INFLUENCE IN PRODUCT SPACES

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Abstract

The theory of influence and sharp threshold is a key tool in probability and probabilistic combinatorics, with numerous applications. One significant aspect of the theory is directed at identifying the level of generality of the product probability space that accommodates the event under study. We derive the influence inequality for a completely general product space, by establishing a relationship to the Lebesgue cube studied by Bourgain, Kahn, Kalai, Katznelson and Linial (BKKKL) in 1992. This resolves one of the assertions of BKKKL. Our conclusion is valid also in the setting of the generalized influences of Keller.

Keywords: Influence; sharp threshold; product space; separable space; measure-space isomorphism

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1. Introduction

A coin shows heads with probability p. We flip it n times, and we observe whether or not some specified event A occurs. In studying the associated probability $P_p(A)$, it is often useful to gain information about the degrees of influence of the individual coin tosses. We make this statement more precise as follows.

Let $(X_e : e \in E)$ be independent Bernoulli variables with parameter p, where $|E| = n < \infty$. Let $A \subseteq \Omega$, where $\Omega = \{0, 1\}^E$. For $\omega \in \Omega$ and $e \in E$, we define the configurations ω^e and ω_e by

$$\omega^e(f) = \begin{cases} \omega(f) & \text{if } f \neq e, \\ 1 & \text{if } f = e, \end{cases} \qquad \omega_e(f) = \begin{cases} \omega(f) & \text{if } f \neq e, \\ 0 & \text{if } f = e. \end{cases}$$

Thus, the configuration ω^e (respectively, ω_e) is derived from ω by 'switching on' (respectively, 'switching off') the variable indexed by e. The *influence* of $e \in E$ on the event A is defined by

$$I_A(e) = P_p(1_A(\omega^e) \neq 1_A(\omega_e)),$$

where 1_A denotes the indicator function of A, and P_p is the appropriate probability measure. That is, the influence of e is the probability that the occurrence of A depends on the value of X_e.

A systematic theory of influence seems to have been developed first by Kahn, Kalai and Linial [12] in 1988, in response to an issue raised by Ben-Or and Linial [2]. There was a later development by Talagrand [21] in 1994. On the other hand, estimates for influences have been key to a number of important results in probability and probabilistic combinatorics that predate these papers, sometimes by many years. Perhaps the most famous such result is the proof by Kesten [16] that the critical probability of bond percolation on the square lattice equals $\frac{1}{2}$.

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There are now several known ways of proving this (see [8, Chapter 5] and [4]), but Kesten's first proof of 1980 used a bespoke theory of influence.

Kahn, Kalai and Linial [12] introduced an inequality for influences in the case $p = \frac{1}{2}$, working thus with uniform measure on the discrete cube $\{0, 1\}^n$. This was extended by Bourgain *et al.* [3] to an influence inequality for the continuous cube $[0, 1]^n$ endowed with Lebesgue measure. Using a discretization argument, this implies an influence inequality for the Bernoulli case with $p \in (0, 1)$. This following formulation of this inequality is a minor perturbation of that of [3] and [12], and is given here in a form suitable for applications (see [8, Theorem 4.29]).

Theorem 1.1. *There exists a universal constant $c > 0$ such that, for any $p \in (0, 1)$, any finite set E, and any event $A \subseteq \{0, 1\}^E$ satisfying $P_p(A) \in (0, 1)$,*

$$\sum_{e \in E} I_A(e) \geq c P_p(A)(1 - P_p(A)) \log\left[\frac{1}{m}\right], \tag{1.1}$$

where $m = \max_e I_A(e)$.

It is immediate that (1.1) implies the existence of some $e \in E$ with

$$I_A(e) \geq c' P_p(A)(1 - P_p(A)) \frac{\log n}{n}, \tag{1.2}$$

where $n = |E|$ and $c' > 0$ is an absolute constant.

There is a slightly extended version of inequality (1.1) due to Talagrand [21], which holds under the further condition that the event in question is *increasing*. Since the set $\{0, 1\}$ is ordered, the product space $\{0, 1\}^E$ is partially ordered. An event A in this space is said to be *increasing* if, whenever $\omega \in A$, $\omega \leq \omega'$, then $\omega' \in A$. It is proved in [21, Theorem 1.1] that (1.1) may be replaced by

$$P_p(A)(1 - P_p(A)) \leq cp(1 - p) \log\left[\frac{2}{p(1 - p)}\right] \sum_{e \in E} \frac{I_A(e)}{\log[1/(p(1 - p)I_A(e))]} \tag{1.3}$$

for an increasing event A. Using the fact that $I_A(e) \leq m := \max_e I_A(e)$, inequality (1.3) implies that

$$\sum_{e \in E} I_A(e) \geq \left(\frac{c^{-1}}{p(1 - p) \log[2/(p(1 - p))]}\right) P_p(A)(1 - P_p(A)) \log\left[\frac{1}{m}\right]. \tag{1.4}$$

Since $0 < p < 1$, it follows that

$$\sum_{e \in E} I_A(e) \geq c' P_p(A)(1 - P_p(A)) \log\left[\frac{1}{m}\right], \tag{1.5}$$

where $c' > 0$ is an absolute constant, in agreement with (1.1) (and assuming A is increasing).

The connection between the influences $I_A(e)$ and the probability $P_p(A)$ is provided by *Russo's formula*,

$$\frac{\mathrm{d}}{\mathrm{d}p} P_p(A) = \sum_{e \in E} I_A(e) \tag{1.6}$$

for any increasing event A. Russo [20] published his formula in 1978, though versions of this natural equality were known earlier to Barlow and Proschan [1, p. 210] and Margulis [17].

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Russo's formula (1.6) may be combined with (1.4) or (1.5) to obtain lower bounds for the derivative of $P_p(A)$ for an increasing event A. Numerous applications of this inequality have been found in areas such as percolation and random graphs.

Since these three early papers [3], [12], [21] on influence, several strands of theory have been developed. One is to seek influence theorems for nonproduct measures, for which we refer the reader to [6] and [7]. Another is towards the question of whether there exists a useful influence inequality for an event in an arbitrary product space, that is, whether an inequality of the form (1.5) holds with the discrete product space $\{0, 1\}^E$ replaced by an arbitrary product probability space. It was asserted in [3] that the latter is indeed true, but the explanation was omitted (a natural argument uses the measure-space isomorphism theorem, which normally requires separability; see Section 3.3). The purpose of the current note is to state and prove a general form of this theorem not requiring separability (see Theorems 2.1 and 2.3).

See [13] for a review of influence and its ramifications, and also [5] and [8, Section 4.5].

2. Statement of results

Let $X = (\Omega, \mathcal{F}, P)$ be a probability space, and let E be a finite set with $|E| = n$. We write $X^E = (\Omega^E, \mathcal{F}^E, \mathbb{P} = P^E)$ for the product space of n copies of X. For an index $e \in E$ and a vector $\psi \in \Omega^{E\setminus\{e\}}$, we define the *fibre*

$$F_\psi = \{\omega \in \Omega^E : \omega(f) = \psi(f) \text{ for } f \neq e\} \simeq \{\psi\} \times \Omega,$$

comprising all $\omega \in \Omega^E$ which agree with ψ off e.

Let $A \in \mathcal{F}^E$ be an event. The *influence* of e on A is defined as

$$I_A(e) = P^{E\setminus\{e\}}(\{\psi \in \Omega^{E\setminus\{e\}} : 0 < P(A \cap F_\psi) < 1\}). \tag{2.1}$$

For economy of notation, the space X is not listed explicitly in $I_A(e)$.

Remark 2.1. Bourgain *et al.* [3] made use of a different definition of influence, which may be expressed in the current context as

$$I'_A(e) = P^{E\setminus\{e\}}(1_A \text{ is not constant on } F_\psi).$$

By comparison with (2.1), we have $I_A(e) \leq I'_A(e)$. Therefore, lower bounds for $I_A(e)$ are stronger than their equivalents for $I'_A(e)$.

An unsatisfactory property of the influence $I'_A(e)$ is that one may have $I'_A(e) \neq I'_{A'}(e)$ for events A and A' that differ by a null set. This observation provoked the revised definition (2.1) introduced in [8]. More general notions of influence have been discussed in [10], [14], and [15], to which we return at (2.2).

Let $\mathcal{L}$ denote the Lebesgue probability space comprising the unit interval $[0, 1]$ endowed with the Borel σ-field $\mathcal{B}[0, 1]$ and Lebesgue measure λ. Our main result for influences as defined in (2.1) is the following. This will be extended to more general influences in Theorem 2.3.

Theorem 2.1. *Let $|E| < \infty$ and $A \in \mathcal{F}^E$. There exists a measurable event B in the Lebesgue product space $\mathcal{L}^E$ such that $\lambda^E(B) = \mathbb{P}(A)$, and $I_B(e) = I_A(e)$ for $e \in E$.*

It follows that the influences of an arbitrary event in the general product space satisfy an inequality whenever such an inequality holds for a general event in the Lebesgue product space. Since X is not generally a partially ordered set, it would be inappropriate to seek results

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restricted to increasing events, and in addition the method of proof will not necessarily respect an existing partial order.

We state one corollary of Theorem 2.1, which may be compared with Theorem 1.1. The proof is presented at the end of Section 4.

Theorem 2.2. *There exists a universal constant $c > 0$ such that, for any probability space $X = (\Omega, \mathcal{F}, P)$, any finite set E, and any event $A \in \mathcal{F}^E$ satisfying $\mathbb{P}(A) \in (0, 1)$,*

$$\sum_{e\in E} I_A(e) \geq c\mathbb{P}(A)(1-\mathbb{P}(A)) \log\left[\frac{1}{m}\right],$$

where $m = \max_e I_A(e)$.

It is immediate, as at (1.2), that there exists $e \in E$ with

$$I_A(e) \geq c'\mathbb{P}(A)(1-\mathbb{P}(A))\frac{\log n}{n},$$

where $n = |E|$ and $c' > 0$ is an absolute constant. By Remark 2.1, this is stronger than Bourgain *et al.*'s [3, Theorem 1].

Our principal Theorem 2.1 may be extended without substantial extra work to a more general notion of influence, introduced by Keller [14]. Let $\mathcal{M}$ be the set of measurable functions $h\colon [0,1] \to [0,1]$. For $h \in \mathcal{M}$, the *h-influence* of $e \in E$ on the event $A \in \mathcal{F}^E$ is defined as

$$I_A^h(e) = P^{E\setminus\{e\}}(h(P(A\cap F_\psi))), \tag{2.2}$$

where $\mu(f)$ denotes the expectation of f under the probability measure μ. Thus, $I_A^h(e) = I_A(e)$ when h is the indicator function $1_{(0,1)}$. The function $h(x) = x(1-x)$ has been considered in [10], and other functions h in [14].

One might define the influence $I_A(e)$ via a conditional expectation rather than the 'pointwise' definitions (2.1) and (2.2). With $\mathcal{F}_e^E$ the sub-σ-field of $\mathcal{F}^E$ generated by $\{\omega(f)\colon f \neq e\}$, (2.2) can be written as

$$I_A^h(e) = P^{E\setminus\{e\}}(h(\mathbb{P}(A \mid \mathcal{F}_e^E))).$$

However, we retain the notation adopted in the prior literature.

Our main theorem for h-influences is as follows.

Theorem 2.3. *Let $h \in \mathcal{M}$ and $A \in \mathcal{F}^E$. There exists a measurable event B in the Lebesgue product space $\mathcal{L}^E$ such that $\lambda^E(B) = \mathbb{P}(A)$, and $I_B^h(e) = I_A^h(e)$ for $e \in E$.*

This extends Theorem 2.1, and yields a positive answer to a question of Keller [14, Footnote 2], asking whether h-influence inequalities may be extended from Lebesgue to general spaces. Theorem 2.3 includes Theorem 2.1, and its proof is presented in Section 4.

3. Discussion

Rather than include here a full discussion of influence and sharp threshold, we draw the attention of the reader to three relevant points.

3.1. Borel or Lebesgue?

We have made no assumption above about the completeness (or not) of the probability space $X^E = (\Omega^E, \mathcal{F}^E, \mathbb{P})$. For events $A, B \in \mathcal{F}^E$ such that $P(A \triangle B) = 0$, we have, from (2.1) and Fubini's theorem, $I_A(e) = I_B(e)$ for $e \in E$. It follows that, when working with definition (2.1), one may use either the product σ-field $\mathcal{F}^E$ or its completion.

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3.2. Form of inequality

There exists a family of influence inequalities, from which one may select one according to the situation under study. By Theorems 2.1 and 2.3, any inequality that is valid for the Lebesgue space has a parallel inequality for a general product space. In these two theorems, no assumption is made of *monotonicity* of the event in question, or about its *invariance* under a group of actions on Ω^E.

3.3. General probability spaces

The probability space of possibly greatest practical value for applications is the Lebesgue space $\mathcal{L}^E$, since many spaces of importance, including the Bernoulli product spaces, may be derived via mappings on $\mathcal{L}^E$. It was implied by Bourgain *et al.* [3] that influence inequalities for an *arbitrary* product space may be derived from those for $\mathcal{L}^E$. A natural route to a proof of such a statement would be to use the measure-space isomorphism theorem (see, for example, [9, Section 40], [11, Appendix A], or [18, Theorem 4.7]). In its usual form, the last theorem places a restriction of separability on the probability space after removal of atoms, and this limits its naïve application in the current situation. The *separable* case is discussed in [8, Section 4.5].

Some probabilists tend to consider nonseparable probability spaces with only limited enthusiasm. The current note was inspired by a desire to understand the assertion of [3], and to resolve a slightly obscure corner of probability theory.

4. Proof of Theorem 2.3

The proof of Theorem 2.3 is achieved via the three lemmas that follow. For probability spaces $X_i = (\Omega_i, \mathcal{F}_i, P_i)$, a mapping $\phi\colon \Omega_1 \to \Omega_2$ is said to be *measure preserving* (from X_1 to X_2) if, for all $B_2 \in \mathcal{F}_2$, the inverse image $B_1 = \phi^{-1}(B_2)$ is measurable and satisfies $P_1(B_1) = P_2(B_2)$.

For a finite set E and a measure-preserving mapping ϕ, the function $\Phi = \phi^E$ is the measure-preserving mapping from X_1^E to X_2^E given by $\Phi((x_e\colon e \in E)) = (\phi(x_e)\colon e \in E)$.

Lemma 4.1. *Let $X_i = (\Omega_i, \mathcal{F}_i, P_i)$, $i = 1, 2$, be probability spaces, and let $\phi\colon \Omega_1 \to \Omega_2$ be measure preserving. Let E be a finite set, and write $\Phi = \phi^E$ as above. If $B_2 \in \mathcal{F}_2^E$ and $B_1 = \Phi^{-1}(B_2)$, then $I_{B_1}^h(e) = I_{B_2}^h(e)$ for all $e \in E$ and $h \in \mathcal{M}$.*

Proof. Let $e \in E$, $h \in \mathcal{M}$, $B_2 \in \mathcal{F}_2$, and $B_1 = \Phi^{-1}(B_2)$. For $\psi \in \Omega_i^{E\setminus\{e\}}$, let F_ψ be the fibre

$$F_\psi = \{\omega \in \Omega_i^E : \omega(f) = \psi(f) \text{ for } f \neq e\} \cong \{\psi\} \times \Omega_i.$$

Suppose that $\nu \in \Omega_1^{E\setminus\{e\}}$ and $\psi \in \Omega_2^{E\setminus\{e\}}$ satisfy $\phi^{E\setminus\{e\}}(\nu) = \psi$. Since ϕ is measure preserving on each component,

$$P_1(\{\nu\} \times \phi^{-1}(B_2 \cap F_\psi)) = P_2(B_2 \cap F_\psi).$$

Now $\{\nu\} \times \phi^{-1}(B_2 \cap F_\psi) = B_1 \cap F_\nu$, so that, for $u \in \mathbb{R}$,

$$P_1^{E\setminus\{e\}}(h(P_1(B_1 \cap F_\nu)) > u) = P_2^{E\setminus\{e\}}(h(P_2(B_2 \cap F_\psi)) > u).$$

We integrate over $u \in [0, \infty)$ to complete the proof. □

A σ-field of subsets of a set Ω is called *countably generated* (or *separable*) if it is generated by some finite or countably infinite collection of subsets of Ω.

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Lemma 4.2. *Let $X = (\Omega, \mathcal{F}, P)$, $|E| < \infty$, and let $A \in \mathcal{F}^E$. There exists a countably generated sub-σ-field $\mathcal{G}$ of $\mathcal{F}$ such that $A \in \mathcal{G}^E$.*

Proof. Let $\{\mathcal{G}_i : i \in I\}$ be the set of all countably generated sub-σ-fields of the σ-field $\mathcal{F}$, and let $\mathcal{H}$ be the union of $\mathcal{G}_i^E$ as i ranges over I. It is easy to see that $\mathcal{H}$ is a σ-field. (To see closure under countable unions, let $A_i \in \mathcal{H}$ for $i = 1, 2, \ldots$. Then $A_i \in \mathcal{G}_{j(i)}^E$ for some $j(i)$. Let $\mathcal{G}_j$ be generated by the countable subset $\mathcal{B}_j$ of $\mathcal{F}$, and let $\mathcal{B} = \bigcup_i \mathcal{B}_{j(i)}$. Then $\mathcal{B}$ is countable, and generates thus some $\mathcal{G}_k$. Hence, $A_i \in \mathcal{G}_{j(i)}^E \subseteq \mathcal{G}_k^E$ for each i, so that $\bigcup_i A_i \in \mathcal{G}_k^E \subseteq \mathcal{H}$.) Furthermore, $\mathcal{H}$ is the smallest σ-field containing every *rectangle* of the form $\prod_{e \in E} F_e$, as the F_e range over $\mathcal{F}$. Therefore, $\mathcal{H} = \mathcal{F}^E$.

Let $A \in \mathcal{F}^E$. Since $A \in \mathcal{H}$, then there exists $a \in I$ such that $A \in \mathcal{G}_a^E$. The proof is complete. □

The remainder of the proof is based upon a concealed version of the measure-space isomorphism theorem. In general terms, this states that (subject to appropriate assumptions) a measure space may be placed in correspondence with the Lebesgue space $\mathcal{L}$. There are two forms of the measure-space isomorphism theorem.

(a) There exists an isomorphism between the measure rings of the measure space and the Lebesgue space (see, for example, [9, Section 40]).

(b) There exists a pointwise bijection between certain derived sample spaces (see, for example, [18, Theorem 4.7]).

We will not appeal to any general theorem here, but instead will construct the required mappings explicitly in a manner requiring no special consideration of the existence (or not) of atoms. This may be achieved either by repeated decimation of subintervals of [0, 1] (see, for example, [19, Section 2.2]), or by way of a mapping to the Cantor set. We choose to follow the second route here. See [11, Appendix A] for a discussion of measure-space isomorphisms.

For $T \subseteq \mathbb{R}^d$, we denote the Borel σ-field of T by $\mathcal{B}(T)$. Let C be the Cantor set of all reals of the form

$$\sum_{k=1}^{\infty} \frac{2}{3^k} a_k, \qquad (a_k : k \in \mathbb{N}) \in \{0, 1\}^{\mathbb{N}}.$$

We shall make use of the fact that C is in one-to-one correspondence with $\{0, 1\}^{\mathbb{N}}$.

Lemma 4.3. *Let $A \in \mathcal{F}^E$, and let $\mathcal{G}$ be a countably generated sub-σ-field of $\mathcal{F}$ such that $A \in \mathcal{G}^E$ (as in Lemma 4.2). There exists a probability space $Z = (C, \mathcal{B}(C), \mu)$ comprising the Cantor set C together with its Borel σ-field and a suitable probability measure μ, such that the following assertions hold.*

(a) *There exists a measure-preserving mapping ψ from X to Z.*

(b) *There exists $G \in \mathcal{B}(C^E)$ such that $A = \Psi^{-1}(G)$, where $\Psi = \psi^E$.*

(c) *There exists a measure-preserving mapping γ from $\mathcal{L}$ to Z.*

This lemma (together with part of the forthcoming proof of Theorem 2.3) may be summarized in the diagrams

$$X \xrightarrow{\psi} Z \xleftarrow{\gamma} \mathcal{L}, \qquad A \xleftarrow{\Psi^{-1}} G \xrightarrow{\Gamma^{-1}} B, \tag{4.1}$$

where $\Gamma = \gamma^E$ and $B = \Gamma^{-1}(G)$.

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Proof of Lemma 4.3. (a) The existence of $\mathcal{G}$ is implied by Lemma 4.2. Since $\mathcal{G}$ is finitely generated, we may find subsets $(B_k : k \in \mathbb{N})$ of Ω that generate $\mathcal{G}$. Define $\psi : \Omega \to C$ by

$$\psi(x) = \sum_{k=1}^{\infty} \frac{2}{3^k} 1_{B_k}(x),$$

where 1_B is the indicator function of B, as usual.

Write $\mathcal{G}' = \{\psi^{-1}(S) : S \in \mathcal{B}(C)\}$. We claim that $\mathcal{G} = \mathcal{G}'$. Since $B_k \in \mathcal{G}'$ for all k, we have $\mathcal{G} \subseteq \mathcal{G}'$. Conversely, since ψ is a sum of $\mathcal{G}$-measurable functions, it is $\mathcal{G}$-measurable, and hence $\mathcal{G}' \subseteq \mathcal{G}$.

Let μ be the probability measure on $(C, \mathcal{B}(C))$ induced by ψ, that is, $\mu(S) = P(\psi^{-1}(S))$ for $S \in \mathcal{B}(C)$. By definition of μ, ψ is measure-preserving from X to $Z = (C, \mathcal{B}(C), \mu)$.

(b) Let $\mathcal{H}$ be the σ-field $\{\Psi^{-1}(S) : S \in \mathcal{B}(C^E)\}$ on Ω^E. By the above, $\mathcal{H} = \mathcal{G}^E$. Consequently, $A \in \mathcal{H}$, and hence $A = \Psi^{-1}(G)$ for some $G \in \mathcal{B}(C^E)$.

(c) Define $\kappa : C \to [0, 1]$ by $\kappa(c) = \mu(C \cap [0, c])$. We may take as the inverse the function

$$\gamma(y) = \inf\{c : \kappa(c) \geq y\}, \qquad y \in [0, 1].$$

Since $\gamma(y) \leq c$ if and only if $y \leq \kappa(c)$, we have

$$\gamma^{-1}(C \cap [0, c]) = [0, \kappa(c)], \qquad c \in C,$$

so that

$$\lambda(\gamma^{-1}(C \cap [0, c])) = \kappa(c) = \mu(C \cap [0, c]).$$

The set $\{C \cap [0, c] : c \in C\}$ is a π-system that generates $\mathcal{B}(C)$, and hence γ is measure-preserving from $\mathcal{L}$ to Z. □

Proof of Theorem 2.3. Let $h \in \mathcal{M}$ and $A \in \mathcal{F}^E$. We shall use the notation introduced in Lemmas 4.2–4.3, and we refer the reader to diagram (4.1). By Lemmas 4.1 and 4.3(a), (b), A and G have equal measure and h-influences. Write $\Gamma = \gamma^E$, and take $B = \Gamma^{-1}(G) \subseteq [0, 1]^E$. Since Γ is measure-preserving, by Lemma 4.1, G and B have equal probability and h-influences. □

Proof of Theorem 2.2. This is an immediate corollary of Theorem 2.3, on applying the corresponding result for the Lebesgue space. The latter result is implied by the work of Bourgain *et al.* [3], and is explicit at [8, Theorem 4.33] (the factor 2 present in the last reference is cosmetic only). □

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