

OPERATOR EQUALITIES AND CHARACTERIZATIONS OF ORTHOGONALITY IN PRE-HILBERT C^* -MODULES

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Abstract In the first part of the paper, we use states on C^* -algebras in order to establish some equivalent statements to equality in the triangle inequality, as well as to the parallelogram identity for elements of a pre-Hilbert C^* -module. We also characterize the equality case in the triangle inequality for adjointable operators on a Hilbert C^* -module. Then we give certain necessary and sufficient conditions to the Pythagoras identity for two vectors in a pre-Hilbert C^* -module under the assumption that their inner product has a negative real part. We introduce the concept of Pythagoras orthogonality and discuss its properties. We describe this notion for Hilbert space operators in terms of the parallelogram law and some limit conditions. We present several examples in order to illustrate the relationship between the Birkhoff–James, Roberts, and Pythagoras orthogonalities, and the usual orthogonality in the framework of Hilbert C^* -modules.

Keywords: Hilbert C^* -module; norm equality; Pythagoras orthogonality; Pythagoras identity

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1. Introduction

Let $S(\mathcal{A})$ be the set of all states of a given C^* -algebra \mathcal{A} . The *numerical range* of an element $a \in \mathcal{A}$ is defined by

$$V(a) = \{\varphi(a) : \varphi \in S(\mathcal{A})\}.$$

If a is a normal element of \mathcal{A} then there exists a state φ on \mathcal{A} such that $|\varphi(a)| = \|a\|$ (cf. [17, Theorem 3.3.6]). The set

$$S_a(\mathcal{A}) = \{\varphi \in S(\mathcal{A}) : |\varphi(a)| = \|a\|\}$$

is nonempty and closed. This set is also convex if a is positive.

A (right) *pre-Hilbert C^* -module* \mathcal{E} over a C^* -algebra \mathcal{A} is a (complex) linear space which is also a right \mathcal{A} -module, having a compatible structure (i.e., $\lambda(xa) = (\lambda x)a = x(\lambda a)$, $\lambda \in \mathbb{C}, a \in \mathcal{A}, x \in \mathcal{E}$), equipped with an \mathcal{A} -valued inner product on \mathcal{E} , i.e., a sesquilinear map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$ with the properties:

$$(a) \quad \langle x, x \rangle \geq 0, \quad x \in \mathcal{E}; \quad \langle x, x \rangle = 0 \text{ if and only if } x = 0.$$

$$(b) \quad \langle x, y \rangle^* = \langle y, x \rangle, \quad x, y \in \mathcal{E}.$$

$$(c) \quad \langle x, ya \rangle = \langle x, y \rangle a, \quad x, y \in \mathcal{E}, \quad a \in \mathcal{A}.$$

The formula

$$\mathcal{E} \ni x \mapsto \|x\| := \|\langle x, x \rangle\|_{\mathcal{A}} \in \mathbb{R}_+$$

defines a norm on \mathcal{E} (for $x \in \mathcal{E}$, the notation $|x| := \langle x, x \rangle^{1/2}$ will be used in the subsequent part of the paper). A pre-Hilbert \mathcal{A} -module which is complete with respect to this norm is called a *Hilbert C^* -module over \mathcal{A}* , or a *Hilbert \mathcal{A} -module*. Every C^* -algebra \mathcal{A} can be regarded as a Hilbert module over itself, the inner product being defined as $\langle a, b \rangle := a^*b$, $a, b \in \mathcal{A}$.

Suppose that \mathcal{E} and \mathcal{F} are Hilbert C^* -modules. Let $\mathcal{L}(\mathcal{E}, \mathcal{F})$ be the set of all maps $T : \mathcal{E} \rightarrow \mathcal{F}$ for which there is an application $T^* : \mathcal{F} \rightarrow \mathcal{E}$ such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle, \quad x \in \mathcal{E}, y \in \mathcal{F}. \quad (1)$$

An operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, called *adjointable*, is \mathcal{A} -linear and bounded, while T^* (the *adjoint* of T) is uniquely determined by (1). The map $T \mapsto T^*$ has the properties of an isometric involution. Moreover, $\mathcal{L}(\mathcal{E}) := \mathcal{L}(\mathcal{E}, \mathcal{E})$ is a C^* -algebra.

Thus, Hilbert C^* -modules are generalization of Hilbert spaces by allowing inner products to take values in a C^* -algebra rather than in the field of complex numbers. Unfortunately, certain basic properties of Hilbert spaces are not valid in general Hilbert C^* -modules. For example, it is not true that any bounded linear operator on a Hilbert C^* -module is adjointable or any closed submodule is orthogonally complemented. Therefore, not only any investigation in the context of Hilbert C^* -modules is non-trivial, but also it is an interesting question to ask under which conditions the results analogous to those for Hilbert spaces can still remain true for Hilbert C^* -modules.

It is known that the equality $\|x + y\| = \|x\| + \|y\|$ holds in a Hilbert space \mathcal{H} if and only if x and y are linearly dependent by positive scalars. Being a starting point in our discussion on Pythagoras identities, one of our goals is to investigate the validity of this equality in the setting of Hilbert C^* -modules. Maybe the first result in this direction is a characterization of Arambašić and Rajić [2] which shows that, for two elements x and y in a pre-Hilbert \mathcal{A} -module \mathcal{E} , $\|x + y\| = \|x\| + \|y\|$ if and only if $\|x\|\|y\| \in V(\langle x, y \rangle)$. The particular situations of Hilbert space operators or of elements in a C^* -algebra have been emphasized earlier by Barraa and Boumazgour [6], respectively by Nakamoto and Takahasi [18]. We show, among others, that the following statements are equivalent: $\| |x|^2 + |y|^2 \| = \|x\|^2 + \|y\|^2$; $\| |x||y| \| = \|x\|\|y\|$; $S_{|x|^2}(\mathcal{A}) \cap S_{|y|^2}(\mathcal{A}) \neq \emptyset$; $\|x\|^2\|y\|^2 \in V(|x|^2|y|^2)$; $\|x\|^2 + \|y\|^2 \in V(|x|^2 + |y|^2)$. We also discuss the ‘triangle equality’ for two adjointable operators s and t on \mathcal{E} . By contrast with the earlier approach,

our result relies on the states of \mathcal{A} and not on the states of the C^* -algebra $\mathcal{L}(\mathcal{E})$. More precisely, we prove that $\|s+t\| = \|s\| + \|t\|$ if and only if there exist sequences $(\varphi_n)_{n \geq 0}$ (of states on \mathcal{A}) and $(x_n)_{n \geq 0}$ (of elements in \mathcal{E}) such that $\varphi_n(|x_n|^2) = 1$, $n \geq 0$ and $\varphi_n(\langle sx_n, tx_n \rangle) \xrightarrow{n \rightarrow \infty} \|s\| \|t\|$.

A norm $\|\cdot\|$ on a vector space \mathcal{X} is induced by a scalar product if and only if the parallelogram identity $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ holds for every $x, y \in \mathcal{X}$. This parallelogram identity is not valid in the general framework of Hilbert C^* -modules. In our attempt to characterize this notion using the language of states, we show that any two of the following statements imply the third one: x and y verify the parallelogram identity; $S_{|x|^2}(\mathcal{A}) \cap S_{|y|^2}(\mathcal{A}) \neq \emptyset$; $S_{|x+y|^2}(\mathcal{A}) \cap S_{|x-y|^2}(\mathcal{A}) \neq \emptyset$.

Our next aim was to characterize an equality of the form $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ (*Pythagoras identity*). This identity has been studied by many authors, in various contexts, starting with James [11]. We prove, under the assumption that the inner product $\langle x, y \rangle$ has negative real part, that the following statements are equivalent: $\|x+y\|^2 = \|x\|^2 + \|y\|^2$; $\|x\|^2 + \|y\|^2 \in V(|x+y|^2)$; there exists $\varphi \in S_{|x|^2}(\mathcal{A}) \cap S_{|y|^2}(\mathcal{A})$ such that $\varphi(\Re(\langle x, y \rangle)) = 0$.

In the general context of (complex) normed linear spaces \mathcal{X} , there were several attempts to extend the notion of orthogonality for two vectors x and y . More exactly, x and y are *orthogonal in the Roberts sense* (in notation, $x \perp_R y$; cf. [20, p. 56]) if $\|x + \lambda y\| = \|x - \lambda y\|$, $\lambda \in \mathbb{C}$. The concept of *Birkhoff–James orthogonality* (in notation, $x \perp_B y$), has been suggested by G. Birkhoff [9] and R.C. James [12] as $\|x + \lambda y\| \geq \|x\|$, $\lambda \in \mathbb{C}$. In the framework of pre-Hilbert C^* -modules, these notions have been studied, for example, in [3, 4, 8, 16].

The main part of this paper is devoted to the study of another concept of orthogonality, namely the *Pythagoras orthogonality*. A vector x is said to be *orthogonal in the Pythagoras sense* to a vector y (in notation, $x \perp_P y$) if

$$\|x + \lambda y\|^2 = \|x\|^2 + |\lambda|^2 \|y\|^2, \quad \lambda \in \mathbb{C}.$$

If $x \perp_P y$ then, clearly, x and y satisfy the parallelogram law, that is

$$\|x + \lambda y\|^2 + \|x - \lambda y\|^2 = 2(\|x\|^2 + |\lambda|^2 \|y\|^2), \quad \lambda \in \mathbb{C}.$$

We start by presenting the main properties of Pythagoras orthogonality and discuss its relationship with the parallelogram law, Roberts orthogonality, Birkhoff–James orthogonality and inner product orthogonality. Pythagoras orthogonality implies both the parallelogram law and Birkhoff–James orthogonality. We show that, for two elements x and y in \mathcal{E} (a pre-Hilbert module over a unital C^* -algebra) such that $|y|^2$ is a positive multiple of the identity, the converse is also true. We finally characterize the Pythagoras orthogonality for two operators A and B in $\mathcal{L}(\mathcal{H})$ (regarded as a Hilbert module over itself) as follows. Under the assumptions that $\text{rank}(A + \alpha_1 B) > 1$ and $\Re(\alpha_2 A^* B) \geq 0$ for certain $\alpha_1, \alpha_2 \in \mathbb{C}$, $\alpha_2 \neq 0$, A and B are orthogonal in the Pythagoras sense if and only if A and B verify the parallelogram law and there exists a sequence $(\xi_n)_{n \geq 0}$ of unit vectors in \mathcal{H} such that $\|A\xi_n\| \xrightarrow{n \rightarrow \infty} \|A\|$, $\|B\xi_n\| \xrightarrow{n \rightarrow \infty} \|B\|$ and $\langle A\xi_n, B\xi_n \rangle \xrightarrow{n \rightarrow \infty} 0$ if and only if A and B verify the parallelogram law and there exists a sequence $(\xi_n)_{n \geq 0}$ of unit vectors in \mathcal{H} such that $\|(A + \lambda B)\xi_n\|^2 \xrightarrow{n \rightarrow \infty} \|A\|^2 + |\lambda|^2 \|B\|^2$ for every $\lambda \in \mathbb{C}$. Several examples are given for illustrative purposes.

2. “Triangle equalities”

We start our work by the observation that the equality case in the triangle inequality for two elements x and y in a normed linear space \mathcal{X} is preserved for their positive multiples αx ($\alpha \geq 0$) and βy ($\beta \geq 0$).

Lemma 2.1 (Abramovich et al. [1, Lemma 2.1]). *Let x and y be two vectors in a normed linear space \mathcal{X} such that $\|x + y\| = \|x\| + \|y\|$. Then $\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\|$ for every $\alpha, \beta \geq 0$.*

An equality of the form $\|\alpha x + \beta y\| = |\alpha|\|x\| + |\beta|\|y\|$ ($\alpha, \beta \in \mathbb{C}$, $\alpha, \beta \neq 0$) can be reformulated for scalars α, β belonging to the unit circle \mathbb{T} . More precisely, the following holds.

Proposition 2.2. *Let x and y be two vectors of a normed linear space \mathcal{X} . The following statements are equivalent:*

- (i) $\|\alpha x + \beta y\| = |\alpha|\|x\| + |\beta|\|y\|$ for some nonzero scalars $\alpha, \beta \in \mathbb{C}$.
- (ii) $\|\alpha x + \beta y\| = \|x\| + \|y\|$ for some $\alpha, \beta \in \mathbb{T}$.

Proof. We only have to prove the implication (i) \Rightarrow (ii), the other one is obvious. Let $x' = \alpha x$ and $y' = \beta y$. Then (i) takes the form $\|x' + y'\| = \|x'\| + \|y'\|$ so, by Lemma 2.1, $\|1/|\alpha|x' + (1/|\beta|)y'\| = 1/|\alpha|\|x'\| + 1/|\beta|\|y'\|$. In other words,

$$\left\| \frac{\alpha}{|\alpha|}x + \frac{\beta}{|\beta|}y \right\| = \|x\| + \|y\|.$$

□

The following result, characterizing the equality case in the triangle inequality for two elements of a pre-Hilbert \mathcal{A} -module has been formulated in [19, Proposition 3] using a representation of \mathcal{A} on a Hilbert space. It will be presented here using the terminology of states. We would also like to mention that the equivalence (i) \Leftrightarrow (iii) has been obtained in [2, Theorem 2.1].

Proposition 2.3 (Popovici [19, Proposition 3]). *Let x, y be two elements in a pre-Hilbert module over a C^* -algebra \mathcal{A} . The following statements are equivalent:*

- (i) $\|x + y\| = \|x\| + \|y\|$.
- (ii) $(\|x\| + \|y\|)^2 \in V(|x + y|^2)$.
- (iii) $\|x\|\|y\| \in V(\langle x, y \rangle)$.

If φ is a given state on \mathcal{A} , then $\varphi(|x + y|^2) = (\|x\| + \|y\|)^2$ if and only if $\varphi(\langle x, y \rangle) = \|x\|\|y\|$. In this case, $\varphi \in S_{|x|^2}(\mathcal{A}) \cap S_{|y|^2}(\mathcal{A})$ and $\varphi(\langle x, y \rangle^ \langle x, y \rangle) = \|x\|^2\|y\|^2$.*

We now describe a triangle ‘equality’ in the context of pre-Hilbert C^* -modules.

Proposition 2.4. *Let x and y be two elements of a pre-Hilbert \mathcal{A} -module. The following statements are equivalent:*

- (i) $\| |x|^2 + |y|^2 \| = \|x\|^2 + \|y\|^2$.
- (ii) $\| |x| |y| \| = \|x\| \|y\|$.
- (iii) $S_{|x|^2}(\mathcal{A}) \cap S_{|y|^2}(\mathcal{A}) \neq \emptyset$.
- (iv) $\|x\|^2 \|y\|^2 \in V(|x|^2 |y|^2)$.
- (v) $\|x\|^2 + \|y\|^2 \in V(|x|^2 + |y|^2)$.

Proof. We only prove (v) \Rightarrow (i), the rest can be concluded from [13, Proposition 3.3], [2, Theorem 2.1], and [18, Theorem 1]). To this end, it is enough to show that if a and b are two positive elements of a C^* -algebra \mathcal{A} , and $\|a\| + \|b\| \in V(a + b)$, then $\|a + b\| = \|a\| + \|b\|$:

Let $\varphi \in S(\mathcal{A})$ be such that $\varphi(a + b) = \|a\| + \|b\|$. Then

$$\|a\| + \|b\| = \varphi(a + b) \leq \|a + b\|.$$

We deduce immediately that $\|a + b\| = \|a\| + \|b\|$, as required. □

We need the next result, which is an immediate consequence of the above proposition.

Corollary 2.5. *Let a and b be two elements of \mathcal{A} . The following statements are equivalent:*

- (i) $\|a^*a + b^*b\| = \|a\|^2 + \|b\|^2$.
- (ii) $\|ab^*\| = \|a\| \|b\|$.

Corollary 2.6. *Let x and y be two elements of a pre-Hilbert \mathcal{A} -module. Then any two of the following statements imply the third one:*

- (i) x and y verify the parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

- (ii) $S_{|x|^2}(\mathcal{A}) \cap S_{|y|^2}(\mathcal{A}) \neq \emptyset$.
- (iii) $S_{|x+y|^2}(\mathcal{A}) \cap S_{|x-y|^2}(\mathcal{A}) \neq \emptyset$.

Proof. By Proposition 2.4, condition (ii) is equivalent to $\| |x|^2 + |y|^2 \| = \|x\|^2 + \|y\|^2$. Similarly, condition (iii) can be replaced by:

$$2\| |x|^2 + |y|^2 \| = \| |x + y|^2 + |x - y|^2 \| = \|x + y\|^2 + \|x - y\|^2.$$

The conclusion then follows easily. □

Our next aim is to describe the equality case in the triangle inequality for two adjointable operators in a Hilbert C^* -module.

Theorem 2.7. Let \mathcal{E} be a Hilbert module over the C^* -algebra \mathcal{A} , and let $s, t \in \mathcal{L}(\mathcal{E})$. The following conditions are equivalent:

- (i) $\|s + t\| = \|s\| + \|t\|$.
- (ii) There exist sequences $(\varphi_n)_{n \geq 0}$ (of states on \mathcal{A}) and $(x_n)_{n \geq 0}$ (of elements in \mathcal{E}) such that $\varphi_n(|x_n|^2) = 1, n \geq 0$ and

$$\varphi_n(\langle sx_n, tx_n \rangle) \xrightarrow{n \rightarrow \infty} \|s\| \|t\|.$$

- (iii) There exist sequences $(\varphi_n)_{n \geq 0}$ (of states on \mathcal{A}) and $(x_n)_{n \geq 0}$ (of elements in \mathcal{E}) such that $\varphi_n(|x_n|^2) \leq 1, n \geq 0$ and

$$\varphi_n(\langle sx_n, tx_n \rangle) \xrightarrow{n \rightarrow \infty} \|s\| \|t\|.$$

Proof. It has been indicated in [14, p. 37] that, for any given state φ of \mathcal{A} and $x \in \mathcal{E}$ with $\varphi(|x|^2) = 1$, the map $s \mapsto \varphi(\langle sx, sx \rangle)$ is a state of $\mathcal{L}(\mathcal{E})$. In addition, for any adjointable operator s on \mathcal{E} ,

$$\|s\|^2 = \sup_{\varphi(|x|^2)=1} \varphi(|sx|^2). \tag{2}$$

(i) \Rightarrow (ii). Let us consider, in view of (2), a sequence $(\varphi_n)_{n \geq 0}$ of states on \mathcal{A} and a sequence $(x_n)_{n \geq 0}$ of elements in \mathcal{E} such that $\varphi_n(|x_n|^2) = 1, n \geq 0$ and

$$\varphi_n(|(s + t)x_n|^2) \xrightarrow{n \rightarrow \infty} \|s + t\|^2.$$

We note that, for any $n \geq 0$,

$$\begin{aligned} \varphi_n(|(s + t)x_n|^2) &= \varphi_n(|sx_n|^2) + \varphi_n(|tx_n|^2) + \varphi_n(\langle sx_n, tx_n \rangle) + \varphi_n(\langle tx_n, sx_n \rangle) \\ &\leq \|s\|^2 + \|t\|^2 + \|s^*t\| + \|t^*s\| \\ &\leq (\|s\| + \|t\|)^2. \end{aligned}$$

We pass to limit (as $n \rightarrow \infty$) to deduce, by (i), that

$$\varphi_n(\langle sx_n, tx_n \rangle) \xrightarrow{n \rightarrow \infty} \|s\| \|t\|,$$

which proves (ii).

The implication (ii) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (i). Let $(\varphi_n)_{n \geq 0}$, and let $(x_n)_{n \geq 0}$ be sequences as in (iii). By passing to limit (as $n \rightarrow \infty$) in the inequalities

$$|\varphi_n(\langle sx_n, tx_n \rangle)| \leq \varphi_n(|sx_n|^2)^{1/2} \varphi_n(|tx_n|^2)^{1/2} \leq \|s\| \|t\| \varphi_n(|x_n|^2) \leq \|s\| \|t\|, \quad n \geq 0,$$

we obtain that

$$\varphi_n(|sx_n|^2) \xrightarrow{n \rightarrow \infty} \|s\|^2 \text{ and } \varphi_n(|tx_n|^2) \xrightarrow{n \rightarrow \infty} \|t\|^2.$$

Hence,

$$\varphi_n(|(s + t)x_n|^2) = \varphi_n(|sx_n|^2) + \varphi_n(|tx_n|^2) + 2\Re\varphi_n(\langle sx_n, tx_n \rangle) \xrightarrow{n \rightarrow \infty} (\|s\| + \|t\|)^2.$$

Letting again $n \rightarrow \infty$ in the inequalities $\varphi_n(|(s + t)x_n|^2) \leq \|s + t\|^2 \leq (\|s\| + \|t\|)^2, n \geq 0$ we finally get the triangle ‘equality’ in (i). □

3. Pythagoras identities

We characterize the Pythagoras identity for two vectors in a pre-Hilbert C^* -modules under the assumption that their inner product has a negative real part.

Proposition 3.1. *Let x and y be two elements in a pre-Hilbert C^* -module \mathcal{E} such that $\Re(\langle x, y \rangle) \leq 0$. The following conditions are equivalent:*

- (i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
- (ii) $\|x\|^2 + 2\Re(\langle x, y \rangle) + \|y\|^2 = \|x\|^2 + \|y\|^2$ and $\|x\| \|y\| = \|x\| \|y\|$.

Proof. Let us firstly note that

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + 2\Re(\langle x, y \rangle) + \|y\|^2 \\ &\leq \|x\|^2 + \|y\|^2 \leq \|x\|^2 + \|y\|^2. \end{aligned} \tag{3}$$

If (i) holds true, then the inequalities in (3) become equalities. Also, by [13, Proposition 3.3], the triangle equality $\|x\|^2 + \|y\|^2 = \|x + y\|^2$ can be written in the form $\|x\| \|y\| = \|x\| \|y\|$, which is exactly the last condition of (ii). The converse follows the same path. □

Theorem 3.2. *Let x and y be two elements in a pre-Hilbert C^* -module \mathcal{E} such that $\Re(\langle x, y \rangle) \leq 0$. The following conditions are equivalent:*

- (i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
- (ii) $\|x\|^2 + \|y\|^2 \in V(|x + y|^2)$.
- (iii) There exists $\varphi \in S_{|x|^2}(\mathcal{A}) \cap S_{|y|^2}(\mathcal{A})$ such that $\varphi(\Re(\langle x, y \rangle)) = 0$.

Proof. The implication (i) \Rightarrow (ii) follows by [17, Theorem 3.3.6].

(ii) \Rightarrow (i). Conversely, if φ is a state on \mathcal{A} as in (ii), then, by (3),

$$\|x + y\|^2 \leq \|x\|^2 + \|y\|^2 = \varphi(|x + y|^2) \leq \|x + y\|^2 = \|x + y\|^2.$$

Consequently, (i) holds true.

(iii) \Rightarrow (ii). Let φ be a state on \mathcal{A} such that

$$\varphi(|x|^2) = \|x\|^2, \quad \varphi(|y|^2) = \|y\|^2 \text{ and } \varphi(\Re(\langle x, y \rangle)) = 0.$$

Then

$$\varphi(|x + y|^2) = \varphi(|x|^2) + 2\varphi(\Re(\langle x, y \rangle)) + \varphi(|y|^2) = \|x\|^2 + \|y\|^2.$$

(ii) \Rightarrow (iii). Conversely, let $\varphi \in S(\mathcal{A})$ be a state which satisfies condition (ii). Then

$$\begin{aligned} \|x\|^2 + \|y\|^2 &= \varphi(|x|^2) + 2\varphi(\Re(\langle x, y \rangle)) + \varphi(|y|^2) \\ &\leq \|x\|^2 + 2\varphi(\Re(\langle x, y \rangle)) + \varphi(|y|^2) \\ &\leq \|x\|^2 + \varphi(|y|^2) \\ &\leq \|x\|^2 + \|y\|^2. \end{aligned}$$

Thus, $\varphi(|x|^2) = \|x\|^2$, $\varphi(|y|^2) = \|y\|^2$ and $\varphi(\Re(\langle x, y \rangle)) = 0$. The statement (iii) is proved. \square

Corollary 3.3. *Let x and y be two elements in a pre-Hilbert C^* -module \mathcal{E} such that $\Re(\langle x, y \rangle) \leq 0$. The following conditions are equivalent:*

- (i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
- (ii) $\|\alpha x + \beta y\|^2 = |\alpha|^2 \|x\|^2 + |\beta|^2 \|y\|^2$ for certain (equivalently, for every) $\alpha, \beta \in \mathbb{C}$ with $\bar{\alpha}\beta > 0$.

Remark 3.4. If $\Re(\langle x, y \rangle) \leq 0$, then, by the Pythagoras identity $\|x + y\|^2 = \|x\|^2 + \|y\|^2$, one can also obtain the following inequality:

$$\|\alpha x + \beta y\|^2 \geq |\alpha|^2 \|x\|^2 + |\beta|^2 \|y\|^2$$

for every complex numbers α, β such that $\bar{\alpha}\beta$ is real.

Indeed, if φ is a state on \mathcal{A} satisfying condition (iii) of Theorem 3.2, then, for every $\alpha, \beta \in \mathbb{C}$ with $\bar{\alpha}\beta \in \mathbb{R}$, it holds

$$\begin{aligned} \|\alpha x + \beta y\|^2 &\geq \varphi(|\alpha x + \beta y|^2) \\ &= |\alpha|^2 \varphi(|x|^2) + 2\alpha\bar{\beta}\varphi(\Re(\langle x, y \rangle)) + |\beta|^2 \varphi(|y|^2) \\ &= |\alpha|^2 \|x\|^2 + |\beta|^2 \|y\|^2. \end{aligned}$$

Under the stronger assumption $\Re(\langle x, y \rangle) = 0$, the Pythagoras identities associated with the pairs (x, y) and, respectively, $(\alpha x, \beta y)$ (for $\bar{\alpha}\beta \in \mathbb{R}^*$) are actually equivalent.

Corollary 3.5. *Let x and y be two elements in a pre-Hilbert C^* -module \mathcal{E} such that $\Re(\langle x, y \rangle) = 0$. The following statements are equivalent:*

- (i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
- (ii) $\|\alpha x + \beta y\|^2 = |\alpha|^2 \|x\|^2 + |\beta|^2 \|y\|^2$ for certain (equivalently, for every) non-null complex numbers α and β with $\bar{\alpha}\beta \in \mathbb{R}$.
- (iii) $\| |x| |y| \| = \|x\| \|y\|$.
- (iv) $S_{|x+y|^2}(\mathcal{A}) = S_{|x|^2}(\mathcal{A}) \cap S_{|y|^2}(\mathcal{A})$.

Proof. Under the assumption $\Re(\langle x, y \rangle) = 0$, condition (i) takes the form $\| |x|^2 + |y|^2 \| = \|x\|^2 + \|y\|^2$. The equivalences between (i), (ii), and (iii) are deduced from Proposition 2.4. By the same result, the statements are also equivalent to $S_{|x|^2}(\mathcal{A}) \cap S_{|y|^2}(\mathcal{A}) \neq \emptyset$. So (iv) implies (i). Finally, if $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ (condition (i) holds true), then,

for any $\varphi \in S_{|x+y|^2}(\mathcal{A})$,

$$\begin{aligned}\|x + y\|^2 &= \varphi(|x + y|^2) \\ &= \varphi(|x|^2) + \varphi(|y|^2) \\ &\leq \|x\|^2 + \|y\|^2.\end{aligned}$$

Hence, $\varphi \in S_{|x|^2}(\mathcal{A}) \cap S_{|y|^2}(\mathcal{A})$. In other words, $S_{|x+y|^2}(\mathcal{A}) \subseteq S_{|x|^2}(\mathcal{A}) \cap S_{|y|^2}(\mathcal{A})$. The converse inclusion is obvious, so the statement (iv) is verified. \square

Corollary 3.6. *Let x and y be two elements in a pre-Hilbert C^* -module \mathcal{E} such that $\langle x, y \rangle = 0$. The following statements are equivalent:*

- (i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
- (ii) $\|\alpha x + \beta y\|^2 = |\alpha|^2 \|x\|^2 + |\beta|^2 \|y\|^2$ for certain (equivalently, for every) nonzero complex numbers α and β .

4. Pythagoras orthogonality

It is our aim in this section to investigate the Pythagoras orthogonality in the context of Hilbert C^* -modules. We list some properties of this notion, as follows:

- (a) If x and y are linearly dependent, then $x \perp_P y$ if and only if $x = 0$ or $y = 0$. Due to this simple remark one may suppose, when trying to describe the concept of Pythagoras orthogonality, that the two vectors x and y are linearly independent. If not stated otherwise, we will make this assumption for the rest of the paper.
- (b) In inner product spaces, $x \perp_P y$ if and only if $\langle x, y \rangle = 0$.
- (c) In pre-Hilbert C^* -modules, if $\langle x, y \rangle = 0$, then $x \perp_P y$ if and only if $\|x + \alpha y\|^2 = \|x\|^2 + |\alpha|^2 \|y\|^2$ for a certain non-null $\alpha \in \mathbb{C}$ if and only if $\| |x| |y| \| = \|x\| \|y\|$ (see Corollaries 3.5 and 3.6).
- (d) In normed $*$ -algebras, $x \perp_P y$ if and only if $x^* \perp_P y^*$.
- (e) $x \perp_P x$ if and only if $x = 0$ (nondegenerate).
- (f) $x \perp_P y$ if and only if $y \perp_P x$ (symmetric).
- (g) If $x \perp_P y$, then $(\alpha x) \perp_P (\beta y)$, $\alpha, \beta \in \mathbb{C}$ (homogeneous).
- (h) If $x \perp_P y$, then $x \perp_R y$, $x \perp_B y$ and $y \perp_B x$.
- (i) If $x \perp_P y$, then x and y satisfy the parallelogram law.
- (j) If $x \perp_R y$, then $x \perp_P y$ if and only if x and y satisfy the parallelogram law.

We describe, in a few examples, for elements x, y in a pre-Hilbert C^* -module, the relationship between the Birkhoff–James, Roberts and Pythagoras orthogonality, the parallelogram law and the equality $\langle x, y \rangle = 0$.

Example 4.1. Let us consider the C^* -algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators on a separable Hilbert space \mathcal{H} ($\mathcal{L}(\mathcal{H})$ is regarded as a Hilbert module over itself). For a given orthonormal basis $(e_n)_{n \geq 1}$ in \mathcal{H} , we define $A, B \in \mathcal{L}(\mathcal{H})$ by

$$Ae_n = \begin{cases} \frac{1}{2}e_1 & \text{if } n = 1, \\ \frac{1}{\sqrt{2^k}}e_2 & \text{if } n = 2k, k \geq 1, \\ 0 & \text{otherwise} \end{cases} \text{ and } Be_n = \begin{cases} \frac{1}{2}e_1 & \text{if } n = 1, \\ \frac{1}{\sqrt{2^k}}e_2 & \text{if } n = 2k + 1, k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|A\| = \|B\| = 1$ and, for any $\lambda \in \mathbb{C}$, we have

$$\|A + \lambda B\|^2 = 1 + |\lambda|^2 = \|A\|^2 + |\lambda|^2 \|B\|^2.$$

Hence, A and B are orthogonal in the Pythagoras sense. Moreover, since $e_1 \in \text{ran } A \cap \text{ran } B$, $\langle A, B \rangle \neq 0$ ($\text{ran } A$ denotes the range of A).

Example 4.2. Suppose that \mathcal{A} is the C^* -algebra $\mathcal{C}[0, 1]$ of all complex valued continuous functions on the closed interval $[0, 1]$ (considered as a Hilbert C^* -module over itself). Let $f, g \in \mathcal{A}$ be defined by

$$f(x) = \begin{cases} \frac{1}{2} - x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}, \text{ and } g(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ x - \frac{1}{2} & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Then $\|f\| = \|g\| = \frac{1}{2}$ and, for $\lambda \in \mathbb{C}$,

$$\|f + \lambda g\| = \max \left\{ \frac{1}{2}, \frac{|\lambda|}{2} \right\}.$$

It follows that $f \perp_B g$, $g \perp_B f$ and $f \perp_R g$. Although $\langle f, g \rangle = 0$, f and g are not orthogonal in the Pythagoras sense, since $\|f + \lambda g\|^2 = \|f\|^2 + |\lambda|^2 \|g\|^2$ if and only if $\lambda = 0$. Equivalently, as $f \perp_R g$, f and g do not satisfy the parallelogram law, either.

Example 4.3. Let S and T be bounded linear operators on \mathcal{H} . If

$$A = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$$

are elements of the C^* -algebra $\mathcal{A} = \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, then $\|A\| = \|S\|$, $\|B\| = \|T\|$ and

$$\|A + \lambda B\|^2 = \|SS^* + |\lambda|^2 TT^*\|, \quad \lambda \in \mathbb{C}.$$

One can immediately verify that $A \perp_B B$, $B \perp_B A$ and $A \perp_R B$. Also, by Corollary 2.5, A and B are orthogonal in the Pythagoras sense if and only if $\|S^*T\| = \|S\|\|T\|$. In particular, this orthogonality condition is satisfied, for example, when S is a scalar multiple of a coisometric operator or of an orthogonal projection P with $\text{ran}(P) \supseteq \text{ran}(T)$.

In addition, one can immediately verify that, A and B satisfy the parallelogram law if and only if they are orthogonal in the Pythagoras sense. On the other hand, $\langle A, B \rangle = A^*B = 0$ if and only if $\text{ran } S \perp \text{ran } T$.

Example 4.4. For given $X \in \mathcal{L}(\mathcal{H}) \setminus \{0\}$ and complex numbers a, b, c, d we define (on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$) the matrix operator

$$M_X(a, b, c, d) = \begin{pmatrix} aI & bX \\ cX^* & dI \end{pmatrix}.$$

Its norm has been computed by Feldman, Krupnik, and Markus in [10, Lemma 1.6] as

$$\|M_X(a, b, c, d)\| = \frac{\sqrt{r-s} + \sqrt{r+s}}{2},$$

where $r = |a|^2 + |d|^2 + (|b|^2 + |c|^2)\|X\|^2$ and $s = 2|ad - bc\|X\|$.

Let $A = M_X(a, 0, 0, d)$ and $B = M_X(0, b, c, 0)$. After direct calculations, one may check that, regardless of the values of a, b, c and d , $A \perp_B B$ and $A \perp_R B$. On the other hand, $B \perp_B A$ if and only if $bc = 0$. We can also verify that *the following conditions are equivalent:*

- (i) A and B are orthogonal in the Pythagoras sense;
- (ii) A and B satisfy the parallelogram law;
- (iii) $ad = bc = 0$.

Finally, $\langle A, B \rangle = 0$ if and only if $ab = cd = 0$.

Example 4.5. We study the concepts of orthogonality presented above for rank one operators, i.e., for operators of the form

$$\mathcal{H} \ni z \mapsto (x \otimes y)(z) := \langle z, y \rangle x \in \mathcal{H},$$

where x and y are given vectors in \mathcal{H} .

Let $x, y, u, v \in \mathcal{H}$, and consider the operators $A = x \otimes y$ and $B = u \otimes v$. Then $\|A\| = \|x\|\|y\|$ and $\|B\| = \|u\|\|v\|$. In this case we have

$$\begin{aligned} \|A + \lambda B\|^2 &= \frac{1}{2} \left(\|x\|^2\|y\|^2 + |\lambda|^2\|u\|^2\|v\|^2 + \lambda\langle u, x \rangle\langle y, v \rangle + \bar{\lambda}\langle x, u \rangle\langle v, y \rangle \right. \\ &\quad \left. + \sqrt{\left(\|x\|^2\|y\|^2 + |\lambda|^2\|u\|^2\|v\|^2 + \lambda\langle u, x \rangle\langle y, v \rangle + \bar{\lambda}\langle x, u \rangle\langle v, y \rangle \right)^2 \right. \\ &\quad \left. - 4|\lambda|^2\|x\|^2\|y\|^2\|u\|^2\|v\|^2 - 4|\lambda|^2|\langle u, x \rangle\langle y, v \rangle|^2 \right. \\ &\quad \left. + 4|\lambda|^2\|x\|^2\|u\|^2|\langle y, v \rangle|^2 + 4|\lambda|^2\|y\|^2\|v\|^2|\langle u, x \rangle|^2 \right). \end{aligned}$$

With the notation $a = \|u\|^2\|v\|^2$, $b = \langle u, x \rangle\langle y, v \rangle$, $c = \|x\|^2\|y\|^2$ and $d = \|x\|^2\|u\|^2|\langle y, v \rangle|^2 + \|y\|^2\|v\|^2|\langle u, x \rangle|^2$, we observe that the Birkhoff–James orthogonality between A and B

takes the form:

$$\sqrt{(c + a|\lambda|^2 + 2\Re(\lambda b))^2 - 4|\lambda|^2(ac + |b|^2 - d)} \geq c - a|\lambda|^2 - 2\Re(\lambda b), \quad \lambda \in \mathbb{C}. \quad (4)$$

Clearly, for λ in a ‘small’ disk $D(0, \varepsilon)$, $a|\lambda|^2 + 2\Re(\lambda b) \leq c$. Then, taking squares, formula (4) becomes:

$$|\lambda|^2(d - |b|^2) + 2c\Re(\lambda b) \geq 0.$$

In particular, for $\mu = \varepsilon/2$ and, successively, $\lambda = \mu, \lambda = -\mu, \lambda = -i\mu$ and $\lambda = i\mu$ and by taking the limit when $\varepsilon \rightarrow 0$, one can deduce that $\Re b \geq 0, \Re b \leq 0, \Im b \geq 0$ and, respectively, $\Im b \leq 0$. This forces $b = 0$.

By symmetry, a similar condition holds true for the Birkhoff–James orthogonality between B and A .

We can write the orthogonality, in the Roberts sense, between A and B as:

$$\begin{aligned} 4\Re(\lambda b) + \sqrt{(c + a|\lambda|^2 + 2\Re(\lambda b))^2 + 4|\lambda|^2 e} \\ = \sqrt{(c + a|\lambda|^2 - 2\Re(\lambda b))^2 + 4|\lambda|^2 e}, \quad \lambda \in \mathbb{C}, \end{aligned}$$

where $e = d - ac - |b|^2$. By squaring the equation, we get:

$$\Re(\lambda b) \left(2\Re(\lambda b) + c + a|\lambda|^2 + \sqrt{(c + a|\lambda|^2 + 2\Re(\lambda b))^2 + 4|\lambda|^2 e} \right) = 0, \quad \lambda \in \mathbb{C}.$$

For $|\lambda|$ small enough, the quantity in the parenthesis becomes strictly positive (as c is strictly positive). Hence, $\Re(\lambda b) = 0$. So, by taking $\lambda \in \mathbb{R}_+$, respectively $\lambda \in i\mathbb{R}_+$, we necessarily have $b = 0$.

It is obvious that $b = 0$ implies the orthogonalities $A \perp_B B, B \perp_B A$ and $A \perp_R B$.

We showed that the following conditions are equivalent:

- (i) $A \perp_B B$.
- (ii) $B \perp_B A$.
- (iii) $A \perp_R B$.
- (iv) $\langle x, u \rangle = 0$ or $\langle y, v \rangle = 0$.

Also, A and B are orthogonal in the Pythagoras sense if and only if one of the following conditions holds true:

- (a) $\{x, u\}$ are linearly dependent and $\langle y, v \rangle = 0$.
- (b) $\{y, v\}$ are linearly dependent and $\langle x, u \rangle = 0$.

Indeed, after some computations for the case $b = 0$ (Roberts and Birkhoff–James orthogonalities are necessary conditions), the orthogonality $A \perp_P B$ becomes:

$$\sqrt{(c + a|\lambda|^2)^2 - 4|\lambda|^2 ac + 4|\lambda|^2 d} = c + a|\lambda|^2, \quad \lambda \in \mathbb{C}.$$

We obtain that $d = ac$, which, in the case $\langle y, v \rangle = 0$, leads us to the equality case in the Cauchy–Schwarz inequality, namely $|\langle x, u \rangle| = \|x\| \|u\|$ (i.e., $\{x, u\}$ are linearly dependent). The situation when $\langle x, u \rangle = 0$ is treated similarly.

Moreover, one can easily see that the parallelogram law:

$$\begin{aligned} &\sqrt{(c + a|\lambda|^2 + 2\Re(\lambda b))^2 + 4|\lambda|^2 e} + \sqrt{(c + a|\lambda|^2 - 2\Re(\lambda b))^2 + 4|\lambda|^2 e} \\ &= 2(c + a|\lambda|^2), \quad \lambda \in \mathbb{C} \end{aligned}$$

can be simplified as $e = 0$, which takes the form:

$$(\|x\|^2 \|u\|^2 - |\langle u, x \rangle|^2) (\|y\|^2 \|v\|^2 - |\langle y, v \rangle|^2) = 0.$$

We deduce that A and B satisfy the parallelogram law if and only if $\{x, u\}$ or $\{y, v\}$ are linearly dependent. Finally, $\langle A, B \rangle = 0$ if and only if $\langle x, u \rangle = 0$.

Let \mathcal{A} be a C^* -algebra. For an element $a \in \mathcal{A}$ we denote by $m(a)$ the minimum modulus of $|a|$, i.e.,

$$m(a) := \inf\{\varphi(|a|) : \varphi \in S(\mathcal{A})\}.$$

Evidently, for a positive element $a \in \mathcal{A}$, $m(a) > 0$ if and only if a is invertible. The following theorem, which relates Birkhoff–James orthogonality with a weaker version of Pythagoras orthogonality, has been formulated in the context of Hilbert spaces by Barraa and Boumazgour in [5, Theorem 3] (see also [16, Theorem 2.9]).

Theorem 4.6. *Let x and y be elements of a pre-Hilbert \mathcal{A} -module. The following conditions are equivalent:*

- (i) *There exists $\varphi \in S_{|x|^2}(\mathcal{A})$ such that $\varphi(\langle x, y \rangle) = 0$.*
- (ii) *$x \perp_B y$.*
- (iii) *$\|x + \lambda y\|^2 \geq \|x\|^2 + |\lambda|^2 m(|y|^2)$, $\lambda \in \mathbb{C}$.*

Proof. The equivalence (i) \Leftrightarrow (ii) has been obtained by Arambašić and Rajić in [3, Theorem 2.7] (see also [8, Theorem 4.4]).

Clearly, (ii) is a consequence of (iii). Conversely, if φ is a state of \mathcal{A} which verifies $\varphi(|x|^2) = \|x\|^2$ and $\varphi(\langle x, y \rangle) = 0$ (by (i)), then, for every $\lambda \in \mathbb{C}$,

$$\begin{aligned} \|x + \lambda y\|^2 &\geq \varphi(|x + \lambda y|^2) \\ &= \varphi(|x|^2) + \bar{\lambda}\varphi(\langle x, y \rangle) + \lambda\varphi(\langle y, x \rangle) + |\lambda|^2\varphi(|y|^2) \\ &= \|x\|^2 + |\lambda|^2\varphi(|y|^2) \\ &\geq \|x\|^2 + |\lambda|^2 m(|y|^2). \end{aligned}$$

Condition (iii) is proved. □

It was noted by James [12, Corollary 2.2] that, for any two elements x and y of a normed linear space \mathcal{X} there exists a number α such that $y \perp_B (x + \alpha y)$. Such a condition is not valid, in full generality, for Pythagoras orthogonality. However, a weaker version might still be formulated. Its operator version can be found in [5, Corollary 4] (see also [16, Corollary 2.11]).

Corollary 4.7. *Let x and y be elements of a pre-Hilbert module over \mathcal{A} such that $m(|y|^2) > 0$. Then there exists a unique $\alpha_0 \in \mathbb{C}$ such that*

$$\|x + \alpha_0 y + \lambda y\|^2 \geq \|x + \alpha_0 y\|^2 + |\lambda|^2 m(|y|^2), \quad \lambda \in \mathbb{C}. \quad (5)$$

Moreover, one can find $\varphi \in S_{|x+\alpha y|^2}(\mathcal{A})$ such that $\varphi(\langle x + \alpha y, y \rangle) = 0$ if and only if $\alpha = \alpha_0$.

Proof. We firstly observe that, since $\lim_{|\alpha| \rightarrow \infty} \|x + \alpha y\| = \infty$,

$$\inf\{\|x + \alpha y\| : \alpha \in \mathbb{C}\} = \inf\{\|x + \alpha y\| : |\alpha| \leq \delta\}$$

for a certain $\delta > 0$. In addition, as the map $\alpha \mapsto \|x + \alpha y\|$ is continuous on the compact set $\{|\alpha| \leq \delta\}$, it attains its minimum at some point $\alpha_0 \in \mathbb{C}$, that is, $\|x + \alpha_0 y + \lambda y\| \geq \|x + \alpha_0 y\|$ for every $\lambda \in \mathbb{C}$. Formula (5) then follows by Theorem 4.6 (iii). If, for some $\alpha_1 \in \mathbb{C}$,

$$\|x + \alpha_1 y + \lambda y\|^2 \geq \|x + \alpha_1 y\|^2 + |\lambda|^2 m(|y|^2), \quad \lambda \in \mathbb{C}$$

then, by taking $\lambda = \alpha_0 - \alpha_1$, we obtain

$$m(|y|^2)|\alpha_1 - \alpha_0|^2 \leq \|x + \alpha_0 y\|^2 - \|x + \alpha_1 y\|^2 \leq 0,$$

so, $\alpha_1 = \alpha_0$. The final statement is a consequence, in view of the uniqueness of α_0 , of Theorem 4.6 ((i) \Leftrightarrow (iii)). \square

As seen earlier, the Pythagoras orthogonality implies both the parallelogram law and Birkhoff–James orthogonality. In certain particular situations, the converse is also true.

Theorem 4.8. *Let \mathcal{A} be a unital C^* -algebra with unit e . If x and y are elements in a pre-Hilbert \mathcal{A} -module such that $|y|^2 = \alpha e$ ($\alpha > 0$ is given), then the following conditions are equivalent:*

- (i) x and y are orthogonal in the Pythagoras sense.
- (ii) x and y satisfy the parallelogram law and are Birkhoff–James orthogonal.

Proof. The direct implication is obvious. Conversely, if (ii) holds true, then, by Theorem 4.6 (iii),

$$\begin{aligned} \|x + \lambda y\|^2 &\geq \|x\|^2 + |\lambda|^2 m(|y|^2) \\ &= \|x\|^2 + \alpha |\lambda|^2 \\ &= \|x\|^2 + |\lambda|^2 \|y\|^2, \quad \lambda \in \mathbb{C}. \end{aligned}$$

In view of the parallelogram law, the inequalities above become equalities. Hence $x \perp_P y$, as required. \square

The operator version of Theorem 4.6 ((i) \Leftrightarrow (ii)) has been obtained by B. Magajna in [15, Lemma 2.2] (see also [7, Remark 3.1]). It states that two bounded linear operators

A and B on \mathcal{H} are orthogonal in the Birkhoff–James sense if and only if there exists a sequence $(\xi_n)_{n \geq 0}$ of unit vectors in \mathcal{H} such that

$$\|A\xi_n\| \xrightarrow{n \rightarrow \infty} \|A\| \text{ and } \langle A\xi_n, B\xi_n \rangle \xrightarrow{n \rightarrow \infty} 0.$$

For Pythagoras orthogonality (a concept which is stronger than Birkhoff–James orthogonality), we must include certain additional conditions. One of the main tools in our developments is the following expression of the $\min_{\lambda \in \mathbb{C}} \|A + \lambda B\|$.

Theorem 4.9 (Arambašić and Rajić [3, Proposition 2.1]). *Let A and B be bounded linear operators on \mathcal{H} . Then*

$$\min_{\lambda \in \mathbb{C}} \|A + \lambda B\|^2 = \sup_{\|\xi\|=1} M_{A,B}(\xi),$$

where

$$M_{A,B}(\xi) = \begin{cases} \|A\xi\|^2 - \frac{|\langle A\xi, B\xi \rangle|^2}{\|B\xi\|^2} & \text{if } B\xi \neq 0, \\ \|A\xi\|^2 & \text{if } B\xi = 0. \end{cases}$$

The following limit characterization provides a step forward in our desired description of Pythagoras orthogonality.

Lemma 4.10. *Let A and B be bounded linear operators acting on \mathcal{H} such that*

$$\|(1 + \lambda_0)A + \lambda_0\alpha B\|^2 = (1 + \lambda_0)^2 \|A\|^2 + \lambda_0^2 |\alpha|^2 \|B\|^2$$

for a certain $\lambda_0 \in \mathbb{R} \setminus \{-1, 0\}$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $(\xi_n)_{n \geq 0}$ a sequence of unit vectors in \mathcal{H} .

1 If $(A + \alpha B)\xi_n \neq 0$, $n \geq 0$ and

$$M_{A,A+\alpha B}(\xi_n) \xrightarrow{n \rightarrow \infty} (1 + \lambda_0)^2 \|A\|^2 + \lambda_0^2 |\alpha|^2 \|B\|^2, \tag{6}$$

and (x_n) is a subsequence of (ξ_n) such that the limits

$$a = \lim_{n \rightarrow \infty} \|Ax_n\|, \quad b = \lim_{n \rightarrow \infty} \|Bx_n\| \text{ and } c = \lim_{n \rightarrow \infty} \langle Ax_n, Bx_n \rangle \tag{7}$$

exist, then a, b , and c satisfy the conditions

$$a^2(\lambda_0 + 1) + c\bar{\alpha}\lambda_0 = -b^2|\alpha|^2\lambda_0 - c\bar{\alpha}(\lambda_0 + 1) = (1 + \lambda_0)^2 \|A\|^2 + \lambda_0^2 |\alpha|^2 \|B\|^2. \tag{8}$$

Moreover,

$$\|(1 + \lambda_0)A + \lambda_0\alpha B\| = \min_{\lambda \in \mathbb{C}} \|(1 + \lambda)A + \lambda\alpha B\|$$

and

$$\|A + \lambda B\|^2 \geq \frac{[(1 + \lambda_0)^2 \|A\|^2 + \lambda_0^2 |\alpha|^2 \|B\|^2][\lambda_0 |\alpha|^2 - (\lambda_0 + 1)|\lambda|^2] - \bar{\alpha}c|\lambda_0\alpha - (\lambda_0 + 1)\lambda|^2}{|\alpha|^2 \lambda_0 (\lambda_0 + 1)}, \quad \lambda \in \mathbb{C}. \tag{9}$$

2 Conversely, if the limits (7) exist, satisfy conditions (8) and $a^2 \neq (1 + \lambda_0)^2 \|A\|^2 + \lambda_0^2 |\alpha|^2 \|B\|^2$, then (6) holds true.

Proof. We may assume, without loss of generality, that $\alpha = 1$ (B can be replaced by $(1/\alpha)B$, if necessary).

Let us now observe that, for every $n \geq 0$, the following inequalities hold true:

$$\begin{aligned} M_{A,A+B}(\xi_n) &= \|(1 + \lambda_0)A + \lambda_0 B\| \xi_n \|^2 - \frac{|\langle [(1 + \lambda_0)A + \lambda_0 B]\xi_n, (A + B)\xi_n \rangle|^2}{\|(A + B)\xi_n\|^2} \\ &\leq \|(1 + \lambda_0)A + \lambda_0 B\| \xi_n \|^2 - \frac{|\langle [(1 + \lambda_0)A + \lambda_0 B]\xi_n, (A + B)\xi_n \rangle|^2}{\|A + B\|^2} \\ &\leq \|(1 + \lambda_0)A + \lambda_0 B\| \xi_n \|^2 \\ &\leq \|(1 + \lambda_0)A + \lambda_0 B\|^2 = (1 + \lambda_0)^2 \|A\|^2 + \lambda_0^2 \|B\|^2. \end{aligned}$$

Letting $n \rightarrow \infty$ we conclude that (6) is equivalent with the following limit conditions:

$$\frac{\langle [(1 + \lambda_0)A + \lambda_0 B]\xi_n, (A + B)\xi_n \rangle}{\|(A + B)\xi_n\|} \xrightarrow{n \rightarrow \infty} 0, \tag{10}$$

$$\langle [(1 + \lambda_0)A + \lambda_0 B]\xi_n, (A + B)\xi_n \rangle \xrightarrow{n \rightarrow \infty} 0 \tag{11}$$

and

$$\|(1 + \lambda_0)A + \lambda_0 B\| \xi_n \|^2 \xrightarrow{n \rightarrow \infty} (1 + \lambda_0)^2 \|A\|^2 + \lambda_0^2 \|B\|^2. \tag{12}$$

Following the notation of (7), one can write (11) as

$$(1 + \lambda_0)a^2 + \lambda_0 b^2 + (1 + \lambda_0)c + \lambda_0 \bar{c} = 0.$$

Similarly, (12) takes the form

$$(1 + \lambda_0)^2 a^2 + \lambda_0^2 b^2 + 2\lambda_0(\lambda_0 + 1)\Re c = (1 + \lambda_0)^2 \|A\|^2 + \lambda_0^2 \|B\|^2.$$

Easy computations then show that (11) and (12) are actually equivalent with (8).

According to these remarks, in order to prove (a), it only remains to let $n \rightarrow \infty$ into the formulas

$$M_{A,A+B}(\xi_n) \leq \min_{\lambda \in \mathbb{C}} \|(1 + \lambda)A + \lambda B\|^2 \leq \|(1 + \lambda_0)A + \lambda_0 B\|^2, \quad n \geq 0$$

and

$$\begin{aligned} \|A + \lambda B\|^2 &\geq \|(A + \lambda B)\xi_n\|^2 \\ &= \|A\xi_n\|^2 + 2\Re \lambda \langle A\xi_n, B\xi_n \rangle + |\lambda|^2 \|B\xi_n\|^2, \quad \lambda \in \mathbb{C}, n \geq 0. \end{aligned} \tag{13}$$

(b) As seen above, (11) and (12) are a consequence of (8). In addition,

$$\|(A + B)\xi_n\|^2 \xrightarrow{n \rightarrow \infty} \frac{a^2 - [(1 + \lambda_0)^2 \|A\|^2 + \lambda_0^2 \|B\|^2]}{\lambda_0^2}.$$

Hence, $\lim_{n \rightarrow \infty} \|(A + B)\xi_n\| > 0$, which shows that (10) also holds true. The proof is completed. \square

Lemma 4.11. *Let A and B be bounded linear operators on \mathcal{H} such that $\text{rank}(A + \alpha_i B) = 1$ for pairwise distinct complex numbers $\alpha_i \in \mathbb{C}, i = 1, 2, 3$. Then $\text{rank}(A + \alpha B) = 1$ for every $\alpha \in \mathbb{C}$.*

Proof. Let x_1, x_2, y_1, y_2 be non-null vectors in \mathcal{H} such that $A + \alpha_1 B = x_1 \otimes y_1$ and $A + \alpha_2 B = x_2 \otimes y_2$. Then

$$A + \lambda B = \frac{\lambda - \alpha_2}{\alpha_1 - \alpha_2} x_1 \otimes y_1 + \frac{\alpha_1 - \lambda}{\alpha_1 - \alpha_2} x_2 \otimes y_2, \quad \lambda \in \mathbb{C}.$$

We distinguish two cases:

(i) $\{x_1, x_2\}$ are linearly independent. Since $\text{rank}(A + \alpha_3 B) = 1$, one can find $\beta_1, \beta_2 \in \mathbb{C}$ (at least one of them is non-null) such that .. Then, for every $z \in \mathcal{H}$, there exists $\mu \in \mathbb{C}$ such that

$$\frac{\alpha_3 - \alpha_2}{\alpha_1 - \alpha_2} \langle z, y_1 \rangle = \mu \beta_1 \text{ and } \frac{\alpha_1 - \alpha_3}{\alpha_1 - \alpha_2} \langle z, y_2 \rangle = \mu \beta_2.$$

Therefore, β_1, β_2 are both non-null and $\bar{\beta}_2(\bar{\alpha}_3 - \bar{\alpha}_2)/\bar{\alpha}_1 - \bar{\alpha}_2 y_1 - \bar{\beta}_1(\bar{\alpha}_1 - \bar{\alpha}_3)/\bar{\alpha}_1 - \bar{\alpha}_2 y_2 = 0$, so $\{y_1, y_2\}$ are linearly dependent. In other words,

$$A + \lambda B = \left[\frac{\lambda - \alpha_2}{\alpha_1 - \alpha_2} x_1 + \frac{\beta_2(\alpha_3 - \alpha_2)(\alpha_1 - \lambda)}{\beta_1(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)} x_2 \right] \otimes y_1, \quad \lambda \in \mathbb{C}.$$

(ii) $\{x_1, x_2\}$ are linearly dependent. In this case, there exists a complex number $\beta \neq 0$ such that $x_2 = \beta x_1$. We conclude that

$$A + \lambda B = x_1 \otimes \left[\frac{\bar{\lambda} - \bar{\alpha}_2}{\bar{\alpha}_1 - \bar{\alpha}_2} y_1 + \frac{\bar{\beta}(\bar{\alpha}_1 - \bar{\lambda})}{\bar{\alpha}_1 - \bar{\alpha}_2} y_2 \right], \quad \lambda \in \mathbb{C}.$$

Since $\{A, B\}$ are linearly independent (as assumed earlier; this also implies that $\{y_1, y_2\}$ are linearly independent), we deduce that $A + \lambda B$ has rank one for every $\lambda \in \mathbb{C}$. \square

We are now ready to present the announced characterization of Pythagoras orthogonality.

Theorem 4.12. *Let A and B be bounded linear operators acting on \mathcal{H} such that $\text{rank}(A + \alpha_1 B) > 1$ and $\Re(\alpha_2 A^* B) \geq 0$ for certain $\alpha_1, \alpha_2 \in \mathbb{C}, \alpha_2 \neq 0$. The following conditions are equivalent:*

- (i) A and B are orthogonal in the Pythagoras sense.
- (ii) A and B verify the parallelogram law and there exists a sequence $(\xi_n)_{n \geq 0}$ of unit vectors in \mathcal{H} such that

$$\|A\xi_n\| \xrightarrow{n \rightarrow \infty} \|A\|, \quad \|B\xi_n\| \xrightarrow{n \rightarrow \infty} \|B\| \text{ and } \langle A\xi_n, B\xi_n \rangle \xrightarrow{n \rightarrow \infty} 0.$$

- (iii) A and B verify the parallelogram law and there exists a sequence $(\xi_n)_{n \geq 0}$ of unit vectors in \mathcal{H} such that

$$\|(A + \lambda B)\xi_n\|^2 \xrightarrow{n \rightarrow \infty} \|A\|^2 + |\lambda|^2 \|B\|^2 \text{ for every } \lambda \in \mathbb{C}.$$

Proof. (i) \Rightarrow (ii). The parallelogram law is obviously weaker than (or, at most equivalent to) Pythagoras orthogonality.

Our next aim is to prove the limit conditions of (ii). Since $\Re(2\alpha_2 A^* B) \geq 0$ and $\Re(3\alpha_2 A^* B) \geq 0$ and, by Lemma 4.11, at least one of the operators $A + \alpha_2 B$, $A + 2\alpha_2 B$ and $A + 3\alpha_2 B$ has rank strictly greater than one, so we can assume that $\alpha_1 = \alpha_2 = \alpha$. As we have previously done, we can also assume that $\alpha = 1$. We firstly observe that, by (i),

$$\|(1 + \lambda)A + \lambda B\|^2 = |1 + \lambda|^2 \|A\|^2 + |\lambda|^2 \|B\|^2, \quad \lambda \in \mathbb{C}.$$

An easy computation then shows that

$$\min_{\lambda \in \mathbb{C}} \|(1 + \lambda)A + \lambda B\|^2 = \frac{\|A\|^2 \|B\|^2}{\|A\|^2 + \|B\|^2}, \tag{14}$$

which is attained for $\lambda_0 = -\|A\|^2 / (\|A\|^2 + \|B\|^2)$. It follows from Theorem 4.9 and (14) that there exists a sequence $(\xi_n)_{n \geq 0}$ of unit vectors in \mathcal{H} such that

$$M_{A,A+B}(\xi_n) \xrightarrow{n \rightarrow \infty} \frac{\|A\|^2 \|B\|^2}{\|A\|^2 + \|B\|^2}. \tag{15}$$

We may suppose, eventually on a subsequence, that $(A + B)\xi_n \neq 0$ for every $n \geq 0$. Indeed, if, otherwise, $(A + B)\xi_n = 0$ for every $n \geq n_0$ and for a certain $n_0 \geq 0$, then (15) takes the form

$$\|A\xi_n\|^2 = \|B\xi_n\|^2 \xrightarrow{n \rightarrow \infty} \frac{\|A\|^2 \|B\|^2}{\|A\|^2 + \|B\|^2} \tag{16}$$

by the definition of $M_{A,A+B}$. As $(A\xi_n)_{n \geq 0}$ is a bounded sequence in \mathcal{H} , it contains a weakly convergent subsequence (denoted also by $(A\xi_n)_{n \geq 0}$) to a vector $w \in \mathcal{H}$. Obviously, $\text{span}\{(A + B)^* w\} \subsetneq \overline{\text{ran}(A + B)^*}$ as the rank of $(A + B)^*$ is strictly greater than 1. Consequently, one can find a unit vector $e \in \overline{\text{ran}(A + B)^*}$, which is orthogonal to $(A + B)^* w$. Then, by setting $u_n = \sqrt{n/(n + 1)}\xi_n + (1/\sqrt{n + 1})e$, $n \geq 0$, we have

$$(A + B)u_n = \sqrt{\frac{n}{n + 1}}(A + B)\xi_n + \frac{1}{\sqrt{n + 1}}(A + B)e = \frac{1}{\sqrt{n + 1}}(A + B)e \neq 0, \quad n \geq n_0,$$

since $e \perp \ker(A + B)$. Moreover, for $n \geq 0$,

$$\begin{aligned} M_{A,A+B}(u_n) &= \|Au_n\|^2 - \frac{|\langle Au_n, (A + B)u_n \rangle|^2}{\|(A + B)u_n\|^2} \\ &= \frac{n}{n + 1} \|A\xi_n\|^2 + \frac{1}{n + 1} \|Ae\|^2 + 2 \frac{\sqrt{n}}{n + 1} \Re(A\xi_n, Ae) \\ &\quad - \frac{\left| \left\langle \sqrt{\frac{n}{n + 1}} A\xi_n + \frac{1}{\sqrt{n + 1}} Ae, \frac{1}{\sqrt{n + 1}} (A + B)e \right\rangle \right|^2}{\left\| \frac{1}{\sqrt{n + 1}} (A + B)e \right\|^2} \end{aligned}$$

$$\begin{aligned}
 &= \frac{n}{n+1} \|A\xi_n\|^2 + \frac{1}{n+1} \|Ae\|^2 + 2\frac{\sqrt{n}}{n+1} \Re\langle A\xi_n, Ae \rangle \\
 &\quad - \frac{\left| \sqrt{\frac{n}{n+1}} \langle A\xi_n, (A+B)e \rangle + \frac{1}{\sqrt{n+1}} \langle Ae, (A+B)e \rangle \right|^2}{\|(A+B)e\|^2}. \tag{17}
 \end{aligned}$$

In view of (16) and the observation that

$$\langle A\xi_n, (A+B)e \rangle \xrightarrow{n \rightarrow \infty} \langle w, (A+B)e \rangle = \langle (A+B)^*w, e \rangle = 0,$$

we deduce, by passing to limit in (17), that

$$M_{A,A+B}(u_n) \xrightarrow{n \rightarrow \infty} \frac{\|A\|^2 \|B\|^2}{\|A\|^2 + \|B\|^2}.$$

One may consider, in this particular situation, the sequence $(u_{n+n_0})_{n \geq 0}$ which will be also denoted by $(\xi_n)_{n \geq 0}$.

The assumptions of Lemma 4.10 (a) are verified. So, the limits (7) satisfy the conditions (equivalent with (8))

$$c = \frac{\|A\|^2(b^2 - \|B\|^2)}{\|B\|^2} = \frac{\|B\|^2(a^2 - \|A\|^2)}{\|A\|^2}. \tag{18}$$

We deduce that $c \leq 0$. Also, by hypothesis (i.e., $\Re(A^*B) \geq 0$),

$$c = \Re c = \lim_{n \rightarrow \infty} \Re\langle \xi_n, A^*B\xi_n \rangle = \lim_{n \rightarrow \infty} \langle \xi_n, \Re(A^*B)\xi_n \rangle \geq 0.$$

This forces $c = 0$ and, by (18), $a = \|A\|$ and $b = \|B\|$.

(ii) \Rightarrow (iii). Clearly, if a sequence $(\xi_n)_{n \geq 0}$ of unit vectors in \mathcal{H} verifies (ii), then

$$\begin{aligned}
 &\|(A + \lambda B)\xi_n\|^2 \\
 &= \|A\xi_n\|^2 + 2\Re\bar{\lambda}\langle A\xi_n, B\xi_n \rangle + |\lambda|^2 \|B\xi_n\|^2 \xrightarrow{n \rightarrow \infty} \|A\|^2 + |\lambda|^2 \|B\|^2, \quad \lambda \in \mathbb{C}.
 \end{aligned}$$

Hence $(\xi_n)_{n \geq 0}$ also verifies (iii).

(iii) \Rightarrow (i). Let $(\xi_n)_{n \geq 0}$ be a sequence of unit vectors in \mathcal{H} such that (iii) holds true. Then

$$\|A + \lambda B\|^2 \geq \|(A + \lambda B)\xi_n\|^2 \xrightarrow{n \rightarrow \infty} \|A\|^2 + |\lambda|^2 \|B\|^2.$$

The proof is finished, as before, by the use of the parallelogram law. □

Remark 4.13. (a) *The necessity of the rank condition.* Let A and B be bounded linear operators on \mathcal{H} such that, for every $\alpha \in \mathbb{C}$, $A + \alpha B$ is a rank one operator. Then, as seen in the proof of Lemma 4.11, one of the following two situations can occur:

(i) $(A, B) = (x \otimes y_A, x \otimes y_B)$, with $x, y_A, y_B \in \mathcal{H}$, $x \neq 0$ and $\{y_A, y_B\}$ linearly independent. Then, by Example 4.5, (A, B) satisfies the parallelogram law. Moreover, $A \perp_P B$ if and only if $\langle y_A, y_B \rangle = 0$. Condition (ii) of Theorem 4.12 takes, for a given sequence $(\xi_n)_{n \geq 0}$ of unit vectors in \mathcal{H} , the form:

$$|\langle \xi_n, y_A \rangle| \xrightarrow{n \rightarrow \infty} \|y_A\|, \quad |\langle \xi_n, y_B \rangle| \xrightarrow{n \rightarrow \infty} \|y_B\| \quad \text{and} \quad \langle \xi_n, y_A \rangle \overline{\langle \xi_n, y_B \rangle} \xrightarrow{n \rightarrow \infty} 0.$$

This is, however, impossible.

(ii) $(A, B) = (x_A \otimes y, x_B \otimes y)$, with $x_A, x_B, y \in \mathcal{H}$, $\{x_A, x_B\}$ linearly independent and $y \neq 0$. Again by Example 4.5, (A, B) satisfies the parallelogram law. We also observe that a sequence $(\xi_n)_{n \geq 0}$ of unit vectors in \mathcal{H} with $|\langle \xi_n, y \rangle| \xrightarrow{n \rightarrow \infty} \|y\|$ satisfies condition (ii) of Theorem 4.12 if and only if $\langle x_A, x_B \rangle = 0$ or, equivalently, $A \perp_P B$. On the other hand, $A \perp_P B$ if and only if $A^* \perp_P B^*$, but condition (ii) of Theorem 4.12 is not verified for the pair (A^*, B^*) (as seen in case (i) above).

(b) *The necessity of the parallelogram law.* Let us consider, as in Example 4.4, $A = M_I(1, 0, 0, 1)$, $B = M_I(0, 1, 1, 0)$ and $h \in \mathcal{H}$ a vector of unit norm. Then A and B do not satisfy the parallelogram law while, for $\xi = (h, 0)$, $\|A\xi\| = \|A\| = 1$, $\|B\xi\| = \|B\| = 1$ and $\langle A\xi, B\xi \rangle = \langle (h, 0), (0, h) \rangle = 0$.

(c) *The importance of the condition $\Re(\alpha A^* B) \geq 0$ for certain nonzero $\alpha \in \mathbb{C}$.* Let x and y be unit vectors in \mathcal{H} such that $\langle x, y \rangle = 0$ (it is assumed that $\dim \mathcal{H} \geq 2$). If $S = x \otimes x + y \otimes y$ and $T = x \otimes y$, then the operators $A = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ are orthogonal in the Pythagoras sense (according to Example 4.3) and $\text{rank}(A + \alpha B) = 2$ for every $\alpha \in \mathbb{C}$. Moreover, for $h_1, h_2 \in \mathcal{H}$ and $\alpha \in \mathbb{C}$, $\alpha \neq 0$, it holds

$$\langle \Re(\alpha A^* B)(h_1, h_2), (h_1, h_2) \rangle = \Re \langle Sh_1, \alpha Th_2 \rangle = \Re(\bar{\alpha} \langle h_1, x \rangle \overline{\langle h_2, y \rangle}).$$

Hence,

$$\langle \Re(\alpha A^* B)(\alpha x, y), (\alpha x, y) \rangle = -\langle \Re(\alpha A^* B)(-\alpha x, y), (-\alpha x, y) \rangle = |\alpha|^2 > 0.$$

Also, condition (ii) of Theorem 4.12 can be expressed by the existence of sequences $(\xi_n)_{n \geq 0}$ and $(\eta_n)_{n \geq 0}$ of vectors in \mathcal{H} with $\|\xi_n\|^2 + \|\eta_n\|^2 = 1$, $n \geq 0$ such that

$$|\langle \xi_n, x \rangle|^2 + |\langle \xi_n, y \rangle|^2 \xrightarrow{n \rightarrow \infty} 1, \quad |\langle \eta_n, y \rangle| \xrightarrow{n \rightarrow \infty} 1 \quad \text{and} \quad \langle \xi_n, x \rangle \overline{\langle \eta_n, y \rangle} \xrightarrow{n \rightarrow \infty} 0.$$

Equivalently,

$$\langle \xi_n, x \rangle \xrightarrow{n \rightarrow \infty} 0, \quad |\langle \xi_n, y \rangle| \xrightarrow{n \rightarrow \infty} 1 \quad \text{and} \quad |\langle \eta_n, y \rangle| \xrightarrow{n \rightarrow \infty} 1.$$

Letting $n \rightarrow \infty$ in the Cauchy–Schwarz inequalities $|\langle \xi_n, y \rangle| \leq \|\xi_n\|$ and $|\langle \eta_n, y \rangle| \leq \|\eta_n\|$ ($n \geq 0$), we deduce that the limits $\lim_{n \rightarrow \infty} \|\xi_n\|$ and $\lim_{n \rightarrow \infty} \|\eta_n\|$ exist and they are both equal to 1. This contradicts, however, the equality $\|\xi_n\|^2 + \|\eta_n\|^2 = 1$, $n \geq 0$.

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