k k'$ is surjective and essentially smooth by base change [EGAIV$_2$, Proposition 6.8.3(iii); EGAIV$_4$, Théorème 17.5.1]. Here, we note that $R \otimes_k k'$ has a dualizing complex by [Har66, (2) on p. 299]. For the rational case, we use the criterion for rational singularities stated in Remark 4.13. Let $f\colon W \to \mathrm{Spec}(R)$ be a resolution of singularities. Since $R \to R \otimes_k k'$ is flat with geometrically regular fibers, the morphism $f'\colon W \otimes_k k' \to \mathrm{Spec}(R \otimes k')$ is a resolution of singularities [Mat89, Theorem 23.7] and satisfies $R^i f'_*\mathcal{O}_{W\otimes_k k'} = 0$ for all $i > 0$ by flat base change. Finally, $R \otimes_k k'$ is normal by [Mat89, Corollary to Theorem 23.9]. □

The main ingredient for terminal and canonical singularities is the following version of $(\mathrm{R_{IV}})$, which follows from the deformation results for complex varieties due to Kawamata [Kaw99] for canonical singularities and Nakayama [Nak04] for terminal singularities. Note that similar results do not hold for Kawamata log terminal or log canonical singularities without assuming that the canonical divisor on $A$ is $\mathbf{Q}$-Cartier; see [Ish18, Example 9.1.7 and Remark 9.1.15].

Proposition 4.15 (cf. [Kaw99, Main Theorem; Nak04, Chapter VI, Theorem 5.2(2)]). *Let $(A, \mathfrak{m})$ be a local ring essentially of finite type over a field $k$ of characteristic zero, and let $t \in \mathfrak{m}$ be a nonzerodivisor. If $A/tA$ has terminal (respectively, canonical) singularities, then $A$ has terminal (respectively, canonical) singularities.*

*Proof.* Note that $A$ is a normal domain by [Sey72, Proposition I.7.4] and [EGAIV$_2$, Proposition 3.4.5].

Step 1. *It suffices to show that if $A'$ is a normal domain of finite type over $k$, and $t' \in A'$ is a non-zero element such that $A'/t'$ has terminal (respectively, canonical) singularities, then $A'$ has terminal (respectively, canonical) singularities along $V(t')$.*



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Since $A$ is essentially of finite type over $k$, there exist a ring $A'$ of finite type over $k$, a prime ideal $\mathfrak{p} \subseteq A'$ such that $A = A'_{\mathfrak{p}}$, and an element $t' \in A'$ such that $t'$ maps to $t \in A$. Since the normal locus in $\operatorname{Spec}(A')$ is open [EGAIV$_2$, Corollaire 6.13.5], we can replace $\operatorname{Spec}(A')$ by an affine open subset containing $\mathfrak{p}$ to assume that $A'$ is normal domain and that $t'$ is non-zero. Since the locus over which $\operatorname{Spec}(A'/t')$ has terminal (respectively, canonical) singularities is open by [Kol13, Corollary 2.12], we can replace $A'$ by an affine open subset again to assume that $A'/t'$ has terminal (respectively, canonical) singularities. Then $A'$ has terminal (respectively, canonical) singularities by assumption, and hence $A'_{\mathfrak{p}}$ has terminal (respectively, canonical) singularities by [Kol13, Corollary 2.12].

Step 2. *It suffices to show that the statement in Step 1 holds when $k = \mathbf{C}$.*

Since $A$ is of finite type over $k$, there exists a subfield $k_0 \subseteq k$ that is a finitely generated field extension of $\mathbf{Q}$ together with a ring $A_0$ of finite type over $k_0$ such that $A \simeq A_0 \otimes_{k_0} k$, and such that there exists an element $t_0 \in A_0$ mapping to $t \in A$ under the homomorphism $A_0 \to A$. Note that $A_0$ is a normal domain by [Mat89, Corollary to Theorem 23.9] and [EGAIV$_2$, Proposition 2.1.14]. Since $A_0/t_0A_0 \otimes_{k_0} k \simeq A/tA$, applying Lemma 4.14, we know that $A_0/t_0A_0 \otimes_{k_0} \mathbf{C}$ has terminal (respectively, canonical) singularities, and, moreover, that $A$ has terminal (respectively, canonical) singularities if $A_0 \otimes_{k_0} \mathbf{C}$ does. Note that the image of $t_0$ in $A_0 \otimes_{k_0} \mathbf{C}$ is a nonzerodivisor by flat base change, and that $A_0 \otimes_{k_0} \mathbf{C}$ is normal by [Mat89, Corollary to Theorem 23.9]. Working one direct factor of $A_0 \otimes_{k_0} \mathbf{C}$ at a time, the statement in Step 1 for $k = \mathbf{C}$ implies that $A_0 \otimes_{k_0} \mathbf{C}$ has terminal (respectively, canonical) singularities.

Step 3. *The statement in Step 1 holds for $k = \mathbf{C}$.*

The terminal case is [Nak04, Chapter VI, Theorem 5.2(2)] and the canonical case is [Kaw99, Main Theorem] (see also [Nak04, Chapter VI, Theorem 5.2(1)]). □

For rational singularities, we prove ($\mathrm{R}_{\mathrm{IV}}$) in greater generality. For the proof, we will work with the notion of pseudo-rational rings defined below, which gives a characteristic-free version of rational singularities.

Definition 4.16 [LT81, § 2]. Let $(A, \mathfrak{m})$ be a noetherian local ring of dimension $d$. We say that $A$ is *pseudo-rational* if:

(i) $A$ is normal;
(ii) $A$ is Cohen–Macaulay;
(iii) the $\mathfrak{m}$-adic completion $\hat{A}$ of $A$ is reduced; and
(iv) for every proper birational morphism $f\colon W \to \operatorname{Spec}(A)$ with $W$ normal, if $E = f^{-1}(\{\mathfrak{m}\})$ is the closed fiber, then the canonical homomorphism

$$H^d_{\mathfrak{m}}(A) = H^d_{\{\mathfrak{m}\}}(f_*\mathcal{O}_W) \xrightarrow{\delta^d_f(\mathcal{O}_W)} H^d_E(\mathcal{O}_W)$$

appearing as the edge map in the Leray–Serre spectral sequence for the composition of functors $\Gamma_{\{\mathfrak{m}\}} \circ f_* = \Gamma_E$ is injective.

We now prove ($\mathrm{R}_{\mathrm{IV}}$) for rational singularities using the strategy in [MS21, Proposition 3.4]. The case when $A$ is essentially of finite type over a field of characteristic zero is due to Elkik [Elk78].

Proposition 4.17 (cf. [Elk78, Théorème 5]). *Let $(A, \mathfrak{m})$ be a quasi-excellent local $\mathbf{Q}$-algebra, and let $t \in \mathfrak{m}$ be a nonzerodivisor. If $A/tA$ has rational singularities, then $A$ has rational singularities.*



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*Proof.* By [Mur21, Remark 7.4], a quasi-excellent local $\mathbf{Q}$-algebra has rational singularities if and only if it is pseudo-rational. Since $A$ is normal and Cohen–Macaulay by [Sey72, Proposition I.7.4] and [BH98, Theorem 2.1.3(a)], and $\hat{A}$ is reduced by [Mat89, Corollary to Theorem 23.9], it suffices to show that for every resolution of singularities $f\colon W \to \operatorname{Spec}(A)$, the homomorphism $\delta_f^d(\mathcal{O}_W)$ is injective, where $d = \dim(A)$.

Let $E = f^{-1}(\{\mathfrak{m}\})$. Set $W_t = f^{-1}(V(t))$, and let $g\colon W_t' \to W_t$ be a resolution of singularities, which exists by [Hir64, Chapter I, § 3, Main Theorem I($n$)]. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(A/tA) & \longrightarrow & H^d_{\mathfrak{m}}(A) & \xrightarrow{t\cdot} & H^d_{\mathfrak{m}}(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \delta_f^d(\mathcal{O}_W)} & & \downarrow{\scriptstyle \delta_f^d(\mathcal{O}_W)} & & \\
0 & \longrightarrow & H^{d-1}_{E}(\mathcal{O}_{W_t}) & \longrightarrow & H^d_{E}(\mathcal{O}_W) & \xrightarrow{t\cdot} & H^d_{E}(\mathcal{O}_W) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & H^{d-1}_{g^{-1}(E)}(\mathcal{O}_{W_t'}) & & & & & &
\end{array}$$

where the top half is obtained from [Mur21, Lemma 3.12]. The top left arrow is injective since the composition in the left column is injective by the hypothesis that $A/tA$ has rational singularities, where we use the fact that the edge maps in Definition 4.16(iv) behave well under composition of morphisms [Smi97, Proposition 1.12]. The rows are exact on the left by the fact that $A$ is Cohen–Macaulay and by the version of Grauert–Riemenschneider vanishing in [Mur21, Theorem B*(i)], respectively.

Now suppose there exists an element $0 \neq \eta \in \ker(\delta_f^d(\mathcal{O}_W))$. Since every element in $H^d_{\mathfrak{m}}(A)$ is annihilated by a power of $t$, after multiplying $\eta$ by a power of $t$ we may assume that $t\eta = 0$, in which case $\eta$ lies in the image of $H^{d-1}_{\mathfrak{m}}(A/tA)$ in the top row. The commutativity of the diagram implies that the composition $H^{d-1}_{\mathfrak{m}}(A/tA) \to H^d_{\mathfrak{m}}(\mathcal{O}_W)$ is injective. Since $\eta \in \ker(\delta_f^d(\mathcal{O}_W))$ by assumption, this shows that $\eta = 0$, which is a contradiction. □

We can now solve Grothendieck's localization problem for terminal, canonical, and rational singularities using [EGAIV$_2$, Proposition 7.9.8]. While our Theorems B and 3.4 apply to positive or mixed characteristic, we are not able to prove this result in these contexts in part because we do not know whether Propositions 4.15 and 4.17 hold in positive or mixed characteristic.

Corollary 4.18. *Let $\varphi\colon A \to B$ be a flat local homomorphism of quasi-excellent local $\mathbf{Q}$-algebras.*

(i) *Suppose that $B$ is essentially of finite type over a field of characteristic zero. If the closed fiber of $\varphi$ has terminal (respectively, canonical) singularities, then all fibers of $\varphi$ have terminal (respectively, canonical) singularities.*
(ii) *If the closed fiber of $\varphi$ has rational singularities, then all fibers of $\varphi$ have rational singularities.*

*Proof.* By Lemma 4.14, we do not need to distinguish between 'has terminal (respectively, canonical, rational) singularities' and 'geometrically has terminal (respectively, canonical, rational) singularities'. We want to apply [EGAIV$_2$, Proposition 7.9.8] where the category $\boldsymbol{C}$ in their notation is the category of schemes essentially of finite type over a field of characteristic zero in situation (i), and the category of schemes locally essentially of finite type over a quasi-excellent local $\mathbf{Q}$-algebra in situation (ii).



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- For [EGAIV$_2$, (7.9.7.1)], the category $\boldsymbol{C}$ is stable under morphisms locally essentially of finite type by definition.
- For [EGAIV$_2$, (7.9.7.2)], the terminal and canonical locus of a scheme $X$ in $\boldsymbol{C}$ is open by [Kol13, Corollary 2.12], and the rational locus of a scheme $X$ in $\boldsymbol{C}$ is open by Remark 4.13.
- The condition [EGAIV$_2$, (7.9.7.3)] is our condition $(\mathrm{R}_{\mathrm{IV}}^{\mathscr{C}})$ when $\mathscr{C}$ is the category of rings whose spectra are in $\boldsymbol{C}$, which holds by Propositions 4.15 and 4.17.
- The condition [EGAIV$_2$, (7.9.7.4)] is a version of our condition $(\mathrm{P}_{\mathrm{IV}}^{\mathscr{C}})$ when $\mathscr{C}$ is the category of rings whose spectra are in $\boldsymbol{C}$, which holds by Lemma 4.14.
- For [EGAIV$_2$, (7.9.8.2)], every module-finite $A$-algebra has a resolution of singularities by [Hir64, Chapter I, § 3, Main Theorem I($n$)]. □

We state a global version of this result as well. A similar statement for the behavior of rational singularities in proper flat families was shown to us by János Kollár; see also [Elk78, Théorème 4].

Corollary 4.19. *Let $f\colon Y \to X$ be a flat morphism of locally noetherian $\mathbf{Q}$-schemes, such that the local rings of $X$ and $Y$ are quasi-excellent.*

(i) *Suppose $f$ maps closed points to closed points. If every closed fiber of $f$ has rational singularities, then all fibers of $f$ have rational singularities. If the local rings of $Y$ are essentially of finite type over fields of characteristic zero, and every closed fiber of $f$ has terminal (respectively, canonical) singularities, then all fibers of $f$ have terminal (respectively, canonical) singularities.*

(ii) *Suppose $f$ is closed. Then the locus of points $x \in X$ over which $f^{-1}(x)$ has rational singularities is stable under generization. If the local rings of $Y$ are essentially of finite type over fields of characteristic zero, then the locus of points $x \in X$ over which $f^{-1}(x)$ has terminal (respectively, canonical) singularities is stable under generization.*

*Proof.* This follows from Corollary 4.18 using the strategies in Propositions 3.8 and 3.9. □

For completeness, we end by showing that pseudo-rationality satisfies $(\mathrm{R}_{\mathrm{II}}^{\mathscr{C}})$ when $\mathscr{C}$ is the category of noetherian rings with residual complexes in the sense of [Har66, Definition on p. 304]. A special case of this statement was shown to us by Karl Schwede. Similar results hold for cyclically pure homomorphisms of $\mathbf{Q}$-algebras whose local rings are quasi-excellent; see [Bou87, Théorème on p. 65; Mur21, Theorem C].

Proposition 4.20. *Let $\varphi\colon (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a flat local homomorphism of noetherian local rings, such that $B$ has a residual complex. If $B$ is pseudo-rational, then $A$ is pseudo-rational.*

*Proof.* First, we know that $A$ is normal [Mat89, Corollary to Theorem 23.9] and Cohen–Macaulay [BH98, Theorem 2.1.7]. To prove that $\hat{A}$ is reduced, we note that

$$\hat{\varphi}\colon \hat{A} \longrightarrow \hat{B}$$

is flat, where $\hat{B}$ is the $\mathfrak{n}$-adic completion of $B$ [Mat89, Theorem 22.4(i)]. But $\hat{B}$ is reduced by assumption, and hence $\hat{A}$ is reduced by [Mat89, Corollary to Theorem 23.9].

It remains to show that condition (iv) holds. Let $\mathfrak{q} \subseteq B$ be a minimal prime lying over $\mathfrak{m}$. The localization $B_{\mathfrak{q}}$ is pseudo-rational since $B$ has a residual complex [LT81, § 4, Corollary of (iii)]. By replacing $B$ with $B_{\mathfrak{q}}$, we reduce to the case where $B/\mathfrak{m}B$ is zero-dimensional, in which case $\sqrt{\mathfrak{m}B} = \mathfrak{n}$. Let $f\colon W \to X := \operatorname{Spec}(A)$ be a proper birational morphism, where $W$ is normal.



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Table 1. Special cases of Problems 1.2 and 1.3. All results except those in the bottom section can be obtained from our methods.

| **R** | Grothendieck's localization problem 1.2 | Local lifting problem 1.3 |
|---|---|---|
| Regular | [And74, Thm. on p. 297] | [Rot79, Thm. 3] |
| $(R_n) + (S_{n+1})$ | [BI84, Prop. 1.5] | [BI84, 2.4(iv)] |
| Normal | [Nis81, Prop. 2.4] | [Nis81, Thm. on p. 154] |
| Weakly normal | Corollary 4.11 | Corollary 4.11 |
| Seminormal | Corollary 4.11 | Corollary 4.11 |
| Reduced | [Nis81, Prop. 2.4] | [Mar75, Prop. 3.6] |
| Complete intersection | [Tab84, Thm. 2] | [Tab84, Thm. 2] + [BI84, Thm. 2.3]$^{\mathrm{N}}$ |
| Gorenstein | [HS78, Thm. 3.3; Mar84, Thm. 3.2] | [Mar84, Thm. 3.2] + [BI84, Thm. 2.3]$^{\mathrm{N}}$ |
| Cohen–Macaulay | [AF94, Thm. 4.1] | [AF94, Thm. 4.1] + [BI84, Thm. 2.3]$^{\mathrm{N}}$ |
| $(CI_n)$ | [CI93, Corollary 2.3; AF94, Thm. 4.5] | [CI93, Corollary 3.5]$^{\mathrm{N}}$ |
| $(G_n)$ | [CI93, Thm. 2.2 and Rem. 2.8; AF94, Thm. 4.5] | [Mar84, Thm. 3.2] + [CI93, Thm. 3.4]$^{\mathrm{N}}$ |
| $(S_n)$ | [AF94, Thm. 4.5] | [AF94, Thm. 4.1] + [CI93, Thm. 3.4]$^{\mathrm{N}}$ |
| C–M + $F$-injective | Corollary 4.3 | Corollary 4.3 |
| Normal + $(R_n)$ | [BI84, Prop. 1.2][1] | [BI84, 2.4(iii)] |
| Domain | Corollary 4.1[2] | |
| Terminal singularities | Corollary 4.18(i)$^{\mathrm{char0}}$ | |
| Canonical singularities | Corollary 4.18(i)$^{\mathrm{char0}}$ | |
| Rational singularities | Corollary 4.18(ii)$^{\mathbf{Q}}$ | |
| $F$-rational | Corollary 4.5[3] | |
| $(R_n)$ | [Ion86, Thm. 1.2][4] | [Ion86, Thm. 1.4]$^{7,\mathrm{N}}$ |
| cid $\leqslant n$ | [AF94, Main Thm. (a)][5] | |
| cmd $\leqslant n$ | [AF94, Main Thm. (b)][6] | |

*Note:* 'C–M' stands for 'Cohen–Macaulay'.
See [AF94, Remark 4.2 and pp. 1–2] for definitions of $(CI_n)$, $(G_n)$, $(S_n)$, cid, and cmd.
We list necessary assumptions for the results in the bottom two sections of the table.
[1] Either $B$ is universally catenary, or $B \otimes_A k$ has geometrically (normal + $(R_n)$) formal fibers.
[2] $A$ is quasi-excellent.
[3] $A$ is quasi-excellent and $B$ is excellent.
[4] $\hat{B}$ is equidimensional.
[5] $A$ has complete intersection formal fibers.
[6] $A$ has Cohen–Macaulay formal fibers.
[7] $A$ is universally catenary.
[N] $A/I$ is Nagata.
[**Q**] $A$ and $B$ are quasi-excellent **Q**-algebras.
[char0] $A$ is a quasi-excellent **Q**-algebra and $B$ is essentially of finite type over a field of characteristic zero.



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Table 2. Properties satisfying Conditions 3.1. The conditions hold under additional assumptions for the properties in the second and third sections of the table.

| **R** | Ascent ($\mathrm{R}_{\mathrm{I}}^{\mathscr{C}\prime}$) | Descent ($\mathrm{R}_{\mathrm{II}}^{\mathscr{C}}$) | Openness ($\mathrm{R}_{\mathrm{III}}$) | Lifting from Cartier divisors ($\mathrm{R}_{\mathrm{IV}}^{\mathscr{C}}$) | Localization ($\mathrm{R}_{\mathrm{V}}^{\mathscr{C}}$) |
|---|---|---|---|---|---|
| Regular | [Mat89, Thm. 23.7] | | [EGAIV$_2$, Thm. 6.12.7] | [EGAIV$_1$, Ch. 0, Cor. 17.1.8] | [Mat89, Thm. 19.3] |
| $(R_n)+(S_{n+1})$ | See $(R_n)$ and $(S_n)$ | | See $(R_n)$ and $(S_n)$ | [BR82, Lem. 0(ii)] | See $(R_n)$ and $(S_n)$ |
| Normal | [Mat89, Cor. to Thm. 23.9] | | [EGAIV$_2$, Cor. 6.13.5] | [Sey72, Prop. I.7.4] | [Bou98, Ch. V, § 1, n° 5, Prop. 16] |
| Weakly normal | [Kol16, Thm. 37] | [Man80, Cor. II.2] | [BF93, Thm. 7.1.3] | Proposition 4.10 | [Man80, Cor. IV.2] |
| Seminormal | [Kol16, Thm. 37] | [GT80, Thm. 1.6] | [GT80, Cor. 2.3] | [Hei08, Main Thm.] | [GT80, Cor. 2.2] |
| Reduced | [Mat89, Cor. to Thm. 23.9] | | [Bou98, Ch. II, § 2, n° 6] | [EGAIV$_2$, Prop. 3.4.6] | [Bou98, Ch. II, § 2, n° 6, Prop. 17] |
| Complete intersection | [Avr75, Thm. 2] | | [GM78, Cor. 3.3] | [BH98, Thm. 2.3.4(a)] | [Avr75, Cor. 1] |
| Gorenstein | [BH98, Cor. 3.3.15] | | [GM78, Cor. 1.5] | [BH98, Prop. 3.1.19(b)] | [BH98, Prop. 3.1.19(a)] |
| Cohen–Macaulay | [BH98, Thm. 2.1.7] | | [EGAIV$_2$, Cor. 6.11.3] | [BH98, Thm. 2.1.3(a)] | [BH98, Thm. 2.1.3(b)] |
| $(CI_n)$ | [CI93, Prop. 1.10] | | [CI93, Prop. 1.10] | [Imb95, Lem. 3.1] | [CI93, Prop. 1.10] |
| $(G_n)$ | [RF72, Prop. 1; Pau73a, Prop. 1] | | [Ooi80, Prop. 18(2)] | [RF72, Prop. 3] | [RF72, Cor. (b) to Prop. 1] |
| $(S_n)$ | [Mat89, Thm. 23.9(iii)] | | [EGAIV$_2$, Prop. 6.11.2(ii)] | [BR82, Lem. 0(i)] | [EGAIV$_2$, Rem. 5.7.3(iv)] |
| C–M + $F$-injective | [Ene09, Thm. 4.3] | [Has10, Lem. 4.6] | [Has10, Cor. 4.18] | [Fed83, Thm. 3.4(1)] | [Has10, Cor. 4.11] |
| Normal + $(R_n)$ | See normal and $(R_n)$ | | See normal and $(R_n)$ | See normal and $(R_n)$[1] | See normal and $(R_n)$ |
| Domain | **false**[2] | [EGAIV$_2$, Prop. 2.1.14] | [Nag59, Lem. 1] | [EGAIV$_2$, Prop. 3.4.5] | [Bou98, Ch. II, § 2, n° 1, Rem. 7] |
| Terminal singularities | | | [Kol13, Cor. 2.12][char0] | [Kaw99, Main Thm.][char0] | [Kol13, Cor. 2.12][char0] |
| Canonical singularities | | | [Kol13, Cor. 2.12][char0] | [Nak04, Ch. VI, Thm. 5.2(2)][char0] | [Kol13, Cor. 2.12][char0] |
| Rational singularities | [Elk78, Thm. 5][Q] | Proposition 4.20[Q] | Remark 4.13[Q] | Proposition 4.17[Q] | [Mur21, Lem. 7.3][Q] |
| $F$-rational | [Ene00, Thm. 2.27][3,e] | [DM20, Prop. A.5] | [Vél95, Thm. 3.5] | [HH94, Thm. 4.2(h)][CM] | [HH94, Thm. 4.2(f)][CM] |
| $(R_n)$ | [Mat89, Thm. 23.9(i),(ii)] | | [EGAIV$_2$, Prop. 6.12.9] | [Ion86, Lem. 2.1][1] | [CI93, Prop. 1.6] |
| cid $\leqslant n$ | [Avr77, Prop. 3.6][4] | | [Avr77, Prop. 3.4] | [Avr77, Prop. 3.10] | [Avr77, Prop. 3.8] |
| cmd $\leqslant n$ | [EGAIV$_2$, Cor. 6.3.2][4] | | [EGAIV$_2$, Prop. 6.11.2(i)] | [EGAIV$_1$, Ch. 0, Prop. 16.4.10(i)] | [EGAIV$_2$, Prop. 6.11.5] |

*Note:* 'C–M' stands for 'Cohen–Macaulay'. See [AF94, Remark 4.2 and pp. 1–2] for definitions of $(CI_n)$, $(G_n)$, $(S_n)$, cid, and cmd.

[1] $(\mathrm{R}_{\mathrm{IV}}^{\mathscr{C}})$ is false when $\mathbf{R}=$ '$(R_n)$' [EGAIV$_2$, Remarque 5.12.6], but holds if one restricts $\mathscr{C}$ to be the category of equidimensional catenary noetherian rings. For 'normal + $(R_n)$', we note that normal local rings are equidimensional.

[2] $(\mathrm{R}_{\mathrm{I}}^{\mathscr{C}\prime})$ is false when $\mathbf{R}=$'domain' [EGAIV$_2$, Remarques 6.5.5(ii) and 6.15.11(ii)].

[3] $(\mathrm{R}_{\mathrm{I}}^{\mathscr{C}\prime})$ holds when $\mathbf{R}=$ '$F$-rational' if $\mathscr{C}$ is the category of excellent rings; cf. [Has01, Theorem 6.4]. Enescu assumes that $B/\mathfrak{m}B\otimes_{A/\mathfrak{m}}F_*^e(A/\mathfrak{m})$ is noetherian for every $e>0$, which is used in the proof of [Ene00, Theorem 2.19]. Enescu states that this latter result holds without this assumption as long as ideals generated by systems of parameters in the rings $B/\mathfrak{m}B\otimes_{A/\mathfrak{m}}F_*^e(A/\mathfrak{m})$ are Frobenius closed in [Ene00, Remark 2.20]. This condition holds by the proof of [DM20, Proposition 3.11].

[4] $(\mathrm{R}_{\mathrm{I}}^{\mathscr{C}\prime})$ holds for complete intersection (respectively, Cohen–Macaulay) homomorphisms when $\mathbf{R}=$ 'cid $\leqslant n$' (respectively, 'cmd $\leqslant n$'). The cited references relate the complete intersection (respectively, Cohen–Macaulay) defects of the domain and fiber to that of the codomain.

[e] This property holds if $\mathscr{C}$ is the category of excellent rings.

[Q] $(\mathrm{R}_{\mathrm{I}}^{\mathscr{C}\prime})$, $(\mathrm{R}_{\mathrm{IV}}^{\mathscr{C}})$, and $(\mathrm{R}_{\mathrm{V}}^{\mathscr{C}})$ hold when $\mathscr{C}$ is the category of $\mathbf{Q}$-algebras whose local rings are quasi-excellent, and $(\mathrm{R}_{\mathrm{III}})$ holds for complete local $\mathbf{Q}$-algebras. For $(\mathrm{R}_{\mathrm{I}}^{\mathscr{C}\prime})$, Elkik works with rings essentially of finite type over a field of characteristic zero, but the same proof works after replacing [Elk78, Théorème 2] with Proposition 4.17.

[rc] This property holds for pseudo-rationality if $\mathscr{C}$ is the category of noetherian rings that have a residual complex in the sense of [Har66, Definition on p. 304].

[CM] This property holds if $\mathscr{C}$ is the category of homomorphic images of Cohen–Macaulay rings.

[char0] $(\mathrm{R}_{\mathrm{III}})$ holds for complete local $\mathbf{Q}$-algebras since resolutions of singularities exist [Hir64, Chapter I, § 3, Main Theorem I($n$)], and the other properties hold if $\mathscr{C}$ is the category of rings essentially of finite type over possibly different fields of characteristic zero. The results for $(\mathrm{R}_{\mathrm{IV}}^{\mathscr{C}})$ require some extra work; see Proposition 4.15.

Set $Y \coloneqq \operatorname{Spec}(B)$, and consider the commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\nu} & W' & \xrightarrow{f'} & Y \\ & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ & & W & \xrightarrow{f} & X \end{array}$$

where the square is cartesian, where $g \coloneqq \operatorname{Spec}(\varphi)$, and where $\nu$ is the normalization of $W' \coloneqq W \times_X Y$. Note that $f'$ is proper by base change and birational by flat base change [EGAI$_{\text{new}}$, Proposition 3.9.9], and hence $\nu$ is finite by [LT81, (iv)$''$ in Remark (a) on pp. 102–103]. We then have the commutative diagram

$$\begin{array}{ccccc} H^d_{(g'\circ\nu)^{-1}(E)}(\mathcal{O}_Z) & \xleftarrow{\delta^d_{f'\circ\nu}(\mathcal{O}_Z)} & H^d_{\{\mathfrak{n}\}}\big((f'\circ\nu)_*\mathcal{O}_Z\big) & = & H^d_{\mathfrak{n}}(B) \\ \uparrow & & \uparrow & & \| \\ H^d_E\big((g'\circ\nu)_*\mathcal{O}_Z\big) & \xleftarrow{\delta^d_f((g'\circ\nu)_*\mathcal{O}_Z)} & H^d_{\{\mathfrak{m}\}}\big((f\circ g'\circ\nu)_*\mathcal{O}_Z\big) & = & H^d_{\mathfrak{m}}(B) \\ \uparrow & & \uparrow & & \uparrow \\ H^d_E(\mathcal{O}_W) & \xleftarrow{\delta^d_f(\mathcal{O}_W)} & H^d_{\{\mathfrak{m}\}}(f_*\mathcal{O}_W) & = & H^d_{\mathfrak{m}}(A) \end{array}$$

where the top half of the diagram is from [Smi97, Proposition 1.12], and the bottom half is from the naturality of $\delta^d_f(-)$ applied to the pullback map $\mathcal{O}_W \to (g'\circ\nu)_*\mathcal{O}_Z$ [Smi97, Lemma 1.11]. The horizontal arrow on the top left is injective by the assumption that $B$ is pseudo-rational, the vertical arrow on the top right is an equality since $\sqrt{\mathfrak{m}B} = \mathfrak{n}$, and the vertical arrow on the bottom right is injective by the flatness of $\varphi$. The commutativity of the diagram shows that $\delta^d_f(\mathcal{O}_W)$ is injective as required. □

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