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Digit frequencies and self-affine sets with non-empty interior

SIMON BAKER

Mathematics institute, University of Warwick, Coventry, CV4 7AL, UK
(e-mail: simonbaker412@gmail.com)

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Abstract. In this paper we study digit frequencies in the setting of expansions in non-integer bases, and self-affine sets with non-empty interior. Within expansions in non-integer bases we show that if $\beta \in (1, 1.787\ldots)$ then every $x \in (0, 1/(\beta - 1))$ has a simply normal β-expansion. We also prove that if $\beta \in (1, (1 + \sqrt{5})/2)$ then every $x \in (0, 1/(\beta - 1))$ has a β-expansion for which the digit frequency does not exist, and a β-expansion with limiting frequency of zeros p, where p is any real number sufficiently close to $1/2$. For a class of planar self-affine sets we show that if the horizontal contraction lies in a certain parameter space and the vertical contractions are sufficiently close to 1, then every non-trivial vertical fibre contains an interval. Our approach lends itself to explicit calculation and gives rise to new examples of self-affine sets with non-empty interior. One particular strength of our approach is that it allows for different rates of contraction in the vertical direction.

Key words: expansions in non-integer bases, digit frequencies, self-affine sets
2010 Mathematics Subject Classification: 11A63 (Primary); 28A80, 11K55 (Secondary)

1. *Introduction*

Let $x \in [0, 1]$. A sequence $(\epsilon_i) \in \{0, 1\}^{\mathbb{N}}$ is called a *binary expansion* of x if

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i}.$$

It is well known that apart from the dyadic rationals (numbers of the form $p/2^n$) every $x \in [0, 1]$ has a unique binary expansion. The exceptional dyadic rationals have precisely two binary expansions. A seemingly innocuous generalisation of these representations is to replace the base 2 with a parameter $\beta \in (1, 2)$. That is, given $x \in \mathbb{R}$, we call a sequence $(\epsilon_i) \in \{0, 1\}^{\mathbb{N}}$ a *β-expansion* of x if

$$x = \pi_\beta((\epsilon_i)) := \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i}.$$

These representations were first introduced in the papers of Parry [**26**] and Rényi [**27**]. It is straightforward to show that x has a β-expansion if and only if $x \in [0, 1/(\beta - 1)]$. In what follows we will let $I_\beta := [0, 1/(\beta - 1)]$.

Despite being a simple generalisation of binary expansions, β-expansions exhibit far more exotic behaviour. In particular, one feature of β-expansions that makes them an interesting object to study is that an $x \in I_\beta$ may have many β-expansions. In fact a result of Sidorov [**29**] states that for any $\beta \in (1, 2)$, Lebesgue almost every $x \in I_\beta$ has a continuum of β-expansions. Moreover, for any $k \in \mathbb{N} \cup \{\aleph_0\}$, there exist $\beta \in (1, 2)$ and $x \in I_\beta$ such that x has precisely k β-expansions; see [**7**, **8**, **14**, **15**, **30**]. Note that the endpoints of I_β always have a unique β-expansion for any $\beta \in (1, 2)$.

A particularly useful technique for studying both binary expansions and β-expansions is to associate a dynamical system to the base. One can then often reinterpret a problem in terms of a property of the dynamical system. The underlying geometry of the dynamical system can then make a problem much more tractable. In this paper we prove results relating to digit frequencies and self-affine sets. These results are of independent interest but also demonstrate the strength of the dynamical approach to β-expansions.

2. *Statement of results*

2.1. *Digit frequencies.*

Let $(\epsilon_i) \in \{0, 1\}^{\mathbb{N}}$. We define the *frequency of zeros of* (ϵ_i) to be the limit

$$\text{freq}_0(\epsilon_i) := \lim_{n\to\infty} \frac{\#\{1 \leq i \leq n : \epsilon_i = 0\}}{n},$$

assuming the limit exists. We call a sequence (ϵ_i) *simply normal* if $\text{freq}_0(\epsilon_i) = 1/2$. For each $x \in [0, 1]$, we let $\text{freq}_0(x)$ denote the frequency of zeros in its binary expansion whenever the limit exists. When the limit does not exist we say $\text{freq}_0(x)$ does not exist. The following results are well known:

(1) Lebesgue almost every $x \in [0, 1]$ has a simply normal binary expansion.

(2) $\dim_H(\{x : \text{freq}_0(x) \text{ does not exist}\}) = 1$.

(3) For each $p \in [0, 1]$ we have

$$\dim_H(\{x : \text{freq}_0(x) = p\}) = \frac{-p\log p - (1-p)\log(1-p)}{\log 2}.$$

In (3) we have adopted the convention $0 \log 0 = 0$. The first statement is a consequence of Borel's normal number theorem [**9**], the second statement appears to be folklore, and the third statement is a result of Eggleston [**13**].

The above results provide part of the motivation for the present work. In particular, we are interested in whether analogues of these results hold for expansions in non-integer bases. Our first result in the setting of β-expansions is the following.

THEOREM 2.1.

(1) *Let $\beta \in (1, \beta_{KL})$. Then every $x \in (0, 1/(\beta - 1))$ has a simply normal β-expansion.*

(2) *Let $\beta \in (1, (1 + \sqrt{5})/2)$. Then every $x \in (0, 1/(\beta - 1))$ has a β-expansion for which the frequency of zeros does not exist.*

(3) *Let $\beta \in (1, (1 + \sqrt{5})/2)$. Then there exists $c = c(\beta) > 0$ such that for every $x \in (0, 1/(\beta - 1))$ and $p \in (1/2 - c, 1/2 + c)$, there exists a β-expansion of x with frequency of zeros equal to p.*

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The quantity $\beta_{KL} \approx 1.787$ appearing in statement 1 of Theorem 2.1 is the Komornik–Loreti constant introduced in [**25**]. In [**25**] Komornik and Loreti proved that β_{KL} is the smallest base for which 1 has a unique β-expansion. It has since been shown to be important for many other reasons. We elaborate on the significance of this constant and its relationship with the Thue–Morse sequence in §3. Note that we can explicitly calculate a lower bound for the quantity c appearing in statement 3 of Theorem 2.1. We include some explicit calculations in §6.

It follows from the results listed above that the set of x whose binary expansion is not simply normal has Hausdorff dimension 1. Our next result shows that as β approaches 2 we see a similar phenomenon.

THEOREM 2.2. *The following statement is true:*

$$\lim_{\beta \nearrow 2} \dim_H(\{x : x \textit{ has no simply normal } \beta\textit{-expansion}\}) = 1.$$

2.2. *Hybrid expansions.* In this section we consider β-expansions where our digit set is $\{-1, 1\}$ instead of $\{0, 1\}$. This is in keeping with the digit set used by Güntürk in [**19**]. Given $\beta \in (1, 2)$ and $x \in [-1/(\beta - 1), 1/(\beta - 1)]$, we say that a sequence $(\epsilon_i) \in \{-1, 1\}^{\mathbb{N}}$ is a *hybrid expansion of* x if

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i}$$

and

$$x = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \epsilon_i.$$

Hybrid expansions were first introduced by Güntürk in [**19**]. Interestingly, the original motivation for studying hybrid expansions was to overcome the problem of analogue to digital conversion where the underling system has background noise. In [**19**] the following result was asserted without proof.

THEOREM 2.3. *There exists $C_1 > 0$ such that, for all $\beta \in (1, 1 + C_1)$, there exists $c = c(\beta) > 0$ such that every $x \in (-c, c)$ has a hybrid expansion.*

A proof was subsequently provided by Dajani, Jiang, and Kempton in [**10**]. They showed that one can take $C_1 \approx 0.327$. We improve upon this theorem in the following way.

THEOREM 2.4. *Let $\beta \in (1, (1 + \sqrt{5})/2)$. Then there exists $c = c(\beta) > 0$ such that every $x \in (-c, c)$ has a hybrid expansion.*

It would be desirable to obtain a result of the form: there exists $C > 0$ such that for every $\beta \in (1, 1 + C)$ every $x \in (-1/(\beta - 1), 1/(\beta - 1))$ has a hybrid expansion. However, it is an immediate consequence of the definition that if x has a hybrid expansion then $x \in [-1, 1]$. Since $[-1, 1] \subsetneq (-1/(\beta - 1), 1/(\beta - 1))$ for all $\beta \in (1, 2)$ it is clear that such a result is not possible. If we normalised by a function that decayed at a slower rate than n^{-1} we would not necessarily have this obstruction. The following result shows that if we

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replace n^{-1} with another normalising function that satisfies a certain growth condition, then we have our desired result.

THEOREM 2.5. *Let* $\beta \in (1, (1+\sqrt{5})/2)$. *Then there exists* $c = c(\beta) > 0$ *such that if* $f : \mathbb{N} \to (0, \infty)$ *is a strictly increasing function which satisfies*

$$\limsup_{n\to\infty} f(n+1) - f(n) < c$$

and

$$\lim_{n\to\infty} f(n) = \infty,$$

then for every $x \in (-1/(\beta-1), 1/(\beta-1))$ *there exists* $(\epsilon_i) \in \{-1, 1\}^{\mathbb{N}}$ *such that*

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i}$$

and

$$x = \lim_{n\to\infty} \frac{1}{f(n)} \sum_{i=1}^{n} \epsilon_i.$$

The following corollary is an immediate consequence of Theorem 2.5.

COROLLARY 2.6. *Let* $\beta \in (1, (1+\sqrt{5})/2)$. *Then for every* $x \in (-1/(\beta-1), 1/(\beta-1))$ *there exists* $(\epsilon_i) \in \{-1, 1\}^{\mathbb{N}}$ *such that*

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i}$$

and

$$x = \lim_{n\to\infty} \frac{1}{n^{1/2}} \sum_{i=1}^{n} \epsilon_i.$$

2.3. *A family of overlapping self-affine sets and simultaneous expansions.* Let $\{S_j\}_{j=1}^m$ be a collection of contracting maps acting on $\mathbb{R}^d$. A result of Hutchinson [**21**] states that there exists a unique non-empty compact set $\Lambda \subseteq \mathbb{R}^d$ such that

$$\Lambda = \bigcup_{j=1}^{m} S_j(\Lambda).$$

We call Λ the *attractor associated to* $\{S_j\}$. Often one is interested in determining the topological properties of Λ. When the collection $\{S_j\}$ consists solely of similarities the attractor Λ is reasonably well understood. However, when the collection $\{S_j\}$ contains affine maps the situation is known to be much more complicated.

In this paper we focus on the following family of self-affine sets. Let $1 < \beta_1, \beta_2, \beta_3 \leq 2$ and

$$S_{-1}(x, y) = \left(\frac{x-1}{\beta_1}, \frac{y-1}{\beta_2}\right) \quad \text{and} \quad S_1(x, y) = \left(\frac{x+1}{\beta_1}, \frac{y+1}{\beta_3}\right).$$

For this collection of contractions we denote the associated attractor by $\Lambda_{\beta_1,\beta_2,\beta_3}$. In Figure 1 we include some examples.

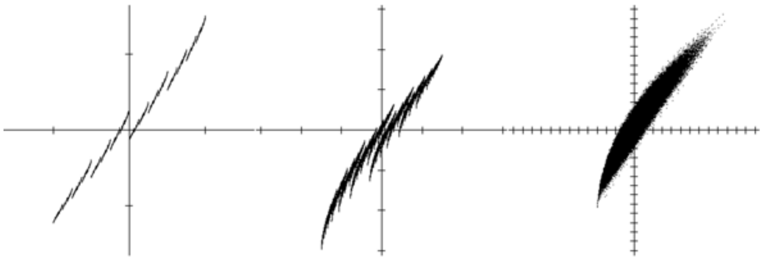

FIGURE 1. A plot of $\Lambda_{2,1.81,1.66}$, $\Lambda_{1.66,1.33,1.53}$, $\Lambda_{1.2,1.11,1.05}$.

When $\beta_2 = \beta_3$ we denote $\Lambda_{\beta_1,\beta_2,\beta_3}$ by $\Lambda_{\beta_1,\beta_2}$. The case where $\beta_2 = \beta_3$ was studied in [**10**, **20**, **28**]. One problem the authors of these papers were particularly interested in was determining those pairs (β_1, β_2) for which the attractor $\Lambda_{\beta_1,\beta_2}$ has non-empty interior. The best result in this direction is the following result due to Hare and Sidorov [**20**].

THEOREM 2.7. *If $\beta_1 \neq \beta_2$ and*

$$\left|\frac{\beta_2^8 - \beta_1^8}{\beta_2^7 - \beta_1^7}\right| + \left|\frac{\beta_2^7\beta_1^7(\beta_2 - \beta_1)}{\beta_2^7 - \beta_1^7}\right| \le 2, \tag{2.1}$$

then $\Lambda_{\beta_1,\beta_2}$ has non-empty interior and $(0, 0) \in \Lambda^{\mathrm{o}}$.

Let π denote the projection from $\mathbb{R}^2$ onto the x-axis. For each $x \in \pi(\Lambda_{\beta_1,\beta_2,\beta_3})$ let

$$\Lambda^x_{\beta_1,\beta_2,\beta_3} := \{y \in \mathbb{R} : (x, y) \in \Lambda_{\beta_1,\beta_2,\beta_3}\}.$$

We call $\Lambda^x_{\beta_1,\beta_2,\beta_3}$ the *fibre of* x. Note that $\pi(\Lambda_{\beta_1,\beta_2,\beta_3}) = [-1/(\beta_1 - 1), 1/(\beta_1 - 1)]$. The following statement is our main result for $\Lambda_{\beta_1,\beta_2,\beta_3}$.

THEOREM 2.8. *Let $\beta_1 \in (1, (1 + \sqrt{5})/2)$. Then there exists $c = c(\beta_1) > 0$ such that for all $\beta_2, \beta_3 \in (1, 1 + c)$ and $x \in (-1/(\beta_1 - 1), 1/(\beta_1 - 1))$ the fibre $\Lambda^x_{\beta_1,\beta_2,\beta_3}$ contains an interval. Moreover, $\Lambda_{\beta_1,\beta_2,\beta_3}$ has non-empty interior and $(0, 0) \in \Lambda^o_{\beta_1,\beta_2,\beta_3}$.*

We emphasise that Theorem 2.8 covers the case where $\beta_2 \neq \beta_3$. Our approach lends itself to explicit calculation, and, following our method, one can obtain a lower bound for the value c appearing in Theorem 2.8. We include some explicit calculations in §6.

Note that for any β_1 sufficiently close to $(1 + \sqrt{5})/2$ the set of $\beta_2 \in (1, 2)$ satisfying (2.1) is empty. Consequently, Theorem 2.8 provides new examples of β_1, β_2 for which $\Lambda^o_{\beta_1,\beta_2}$ is non-empty. Theorem 2.8 is also optimal in the following sense. For any $\beta_1 \in [(1 + \sqrt{5})/2, 2)$ and $\beta_2, \beta_3 \in (1, 2)$, there exists $x \in (-1/(\beta_1 - 1), 1/(\beta_1 - 1))$ such that the fibre $\Lambda^x_{\beta_1,\beta_2,\beta_3}$ is countable and therefore does not contain an interval. We explain why this is the case in §7.

It is natural to ask whether the property $\Lambda^x_{\beta_1,\beta_2,\beta_3}$ contains an interval for every $x \in (-1/(\beta_1 - 1), 1/(\beta_1 - 1))$ is stronger than the property $\Lambda^o_{\beta_1,\beta_2,\beta_3} \neq \emptyset$. This is in fact the case and is a consequence of the following proposition.

PROPOSITION 2.9. *$\Lambda_{\beta_1,\beta_2,\beta_3}$ has non-empty interior if and only if $\{x : \Lambda^x_{\beta_1,\beta_2,\beta_3}$ contains an interval$\}$ contains an open dense subset of* $[-1/(\beta_1 - 1), 1/(\beta_1 - 1)]$.

Proof. Let us start by introducing some notation. Let $F := \{x : \Lambda^x_{\beta_1,\beta_2,\beta_3}$ contains an interval$\}$. Suppose $\Lambda^o_{\beta_1,\beta_2,\beta_3} \neq \emptyset$. Then there exist two non-trivial open intervals I and J such that $I \times J \subseteq \Lambda_{\beta_1,\beta_2,\beta_3}$. Let $\phi_{-1}(x) = (x-1)/\beta_1$ and $\phi_1(x) = (x+1)/\beta_1$. Since $S_{-1}(I \times J)$ is an open rectangle contained in $\Lambda_{\beta_1,\beta_2,\beta_3}$, it follows that $\phi_{-1}(I) \subseteq F$. Similarly, $\phi_1(I) \subseteq F$. Repeating this argument, it follows that all images of I under finite concatenations of ϕ_{-1} and ϕ_1 are contained in F. The union of these images of I is an open dense subset of $[-1/(\beta_1-1), 1/(\beta_1-1)]$. It follows that F contains an open dense subset of $[-1/(\beta_1-1), 1/(\beta_1-1)]$.

It remains to prove the leftward implication. We start by partitioning the set F. Given $(a, b, c, d) \in \mathbb{Z}^4$, let

$$F_{a,b,c,d} := \left\{x : \left[\frac{a}{b}, \frac{c}{d}\right] \subseteq \Lambda^x_{b_1,b_2,b_3}\right\}.$$

Importantly, we have

$$F = \bigcup_{(a,b,c,d)\in\mathbb{Z}^4,\ a/b<c/d} F_{a,b,c,d}.$$

Suppose $F_{a,b,c,d}$ is nowhere dense for all $(a, b, c, d) \in \mathbb{Z}^4$. Since F contains an open dense set its complement is a nowhere dense set. It follows that $[-1/(\beta_1-1), 1/(\beta_1-1)]$ is the countable union of nowhere dense sets. By the Baire category theorem this is not possible. Therefore there must exist $(a', b', c', d') \in \mathbb{Z}^4$ such that $a'/b' < c'/d'$ and $F_{a',b',c',d'}$ is dense in some non-trivial interval I'. Since $\Lambda_{\beta_1,\beta_2,\beta_3}$ is closed it follows that

$$I' \times [a'/b', c'/d'] \subseteq \Lambda_{\beta_1,\beta_2,\beta_3}$$

and $\Lambda_{\beta_1,\beta_2,\beta_3}$ has non-empty interior. □

Interestingly, computer simulations suggest that there exist examples where $\Lambda_{\beta_1,\beta_2,\beta_3}$ has non-empty interior yet $\{x : \Lambda^x_{\beta_1,\beta_2,\beta_3}$ is a singleton$\}$ is infinite and even has positive Hausdorff dimension. See Figure 2 for such an example.

In [**19**], in addition to the notion of a hybrid expansion, Güntürk introduced the notion of a simultaneous expansion. These are defined as follows. Given $x \in [-1/(\beta_1-1), 1/(\beta_1-1)]$ and $\beta_1, \beta_2 \in (1, 2)$, we say that a sequence $(\epsilon_i) \in \{-1, 1\}^{\mathbb{N}}$ is a simultaneous (β_1, β_2)-expansion for x if

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta_1^i} = \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta_2^i}.$$

These expansions relate to our self-affine sets via the following observation. If $\beta_2 = \beta_3$ then

$$\Lambda_{\beta_1,\beta_2} = \left\{\left(\sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta_1^i}, \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta_2^i}\right) : (\epsilon_i) \in \{-1, 1\}^{\mathbb{N}}\right\}.$$

Therefore

$$\{(x, x) : x \text{ has a simultaneous } (\beta_1, \beta_2)\text{-expansion}\} = \Lambda_{\beta_1,\beta_2} \cap \{(x, x) : x \in \mathbb{R}\}.$$

In [**19**] it was asserted by Güntürk that there exists $C > 0$ such that, for $1 < \beta_1 < \beta_2 < 1 + C$, there exists $c = c(\beta_1, \beta_2) > 0$ such that every $x \in (-c, c)$ has a simultaneous (β_1, β_2)-expansion. Note that the existence of $C > 0$ satisfying the above follows if one can show

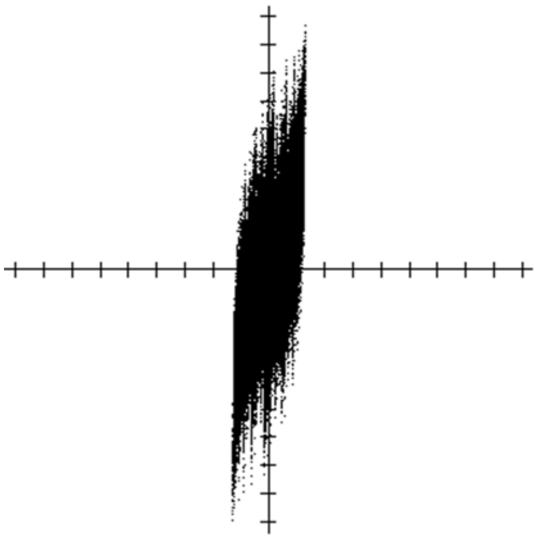

FIGURE 2. A plot of $\Lambda_{1.8,1.05}$. For this choice of β_1 and β_2 it can be shown that $\{x : \Lambda^x_{\beta_1,\beta_2}$ is a singleton$\}$ has positive Hausdorff dimension.

that for $1 < \beta_1 < \beta_2 < 1 + C$ the attractor $\Lambda_{\beta_1,\beta_2}$ contains $(0, 0)$ in its interior. Using this observation, Güntürk's assertion was proved to be correct in [**10**]. The largest parameter space for which it is known that $(0, 0) \in \Lambda^o_{\beta_1,\beta_2}$, and consequently that any x sufficiently close to zero has a simultaneous (β_1, β_2)-expansion, is that stated in Theorem 2.7. Our contribution in this direction is the following theorem which follows as an immediate consequence of Theorem 2.8 by taking $\beta_2 = \beta_3$.

THEOREM 2.10. *Let $\beta_1 \in (1, (1+\sqrt{5})/2)$. Then there exists $C = C(\beta_1) > 0$ such that, if $\beta_2 \in (1, 1 + C)$, then every x sufficiently small has a simultaneous (β_1, β_2)-expansion.*

Before moving onto our proofs we say a few words about the methods used in this paper and compare them with those used in [**10**, **20**]. In these papers the authors show that $(0, 0) \in \Lambda^o_{\beta_1,\beta_2}$ by constructing a polynomial $P(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$ which satisfies:

(1) $P(\beta_1) = P(\beta_2) = 0$;
(2) $\sum_{j=0}^{n-1} |b_j| \leq 2$;
(3) $b_1 = 0$;
(4) $b_0 \neq 0$.

Once the existence of this polynomial is established, one can devise an algorithm which can be applied to any x_1, x_2 sufficiently small. This algorithm then yields an $(\epsilon_i) \in \{-1, 1\}^{\mathbb{N}}$ such that $(x_1, x_2) = (\sum_{i=1}^{\infty} \epsilon_i \beta_1^{-i}, \sum_{i=1}^{\infty} \epsilon_i \beta_2^{-i})$.

This approach is somewhat unsatisfactory. The existence of the polynomial and the algorithm used to construct the (ϵ_i) provide little intuition as to why $(0, 0)$ should be in the interior of $\Lambda_{\beta_1,\beta_2}$. Our approach, as well as allowing for different rates of contraction in the vertical direction, is more intuitive and explicitly constructs the interval appearing in each fibre of $\Lambda_{\beta_1,\beta_2,\beta_3}$.

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The rest of this paper is arranged as follows. In §3 we recall and prove some technical results that are required to prove our theorems. In §4 we prove our theorems relating to digit frequencies. In §5 we prove Theorem 2.8. In §6 we include an example where we explicitly calculate some of the parameters appearing in our theorems. In §7 we include some general discussion and pose some questions.

3. *Preliminaries*

In this section we prove some useful technical results and recall some background material. Let us start by introducing the maps $T_{-1}(x) = \beta x + 1$, $T_0(x) = \beta x$ and $T_1(x) = \beta x - 1$. Given an $x \in I_\beta$, we let

$$\Sigma_\beta(x) := \left\{ (\epsilon_i) \in \{0, 1\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} = x \right\}$$

and

$$\Omega_\beta(x) := \{(a_i) \in \{T_0, T_1\}^{\mathbb{N}} : (a_n \circ \cdots \circ a_1)(x) \in I_\beta \text{ for all } n \in \mathbb{N}\}.$$

Similarly, given $x \in \widetilde{I}_\beta := [-1/(\beta - 1), 1/(\beta - 1)]$, let

$$\widetilde{\Sigma}_\beta(x) := \left\{ (\epsilon_i) \in \{-1, 1\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} = x \right\}$$

and

$$\widetilde{\Omega}_\beta(x) := \{(a_i) \in \{T_{-1}, T_1\}^{\mathbb{N}} : (a_n \circ \cdots \circ a_1)(x) \in \widetilde{I}_\beta \text{ for all } n \in \mathbb{N}\}.$$

The dynamical interpretation of β-expansions is best seen through the following result.

LEMMA 3.1. *For any $x \in I_\beta$ ($x \in \widetilde{I}_\beta$) we have* Card $\Sigma_\beta(x) =$ Card $\Omega_\beta(x)$ (Card $\widetilde{\Sigma}_\beta(x) =$ Card $\widetilde{\Omega}_\beta(x)$). *Moreover, the map which sends (ϵ_i) to (T_{ϵ_i}) is a bijection between $\Sigma_\beta(x)$ and $\Omega_\beta(x)$ ($\widetilde{\Sigma}_\beta(x)$ and $\widetilde{\Omega}_\beta(x)$).*

Lemma 3.1 was originally proved in [**6**] for an arbitrary digit set of the form $\{0, \ldots, m\}$. The proof easily extends to the digit set $\{-1, 1\}$.

Lemma 3.1 allows us to reinterpret problems from β-expansions in terms of the allowable trajectories that can occur within a dynamical system. This interpretation has its origins in the work of Dajani and Kraaikamp [**11**]. In Figure 3 we include a graph of T_0 and T_1 acting on I_β. One can see from this picture, or check by hand, that if $x \in [1/\beta, 1/(\beta(\beta - 1))]$ then both T_0 and T_1 map x into I_β. Therefore, by Lemma 3.1, this x has at least two β-expansions. More generally, if there exists a sequence of T_0s and T_1s that map x into $[1/\beta, 1/(\beta(\beta - 1))]$, then x has at least two β-expansions.

The interval $[1/\beta, 1/(\beta(\beta - 1))]$ is clearly important when it comes to studying $\Sigma_\beta(x)$ and $\Omega_\beta(x)$. In what follows we let

$$\mathcal{S}_\beta := \left[\frac{1}{\beta}, \frac{1}{\beta(\beta - 1)} \right].$$

Another particularly useful interval for studying β-expansions is

$$\mathcal{O}_\beta := \left[\frac{1}{\beta^2 - 1}, \frac{\beta}{\beta^2 - 1} \right].$$

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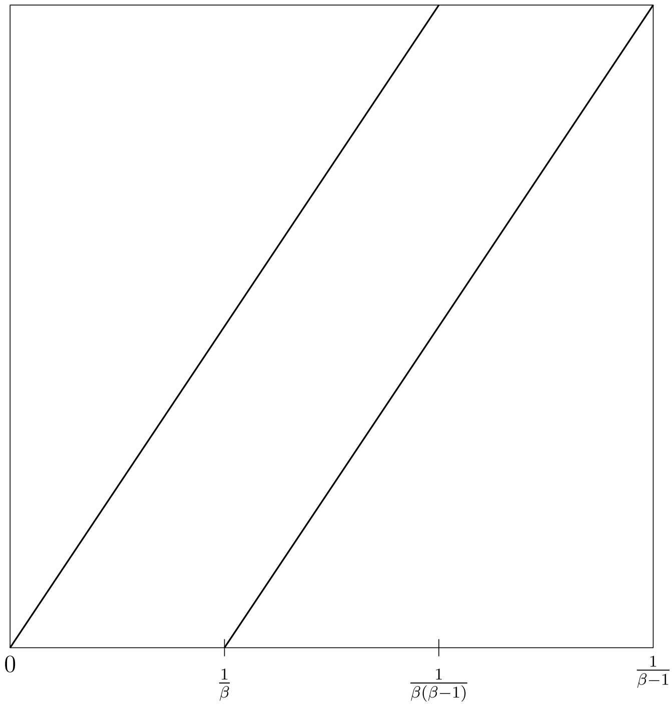

FIGURE 3. The overlapping graphs of T_0 and T_1.

The analogues of $\mathcal{S}_\beta$ and $\mathcal{O}_\beta$ for the digit set $\{-1, 1\}$ are

$$\widetilde{\mathcal{S}}_\beta := \left[\frac{\beta-2}{\beta(\beta-1)}, \frac{2-\beta}{\beta(\beta-1)}\right] \quad \text{and} \quad \widetilde{\mathcal{O}}_\beta := \left[\frac{1-\beta}{\beta^2-1}, \frac{\beta-1}{\beta^2-1}\right].$$

The intervals $\mathcal{O}_\beta$ and $\widetilde{\mathcal{O}}_\beta$ are important because of the following lemma.

LEMMA 3.2. *For any $\beta \in (1, 2)$ we have*

$$T_0\left(\frac{1}{\beta^2-1}\right) = \frac{\beta}{\beta^2-1} \quad \textit{and} \quad T_1\left(\frac{\beta}{\beta^2-1}\right) = \frac{1}{\beta^2-1}, \tag{3.1}$$

and

$$T_{-1}\left(\frac{1-\beta}{\beta^2-1}\right) = \frac{\beta-1}{\beta^2-1} \quad \textit{and} \quad T_1\left(\frac{\beta-1}{\beta^2-1}\right) = \frac{1-\beta}{\beta^2-1}. \tag{3.2}$$

Moreover, for any $x \in (0, 1/(\beta-1))$ ($x \in (-1/(\beta-1), 1/(\beta-1))$), there exists a sequence of T_0s or T_1s (T_{-1}s or T_1s) that map x into $\mathcal{O}_\beta$ ($\widetilde{\mathcal{O}}_\beta$).

Proof. Verifying (3.1) and (3.2) is a simple calculation. These equations tell us that it is not possible for an x to be mapped over $\mathcal{O}_\beta$ or $\widetilde{\mathcal{O}}_\beta$ via an application of one of our maps. Note that the endpoints of the interval I_β are the fixed points of the maps T_0 and T_1. Similarly, the endpoints of the interval $\widetilde{I}_\beta$ are the fixed points of the maps T_{-1} and T_1. Combining these observations with the expansivity of our maps implies the second half of our lemma. □

Several of our theorems will rely on the following proposition. Loosely speaking, it states that for $\beta \in (1, (1+\sqrt{5})/2)$, for any $x \in \mathcal{O}_\beta$ $(x \in \widetilde{\mathcal{O}}_\beta)$ there exists a method for generating expansions of x such that we have a lot of control over the digits that appear. Before we state this result it is useful to introduce some notation.

In what follows we let $\{T_0, T_1\}^* = \bigcup_{n=0}^{\infty}\{T_0, T_1\}^n$. Given $\omega = (\omega_1, \ldots, \omega_n) \in \{T_0, T_1\}^*$, let $\omega(x) = (\omega_n \circ \cdots \circ \omega_1)(x)$. We let $|\omega|$ denote the length of ω. We also let

$$|\omega|_0 = \#\{1 \leq i \leq |\omega| : \omega_i = T_0\}$$

and

$$|\omega|_1 = \#\{1 \leq i \leq |\omega| : \omega_i = T_1\}.$$

For a finite word $\omega \in \{T_0, T_1\}^*$ we denote by ω^k its k-fold concatenation with itself and by ω^∞ the infinite sequence obtained by concatenating ω indefinitely. The above notions translate over in the obvious way to sequences of maps whose components are from the set $\{T_{-1}, T_1\}$. We also define $|\cdot|_{-1}$ in the obvious way.

PROPOSITION 3.3. *Let $\beta \in (1, (1+\sqrt{5})/2)$. There exists $n(\beta) \in \mathbb{N}$ such that if $x \in \mathcal{O}_\beta$ $(x \in \widetilde{\mathcal{O}}_\beta)$ then there exists $\omega^0, \omega^1 \in \{T_0, T_1\}^*$ $(\omega^{-1}, \omega^1 \in \{T_{-1}, T_1\}^*)$ satisfying the following:*
- *$|\omega^0| \leq n(\beta)$ and $|\omega^1| \leq n(\beta)$ $(|\omega^{-1}| \leq n(\beta)$ and $|\omega^1| \leq n(\beta))$;*
- *$\omega^0(x) \in \mathcal{O}_\beta$ and $\omega^1(x) \in \mathcal{O}_\beta$ $(\omega^{-1}(x) \in \widetilde{\mathcal{O}}_\beta$ and $\omega^1(x) \in \widetilde{\mathcal{O}}_\beta)$;*
- *$|\omega^0|_0 > |\omega^0|_1$ $(|\omega^{-1}|_{-1} > |\omega^{-1}|_1)$;*
- *$|\omega^1|_1 > |\omega^1|_0$ $(|\omega^1|_1 > |\omega^1|_{-1})$.*

Let us take the opportunity to emphasise that ω^{-1} is not an inverse map.

We will only give a proof of Proposition 3.3 for the digit set $\{0, 1\}$. The case where the digit set is $\{-1, 1\}$ is dealt with similarly. Before giving a proof of Proposition 3.3 for the digit set $\{0, 1\}$ it is useful to define two more intervals and state some basic facts. For any $\beta \in (1, 2)$ let

$$\mathcal{I}_\beta := \left[\frac{1}{2}\left(\frac{1}{\beta} + \frac{1}{\beta^2 - 1}\right), \frac{1}{2}\left(\frac{\beta}{\beta^2 - 1} + \frac{1}{\beta(\beta - 1)}\right)\right]$$

and

$$\begin{aligned}\mathcal{J}_\beta := &\left[T_1\left(\frac{1}{2}\left(\frac{1}{\beta} + \frac{1}{\beta^2 - 1}\right)\right), T_0\left(\frac{1}{2}\left(\frac{\beta}{\beta^2 - 1} + \frac{1}{\beta(\beta - 1)}\right)\right)\right] \\ = &\left[\frac{\beta}{2}\left(\frac{1}{\beta} + \frac{1}{\beta^2 - 1}\right) - 1, \frac{\beta}{2}\left(\frac{\beta}{\beta^2 - 1} + \frac{1}{\beta(\beta - 1)}\right)\right].\end{aligned}$$

For any $\beta \in (1, 2)$ these intervals are well defined and non-trivial. Note that the left endpoint of the interval of $\mathcal{I}_\beta$ is the midpoint of the left endpoints of $\mathcal{S}_\beta$ and $\mathcal{O}_\beta$, and the right endpoint of $\mathcal{I}_\beta$ is the midpoint of the right endpoints of $\mathcal{S}_\beta$ and $\mathcal{O}_\beta$.

LEMMA 3.4. *For any $\beta \in (1, (1+\sqrt{5})/2)$ we have $\mathcal{O}_\beta \subsetneq \mathcal{I}_\beta \subsetneq \mathcal{S}_\beta$ and $\mathcal{J}_\beta \subseteq (0, 1/(\beta - 1))$.*

The proof of Lemma 3.4 is straightforward and therefore omitted.

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LEMMA 3.5. *Let $\beta \in (1, (1+\sqrt{5})/2)$. There exists $n_1(\beta) \in \mathbb{N}$ such that:*

- *if*
$$x \in \left[\frac{\beta}{2}\left(\frac{1}{\beta} + \frac{1}{\beta^2-1}\right) - 1, \frac{1}{\beta^2-1}\right]$$
then $T_0^i(x) \in \mathcal{O}_\beta$ for some $1 \le i \le n_1(\beta)$;
- *if*
$$x \in \left[\frac{\beta}{\beta^2-1}, \frac{\beta}{2}\left(\frac{\beta}{\beta^2-1} + \frac{1}{\beta(\beta-1)}\right)\right]$$
then $T_1^i(x) \in \mathcal{O}_\beta$ for some $1 \le i \le n_1(\beta)$.

Proof. We begin our proof by pointing out that the left endpoint of I_β is the fixed point of T_0 and the right endpoint of I_β is the fixed point of T_1. Moreover, the maps T_0 and T_1 expand distances from their respective fixed points in the following way:

$$T_0(x) - 0 = \beta(x-0) \quad \text{and} \quad T_1(x) - \frac{1}{\beta-1} = \beta\left(x - \frac{1}{\beta-1}\right). \tag{3.3}$$

Let us fix
$$x \in \left[\frac{\beta}{2}\left(\frac{1}{\beta} + \frac{1}{\beta^2-1}\right) - 1, \frac{1}{\beta^2-1}\right].$$
The second case is dealt with similarly. Lemma 3.4 guarantees
$$\frac{\beta}{2}\left(\frac{1}{\beta} + \frac{1}{\beta^2-1}\right) - 1 > 0.$$
Let $n_1(\beta) \in \mathbb{N}$ be the unique natural number which satisfies
$$\beta^{n_1(\beta)-1}\left(\frac{\beta}{2}\left(\frac{1}{\beta} + \frac{1}{\beta^2-1}\right) - 1\right) \le \frac{1}{\beta^2-1} < \beta^{n_1(\beta)}\left(\frac{\beta}{2}\left(\frac{1}{\beta} + \frac{1}{\beta^2-1}\right) - 1\right).$$
Then by (3.3), the monotonicity of T_0 and the first part of Lemma 3.2, there must exist $1 \le i \le n_1(\beta)$ such that $T_0^i(x) \in \mathcal{O}_\beta$. □

Equipped with Lemma 3.5, we are now in a position to prove Proposition 3.3.

Proof of Proposition 3.3. Let $\beta \in (1, (1+\sqrt{5})/2)$ and $x \in \mathcal{O}_\beta$. We only show how to construct ω^0. The construction of ω^1 follows from an analogous argument. Alternatively, one could consider $x' = 1/(\beta-1) - x$. It can be shown that if we took the corresponding ω^0 for x' and replaced every occurrence of T_0 with T_1 and T_1 with T_0, then the resulting sequence would have the desired properties of an ω^1 for our original x.

Let us start by considering the image of x under $T_1(x)$. By Lemma 3.2 we know that $T_1(x) \in [\beta/(\beta^2-1) - 1, 1/(\beta^2-1)]$. There are two cases to consider, either $T_1(x) \notin \mathcal{I}_\beta$ or $T_1(x) \in \mathcal{I}_\beta$.

Case 1. Suppose $T_1(x) \notin \mathcal{I}_\beta$. Then
$$T_1(x) < \frac{1}{2}\left(\frac{1}{\beta} + \frac{1}{\beta^2-1}\right)$$
and consequently
$$x < \frac{\beta}{\beta^2-1} - \delta(\beta),$$

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where

$$\delta(\beta) := \frac{\beta}{\beta^2-1} - T_1^{-1}\bigg(\frac{1}{2}\bigg(\frac{1}{\beta} + \frac{1}{\beta^2-1}\bigg)\bigg) = \frac{\beta}{\beta^2-1} - \frac{1}{\beta} - \frac{1}{2\beta}\bigg(\frac{1}{\beta} + \frac{1}{\beta^2-1}\bigg) > 0.$$

Importantly, $\delta = \delta(\beta)$ only depends upon β.

We repeatedly apply T_0 to $T_1(x)$ until $(T_0^{i_1} \circ T_1)(x) \in \mathcal{O}_\beta$. This is permissible by Lemma 3.2. It is a consequence of Lemma 3.5 that $i_1 \leq n_1(\beta)$ for some $n_1(\beta)$ that only depends upon β. If $i_1 > 1$ then we stop and take $\omega^0 = (T_1, (T_0)^{i_1})$.

If $i_1 = 1$ then

$$(T_0 \circ T_1)(x) \in \mathcal{O}_\beta$$

and

$$(T_0 \circ T_1)(x) < \frac{\beta}{\beta^2-1} - \beta^2\delta(\beta). \tag{3.4}$$

Equation (3.4) follows because $x < \beta/(\beta^2-1) - \delta(\beta)$ and $T_0 \circ T_1$ is orientation preserving.

Let $n_2(\beta)$ be the unique natural number satisfying

$$\beta^{2(n_2(\beta)-1)}\delta(\beta) \leq \frac{\beta}{\beta^2-1} - \frac{1}{\beta^2-1} < \beta^{2n_2(\beta)}\delta(\beta),$$

and let $k \geq 1$ be the smallest integer such that

$$(\overbrace{(T_0 \circ T_1) \circ \cdots \circ (T_0 \circ T_1)}^{k \text{ times}})(x) \notin \mathcal{O}_\beta.$$

Observe that

$$(\overbrace{(T_0 \circ T_1) \circ \cdots \circ (T_0 \circ T_1)}^{k \text{ times}})(x) < \frac{\beta}{\beta^2-1} - \beta^{2k}\delta(\beta)$$

and thus $k \leq n_2(\beta)$. By Lemma 3.5 there exists $1 \leq i_2 \leq n_1(\beta)$ such that

$$T_0^{i_2} \circ (\overbrace{(T_0 \circ T_1) \circ \cdots \circ (T_0 \circ T_1)}^{k \text{ times}})(x) \in \mathcal{O}_\beta,$$

and one can take $\omega^0 = (\overbrace{T_1, T_0, \ldots, T_1, T_0}^{k \text{ times}}, T_0^{i_2})$.

Case 2. If $T_1(x) \in \mathcal{I}_\beta$ then

$$T_1(x) \in \bigg[\frac{1}{2}\bigg(\frac{1}{\beta} + \frac{1}{\beta^2-1}\bigg), \frac{1}{\beta^2-1}\bigg].$$

We consider $T_1^2(x)$ and repeatedly apply T_0 until $(T_0^{i_1} \circ T_1^2)(x) \in \mathcal{O}_\beta$. We cannot have $i_1 = 1$, since by the monotonicity of our maps we would then have

$$(T_0 \circ T_1)\bigg(\frac{1}{\beta^2-1}\bigg) \geq \frac{1}{\beta^2-1},$$

which is not possible since $T_0 \circ T_1$ expands the distance from the fixed point $\beta/(\beta^2-1)$ by a factor β^2. By Lemma 3.5 we must have $i_1 \leq n_1(\beta)$. If $i_1 > 2$ then we may stop and take $\omega^0 = (T_1, T_1, (T_0)^{i_1})$.

If $i_1 = 2$ then

$$(T_0 \circ T_0 \circ T_1 \circ T_1)(x) \in \left[\frac{1}{\beta^2 - 1}, (T_0 \circ T_0 \circ T_1 \circ T_1)\left(\frac{\beta}{\beta^2 - 1}\right)\right].$$

But for any $\beta \in (1, 2)$ it can be shown that

$$(T_0 \circ T_0 \circ T_1 \circ T_1)\left(\frac{\beta}{\beta^2 - 1}\right) = \frac{\beta^3 + \beta^2 - \beta^4}{\beta^2 - 1} < \frac{\beta}{\beta^2 - 1}.$$

Therefore if $i_1 = 2$ then

$$(T_0 \circ T_0 \circ T_1 \circ T_1)(x) \leq \frac{\beta}{\beta^2 - 1} - \delta'(\beta),$$

where

$$\delta'(\beta) := \frac{\beta}{\beta^2 - 1} - \frac{\beta^3 + \beta^2 - \beta^4}{\beta^2 - 1} > 0.$$

We are now in a position where we can replicate the arguments used in Case 1 with x replaced by $(T_0 \circ T_0 \circ T_1 \circ T_1)(x)$. Repeating these arguments implies the existence of the required ω^0 and $n(\beta)$. □

Proposition 3.3 allows us to effectively handle the parameter space $(1, (1 + \sqrt{5})/2)$. To prove results within the interval $[(1 + \sqrt{5})/2, 2)$, we need to recall some background results from unique expansions. Given $\beta \in (1, 2)$, let

$$\mathcal{U}_\beta := \left\{x \in \left[0, \frac{1}{\beta - 1}\right] : x \text{ has a unique } \beta\text{-expansion}\right\}$$

and

$$\widetilde{\mathcal{U}}_\beta := \left\{(\epsilon_i) \in \{0, 1\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} \in \mathcal{U}_\beta\right\}.$$

We call $\mathcal{U}_\beta$ the *univoque set* and $\widetilde{\mathcal{U}}_\beta$ the *univoque sequences*. By definition there is a bijection between these two sets. The study of these sets is classical. For more on these sets we refer the reader to [**1**, **12**, **24**] and the references therein.

A useful tool for studying univoque sequences is the lexicographic ordering. This is defined as follows. Given $(\epsilon_i), (\delta_i) \in \{0, 1\}^{\mathbb{N}}$, we say that $(\epsilon_i) \prec (\delta_i)$ if $\epsilon_1 < \delta_1$, or if there exists $n \in \mathbb{N}$ such that $\epsilon_{n+1} < \delta_{n+1}$ and $\epsilon_i = \delta_i$ for $1 \leq i \leq n$. One can also define $\preceq, \succ, \succeq$ in the natural way. We also let $\overline{\epsilon_i} = 1 - \epsilon_i$. When studying univoque sequences an important role is played by the *quasi-greedy expansion* of 1. This sequence is defined to be the lexicographically largest infinite β-expansion of 1. We call a sequence infinite if it does not end in an infinite tail of zeros. Given $\beta \in (1, 2)$, we denote the quasi-greedy expansion of 1 by $\alpha(\beta) = (\alpha_i(\beta))$. The following characterisation of quasi-greedy expansions is due to Baiocchi and Komornik [**5**, Theorem 2.2].

LEMMA 3.6. *The map which sends β to $\alpha(\beta)$ is a strictly increasing bijection from* $(1, 2]$ *onto the set of sequences* $(\alpha_i) \in \{0, 1\}^{\mathbb{N}}$ *which satisfy*

$$(\alpha_{n+i}) \preceq (\alpha_i) \quad \textit{whenever } \alpha_n = 0. \tag{3.5}$$

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We remark that if $x \in \mathcal{U}_\beta$ and $x \notin \{0, 1/(\beta-1)\}$ then x is eventually mapped into $((2-\beta)/(\beta-1), 1)$. This is essentially a consequence of the fact that under these assumptions the orbit of x is non-trivial and must avoid $\mathcal{S}_\beta$. Moreover, it is a consequence of being in $\mathcal{U}_\beta$ that once x is mapped into $((2-\beta)/(\beta-1), 1)$, it cannot be mapped outside of $((2-\beta)/(\beta-1), 1)$. Consequently, the following sets can be thought of as attractors for $\mathcal{U}_\beta$ and $\widetilde{\mathcal{U}}_\beta$. Let

$$\mathcal{A}_\beta := \left\{x \in \left(\frac{2-\beta}{\beta-1}, 1\right) : x \text{ has a unique } \beta\text{-expansion}\right\}$$

and

$$\widetilde{\mathcal{A}}_\beta := \left\{(\epsilon_i) \in \{0,1\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} \in \mathcal{A}_\beta\right\}.$$

The above observation and the following lemma are due to Glendinning and Sidorov [**18**, Lemma 4].

LEMMA 3.7.

$$\widetilde{\mathcal{A}}_\beta = \{(\epsilon_i) \in \{0,1\}^{\mathbb{N}} : (\overline{\alpha_i(\beta)}) \prec (\epsilon_{n+i}) \prec (\alpha_i(\beta)) \textit{ for all } n \in \mathbb{N}\}.$$

Lemma 3.7 demonstrates the importance of the sequences $\alpha(\beta)$ when studying univoque sequences. The following lemma is a consequence of Lemmas 3.6 and 3.7.

LEMMA 3.8. *$\widetilde{\mathcal{A}}_\beta \subseteq \widetilde{\mathcal{A}}_{\beta'}$ for $\beta < \beta'$.*

To extend our frequency results to the parameter space $((1+\sqrt{5})/2, \beta_{KL})$ it is instructive to recall some properties of the Thue–Morse sequence. There are various ways to define the Thue–Morse sequence; we choose the following way defined via an iterative reflection process. Let $\tau^0 = 0$ and define τ^1 to be τ^0 concatenated with $\overline{\tau^0}$, that is, $\tau^1 = \tau^0\overline{\tau^0} = 01$. We then define τ^2 to be $\tau^2 := \tau^1\overline{\tau^1}$. We continue this process inductively: given τ^k, let $\tau^{k+1} = \tau^k\overline{\tau^k}$. We can repeat this process indefinitely, and in doing so we obtain an infinite limit sequence $\tau := (\tau_i)_{i=0}^{\infty}$. This τ is the Thue–Morse sequence. The first few τ^k and the initial digits of τ are listed below:

$$\tau^0 = 0, \quad \tau^1 = 01, \quad \tau^2 = 0110, \quad \tau^3 = 01101001$$
$$\tau = 0110\,1001\,1001\,0110 \cdots .$$

For more on the Thue–Morse sequence we refer the reader to [**4**]. The significance of the Thue–Morse sequence within expansions in non-integer bases is that the Komornik–Loreti constant $\beta_{KL} \approx 1.787$, that is, the smallest $\beta \in (1,2)$ such that 1 has a unique β-expansion, is the unique solution to the equation

$$1 = \sum_{i=1}^{\infty} \frac{\tau_i}{\beta^i}.$$

For a proof of this fact, see [**25**]. In [**3**] it was shown that β_{KL} is transcendental.

Of particular importance to us are the sequences

$$\upsilon^n = (\upsilon_i^n)_{i=1}^{\infty} := (\tau_1, \ldots, \tau_{2^n-1}, 0)^{\infty}$$

and

$$\kappa^n := (T_{\tau_0}, T_{\tau_1}, \ldots, T_{\tau_{2^n-1}}) \in \{T_0, T_1\}^{2^n}.$$

It can be shown that the sequences υ^n all satisfy (3.5). Therefore by Lemma 3.6 for each $n \in \mathbb{N}$ there exists $\beta_n \in (1, 2)$ such that $\alpha(\beta_n) = \upsilon^n$. It follows from the definitions that $\beta_1 = (1 + \sqrt{5})/2$ and $\beta_n \nearrow \beta_{KL}$. Moreover, for any $\beta \in [\beta_n, \beta_{n+1})$ we have the following properties:

$$\pi_\beta((\tau^1)^\infty) < \pi_\beta((\tau^2)^\infty) < \cdots < \pi_\beta((\tau^n)^\infty) \le \frac{1}{\beta} < \pi_\beta((\tau^{n+1})^\infty), \tag{3.6}$$

$$\pi_\beta((\overline{\tau^{n+1}})^\infty) < \frac{1}{\beta(\beta-1)} \le \pi_\beta((\overline{\tau^n})^\infty) < \cdots < \pi_\beta((\overline{\tau^2})^\infty) < \pi_\beta((\overline{\tau^1})^\infty) \tag{3.7}$$

and

$$\pi_\beta((\tau^{n+1})^\infty) < \pi_\beta((\overline{\tau^{n+1}})^\infty). \tag{3.8}$$

Equations (3.6) and (3.7) are a consequence of the main result of [**2**]; see, in particular, Theorem 1.3 and Proposition 2.16 in that paper. Proving equation (3.8) holds is a straightforward calculation.

We also highlight the following facts which are a consequence of the Thue–Morse construction. For all $\beta \in (1, 2)$,

$$\kappa^n(\pi_\beta((\tau^n)^\infty)) = \pi_\beta((\tau^n)^\infty) \quad \text{and} \quad \overline{\kappa^n}(\pi_\beta((\overline{\tau^n})^\infty)) = \pi_\beta((\overline{\tau^n})^\infty), \tag{3.9}$$

$$\kappa^n(\pi_\beta((\tau^{n+1})^\infty)) = \pi_\beta((\overline{\tau^{n+1}})^\infty) \quad \text{and} \quad \overline{\kappa^n}(\pi_\beta((\overline{\tau^{n+1}})^\infty)) = \pi_\beta((\tau^{n+1})^\infty). \tag{3.10}$$

In (3.9) and (3.10) we have used $\overline{\kappa^n}$ to denote the sequence of maps obtained by replacing each T_0 in κ^n with T_1, and each T_1 in κ^n with T_0. Observe that (3.9) asserts that $\pi_\beta((\tau^n)^\infty)$ and $\pi_\beta((\overline{\tau^n})^\infty)$ are the fixed points of κ^n and $\overline{\kappa^n}$ respectively, and (3.10) states that $\pi_\beta((\tau^{n+1})^\infty)$ and $\pi_\beta((\overline{\tau^{n+1}})^\infty)$ are mapped from one to the other by κ^n and $\overline{\kappa^n}$, respectively. As we will see, these points will play a similar role to that played by the endpoints of I_β and $\mathcal{O}_\beta$ within the parameter space $(1, (1 + \sqrt{5})/2)$.

The following lemma is a consequence of the construction of the Thue–Morse sequence described above.

LEMMA 3.9. *For all $n \ge 1$ we have*

$$\frac{\#\{0 \le i \le |\tau^n| - 1 : \tau_i^n = 0\}}{|\tau^n|} = \frac{1}{2} \quad \textit{and} \quad \frac{\#\{0 \le i \le |\overline{\tau^n}| - 1 : \overline{\tau_i^n} = 0\}}{|\overline{\tau^n}|} = \frac{1}{2}.$$

Consequently, $(\tau^n)^\infty$ and $(\overline{\tau^n})^\infty$ are simply normal for all $n \in \mathbb{N}$. Similarly, for all $n \ge 1$ we have

$$\frac{|\kappa^n|_0}{|\kappa^n|} = \frac{1}{2} \quad \textit{and} \quad \frac{|\kappa^n|_1}{|\kappa^n|} = \frac{1}{2}.$$

Lemma 3.9 implies that if x can be mapped onto $\pi_\beta((\tau^n)^\infty)$ or $\pi_\beta((\overline{\tau^n})^\infty)$ then x must have a simply normal β-expansion. This observation will be used in our proof of Theorem 2.1.

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4. *Proofs of our digit frequency statements*

4.1. Proofs of Theorems 2.1 and 2.4. We start this section by proving a proposition that implies statements 2 and 3 of Theorem 2.1, and statement 1 of Theorem 2.1 for the parameter space $(1, (1+\sqrt{5})/2)$. This proposition will also allow us to prove Theorem 2.4.

PROPOSITION 4.1. *Let (p_k) be an arbitrary sequence in $J = (\frac{1}{2} - 1/(2n(\beta)), \frac{1}{2} + 1/(2n(\beta)))$. Then for every $x \in \mathcal{O}_\beta$ ($x \in \widetilde{\mathcal{O}}_\beta$) there exist $(\epsilon_i) \in \Sigma_\beta(x)$ ($(\epsilon_i) \in \widetilde{\Sigma}_\beta(x)$) and a sequence (n_k) such that*

$$||(\epsilon_i)_{i=1}^{n_k}|_0 - p_k|(\epsilon_i)_{i=1}^{n_k}|| \leq n(\beta) \tag{4.1}$$

($||(\epsilon_i)_{i=1}^{n_k}|_{-1} - p_k|(\epsilon_i)_{i=1}^{n_k}|| \leq n(\beta)$) and

$$0 \leq n_{k+1} - n_k \leq \frac{|p_k - p_{k+1}|n_k + n(\beta)}{\min\{p_{k+1} - 1/2 + 1/(2n(\beta)), 1/2 + 1/(2n(\beta)) - p_{k+1}\}} n(\beta) \tag{4.2}$$

for $k \geq 1$.

Proof. Fix $x \in \mathcal{O}_\beta$. We define a sequence $\lambda^k \in \{T_0, T_1\}^*$ such that $\lambda^k(x) \in \mathcal{O}_\beta$ and λ^k corresponds to $(\epsilon_i)_{i=1}^{n_k}$ using Lemma 3.1.

It is a simple exercise to show that any ω^0 and ω^1 whose existence is asserted by Proposition 3.3 must satisfy

$$\frac{|\omega^0|_0}{|\omega^0|} \geq \frac{1}{2} + \frac{1}{2n(\beta)} \quad \text{and} \quad \frac{|\omega^1|_0}{|\omega^0|} \leq \frac{1}{2} - \frac{1}{2n(\beta)}. \tag{4.3}$$

If $p_1 \in [\frac{1}{2}, \frac{1}{2} + 1/(2n(\beta)))$ then let $\lambda^1 = \omega^0$ where ω^0 is as in Proposition 3.3, and thus $\lambda^1(x) \in \mathcal{O}_\beta$. Equation (4.3) implies

$$0 \leq \left(\frac{1}{2} + \frac{1}{2n(\beta)}\right)|\lambda^1| - p_1|\lambda^1| \leq |\lambda^1|_0 - p_1|\lambda^1| \leq n(\beta),$$

which implies (4.1) for $k = 1$.

Assume that we have constructed λ^k such that (4.1) and (4.2) hold. If $|\lambda^k|_0 - p_{k+1}|\lambda^k| \geq 0$ then let j be the smallest natural number such that

$$|(\lambda^k, \overbrace{\omega^1, \ldots, \omega^1}^{j \text{ times}})|_0 - p_{k+1}|(\lambda^k, \overbrace{\omega^1, \ldots, \omega^1}^{j \text{ times}})| < 0.$$

Here the lth ω^1 is the map defined in Proposition 3.3 for $(\lambda^k, \overbrace{\omega^1, \ldots, \omega^1}^{l-1 \text{ times}})(x) \in \mathcal{O}_\beta$. Set $\lambda^{k+1} := (\lambda^k, \overbrace{\omega^1, \ldots, \omega^1}^{j \text{ times}})$. Since

$$\begin{aligned}|\lambda^{k+1}|_0 - p_{k+1}|\lambda^{k+1}| &= |\lambda^k|_0 - p_{k+1}|\lambda^k| + \sum(|\omega^1|_0 - p_{k+1}|\omega^1|)\\ &\overset{(4.3)}{\leq} |\lambda^k|_0 - p_{k+1}|\lambda^k| + \sum\left(\frac{1}{2} - \frac{1}{2n(\beta)} - p_{k+1}\right)|\omega^1|\\ &\leq |\lambda^k|_0 - p_{k+1}|\lambda^k| + j\left(\frac{1}{2} - \frac{1}{2n(\beta)} - p_{k+1}\right),\end{aligned}$$

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we have that

$$j \leq \frac{|\lambda^k|_0 - p_{k+1}|\lambda^k|}{p_{k+1} - 1/2 + 1/(2n(\beta))} \leq \frac{(p_k - p_{k+1})|\lambda^k| + n(\beta)}{p_{k+1} - 1/2 + 1/(2n(\beta))},$$

where we use (4.1) for k in the second inequality. Therefore,

$$|\lambda^{k+1}| - |\lambda^k| = \sum |\omega^1| \leq jn(\beta) \leq \frac{(p_k - p_{k+1})|\lambda^k| + n(\beta)}{p_{k+1} - 1/2 + 1/(2n(\beta))} n(\beta).$$

We also have

$$\begin{aligned} 0 > |\lambda^{k+1}|_0 - p_{k+1}|\lambda^{k+1}| &= |(\lambda^k, \overbrace{\omega^1, \ldots, \omega^1}^{j-1 \text{ times}})|_0 \\ &\quad - p_{k+1}|(\lambda^k, \overbrace{\omega^1, \ldots, \omega^1}^{j-1 \text{ times}})| + |\omega^1|_0 - p_{k+1}|\omega^1| \\ &\geq |\omega^1|_0 - p_{k+1}|\omega^1| \geq -p_{k+1}|\omega^1| \geq -n(\beta). \end{aligned}$$

Therefore (4.1) and (4.2) hold for $k+1$. If $|\lambda^k|_0 - p_{k+1}|\lambda^k| < 0$ then we repeat the above argument but for $\lambda^{k+1} := (\lambda^k, \overbrace{\omega^1, \ldots, \omega^1}^{j \text{ times}})$ where j is the smallest integer such that

$$|(\lambda^k, \overbrace{\omega^1, \ldots, \omega^1}^{j \text{ times}})|_0 - p_{k+1}|(\lambda^k, \overbrace{\omega^1, \ldots, \omega^1}^{j \text{ times}})| \geq 0,$$

where the lth ω^0 is the map defined in Proposition 3.3 for $(\lambda^k, \overbrace{\omega^1, \ldots, \omega^1}^{(l-1) \text{ times}})(x) \in \mathcal{O}_\beta$. □

Using Proposition 4.1, we obtain Theorem 2.4 almost immediately. For completeness' sake we include a proof of this theorem.

Proof of Theorem 2.4. Let $\beta \in (1, (1+\sqrt{5})/2)$ and $x \in (-1/(n(\beta)), 1/(n(\beta)))$. We apply Proposition 4.1 with $p_k \equiv (1-x)/2$. Let $(\epsilon_i) \in \tilde{\Sigma}_\beta(x)$ and (n_k) be the sequences guaranteed by this proposition. For this choice of (p_k) the right-hand side of (4.2) does not depend upon k and so $n_{k+1} - n_k$ is uniformly bounded from above. It follows from (4.1) and the fact that $n_{k+1} - n_k$ is uniformly bounded from above that (ϵ_i) satisfies

$$\lim_{n\to\infty} \frac{\#\{1 \leq i \leq n : \epsilon_i = 1\}}{n} = \frac{1+x}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{\#\{1 \leq i \leq n : \epsilon_i = -1\}}{n} = \frac{1-x}{2}.$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \epsilon_i &= \lim_{n\to\infty}\bigg(\frac{\#\{1 \leq i \leq n : \epsilon_i = 1\}}{n} - \frac{\#\{1 \leq i \leq n : \epsilon_i = -1\}}{n}\bigg) \\ &= \frac{1+x}{2} - \frac{1-x}{2} \\ &= x. \end{aligned}$$

Consequently, (ϵ_i) is a hybrid expansion of x.

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We now explain how Proposition 4.1 is used to deduce most of Theorem 2.1. By Lemma 3.2 we can assume without loss of generality that x is contained $\mathcal{O}_\beta$. Statement 2 from Theorem 2.1 follows from Proposition 4.1 by considering a sequence of (p_k) with more than one accumulation point. Statement 1 of Theorem 2.1 for the parameter space $(1, (1+\sqrt{5})/2)$ follows by applying Proposition 4.1 with $p_k \equiv 1/2$. Similarly, statement 3 follows from Proposition 4.1 by fixing $p_k \equiv p$ where $p \in (\frac{1}{2} - 1/(2n(\beta)), \frac{1}{2} + 1/(2n(\beta)))$.

Now we prove statement 1 from Theorem 2.1 for the parameter space $[(1+\sqrt{5})/2, \beta_{KL})$.

Proof of statement 1 *of Theorem 2.1 within the parameter space* $[(1+\sqrt{5})/2, \beta_{KL})$. Let us start by fixing $\beta \in [(1+\sqrt{5})/2, \beta_{KL})$. Then there exists $n \in \mathbb{N}$ such that $\beta \in [\beta_n, \beta_{n+1})$. Recall that the sequence (β_n) is defined in §3.

For each $1 \leq i \leq n+1$ let

$$\mathcal{I}_i := [\pi_\beta((\tau^i)^\infty), \pi_\beta((\overline{\tau^i})^\infty)].$$

Here the τ^i are the finite sequences appearing in the construction of the Thue–Morse sequence in §3. By (3.6) and (3.7) we know that these intervals are well defined and

$$\mathcal{I}_{n+1} \subseteq \mathcal{S}_\beta \subseteq \mathcal{I}_n \subseteq \cdots \subseteq \mathcal{I}_1 = \mathcal{O}_\beta. \tag{4.4}$$

Moreover, by (3.6) and (3.7) we know that $\mathcal{I}_{n+1}$ is a proper subinterval of $\mathcal{S}_\beta$. Therefore

$$[T_1(\pi_\beta((\tau^{n+1})^\infty)), T_0(\pi_\beta((\overline{\tau^{n+1}})^\infty))] \subseteq \left(0, \frac{1}{\beta-1}\right).$$

It follows from this observation, Lemma 3.2 and the expansivity of the maps T_0 and T_1 that if $x \in [\pi_\beta((\tau^i)^\infty), \pi_\beta((\overline{\tau^i})^\infty)]$, then $T_0(x)$ and $T_1(x)$ can both be mapped back into $\mathcal{O}_\beta$ using at most $l(\beta) \in \mathbb{N}$ iterations of T_1 or T_0, respectively. Importantly, $l(\beta)$ is a natural number that only depends upon β.

Now let us fix x. Without loss of generality we may assume $x \in \mathcal{O}_\beta$. If x is a preimage of an endpoint of an $\mathcal{I}_i$, then by Lemma 3.9 we know that x has a simply normal expansion. Therefore to prove our result it suffices to consider those x that are not preimages of an endpoint of an $\mathcal{I}_i$. We now give an algorithm which shows how one can construct a simply normal expansion for any x satisfying this condition.

Step 1. By (4.4) and our assumption that x is not a preimage of an endpoint of an $\mathcal{I}_i$, we know that x satisfies one of the following:

$$x \in (\pi_\beta((\tau^{n+1})^\infty), \pi_\beta((\overline{\tau^{n+1}})^\infty)), \quad x \in (\pi_\beta((\tau^i)^\infty), \pi_\beta((\tau^{i+1})^\infty))$$

or

$$x \in (\pi_\beta((\overline{\tau^{i+1}})^\infty), \pi_\beta((\overline{\tau^i})^\infty))$$

for some $1 \leq i \leq n$. If $x \in (\pi_\beta((\tau^{n+1})^\infty), \pi_\beta((\overline{\tau^{n+1}})^\infty))$, apply T_0 to x and then T_1 until $(T_1^j \circ T_0)(x) \in \mathcal{O}_\beta$. By our previous remarks we know that $j \leq l(\beta)$. Let $\lambda^1 = (T_0, (T_1)^j)$. Then

$$||(\lambda^1)_{i=1}^m|_0 - |(\lambda^1)_{i=1}^m|_1| \leq l(\beta)$$

for all $1 \leq m \leq |\lambda^1|$.

If $x \in (\pi_\beta((\tau^i)^\infty), \pi_\beta((\tau^{i+1})^\infty))$, then we repeatedly apply κ^i to x until x is mapped into $(\pi_\beta((\tau^{i+1})^\infty), \pi_\beta((\overline{\tau^{i+1}})^\infty))$. The fact that we can do this follows from (3.10), our assumption that x is not a preimage of an endpoint of an $\mathcal{I}_i$, and the fact that $\pi_\beta((\tau^i)^\infty)$ is the unique fixed point of κ^i and κ^i scales distances by some factor strictly greater than one. Likewise, if $x \in (\pi_\beta((\overline{\tau^{i+1}})^\infty), \pi_\beta((\overline{\tau^i})^\infty))$, then by repeatedly applying $\overline{\kappa}^i$ the point x is mapped into $(\pi_\beta((\tau^{i+1})^\infty), \pi_\beta((\overline{\tau^{i+1}})^\infty))$. In either case we let x^1 denote the image point of x in $(\pi_\beta((\tau^{i+1})^\infty), \pi_\beta((\overline{\tau^{i+1}})^\infty))$. If $x^1 \notin (\pi_\beta((\tau^{n+1})^\infty), \pi_\beta((\overline{\tau^{n+1}})^\infty))$ then

$$x^1 \in (\pi_\beta((\tau^{i_1})^\infty), \pi_\beta((\tau^{i_1+1})^\infty)) \cup (\pi_\beta((\overline{\tau^{i_1+1}})^\infty), \pi_\beta((\overline{\tau^{i_1}})^\infty)) \tag{4.5}$$

for some $i_1 > i$.

Repeating the above argument, we see that if $x^1 \notin (\pi_\beta((\tau^{n+1})^\infty), \pi_\beta((\overline{\tau^{n+1}})^\infty))$, then by repeatedly applying either κ^{i_1} or $\overline{\kappa}^{i_1}$ to x^1 our orbit is eventually mapped into $(\pi_\beta((\tau^{i_1+1})^\infty), \pi_\beta((\overline{\tau^{i_1+1}}^\infty)))$. We can repeat this procedure until our orbit is eventually mapped into $(\pi_\beta((\tau^{n+1})^\infty), \pi_\beta((\overline{\tau^{n+1}})^\infty))$. Therefore we may conclude that there exists a sequence of maps $\kappa^* \in \{T_0, T_1\}^*$ such that

$$\kappa^*(x) \in (\pi_\beta((\tau^{n+1})^\infty), \pi_\beta((\overline{\tau^{n+1}})^\infty)).$$

Moreover, the sequence of maps κ^* is the concatenation of finitely many blocks all of length at most 2^n, where each of these blocks has the same number of T_0s and T_1s by Lemma 3.9. Therefore

$$|\kappa^*|_0 = |\kappa^*|_1 \quad \text{and} \quad ||(\kappa^*)_{i=1}^m|_0 - |(\kappa^*)_{i=1}^m|_1| \le 2^n$$

for all $1 \le m \le |\kappa^*|$. We now apply T_0 to $\kappa^*(x)$ and then apply T_1 until $(T_1^j \circ T_0 \circ \kappa^*)(x) \in \mathcal{O}_\beta$. Let $\lambda^1 = (\kappa^*, T_0, (T_1)^j)$. Then $\lambda^1(x) \in \mathcal{O}_\beta$ and

$$||(\lambda_i^1)_{i=1}^m|_0 - |(\lambda_i^1)_{i=1}^m|_1| \le 2^n$$

if $1 \le m \le |\kappa^*|$. Moreover,

$$||(\lambda_i^1)_{i=1}^m|_0 - |(\lambda_i^1)_{i=1}^m|_1| \le l(\beta)$$

if $|\kappa^*| < m \le |\lambda^1|$ since $|\kappa^*|_0 = |\kappa^*|_1$ and $j \le l(\beta)$.

It follows from the above that we have constructed $\lambda^1 \in \{T_0, T_1\}^*$ such that $\lambda^1(x) \in \mathcal{O}_\beta$,

$$||(\lambda_i^1)_{i=1}^m|_0 - |(\lambda_i^1)_{i=1}^m|_1| \le 2^n + l(\beta) \tag{4.6}$$

for all $1 \le m \le |\lambda^1|$, and

$$||\lambda^1|_0 - |\lambda^1|_1| \le l(\beta). \tag{4.7}$$

Step $k+1$. Suppose we have constructed $\lambda^k \in \{T_0, T_1\}^*$ such that $\lambda^k(x) \in \mathcal{O}_\beta$,

$$||(\lambda_i^k)_{i=1}^m|_0 - |(\lambda_i^k)_{i=1}^m|_1| \le 2^n + l(\beta) \tag{4.8}$$

for all $1 \le m \le |\lambda^k|$, and

$$||\lambda^k|_0 - |\lambda^k|_1| \le l(\beta). \tag{4.9}$$

We now show how to construct λ^{k+1} satisfying $\lambda^{k+1}(x) \in \mathcal{O}_\beta$, (4.8) and (4.9). There are two cases to consider. Either

$$|\lambda^k|_0 - |\lambda^k|_1$$

is positive or it is negative. Let us assume it is positive. The negative case is handled similarly. By the same argument used in Step 1, if $\lambda^k(x) \notin (\pi_\beta((\tau^{n+1})^\infty), \pi_\beta((\overline{\tau^{n+1}})^\infty))$, then there exists $\kappa^* \in \{T_0, T_1\}^*$ such that $|\kappa^*|_0 = |\kappa^*|_1$ and $(\kappa^* \circ \lambda^k)(x) \in (\pi_\beta((\tau^{n+1})^\infty), \pi_\beta((\overline{\tau^{n+1}})^\infty))$. Moreover, κ^* is the concatenation of finitely many blocks of length at most 2^n, and each block has the same number of T_0s and T_1s. We then apply T_0 and T_1 until $(T_1^j \circ T_0 \circ \kappa^* \circ \lambda^k)(x) \in \mathcal{O}_\beta$. At this point we set $\lambda^{k+1} = (\lambda^k, \kappa^*, T_0, T_1^j)$. Then

$$||(\lambda_i^{k+1})_{i=1}^m|_0 - |(\lambda_i^{k+1})_{i=1}^m|_1| \leq 2^n + l(\beta)$$

if $1 \leq m \leq |\lambda^k|$ by (4.8). If $|\lambda^k| < m \leq |\lambda^k| + |\kappa^*|$ then

$$||(\lambda_i^{k+1})_{i=1}^m|_0 - |(\lambda_i^{k+1})_{i=1}^m|_1| \leq 2^n + l(\beta).$$

This is a consequence of (4.9) and the fact that κ^* is the concatenation of finitely many blocks of length at most 2^n, where each block has the same number of T_0s as T_1s. If $|\lambda^k| + |\kappa^*| < m \leq |\lambda^{k+1}|$ then

$$\begin{aligned} |(\lambda_i^{k+1})_{i=1}^m|_0 - |(\lambda_i^{k+1})_{i=1}^m|_1 = {} & |\lambda^k|_0 - |\lambda^k|_1 + |\kappa^*|_0 - |\kappa^*|_1 \\ & + 1 - (m - |\lambda^k| - |\kappa^*| - 1). \end{aligned} \tag{4.10}$$

Using the fact that $|\kappa^*|_0 = |\kappa^*|_1$ and (4.9), we see that (4.10) implies

$$|(\lambda_i^k)_{i=1}^m|_0 - |(\lambda_i^k)_{i=1}^m|_1 \leq l(\beta) + 1$$

if $|\lambda^k| + |\kappa^*| < m \leq |\lambda^{k+1}|$. Using the assumption that $|\lambda^k|_0 - |\lambda^k|_1$ is positive, along with $|\kappa^*|_0 = |\kappa^*|_1$ and $j \leq l(\beta)$, we see that (4.10) also implies

$$-l(\beta) \leq |(\lambda_i^k)_{i=1}^m|_0 - |(\lambda_i^k)_{i=1}^m|_1$$

if $|\lambda^k| + |\kappa^*| < m \leq |\lambda^{k+1}|$. Therefore

$$||(\lambda_i^k)_{i=1}^m|_0 - |(\lambda_i^k)_{i=1}^m|_1| \leq 2^n + l(\beta)$$

if $|\lambda^k| + |\kappa^*| < m \leq |\lambda^{k+1}|$. Moreover, since $j \geq 1$, we see that (4.10) implies

$$|\lambda^{k+1}|_0 - |\lambda^{k+1}|_1 \leq |\lambda^k|_0 - |\lambda^k|_1 \leq l(\beta).$$

Therefore $\lambda^{k+1}(x) \in \mathcal{O}_\beta$ and λ^{k+1} satisfies (4.8) and (4.9). We have completed our inductive step when $\lambda^k(x) \notin (\pi_\beta((\tau^{n+1})^\infty), \pi_\beta((\overline{\tau^{n+1}})^\infty))$. When $\lambda^k(x) \in (\pi_\beta((\tau^{n+1})^\infty), \pi_\beta((\overline{\tau^{n+1}})^\infty))$ the construction of λ^{k+1} is the same as above except that we do not need to construct the initial sequence of maps κ^*.

Repeating this procedure indefinitely gives rise to an infinite sequence $\lambda \in \Omega_\beta(x)$ such that

$$||(\lambda_i)_{i=1}^m|_0 - |(\lambda_i)_{i=1}^m|_1| \leq 2^n + l(\beta) \tag{4.11}$$

for all $m \in \mathbb{N}$. It follows from (4.11) that within λ the map T_0 appears with frequency $1/2$ and the map T_1 appears with frequency $1/2$. By Lemma 3.1 there exists $(\epsilon_i) \in \Sigma_\beta(x)$ which is simply normal. $\square$

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4.2. *Proofs of Theorems 2.2 and 2.5.* We now give a proof of Theorem 2.2. Ideas similar to those used in this proof appeared in [**22**].

Proof of Theorem 2.2. Recall from Lemma 3.7 that

$$\widetilde{\mathcal{A}}_\beta = \{(\epsilon_i) \in \{0,1\}^{\mathbb{N}} : (\overline{\alpha_i(\beta)}) \prec (\epsilon_{n+i}) \prec (\alpha_i(\beta)) \text{ for all } n \in \mathbb{N}\}. \tag{4.12}$$

Let β_n be the unique positive solution to the equation

$$x^{n+1} = x^n + x^{n-1} + \cdots + x + 1$$

with modulus larger than 1. The number β_n is commonly referred to as the nth multinacci number. Note that $\beta_n \nearrow 2$ as $n \to \infty$. It is a consequence of Lemma 3.6 that

$$\alpha(\beta_n) = ((1)^n, 0)^\infty.$$

It follows from (4.12) that

$$\{(\epsilon_i) \in \{0,1\}^{\mathbb{N}} : (\epsilon_i) \text{ does not contain } n \text{ consecutive 0s or 1s}\} \subseteq \widetilde{\mathcal{A}}_{\beta_n}. \tag{4.13}$$

Let

$$W_n = \{(\epsilon_i) \in \{0,1\}^n : |(\epsilon_i)|_1 > |(\epsilon_i)|_0 \text{ and } (\epsilon_i) \neq (1)^n\}.$$

Consider the case where $n = 2k+1$. Any element of $\{0,1\}^{2k+1}$ satisfies either $|(\epsilon_i)|_1 > |(\epsilon_i)|_0$ or $|(\epsilon_i)|_1 < |(\epsilon_i)|_0$. It follows that

$$\#W_{2k+1} = 2^{2k} - 1. \tag{4.14}$$

Let $T_{2k+1} := W_{2k+1}^{\mathbb{N}}$. Each element of T_{2k+1} fails to be simply normal. This is because the number of 1s in each successive block of length $2k+1$ is at least k. What is more, any element of T_{2k+1} cannot contain $2(2k+1)$ consecutive 0s or 1s. Therefore $T_{2k+1} \subseteq \widetilde{\mathcal{A}}_{\beta_{2(2k+1)}}$ by (4.13). By Lemma 3.8 we also know that $T_{2k+1} \subseteq \widetilde{\mathcal{A}}_\beta$ for any $\beta \in (\beta_{2(2k+1)}, 2)$.

We now compute the Hausdorff dimension of the set $\pi_\beta(T_{2k+1})$ for $\beta \in (\beta_{2(2k+1)}, 2)$. Since every element of T_{2k+1} fails to be simply normal and each element of $\pi_\beta(T_{2k+1})$ has a unique β-expansion, the Hausdorff dimension of $\pi_\beta(T_{2k+1})$ will give a lower bound for the Hausdorff dimension of those x without a simply normal β-expansion.

Let us now fix $\beta \in (\beta_{2(2k+1)}, 2)$. Notice that $\pi_\beta(T_{2k+1})$ satisfies the similarity relation

$$\pi_\beta(T_{2k+1}) = \bigcup_{(\epsilon_i)_{i=1}^{2k+1} \in W_{2k+1}^1} (T_{\epsilon_1}^{-1} \circ \cdots \circ T_{\epsilon_{2k+1}}^{-1})(\pi_\beta(T_{2k+1})). \tag{4.15}$$

Each map on the right-hand side of (4.15) is a contracting similarity that scales by a factor β^{-2k-1}. Therefore $\pi_\beta(T_{2k+1})$ is a self-similar set. It is a consequence of each element of $\pi_\beta(T_{2k+1})$ having a unique β-expansion that the union in (4.15) is disjoint. Therefore $\pi_\beta(T_{2k+1})$ is a self-similar set and the set of contractions generating it satisfy the strong separation condition. The well-known formula for the Hausdorff dimension of a self-similar set satisfying the strong separation condition (see, for example, [**17**, Theorem 9.3]) implies that $\dim_H(\pi_\beta(T_{2k+1}))$ satisfies

$$1 = \#W_{2k+1} \cdot \beta^{-(2k+1)\dim_H(\pi_\beta(T_{2k+1}))}.$$

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Rearranging this equation and appealing to (4.14), we obtain

$$\dim_H(\pi_\beta(T_{2k+1})) = \frac{\log 2^{2k}-1}{\log \beta^{2k+1}} > \frac{\log 2^{2k}-1}{\log 2^{2k+1}} \geq \frac{2k-1}{2k+1}$$

for any $\beta \in (\beta_{2(2k+1)}, 2)$. Since k is arbitrary it follows that

$$\lim_{\beta \nearrow 2} \dim_H(\{x : x \text{ has no simply normal } \beta\text{-expansion}\}) = 1. \qquad \square$$

We now give a proof of Theorem 2.5. In the proof of this theorem we will require the interpretation of Proposition 3.3 when the digit set is $\{-1, 1\}$ not $\{0, 1\}$.

Proof of Theorem 2.5. Let us start by fixing $\beta \in (1, (1+\sqrt{5})/2)$ and $x \in (-1/(\beta-1), 1/(\beta-1))$. Suppose that $f : \mathbb{N} \to \mathbb{R}$ is a strictly increasing function satisfying

$$\lim_{n\to\infty} f(n) = \infty$$

and

$$f(n+1) - f(n) < \frac{\beta-1}{n(\beta)} \tag{4.16}$$

for all $n \geq N$, where N is some large natural number. Here $n(\beta)$ is as in the statement of Proposition 3.3. We now describe an algorithm which yields an expansion of x with the desired properties.

Step 1. The first step in our construction is to pick an arbitrary sequence $\lambda^0 \in \{T_{-1}, T_1\}^N$ such that $\lambda^0(x) \in \widetilde{\mathcal{O}}_\beta$. We can do this by Lemma 3.2 and replacing our value of N with a larger value if necessary. At this point we consider the sign of the quantity

$$|\lambda^0|_1 - |\lambda^0|_{-1} - f(N)x. \tag{4.17}$$

Let us start by assuming this quantity is negative. The positive case is handled similarly. Since $\lambda^0(x) \in \widetilde{\mathcal{O}}_\beta$, we can apply Proposition 3.3 to assert that there exists ω^1 satisfying $(\omega^1 \circ \lambda^0)(x) \in \widetilde{\mathcal{O}}_\beta$, $|\omega^1| \leq n(\beta)$ and $|\omega^1|_{-1} < |\omega^1|_1$. Let $\lambda^{0,1} = (\lambda^0, \omega^1)$. Consider the quantity

$$|\lambda^{0,1}|_1 - |\lambda^{0,1}|_{-1} - f(|\lambda^{0,1}|)x.$$

If this term is greater than or equal to zero then there has been a sign change. In which case let $\lambda^1 = \lambda^{0,1}$. Using (4.16) and the fact $|\omega^1| \leq n(\beta)$, it can be shown that

$$0 \leq |\lambda^1|_1 - |\lambda^1|_{-1} - f(|\lambda^1|)x \leq n(\beta) + 1 \tag{4.18}$$

if there has been a sign change. Suppose that we do not see a sign change. By Proposition 3.3 there exists ω^1 satisfying $|\omega^1| \leq n(\beta)$, $|\omega^1|_{-1} < |\omega^1|_1$ and $\lambda^{0,2}(x) \in \widetilde{\mathcal{O}}_\beta$, where $\lambda^{0,2} = (\lambda^{0,1}, \omega^1)$. We consider the quantity

$$|\lambda^{0,2}|_1 - |\lambda^{0,2}|_{-1} - f(|\lambda^{0,2}|)x,$$

and ask whether there has been a sign change. If there has been a sign change we let $\lambda^1 = \lambda^{0,2}$. If not we concatenate $\lambda^{0,2}$ with the ω^1 guaranteed by Proposition 3.3. We repeat this procedure and obtain a sequence $(\lambda^{0,j})$. Note that for all $j \geq 1$ we have

$$(|\lambda^{0,j+1}|_1 - |\lambda^{0,j+1}|_{-1}) - (|\lambda^{0,j}|_1 - |\lambda^{0,j}|_{-1}) \geq 1. \tag{4.19}$$

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What is more,

$$
\begin{aligned}
&|xf(|\lambda^{0,j+1}|) - xf(|\lambda^{0,j}|)| \\
&\quad = \left| x\left(\sum_{i=0}^{|\lambda^{0,j+1}|-|\lambda^{0,j}|-1} f(|\lambda^{0,j+1}| - i) - f(|\lambda^{0,j+1}| - i - 1) \right) \right| \\
&\quad < \left| \frac{x(\beta-1)n(\beta)}{n(\beta)} \right| \\
&\quad < c
\end{aligned} \tag{4.20}
$$

for some $c < 1$ depending on x. Combining equations (4.19) and (4.20), we obtain

$$
|\lambda^{0,j+1}|_1 - |\lambda^{0,j+1}|_{-1} - xf(|\lambda^{0,j+1}|) > |\lambda^{0,j}|_1 - |\lambda^{0,j}|_{-1} - xf(|\lambda^{0,j}|) + (1-c). \tag{4.21}
$$

Repeatedly applying (4.21), we observe that

$$
|\lambda^{0,j}|_1 - |\lambda^{0,j}|_{-1} - xf(|\lambda^{0,j}|) > |\lambda^{0}|_1 - |\lambda^{0}|_{-1} - xf(|\lambda^{0}|) + j(1-c). \tag{4.22}
$$

Since $(1-c) > 0$, equation (4.22) implies that we must observe a sign change after finitely many steps. Let $\lambda^1 = \lambda^{0,j^*}$ where j^* is the smallest $j^* \in \mathbb{N}$ such that

$$
|\lambda^{0,j^*}|_1 - |\lambda^{0,j^*}|_{-1} - f(|\lambda^{0,j^*}|)x \geq 0.
$$

This λ^1 then satisfies (4.18) and $\lambda^1(x) \in \mathcal{O}_\beta$.

When (4.17) is positive we can obtain an appropriate analogue of (4.18). In either case we will have constructed $\lambda^1 \in \{T_{-1}, T_1\}^*$ such that $\lambda^1(x) \in \widetilde{\mathcal{O}}_\beta$ and

$$
||\lambda^1|_1 - |\lambda^1|_{-1} - f(|\lambda^1|)x| \leq n(\beta) + 1. \tag{4.23}
$$

Step $k+1$. Suppose that we have constructed $\lambda^k \in \{T_{-1}, T_1\}^*$ such that $\lambda^k(x) \in \widetilde{\mathcal{O}}_\beta$ and

$$
||\lambda^k|_1 - |\lambda^k|_{-1} - f(|\lambda^k|)x| \leq n(\beta) + 1. \tag{4.24}
$$

We now show how to construct λ^{k+1} such that $\lambda^{k+1}(x) \in \widetilde{\mathcal{O}}_\beta$ and (4.24) is still satisfied. Consider the term appearing within the modulus signs in (4.24). If this term is positive then we concatenate λ^k with ω^{-1}, if it is negative then we concatenate λ^k with ω^1. Here ω^{-1} and ω^1 are as in Proposition 3.3. In either case we call our new sequence λ^{k+1}. By Proposition 3.3 we have $\lambda^{k+1}(x) \in \widetilde{\mathcal{O}}_\beta$. By similar arguments to those used in Step 1, it can be shown that λ^{k+1} satisfies

$$
||\lambda^{k+1}|_1 - |\lambda^{k+1}|_{-1} - f(|\lambda^{k+1}|)x| \leq n(\beta) + 1.
$$

This completes our inductive step.

Note that it is a consequence of our construction that

$$
|\lambda^{k+1}| - |\lambda^k| \leq n(\beta) \tag{4.25}
$$

for all $k \geq 1$. Now let $\lambda \in \widetilde{\Omega}_\beta(x)$ denote the infinite sequence of transformations we obtain by repeating step $k+1$ indefinitely. It is a consequence of (4.16), (4.24) and (4.25) that

$$
||(\lambda_i)_{i=1}^n|_1 - |(\lambda_i)_{i=1}^n|_{-1} - f(n)x| \leq C(\beta) \tag{4.26}
$$

for all $n \geq |\lambda^1|$, where $C(\beta)$ is a constant that only depends upon β.

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Let (ϵ_i) be the element of $\widetilde{\Sigma}_\beta(x)$ obtained by applying the bijection in Lemma 3.1 to λ. Using the simple identity

$$\sum_{i=1}^{n} \epsilon_i = |(\lambda_i)_{i=1}^n|_1 - |(\lambda_i)_{i=1}^n|_{-1}$$

and (4.26), we obtain

$$\left|\sum_{i=1}^{n} \epsilon_i - f(n)x\right| \leq C(\beta)$$

for all $n \geq |\lambda^1|$. Since $f(n) \to \infty$ we must have

$$\lim_{n\to\infty} \frac{1}{f(n)} \sum_{i=1}^{n} \epsilon_i \to x$$

as required. □

5. *Self-affine sets with non-empty interior*

In this section we prove Theorem 2.8. As we will see in §6, one can explicitly calculate a lower bound for the value of δ appearing in the statement of this theorem. We start by introducing some notation and proving a technical proposition.

Note that if $x \in \widetilde{\mathcal{O}}_\beta$ then $\omega^1(x) \in \widetilde{\mathcal{O}}_\beta$ and $\omega^{-1}(x) \in \widetilde{\mathcal{O}}_\beta$, where ω^1 and ω^{-1} are as in Proposition 3.3. Applying Proposition 3.3 again, we know that there exist $\omega^{1'}$ and $\omega^{-1'}(x)$ such that $(\omega^{1'} \circ \omega^1)(x) \in \widetilde{\mathcal{O}}_\beta$ and $(\omega^{-1'} \circ \omega^{-1})(x) \in \widetilde{\mathcal{O}}_\beta$. Clearly we can apply Proposition 3.3 repeatedly to x and its successive images. By an abuse of notation, we let $(\omega_i^1)_{i=1}^\infty \in \widetilde{\Omega}_\beta(x)$ denote the infinite sequence we obtain by repeatedly applying ω^1. Similarly, $(\omega_i^{-1})_{i=1}^\infty \in \widetilde{\Omega}_\beta(x)$ will denote the infinite sequence we obtain by repeatedly applying ω^{-1}. Moreover, given an $x \in \widetilde{\mathcal{O}}_\beta$, a sequence whose entries consist of ω^{-1}s and ω^1s will represent the element of $\widetilde{\Omega}_\beta(x)$ obtained by repeatedly applying Proposition 3.3 and applying ω^{-1} and ω^1 in accordance with the order they appear in that sequence. In what follows we let $B : \{T_{-1}, T_1\}^{\mathbb{N}} \to \{-1, 1\}^{\mathbb{N}}$ be the map which sends (T_{ϵ_i}) to (ϵ_i). Note that B is a bijection between $\widetilde{\Omega}_\beta(x)$ and $\widetilde{\Sigma}_\beta(x)$ by Lemma 3.1. By an abuse of notation we also let B denote the map $B : \{T_{-1}, T_1\}^n \to \{-1, 1\}^n$ which sends $(T_{\epsilon_i})_{i=1}^n$ to $(\epsilon_i)_{i=1}^n$.

Returning to our self-affine sets, one can verify that $\Lambda_{\beta_1,\beta_2,\beta_3}$ has the closed from

$$\Lambda_{\beta_1,\beta_2,\beta_3} = \left\{\left(\sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta_1^i}, \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta_2^{|(\epsilon_j)_{j=1}^i|_{-1}} \beta_3^{|(\epsilon_j)_{j=1}^i|_1}}\right) : (\epsilon_i) \in \{-1, 1\}^{\mathbb{N}}\right\}.$$

In what follows we let $\pi_{\beta_2,\beta_3} : \{-1, 1\}^{\mathbb{N}} \cup \{-1, 1\}^* \to \mathbb{R}$ denote the map

$$\pi_{\beta_2,\beta_3}((\epsilon_i)) = \sum_{i=1}^{|(\epsilon_i)|} \frac{\epsilon_i}{\beta_2^{|(\epsilon_j)_{j=1}^i|_{-1}} \beta_3^{|(\epsilon_j)_{j=1}^i|_1}}.$$

The equality

$$\pi_{\beta_2,\beta_3}(\widetilde{\Sigma}_{\beta_1}(x)) = \Lambda^x_{\beta_1,\beta_2,\beta_3} \tag{5.1}$$

holds for any $x \in [-1/(\beta_1 - 1), 1/(\beta_1 - 1)]$. Equation (5.1) shows the connection between the set of β_1-expansions of a given x and its vertical fibre. This connection is what allows us to prove Theorem 2.8.

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PROPOSITION 5.1. *Let $\beta_1 \in (1, (1+\sqrt{5})/2)$. Then there exists $\delta = \delta(\beta_1) > 0$ such that for any $\beta_2, \beta_3 \in (1, 1+\delta)$ and $x \in \widetilde{\mathcal{O}}_{\beta_1}$ we have*

$$\pi_{\beta_2,\beta_3}(B(\omega^1, (\omega_i^{-1})_{i=1}^\infty)) < 0 \quad \text{and} \quad 0 < \pi_{\beta_2,\beta_3}(B(\omega^{-1}, (\omega_i^1)_{i=1}^\infty)).$$

Proof. Let us start by fixing $\beta_1 \in (1, (1+\sqrt{5})/2)$ and let $n(\beta_1)$ be as in the statement of Proposition 3.3. Let $\delta' > 0$ be sufficiently small such that if $\beta_2, \beta_3 \in (1, 1+\delta')$. Then

$$\sum_{i=1}^{|(\epsilon_i)|} \frac{\epsilon_i}{\beta_2^{|(\epsilon_j)_{j=1}^i|_{-1}} \beta_3^{|(\epsilon_j)_{j=1}^i|_1}} \geq \frac{1}{2} \tag{5.2}$$

whenever $(\epsilon_i) \in \{-1, 1\}^*$ satisfies $|(\epsilon_i)| \leq n(\beta)$ and $|(\epsilon_i)|_1 > |(\epsilon_i)|_{-1}$. Such a δ' exists since for any (ϵ_i) satisfying these properties we have

$$\sum_{i=1}^{|(\epsilon_i)|} \frac{\epsilon_i}{1^{|(\epsilon_j)_{j=1}^i|_{-1}} 1^{|(\epsilon_j)_{j=1}^i|_1}} = \sum_{i=1}^{|(\epsilon_i)|} \epsilon_i = |(\epsilon_i)|_1 - |(\epsilon_i)|_{-1} \geq 1 > \frac{1}{2},$$

and strict inequality is preserved in a neighbourhood of 1. For the same value of δ' we have

$$\sum_{i=1}^{|(\epsilon_i)|} \frac{\epsilon_i}{\beta_2^{|(\epsilon_j)_{j=1}^i|_{-1}} \beta_3^{|(\epsilon_j)_{j=1}^i|_1}} \leq -\frac{1}{2} \tag{5.3}$$

whenever $(\epsilon_i) \in \{-1, 1\}^*$ satisfies $|(\epsilon_i)| \leq n(\beta)$ and $|(\epsilon_i)|_{-1} > |(\epsilon_i)|_1$. Suppose that $\beta_2, \beta_3 \in (1, 1+\delta')$. Then

$$\begin{aligned}
\pi_{\beta_2,\beta_3}(B(\omega^{-1}, (\omega_i^1)_{i=1}^\infty)) &= \pi_{\beta_2,\beta_3}(B(\omega^{-1})) + \sum_{i=0}^{\infty} \frac{\pi_{\beta_2,\beta_3}(B(\omega_{i+1}^1))}{\beta_2^{|\omega^{-1}|_{-1}+\sum_{j=0}^i |\omega_j^1|_{-1}} \beta_3^{|\omega^{-1}|_1+\sum_{j=0}^i |\omega_j^1|_1}} \\
&\geq -n(\beta_1) + \sum_{i=0}^{\infty} \frac{1}{2\beta_2^{|\omega^{-1}|_{-1}+\sum_{j=0}^i |\omega_j^1|_{-1}} \beta_3^{|\omega^{-1}|_1+\sum_{j=0}^i |\omega_j^1|_1}} \\
&\geq -n(\beta_1) + \sum_{i=0}^{\infty} \frac{1}{2\max(\beta_2, \beta_3)^{|\omega^{-1}|+\sum_{j=0}^i |\omega_j^1|}} \\
&\geq -n(\beta_1) + \sum_{i=0}^{\infty} \frac{1}{2\max(\beta_2, \beta_3)^{(i+1)n(\beta_1)}} \\
&\geq -n(\beta_1) + \frac{1}{2(\max(\beta_2, \beta_3)^{n(\beta_1)} - 1)}.
\end{aligned}$$

In the first inequality we used (5.2) and the fact that $|\omega^{-1}| \leq n(\beta_1)$. In the third inequality we used $|\omega^{-1}| \leq n(\beta_1)$, and $|\omega_i^1| \leq n(\beta_1)$ for all i. Summarising the above, we have

$$-n(\beta_1) + \frac{1}{2(\max(\beta_2, \beta_3)^{n(\beta_1)} - 1)} \leq \pi_{\beta_2,\beta_3}(B(\omega^{-1}, (\omega_i^1)_{i=1}^\infty)) \tag{5.4}$$

whenever $\beta_2, \beta_3 \in (1, 1+\delta')$. Similarly, one can show that if $\beta_2, \beta_3 \in (1, 1+\delta')$ then

$$\pi_{\beta_2,\beta_3}(B(\omega^1, (\omega_i^{-1})_{i=1}^\infty)) \leq n(\beta_1) - \frac{1}{2(\max(\beta_2, \beta_3)^{n(\beta_1)} - 1)}. \tag{5.5}$$

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There exists $\delta'' > 0$ such that for $\beta_2, \beta_3 \in (1, 1+\delta'')$ we have

$$n(\beta_1) - \frac{1}{2(\max(\beta_2, \beta_3)^{n(\beta_1)} - 1)} < 0 \quad \text{and} \quad 0 < -n(\beta_1) + \frac{1}{2(\max(\beta_2, \beta_3)^{n(\beta_1)} - 1)}. \tag{5.6}$$

Taking $\delta = \min(\delta', \delta'')$, we see that (5.4), (5.5) and (5.6) imply that for $\beta_2, \beta_3 \in (1+\delta)$ we have

$$\pi_{\beta_2,\beta_3}(B(\omega^1, (\omega_i^{-1})_{i=1}^{\infty})) < 0 \quad \text{and} \quad 0 < \pi_{\beta_2,\beta_3}(B(\omega^{-1}, (\omega_i^1)_{i=1}^{\infty})).$$

This completes our proof. □

In the proof of Proposition 5.1 the parameter $1/2$ appearing in (5.2) and (5.3) is an arbitrary choice. We could have replaced $1/2$ with any $c \in (0, 1)$. It is not clear what an optimal choice of c would be. What is more, the quantity $n(\beta_1)$ appearing in (5.4) and (5.5) is not necessarily optimal. In §6 we see that for an explicit choice of β_1 these parameters can be improved upon to give a larger value of δ.

The following corollary follows immediately from Proposition 5.1.

COROLLARY 5.2. *Let $\beta_1 \in (1, (1+\sqrt{5})/2)$. Then there exists $\delta = \delta(\beta_1) > 0$ such that for any $\beta_2, \beta_3 \in (1, 1+\delta)$ and $x \in \widetilde{\mathcal{O}}_{\beta_1}$ we have*

$$\pi_{\beta_2,\beta_3}(B(\lambda, \omega^1, (\omega_i^{-1})_{i=1}^{\infty})) < \pi_{\beta_2,\beta_3}(B(\lambda, \omega^{-1}, (\omega_i^1)_{i=1}^{\infty}))$$

for all $\lambda \in \{T_0, T_1\}^*$.

We are now in a position to prove Theorem 2.8.

Proof of Theorem 2.8. Let us fix $\beta_1 \in (1, (1+\sqrt{5})/2)$ and let $\delta > 0$ be as in the statement of Proposition 5.1. Fix $\beta_2, \beta_3 \in (1, 1+\delta)$ and $x \in (-1/(\beta_1 - 1), 1/(\beta_1 - 1))$. By Lemma 3.2 there exists $\lambda^0 \in \{T_{-1}, T_1\}^*$ such that $\lambda^0(x) \in \widetilde{\mathcal{O}}_{\beta_1}$. Consider the interval

$$[\pi_{\beta_2,\beta_3}(B(\lambda^0, (\omega_i^{-1})_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B(\lambda^0, (\omega_i^1)_{i=1}^{\infty}))],$$

where $(\omega_i^{-1})_{i=1}^{\infty}$ and $(\omega_i^1)_{i=1}^{\infty}$ are obtained by repeatedly applying Proposition 3.3 to $\lambda^0(x)$ and its images. Recalling the proof of Proposition 5.1, for this choice of δ we have $\pi_{\beta_2,\beta_3}(B(\omega_i^1)) > 1/2$ and $\pi_{\beta_2,\beta_3}(B(\omega_i^{-1})) < -1/2$; as such the above interval is well defined and non-trivial. We will now show that this interval is contained within the fibre $\Lambda^x_{\beta_1,\beta_2,\beta_3}$. Let us fix

$$y \in [\pi_{\beta_2,\beta_3}(B(\lambda^0, (\omega_i^{-1})_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B(\lambda^0, (\omega_i^1)_{i=1}^{\infty}))].$$

There are two cases to consider: either

$$y \in [\pi_{\beta_2,\beta_3}(B(\lambda^0, (\omega_i^{-1})_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B(\lambda^0, \omega_1^{-1}, (\omega_i^1)_{i=1}^{\infty}))]$$

or

$$y \in [\pi_{\beta_2,\beta_3}(B(\lambda^0, \omega_1^{-1}, (\omega_i^1)_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B(\lambda^0, (\omega_i^1)_{i=1}^{\infty}))].$$

The first interval is well defined and non-trivial by the same reasoning as that given above. The second interval is not necessarily well defined. However, when it is not well defined, that is, $\pi_{\beta_2,\beta_3}(B(\lambda^0, \omega_1^{-1}, (\omega_i^1)_{i=1}^{\infty})) > \pi_{\beta_2,\beta_3}(B(\lambda^0, (\omega_i^1)_{i=1}^{\infty}))$, y is contained in the first

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interval. As such we can overlook this technicality. In the first case we let $\lambda^1 = (\lambda^0, \omega^{-1})$, in the second case we let $\lambda^1 = (\lambda^0, \omega^1)$. For the first case it is immediate that

$$y \in [\pi_{\beta_2,\beta_3}(\lambda^1, (\omega_i^{-1})_{i=1}^{\infty}), \pi_{\beta_2,\beta_3}(\lambda^1, (\omega_i^1)_{i=1}^{\infty})].$$

By Corollary 5.2 we know that

$$\pi_{\beta_2,\beta_3}(B(\lambda^0, \omega^1, (\omega_i^{-1})_{i=1}^{\infty})) < \pi_{\beta_2,\beta_3}(B(\lambda^0, \omega^{-1}, (\omega_i^1)_{i=1}^{\infty})).$$

Therefore for the second case we also have

$$y \in [\pi_{\beta_2,\beta_3}(\lambda^1, (\omega_i^{-1})_{i=1}^{\infty}), \pi_{\beta_2,\beta_3}(\lambda^1, (\omega_i^1)_{i=1}^{\infty})].$$

Now suppose that we have constructed a sequence $\lambda^k \in \{T_{-1}, T_1\}^*$ such that

$$y \in [\pi_{\beta_2,\beta_3}(B(\lambda^k, (\omega_i^{-1})_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B(\lambda^k, (\omega_i^1)_{i=1}^{\infty}))]. \tag{5.7}$$

We now show how to construct λ^{k+1} satisfying (5.7). Again there are two cases to consider: either

$$y \in [\pi_{\beta_2,\beta_3}(B(\lambda^k, (\omega_i^{-1})_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B(\lambda^k, \omega_1^{-1}, (\omega_i^1)_{i=1}^{\infty}))]$$

or

$$y \in [\pi_{\beta_2,\beta_3}(B(\lambda^k, \omega_1^{-1}, (\omega_i^1)_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B(\lambda^k, (\omega_i^1)_{i=1}^{\infty}))].$$

The first interval is still well defined and non-trivial. The second interval is not necessarily well defined but this technicality can be overlooked for the same reason as that given before. In the first case we take $\lambda^{k+1} = (\lambda^k, \omega^{-1})$, then we automatically have

$$y \in [\pi_{\beta_2,\beta_3}(B(\lambda^{k+1}, (\omega_i^{-1})_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B(\lambda^{k+1}, (\omega_i^1)_{i=1}^{\infty}))].$$

In the second case we take $\lambda^{k+1} = (\lambda^k, \omega^1)$. Applying Corollary 5.2 as above, we then have

$$y \in [\pi_{\beta_2,\beta_3}(B(\lambda^{k+1}, (\omega_i^{-1})_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B(\lambda^{k+1}, (\omega_i^1)_{i=1}^{\infty}))].$$

Thus we have completed our inductive step.

Continuing in this manner yields an infinite sequence $\lambda \in \Omega_{\beta_1}(x)$. Since the diameter of the interval appearing in (5.7) tends to zero as $k \to \infty$, it follows that

$$y = \pi_{\beta_2,\beta_3}(B(\lambda)).$$

Since y was arbitrary it follows that

$$[\pi_{\beta_2,\beta_3}(B(\lambda^0, (\omega_i^{-1})_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B(\lambda^0, (\omega_i^1)_{i=1}^{\infty}))] \subseteq \pi_{\beta_2,\beta_3}(B(\widetilde{\Omega}_{\beta_1}(x))).$$

By (5.1) and Lemma 3.1 it follows that

$$[\pi_{\beta_2,\beta_3}(B(\lambda^0, (\omega_i^{-1})_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B(\lambda^0, (\omega_i^1)_{i=1}^{\infty}))] \subseteq \Lambda^x_{\beta_1,\beta_2,\beta_3}$$

as required.

To see that $(0, 0) \in \Lambda^0_{\beta_1,\beta_2,\beta_3}$, we remark that if $x \in \widetilde{\mathcal{O}}_{\beta_1}$ then we do not require the initial map λ^0 which maps x into $\widetilde{\mathcal{O}}_{\beta_1}$. Consequently, for every $x \in \widetilde{\mathcal{O}}_{\beta_1}$ the fibre $\Lambda^x_{\beta_1,\beta_2,\beta_3}$ contains the interval

$$[\pi_{\beta_2,\beta_3}(B((\omega_i^{-1})_{i=1}^{\infty})), \pi_{\beta_2,\beta_3}(B((\omega_i^1)_{i=1}^{\infty}))].$$

By Proposition 5.1 this interval contains a neighbourhood of zero. Since 0 is contained in the interior of $\widetilde{\mathcal{O}}_{\beta_1}$ it follows that $(0, 0) \in \Lambda^0_{\beta_1,\beta_2,\beta_3}$. □

TABLE 1. A partition of the interval $\mathcal{O}_{\beta^*}$ and the corresponding ω^0 and ω^1.

Interval	ω^0	Interval	ω^1
$[0.872, \ldots, 0.959\ldots]$	(T_1, T_0, T_0, T_0)	$[1.188, \ldots, 1.276\ldots]$	(T_0, T_1, T_1, T_1)
$[0.959, \ldots, 1.087\ldots]$	(T_1, T_0, T_0)	$[1.061, \ldots, 1.188\ldots]$	(T_0, T_1, T_1)
$[1.087, \ldots, 1.128\ldots]$	$(T_1, T_0, T_1, T_0, T_0, T_0)$	$[1.020, \ldots, 1.061\ldots]$	$(T_0, T_1, T_0, T_1, T_1, T_1)$
$[1.128, \ldots, 1.188\ldots]$	$(T_1, T_0, T_1, T_0, T_0)$	$[0.960, \ldots, 1.020\ldots]$	$(T_0, T_1, T_0, T_1, T_1)$
$[1.188, \ldots, 1.208\ldots]$	$(T_1, T_1, (T_0)^6)$	$[0.940, \ldots, 0.960\ldots]$	$(T_0, T_0, (T_1)^6)$
$[1.208, \ldots, 1.236\ldots]$	$(T_1, T_1, (T_0)^5)$	$[0.912, \ldots, 0.940\ldots]$	$(T_0, T_0, (T_1)^5)$
$[1.236, \ldots, 1.276\ldots]$	$(T_1, T_1, (T_0)^4)$	$[0.872, \ldots, 0.912\ldots]$	$(T_0, T_0, (T_1)^4)$

6. *An explicit calculation*

In this section we fix $\beta^* \approx 1.4656$, the appropriate root of $x^3 - x^2 - 1 = 0$. A simple calculation yields

$$\mathcal{O}_{\beta^*} = [0.872\ldots, 1.276\ldots].$$

In Table 1 we include a list of intervals that partition $\mathcal{O}_{\beta^*}$ along with the corresponding sequences ω^0 and ω^1 which satisfy the conclusions of Proposition 3.3 for those elements within each interval.

Upon examination of Table 1 we observe that $|\omega^0| \le 8$ and $|\omega^1| \le 8$ for all ω^0 and ω^1. As such we can take $n(\beta^*) = 8$. It follows from Proposition 4.1 that for any $x \in (0, 1/(\beta^* - 1))$ and $p \in (7/16, 9/16)$, there exists an expansion of x in base β^* such that the digit zero occurs with frequency p.

We now consider Theorem 2.8 and show how one can explicitly calculate the parameter δ appearing in its statement. Note that

$$\widetilde{\mathcal{O}}_{\beta^*} = [-0.403\ldots, 0.403\ldots].$$

We start by pointing out Table 2. This table lists a collection of intervals that partition $\widetilde{\mathcal{O}}_{\beta^*}$ along with the corresponding sequences ω^{-1} and ω^1 which satisfy the conclusions of Proposition 3.3 for those elements within each interval. We remark that Table 2 can be obtained from Table 1 by a simple change of coordinates.

The crucial step in the proof of Theorem 2.8 is Proposition 5.1. The δ appearing in this statement is the same δ appearing in the statement of Theorem 2.8. As such, to determine a δ so that the conclusions of Theorem 2.8 are satisfied, we need to calculate a δ such that if $\beta_2, \beta_3 \in (1, 1+\delta)$ and $x \in \widetilde{\mathcal{O}}_{\beta^*}$ then

$$\pi_{\beta_2,\beta_3}(B(\omega^1, (\omega_i^{-1})_{i=1}^{\infty})) < 0 \quad \text{and} \quad 0 < \pi_{\beta_2,\beta_3}(B(\omega^{-1}, (\omega_i^1)_{i=1}^{\infty})). \tag{6.1}$$

Let

$$A_{-1} = \{\omega^{-1} : \omega^{-1} \text{ appears in Table 2}\} \quad \text{and} \quad A_1 = \{\omega^1 : \omega^1 \text{ appears in Table 2}\}.$$

We will explicitly construct a δ such that if $\beta_2, \beta_3 \in (1, 1+\delta)$ then

$$\pi_{\beta_2,\beta_3}(B(a, (b_i)_{i=1}^{\infty})) < 0 \quad \text{and} \quad 0 < \pi_{\beta_2,\beta_3}(B(c, (d_i)_{i=1}^{\infty})), \tag{6.2}$$

for any $a \in A_1$ and $(b_i) \in A_{-1}^{\mathbb{N}}$, and for any $c \in A_{-1}$ and $(d_i) \in A_1^{\mathbb{N}}$. Clearly (6.2) implies (6.1).

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TABLE 2. A partition of the interval $\widetilde{\mathcal{O}}_{\beta^*}$ and the corresponding ω^{-1} and ω^1.

Interval	ω^{-1}	Interval	ω^1
$[-0.403, \ldots, -0.229, \ldots]$	$(T_1, T_{-1}, T_{-1}, T_{-1})$	$[0.229, \ldots, 0.403, \ldots]$	(T_{-1}, T_1, T_1, T_1)
$[-0.229, \ldots, 0.026, \ldots]$	(T_1, T_{-1}, T_{-1})	$[-0.026, \ldots, 0.229, \ldots]$	(T_{-1}, T_1, T_1)
$[0.026, \ldots, 0.108, \ldots]$	$(T_1, T_{-1}, T_1, T_{-1}, T_{-1}, T_{-1})$	$[-0.108, \ldots, -0.026, \ldots]$	$(T_{-1}, T_1, T_{-1}, T_1, T_1, T_1)$
$[0.108, \ldots, 0.228, \ldots]$	$(T_1, T_{-1}, T_1, T_{-1}, T_{-1})$	$[-0.228, \ldots, -0.108, \ldots]$	$(T_{-1}, T_1, T_{-1}, T_1, T_1)$
$[0.228, \ldots, 0.268, \ldots]$	$(T_1, T_1, (T_{-1})^6)$	$[-0.268, \ldots, -0.228, \ldots]$	$(T_{-1}, T_{-1}, (T_1)^6)$
$[0.268, \ldots, 0.324, \ldots]$	$(T_1, T_1, (T_{-1})^5)$	$[-0.324, \ldots, -0.268, \ldots]$	$(T_{-1}, T_{-1}, (T_1)^5)$
$[0.324, \ldots, 0.403, \ldots]$	$(T_1, T_1, (T_{-1})^4)$	$[-0.403, \ldots, -0.324, \ldots]$	$(T_{-1}, T_{-1}, (T_1)^4)$

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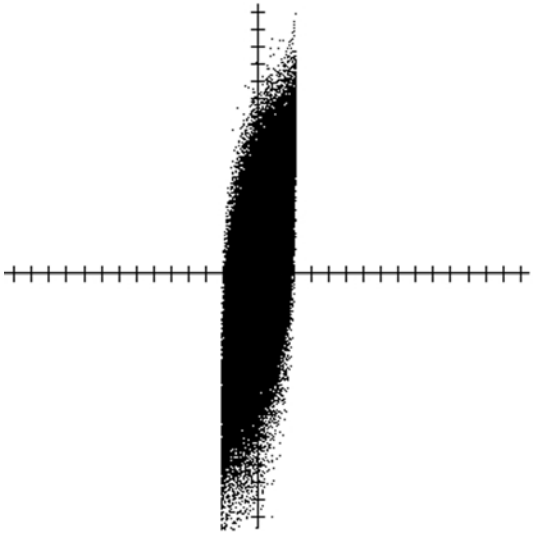

FIGURE 4. A plot of $\Lambda_{\beta^*,1.03,1.04}$. For each $x \in (-1/(\beta^*-1), 1/(\beta^*-1))$ the fibre $\Lambda^x_{\beta^*,1.03,1.04}$ contains an interval.

The following lemma makes determining a δ for which (6.2) holds straightforward.

LEMMA 6.1. *Let $\mathcal{D} = \{\kappa_l\} \subseteq \{-1, 1\}^*$ be a finite set consisting of strings of digits (possibly of different lengths). Then*

$$\min_l \pi_{\beta_2,\beta_3}((\kappa_l)^\infty) \leq \pi_{\beta_2,\beta_3}((a_i)_{i=1}^\infty) \leq \max_l \pi_{\beta_2,\beta_3}((\kappa_l)^\infty)$$

for any $(a_i) \in \mathcal{D}^\mathbb{N}$.

The proof of Lemma 6.1 is straightforward and therefore omitted.

It is a consequence of Lemma 6.1 that to calculate a δ such that (6.2) holds, it suffices to determine a δ such that for all $\beta_2, \beta_3 \in (1, 1+\delta)$ we have

$$\max_{a\in A_1, b\in A_{-1}} \pi_{\beta_2,\beta_3}(B(a, (b)^\infty)) < 0 \quad \text{and} \quad 0 < \min_{c\in A_{-1}, d\in A_1} \pi_{\beta_2,\beta_3}(B(c, (d)^\infty)). \tag{6.3}$$

Since there are only finitely many elements in A_{-1} and A_1, to determine a δ for which (6.3) holds one only has to consider finitely many inequalities. Inputting each of these inequalities into a computer yields $\delta = 0.041$. Consequently, if $\beta_2, \beta_3 \in (1, 1.041)$ then (6.3) holds and by Proposition 5.1 and Theorem 2.8 the fibre $\Lambda^x_{\beta^*,\beta_2,\beta_3}$ contains an interval for all $x \in (-1/(\beta^*-1), 1/(\beta-1))$. In Figure 4 we include a plot of $\Lambda_{\beta^*,1.03,1.04}$.

7. *Remarks*

We finish this paper by making some remarks and posing questions.

Remark 7.1. Note that when $\beta = (1+\sqrt{5})/2$ it can be shown that

$$\Sigma_{(1+\sqrt{5})/2}(1) = \{(10)^\infty, ((10)^k 0(1)^\infty), (10)^k 11(0)^\infty : k \geq 0\}.$$

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Moreover, for $\beta \in ((1+\sqrt{5})/2, 2)$ it can be shown that

$$\Sigma_\beta\left(\frac{\beta}{\beta^2-1}\right) = \{(10)^\infty\}.$$

Consequently, we see that statements 2 and 3 from Theorem 2.1 cannot be extended past the parameter $(1+\sqrt{5})/2$. Thus these statements are optimal.

Similarly, for the digit set $\{-1, 1\}$ one can construct non-trivial x such that $\widetilde{\Omega}_{(1+\sqrt{5})/2}(x)$ is infinite countable, and for $\beta_1 \in ((1+\sqrt{5})/2, 2)$ examples of non-trivial x for which $\widetilde{\Omega}_{(1+\sqrt{5})/2}(x)$ is a singleton set. For these particular choices of x it is clear that the vertical fibre $\Lambda^x_{\beta_1,\beta_2,\beta_3}$ cannot contain an interval. Consequently, one cannot improve upon the interval $(1, (1+\sqrt{5})/2)$ appearing in the statement of Theorem 2.8.

Remark 7.2. In this paper we have only considered simply normal expansions. It is natural to wonder about normal expansions. Recall that a sequence $(\epsilon_i) \in \{0, 1\}^{\mathbb{N}}$ is normal if for every finite block $(\delta_1, \ldots, \delta_k) \in \{0, 1\}^*$ we have

$$\lim_{n\to\infty} \frac{\#\{1 \le i \le n : \epsilon_i = \delta_1, \ldots, \epsilon_{i+k-1} = \delta_k\}}{n} = \frac{1}{2^k}.$$

A natural question to ask is whether there exists $c > 0$ such that if $\beta \in (1, 1+c)$ then every $x \in (0, 1/(\beta-1))$ has a normal expansion. This question was originally posed to the author by Kempton [**23**]. The author suspects that such a c does not exist, but we cannot prove this. A natural obstruction to proving the non-existence of such a c is that there exists $c' > 0$ such that for any $\beta \in (1, 1+c')$, every $x \in (0, 1/(\beta-1))$ has a β-expansion that contains all finite blocks of digits. The existence of such a c' was originally proved by Erdős and Komornik [**16**]. Consequently, to prove the non-existence of such a c one would have to prove that for every β sufficiently close to one, there exists an $x \in (0, 1/(\beta-1))$ such that for every $(\epsilon_i) \in \Sigma_\beta(x)$ there exists a block of digits which do not occur with the desired frequency. This appears to be a difficult problem.

Remark 7.3. In Theorem 2.2 we proved that

$$\lim_{\beta \nearrow 2} \dim_H(\{x : x \text{ has no simply normal } \beta\text{-expansion}\}) = 1.$$

It is natural to ask whether

$$\dim_H(\{x : x \text{ has no simply normal } \beta\text{-expansion}\}) < 1$$

for all $\beta \in (1, 2)$. A solution to this question would likely involve the study of those x for which $\Sigma_\beta(x)$ is uncountable yet every element of $\Sigma_\beta(x)$ fails to be simply normal. Studying this set appears to be a difficult task. Indeed for β close to 2 it is unclear whether this set is non-empty.

Remark 7.4. In the proof of Theorem 2.8 we explicitly constructed an interval appearing in the fibre $\Lambda^x_{\beta_1,\beta_2,\beta_3}$. For each y in this interval we construct a $\lambda \in \widetilde{\Omega}_{\beta_1}(x)$ such that $\pi_{\beta_2,\beta_3}(B(\lambda)) = y$. The method by which we construct this λ bears a strong resemblance to the way one normally constructs β-expansions. The author wonders whether a refinement of the argument given in the proof of Proposition 3.3 would yield an algorithm by which we can construct many $\lambda \in \widetilde{\Omega}_{\beta_1}(x)$ such that $\pi_{\beta_2,\beta_3}(B(\lambda)) = y$, and for which we have a lot of control over the frequency of the T_{-1}s and T_1s that appear in λ. With such an algorithm the author expects one could adapt the proof of Theorem 2.8 to give new examples of self-affine sets in three dimensions with non-empty interior. Possibly this method could be extended to n-dimensional self-affine sets.

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