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DYNAMIC GRADED EPISTEMIC LOGIC

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Abstract. Graded epistemic logic is a logic for reasoning about uncertainties. Graded epistemic logic is interpreted on graded models. These models are generalizations of Kripke models. We obtain completeness of some graded epistemic logics. We further develop dynamic extensions of graded epistemic logics, along the framework of dynamic epistemic logic. We give an extension with public announcements, i.e., public events, and an extension with graded event models, a generalization also including nonpublic events. We present complete axiomatizations for both logics.

§1. Introduction. Graded modal logic was introduced in Fine (1972) and Goble (1970), further developed, in, e.g., de Caro (1988) and Fattorosi-Barnaba & de Caro (1985), and employed in van der Hoek (1992) and van der Hoek & Meyer (1992) as a quantitative approach to deal with the problem of expressing an agent's confidence in her beliefs. Consider the following example:

> *Consider an agent getting input from three sources w_1, w_2, and w_3. Suppose furthermore, that two types of information are relevant for this agent, say p and q. All the sources agree on p: the agent is confident that p is true. On the other hand, in w_1 and w_2, q is true, whereas in w_3, it is false: the agent is more confident that q is true than that q is false.*

Using the standard multimodal logic S5, one cannot express that the agent has more confidence in q than in $\neg q$. For expressing such a difference, van der Hoek (1992) and van der Hoek & Meyer (1992) use graded modalities and the resulting logic is *graded epistemic logic*. Intuitively, for an agent a, the graded modality $\langle a\rangle_n\varphi$ represents agent a's confidence in the truth of φ by a natural number n. Similarly, $\langle a\rangle_n\neg\varphi$ represents agent a's confidence in the truth of $\neg\varphi$. The agent can compare his beliefs with his disbeliefs by comparing these figures.

Graded modalities are interpreted in Kripke models as counting the number of accessible states of the current state. The logic of graded modalities is an extension of the standard modal logic. There are many applications of those modalities in the literature. For example, in van der Hoek & de Rijke (1995), graded modalities are used in knowledge representation theory to count objects.

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A distinction is generally made between logics with *graded modalities* such as the abovementioned Fine (1972) and van der Hoek & Meyer (1992) and logics with *modalities for degrees of belief*. Logics of degrees of belief go back to Grove (1988) and Spohn (1988), although these could more properly be said to be semantic frameworks to model degrees of belief. Logics of degrees of belief have seen some popularity in artificial intelligence and AGM style belief revision, see, e.g., Laverny (2006) and van der Hoek (1993). Belief revision based on degrees of belief has been investigated in Andersen, Bolander, van Ditmarsch, & Jensen (2017), Aucher (2003), and van Ditmarsch (2005).

The current movement of epistemic logic is towards the description of the logical dynamics of information and interaction. Various so-called *dynamic epistemic logics*, in, e.g., van Ditmarsch, van der Hoek, & Kooi (2007) and van Ditmarsch, Halpern, van der Hoek, & Kooi (editors) (2015), have been developed for this purpose. However, to the best knowledge of the authors, dynamic extensions of logics with graded modalities have not been developed. The aim of this article is to study dynamic graded epistemic logics.

This article is divided into two parts. In the first part, we generalize Kripke models to graded models, and prove the completeness results for some graded epistemic logics. These logics are static logics for reasoning about gradations of epistemic uncertainty. In the second part, we introduce dynamic extensions of graded epistemic logics.

§2. Graded epistemic logic. Let $\mathcal{A}$ be a finite set of agents. The language of graded epistemic logic (as we present various semantics focussing on knowledge and belief, we use this term rather than 'graded modal logic') consists of a denumerable set of propositional variables Prop, propositional connectives $\neg$ and $\vee$, and graded modalities $\langle a\rangle_n$, where $n \in \mathbb{N}$ is a natural number and $a \in \mathcal{A}$. The number n in a graded modality $\langle a\rangle_n$ represents the grade of the modality. The set of all *graded epistemic formulae* $\mathcal{L}^g_{\mathrm{EL}}$ is defined inductively by the following rule:

$$\mathcal{L}^g_{\mathrm{EL}} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \langle a\rangle_n\varphi,$$

where $p \in \mathsf{Prop}$, $n \in \mathbb{N}$, and $a \in \mathcal{A}$. The *complexity* of a formula $\varphi \in \mathcal{L}^g_{\mathrm{EL}}$ is the number of connectives occurring in φ.

Other propositional connectives $\bot, \top, \wedge, \to$, and $\leftrightarrow$ are defined as usual. The dual of $\langle a\rangle_n$ is defined as $[a]_n\varphi := \neg\langle a\rangle_n\neg\varphi$. In particular, define $\langle a\rangle\varphi := \langle a\rangle_1\varphi$ and $[a]\varphi := [a]_1\varphi$. Define $\langle a\rangle_{!n}\varphi := \langle a\rangle_n\varphi \wedge \neg\langle a\rangle_{n+1}\varphi$.

2.1. Semantics of graded epistemic logic. In this work, sum and product operations and the greater than relation are defined over natural numbers $\mathbb{N}$ plus ω, the number greater than any natural number. For $\mathbb{N}\cup\{\omega\}$ we may write $\mathbb{N}^\omega$. Variables n, m etc. vary over natural numbers, not over $\mathbb{N}^\omega$. We note that for all $n \in \mathbb{N}$: $n < \omega$, if $n \neq 0$ then $n \cdot \omega = \omega$ and $0 \cdot \omega = 0$, and $n + \omega = \omega$.

DEFINITION 2.1. *A* graded frame *is a pair* $\mathfrak{F} = (W, \{\sigma_a\}_{a\in\mathcal{A}})$, *where* $W \neq \varnothing$ *is a set of epistemic states, and* $\sigma_a : W \to (W \to \mathbb{N}^\omega)$ *is a function which assigns a natural number or* ω *to each pair of states.*

A graded model *is a tuple* $\mathfrak{M} = (W, \{\sigma_a\}_{a\in\mathcal{A}}, V)$ *where* $(W, \{\sigma_a\}_{a\in\mathcal{A}})$ *is a graded frame, and* $V : \mathsf{Prop} \to \mathcal{P}(W)$ *is a valuation from* Prop *to the powerset of* W.

For $X \subseteq W$ and $w \in W$, define $\sigma_a(w)(X)$ as $\sum_{u\in X}\sigma_a(w)(u)$ (possibly ω, and where $\sigma_a(w)(\emptyset) = 0$). The notation $X \subseteq_{<\omega} W$ represents that X is a finite subset of W. Let $\mathcal{P}^+(W)$ be the set of all nonempty finite subsets of W. For any subset $Y \subseteq W$ and $n \in \mathbb{N}$, it is obvious that the following conditions are equivalent:

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— $\sigma_a(w)(Y) \geq n$
— There is $X \subseteq_{<\omega} Y$ such that $\sigma_a(w)(X) \geq n$.
— There is $X \in \mathcal{P}^+(Y)$ such that $\sigma_a(w)(X) \geq n$.

Henceforth, these conditions are used without mention of their equivalence.

DEFINITION 2.2. *The* truth *of a formula* $\varphi \in \mathcal{L}^g_{EL}$ *at a state* w *in a graded model* $\mathfrak{M} = (W, \{\sigma_a\}_{a\in\mathcal{A}}, V)$*, notation* $\mathfrak{M}, w \Vdash_g \varphi$*, is defined recursively as below*:

$\mathfrak{M}, w \Vdash_g p$ *iff* $w \in V(p)$*, for each* $p \in \mathsf{Prop}$.
$\mathfrak{M}, w \Vdash_g \neg\varphi$ *iff* $\mathfrak{M}, w \nVdash_g \varphi$.
$\mathfrak{M}, w \Vdash_g \varphi \vee \psi$ *iff* $\mathfrak{M}, w \Vdash_g \varphi$ *or* $\mathfrak{M}, w \Vdash_g \psi$.
$\mathfrak{M}, w \Vdash_g \langle a\rangle_n\varphi$ *iff* $\exists X \subseteq_{<\omega} W(\sigma_a(w)(X) \geq n \;\&\; X \subseteq [\![\varphi]\!]_{\mathfrak{M}})$.

For the dual modality, we have the following derived semantic clause:

$$\mathfrak{M}, w \Vdash_g [a]_n\varphi \text{ iff } \forall X \subseteq_{<\omega} W(\sigma_a(w)(X) \geq n \;\Rightarrow\; \exists u \in X(u \in [\![\varphi]\!]_{\mathfrak{M}})).$$

The notation $[\![\varphi]\!]_{\mathfrak{M}}$ *stands for the* truth set *of* φ *in* $\mathfrak{M}$*, i.e.,* $[\![\varphi]\!]_{\mathfrak{M}} = \{u \in W \mid \mathfrak{M}, u \Vdash_g \varphi\}$. *For any set of formulae* Γ*, define* $[\![\Gamma]\!]_{\mathfrak{M}} = \bigcap\{[\![\varphi]\!]_{\mathfrak{M}} \mid \varphi \in \Gamma\}$.

A formula φ *is* true *in* $\mathfrak{M}$*, notation* $\mathfrak{M} \Vdash_g \varphi$*, if* $[\![\varphi]\!]_{\mathfrak{M}} = W$. *A formula* φ *is* valid at a state w *in a graded frame* $\mathfrak{F} = (W, \{\sigma_a\}_{a\in\mathcal{A}})$*, notation* $\mathfrak{F}, w \Vdash_g \varphi$*, if* $\mathfrak{F}, V, w \Vdash_g \varphi$ *for any valuation* V *in* $\mathfrak{F}$. *A formula* φ *is* valid *in* $\mathfrak{F}$*, notation* $\mathfrak{F} \Vdash_g \varphi$*, if* $\mathfrak{F}, w \Vdash_g \varphi$ *for any state* $w \in W$.

Obviously, the following formulae are valid in any graded frame: $\langle a\rangle_0\varphi \leftrightarrow \top$; $[a]_0\varphi \leftrightarrow \bot$; $\langle a\rangle_!\varphi \leftrightarrow [a]\neg\varphi$.

2.2. Comparison between graded models and Kripke models. A *Kripke frame* is a pair $\mathcal{F} = (W, \{R_a\}_{a\in\mathcal{A}})$, where W is a nonempty set of states, and each $R_a \subseteq W \times W$. Similarly, a *Kripke model* is a tuple $\mathcal{M} = (W, \{R_a\}_{a\in\mathcal{A}}, V)$, where $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation and where $(W, \{R_a\}_{a\in\mathcal{A}})$ is a Kripke frame. For any $w \in W$, define $R_a(w) = \{u \in W \mid wR_au\}$. For any $X \subseteq W$, let $|X|$ denote the cardinality of X.

DEFINITION 2.3. *The satisfiability relation* $\mathcal{M}, w \Vdash_K \varphi$ *in a Kripke model* $\mathcal{M} = (W, \{R_a\}_{a\in\mathcal{A}}, V)$ *is defined recursively as follows*:

$\mathcal{M}, w \Vdash_K p$ *iff* $w \in V(p)$*, for each* $p \in \mathsf{Prop}$.
$\mathcal{M}, w \Vdash_K \neg\varphi$ *iff* $\mathcal{M}, w \nVdash_K \varphi$.
$\mathcal{M}, w \Vdash_K \varphi \vee \psi$ *iff* $\mathcal{M}, w \Vdash_K \varphi$ *or* $\mathcal{M}, w \Vdash_K \psi$.
$\mathcal{M}, w \Vdash_K \langle a\rangle_n\varphi$ *iff* $|R_a(w) \cap [\![\varphi]\!]_{\mathcal{M}}| \geq n$.

Truth in a model and validity are defined as usual.

There is a strong connection between graded frames and Kripke frames. Now we will show that each Kripke frame can be transformed into a graded frame, and vice versa.

DEFINITION 2.4. *Given a Kripke frame* $\mathcal{F} = (W, \{R_a\}_{a\in\mathcal{A}})$*, define the graded frame* $\mathcal{F}^\circ = (W, \{\sigma^R_a\}_{a\in\mathcal{A}})$ *by setting*

$$\sigma^R_a(w)(u) = \begin{cases} 1, & \text{if } wR_au. \\ 0, & \text{otherwise}. \end{cases}$$

For a Kripke model $\mathcal{M} = (\mathcal{F}, V)$*, let* $\mathcal{M}^\circ = (\mathcal{F}^\circ, V)$.

Given a graded frame $\mathfrak{F} = (W, \{\sigma_a\}_{a\in\mathcal{A}})$*, define the Kripke frame* $\mathfrak{F}_\circ = (W_\circ, \{R^\sigma_a\}_{a\in\mathcal{A}})$ *by setting*

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$W_\circ = \{(w, i) \mid w \in W \ \&\ i \in \mathbb{N}^\omega\}$;
$(w, i)R^\sigma_a(u, j)$ *iff* $\sigma_a(w)(u) \geq j > 0$.

For a graded model $\mathfrak{M} = (\mathfrak{F}, V)$*, define* $\mathfrak{M}_\circ = (\mathfrak{F}_\circ, V_\circ)$ *where* $V_\circ(p) = \{(w, i) \in W_\circ \mid w \in V(p)\}$ *for each* $p \in \mathsf{Prop}$.

PROPOSITION 2.5. *Given a Kripke model* $\mathcal{M} = (W, \{R_a\}_{a\in\mathcal{A}}, V)$ *where* $\mathcal{F} = (W, \{R_a\}_{a\in\mathcal{A}})$ *is a Kripke frame, for any* $w \in W$ *and formula* $\varphi \in \mathcal{L}^g_{EL}$, (1) $\mathcal{M}, w \Vdash_K \varphi$ *iff* $\mathcal{M}^\circ, w \Vdash_g \varphi$*;* (2) $\mathcal{M} \Vdash_K \varphi$ *iff* $\mathcal{M}^\circ \Vdash_g \varphi$*;* (3) $\mathcal{F}, w \Vdash_K \varphi$ *iff* $\mathcal{F}^\circ, w \Vdash_g \varphi$*;* (4) $\mathcal{F} \Vdash_K \varphi$ *iff* $\mathcal{F}^\circ \Vdash_g \varphi$.

Proof. The items (2)–(4) follow from (1). One can verify (1) by induction on the complexity of φ. We sketch only the proof of the modal case $\varphi := \langle a\rangle_n\psi$ for $n > 0$. Assume $\mathcal{M}, w \Vdash_K \langle a\rangle_n\psi$. Then, there is a nonempty finite set $X = \{u_1, \dots, u_n\}$ such that wR_au_i and $\mathcal{M}, u_i \Vdash_K \psi$ for $1 \leq i \leq n$. By the construction, $\sigma^R_a(w)(X) = n$. By induction hypothesis, $\mathcal{M}^\circ, u_i \Vdash_g \psi$ for $1 \leq i \leq n$. Hence $\mathcal{M}^\circ, w \Vdash_g \langle a\rangle_n\psi$. Conversely, assume $\mathcal{M}^\circ, w \Vdash_g \langle a\rangle_n\psi$. Then, there is $X \in \mathcal{P}^+(W)$ such that $\sigma^R_a(w)(X) \geq n$ and $\mathcal{M}^\circ, u \Vdash_g \psi$ for all $u \in X$. By the construction, $X \subseteq R_a(w)$ and $|X| \geq n$. By induction hypothesis, $\mathcal{M}, u \Vdash_K \psi$ for all $u \in X$. Hence $\mathcal{M}, w \Vdash_K \langle a\rangle_n\psi$. □

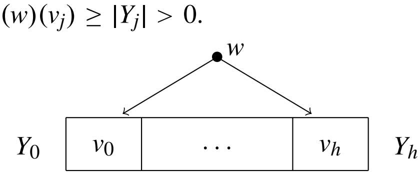
PROPOSITION 2.6. *Given a graded model* $\mathfrak{M} = (\mathfrak{F}, V)$ *with a underlying graded frame* $\mathfrak{F} = (W, \{\sigma_a\}_{a\in\mathcal{A}})$*, for any state* $w \in W$ *and formula* $\varphi \in \mathcal{L}^g_{EL}$, (1) $\mathfrak{M}, w \Vdash_g \varphi$ *iff* $\mathfrak{M}_\circ, (w, 0) \Vdash_K \varphi$*;* (2) $\mathfrak{M} \Vdash_g \varphi$ *iff* $\mathfrak{M}_\circ \Vdash_K \varphi$*;* (3) *if* $\mathfrak{F}_\circ, (w, 0) \Vdash_K \varphi$*, then* $\mathfrak{F}, w \Vdash_g \varphi$*;* (4) *if* $\mathfrak{F}_\circ \Vdash_K \varphi$*, then* $\mathfrak{F} \Vdash_g \varphi$.

Proof. The items (2)–(4) follow from (1). It suffices to show (1) by induction on the complexity of φ. We sketch only the proof of the modal case $\varphi := \langle a\rangle_k\psi$ for $k > 0$. Assume $\mathfrak{M}, w \Vdash_g \langle a\rangle_k\psi$. We have the following cases:

Case 1. $\exists u \in W(\sigma_a(w)(u) = \omega \ \&\ \mathfrak{M}, u \Vdash_g \psi)$. Then $(w, 0)R^\sigma_a(u, m)$ for all $m \in \mathbb{N}$. By induction hypothesis, $\mathfrak{M}, (w, m) \Vdash_K \psi$ for all $m \in \mathbb{N}$. Then we have $\mathfrak{M}, (w, 0) \Vdash_K \langle a\rangle_n\psi$.

Case 2. $\forall u \in W(\mathfrak{M}, u \Vdash_g \psi \Rightarrow \sigma_a(w)(u) < \omega)$. Then there are states $u_0, \dots, u_{m-1}$ for some $m > 0$ such that $\sigma_a(w)(u_i) = n_i > 0$ and $\mathfrak{M}, u_i \Vdash_g \psi$ $(i < m)$ and $n_1 + \cdots + n_m \geq k$. There are at least k copies of ψ-states in the model $\mathfrak{M}_\circ$ which are successors of (w, n). Then $\mathfrak{M}_\circ, (w, 0) \Vdash_K \langle a\rangle_k\psi$.

Conversely, assume $\mathfrak{M}_\circ, (w, 0) \Vdash_K \langle a\rangle_k\psi$. There are k-pairs $(u_0, n_0), \dots, (u_{k-1}, n_{k-1})$ such that $\sigma_a(w)(u_i) \geq n_i > 0$ and $\mathfrak{M}_\circ, (u_i, n_i) \Vdash_K \psi$ for $i < k$. By inductive hypothesis, $\mathfrak{M}, u_i \Vdash_g \psi$ for $i < k$. Let $Y = \{v_0, \dots, v_h\}$ where $v_0, \dots, v_h$ are the states that occur in the pairs $(u_0, n_0), \dots, (u_{k-1}, n_{k-1})$. Then $(u_0, n_0), \dots, (u_{k-1}, n_{k-1})$ are classified into $Y_1, \dots, Y_h$ where, for $1 \leq j \leq h$, Y_j consists of pairs which have the same first-order coordinate v_j. Clearly $\sigma_a(w)(v_j) \geq |Y_j| > 0$.



Then $\sigma_a(w)(Y) \geq \Sigma_{1\leq j\leq h}|Y_j| = k$. Hence $\mathfrak{M}, x \Vdash_g \langle a\rangle_k\psi$. □

COROLLARY 2.7. *For any graded model* $\mathfrak{M} = (\mathfrak{F}, V)$ *with domain* W*,* $w \in W$ *and formula* $\varphi \in \mathcal{L}^g_{EL}$*,* $\mathfrak{M}, w \Vdash_g \varphi$ *iff* $(\mathfrak{M}_\circ)^\circ, (w, 0) \Vdash_g \varphi$.

Proof. Directly from Propositions 2.6(1) and 2.5(1). □

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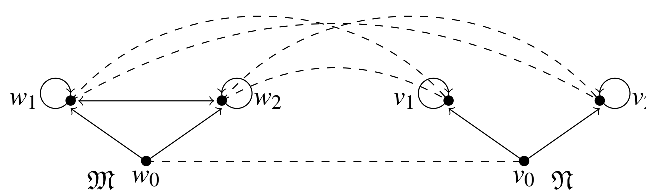
2.3. Graded bisimulation. Bisimulation is a powerful tool for understanding the expressive power of a modal language. A concept of graded bisimulation between Kripke models was introduced by de Rijke (2000). He proved that graded modal logic is the graded bisimulation invariant fragment of first-order logic with identity. Clearly, the standard notion of bisimulation would have been unsuitable. The example given in de Rijke (2000) is illuminating.

EXAMPLE 2.8. *Consider Kripke models $\mathfrak{M}$ and $\mathfrak{N}$ below, with p true everywhere, and let Z be the dashed relation. Relation Z is a standard bisimulation. Although $(w_1, v_1) \in Z$, $\mathfrak{M}, w_1 \Vdash_g \langle a \rangle_2 p$ but $\mathfrak{N}, v_1 \nVdash_g \langle a \rangle_2 p$. Standard bisimulation therefore does not guarantee logical equivalence in graded epistemic logic.*



The graded bisimulation defined in de Rijke (2000) is based on Kripke models. His definition consists of seven different clauses. It is therefore rather involved. Aceto, Ingólfsdóttir, & Sack (2010) showed a perfect correspondence between De Rijke's notion and a different notion called *resource bisimulation*, proposed by Corradini, De Nicola, & Labella (1999), that is rather elegant. Corradini *et al.*,'s notion is what we will now define as graded bisimulation, although with a minor difference: in Aceto *et al.*, (2010) and Corradini *et al.*, (1999) agreement of propositional variables is not part of the definition.

Given a relation $Z \subseteq W \times W'$, the *lifting* of Z is the relation $\widehat{Z} \subseteq \mathcal{P}(W) \times \mathcal{P}(W')$ defined as: $X\widehat{Z}X'$ iff $\forall x \in X \exists x' \in X'(xZx')$ and $\forall x' \in X' \exists x \in X(xZx')$.

DEFINITION 2.9 (Graded bisimulation). *Let $\mathfrak{M} = (W, \{\sigma_a\}_{a\in\mathcal{A}}, V)$ and $\mathfrak{M}' = (W', \{\sigma'_a\}_{a\in\mathcal{A}}, V')$ be graded models. A nonempty relation $Z \subseteq W \times W'$ is called a* g-bisimulation *between $\mathfrak{M}$ and $\mathfrak{M}'$ (notation: $Z : \mathfrak{M} \rightleftharpoons_g \mathfrak{M}'$), if the following conditions hold for all $(w, w') \in Z$ and (n $\in \mathbb{N}$ with) $n > 0$:*

> (Atomic) *w and w' satisfy the same proposition variables.*
> (Forth) *if $\sigma_a(w)(X) \geq n$ and $\forall v \in X$, $\sigma_a(w)(v) > 0$, then there exists $X' \in \mathcal{P}(W')$ with $\sigma'_a(w')(X') \geq n$, $\forall v' \in X'$, $\sigma'_a(w')(v') > 0$, and $X\widehat{Z}X'$.*
> (Back) *if $\sigma'_a(w')(X') \geq n$ and $\forall v' \in X'$, $\sigma'_a(w')(v') > 0$, then there exists $X \in \mathcal{P}(W)$ with $\sigma_a(w)(X) \geq n$, $\forall v \in X$, $\sigma_a(w)(v) > 0$, and $X\widehat{Z}X'$.*

If there is a g-bisimulation $Z : \mathfrak{M} \rightleftharpoons_g \mathfrak{M}'$ with wZw', then w and w' are called g-bisimilar *(notation: $\mathfrak{M}, w \rightleftharpoons_g \mathfrak{M}', w'$). If $\forall w \in W \exists w' \in W'(wZw')$, then Z is called* surjective. *Z is called* global *if both Z and Z^{-1} are surjective.*

We note that a graded bisimulation on a graded model where all weights are 0 or 1 is a standard bisimulation. The *graded modal equivalence relation* between graded models $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$, notation $\mathfrak{M}, w \equiv_g \mathfrak{M}', w'$, is defined by

$$\mathfrak{M}, w \equiv_g \mathfrak{M}', w' \text{ iff } \forall \varphi \in \mathcal{L}^g_{\mathrm{EL}}(\mathfrak{M}, w \Vdash_g \varphi \Leftrightarrow \mathfrak{M}', w' \Vdash_g \varphi).$$

A graded model $\mathfrak{M} = (W, \{\sigma_a\}_{a\in\mathcal{A}}, V)$ is *image finite* if for all $w \in W$ and $a \in \mathcal{A}$, $|\{u \in W \mid \sigma_a(w)(u) > 0\}| < \omega$. The following result, which is also known as the

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Hennessy–Milner property, can now be obtained for graded bisimulation. The direction that bisimilarity implies modal equivalence also holds for models that are not image finite. We refer to Aceto *et al.*, (2010) for proof details.

THEOREM 2.10 (Aceto *et al.*, 2010, Prop. 4.11). *Let image-finite graded models* $(\mathfrak{M}, w)$ *and* $(\mathfrak{M}', w')$ *be given. Then* $\mathfrak{M}, w \rightleftharpoons_g \mathfrak{M}', w'$ *iff* $\mathfrak{M}, w \equiv_g \mathfrak{M}', w'$.

As in de Rijke (2000), we can obtain a similar result for modally saturated models. A graded model $\mathfrak{M} = (W, \{\sigma_a\}_{a\in\mathcal{A}}, V)$ is *graded modally saturated* if for any $\Gamma \subseteq \mathcal{L}^g_{\text{EL}}$, $w \in W$, $n > 0$, and $a \in \mathcal{A}$:

$$\text{if } \sigma_a(w)([\![\Delta]\!]_{\mathfrak{M}}) \geq n \text{ for any } \Delta \in \mathcal{P}^+(\Gamma), \text{ then } \sigma_a(w)([\![\Gamma]\!]_{\mathfrak{M}}) \geq n.$$

THEOREM 2.11. *Let graded modally saturated models* $(\mathfrak{M}, w)$ *and* $(\mathfrak{M}', w')$ *be given. Then* $\mathfrak{M}, w \rightleftharpoons_g \mathfrak{M}', w'$ *iff* $\mathfrak{M}, w \equiv_g \mathfrak{M}', w'$.

Proof. The direction from bisimilarity to modal equivalence is elementary, and as in the previous theorem. For the other direction, it suffices to show that the graded modal equivalence relation $\equiv_g$ is a graded bisimulation. We only show the forth condition. The back condition can be shown similarly.

Assume that $\mathfrak{M}, w \equiv_g \mathfrak{M}', w'$, $\sigma_a(w)(X) \geq n$ and $\forall z \in X$, $\sigma_a(w)(z) > 0$. Let $n_x = \Sigma\{\sigma_a(w)(z) \mid \mathfrak{M}, x \equiv_g \mathfrak{M}, z\}$. Then $\Sigma_{x\in X} n_x \geq n$. Let now $\Gamma_x = \{\varphi \in \mathcal{L}^g_{\text{EL}} : \mathfrak{M}, x \Vdash_g \varphi\}$. For each $\Delta \subseteq_{<\omega} \Gamma_x$, $\mathfrak{M}, w \Vdash_g \langle a\rangle_{n_x} \bigwedge \Delta$, and with $\mathfrak{M}, w \equiv_g \mathfrak{M}', w'$ we get $\mathfrak{M}', w' \Vdash_g \langle a\rangle_{n_x} \bigwedge \Delta$. Hence $\sigma'_a(w')([\![\Delta]\!]_{\mathfrak{M}'}) \geq n_x$. From graded modal saturation now follows that $\sigma'_a(w')([\![\Gamma_x]\!]_{\mathfrak{M}'}) \geq n_x$. Clearly, whenever $\mathfrak{M}, x \not\equiv_g \mathfrak{M}, y$, we have $[\![\Gamma_x]\!]_{\mathfrak{M}'} \cap [\![\Gamma_y]\!]_{\mathfrak{M}'} = \emptyset$. Therefore, for $X' = \bigcup_{x\in X}[\![\Gamma_x]\!]_{\mathfrak{M}'}$, we must have $\sigma'_a(w')(X') \geq n$. Moreover, for every $x \in X$, there exists $x' \in [\![\Gamma_x]\!]_{\mathfrak{M}'}$. For that x' we obviously that $\mathfrak{M}, x \equiv_g \mathfrak{M}', x'$. Conversely, for any $x' \in X'$, $x' \in [\![\Gamma_x]\!]_{\mathfrak{M}'}$ for some $x \in X$. Hence, we again establish $\mathfrak{M}, x \equiv_g \mathfrak{M}', x'$. Therefore $\mathfrak{M}, X \widehat{\equiv}_g \mathfrak{M}', X'$. □

We close this subsection with an obvious sanity requirement for our translations into and from Kripke models.

PROPOSITION 2.12. *Let graded model* $\mathfrak{M}$ *be given. Then* $\mathfrak{M} \rightleftharpoons_g (\mathfrak{M}_\circ)^\circ$.

Proof. Let $\mathfrak{M} = (W, \{\sigma_a\}_{a\in A}, V)$. Applying Definition 2.4, we get that $(\mathfrak{M}_\circ)^\circ = (W_\circ, \{\sigma'_a\}_{a\in A}, V_\circ)$, where $W_\circ = W \times \mathbb{N}^\omega$, $\sigma'_a(w,i)(u,j) = 1$ if $\sigma_a(w)(u) \geq j > 0$ and $\sigma'_a(w,i)(u,j) = 0$ if $\sigma_a(w)(u) = 0$, and $V_\circ(p) = V(p) \times \mathbb{N}^\omega$. Define relation $Z \subseteq W \times (W \times \mathbb{N}^\omega)$ as below:

$$Z = \{(w,(w,i)) \mid w \in W \;\&\; i \in \mathbb{N}^\omega\}.$$

We show that Z is a graded bisimulation. The atomic condition is obvious as $w \in V(p)$ iff $(w,i) \in V_\circ(p)$.

(Forth) Let $\sigma_a(w)(X) \geq n$ and $\forall v \in X(\sigma_a(w)(v) > 0)$. Consider $X' = \{(v,j) \mid v \in X, \sigma_a(w)(v) \geq j > 0\}$. If there is a $v \in X$ with $\sigma_a(w)(v) = \omega$, then $|X'| = \omega$ and so $\sigma'_a(w,i)(X') = \omega \geq n$. Otherwise, $\sigma'_a(w,i)(v,j) = 1$ for all (v,j) with $\sigma_a(w)(v) \geq j > 0$, so that $\Sigma_{j=1}^{\sigma_a(w)(v)} \sigma'_a(w,i)(v,j) = \sigma_a(w)(v)$ and thus

$$\begin{aligned}\sigma_a(w,i)(X') &= \Sigma_{v\in X}\Sigma_{j=1}^{\sigma_a(w)(v)}\sigma'_a(w,i)(v,j)\\ &= \Sigma_{v\in X}\sigma_a(w)(v)\\ &= \sigma_a(w)(X) \geq n.\end{aligned}$$

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(Back) Let $X' \subseteq W \times \mathbb{N}^{\omega}$ be such that $\sigma'_a(w,i)(X') \geq n$ and $\forall(v,j) \in X', \sigma'_a(w,i)(v,j) > 0$. Consider $X = \{v \in W \mid \exists j(v,j) \in X'\}$. If X' contains a member (v,ω), then $\sigma_a(w)(v) = \omega$, so $\sigma_a(w)(X) \geq \sigma_a(w)(v) \geq n$. Otherwise, $\sigma_a(w)(X) = \sigma'_a(w,i)(X'') \geq \sigma'_a(w,i)(X') \geq n$, where $X'' = \{(v,j) \mid \sigma_a(w)(v) \geq j > 0\}$. (Set X' may be a strict subset of X''.) □

2.4. Axiomatization and completeness. In this section, we consider the axiomatization of graded epistemic logic. In the next section, we present graded epistemic logic versions for the standard logics of knowledge and belief. The axiomatization $\mathtt{K_g}$ presented in Definition 2.13 is equivalent to the Hilbert-style axiomatic system given in de Caro (1988) and Fattorosi-Barnaba & de Caro (1985). It is known as *minimal graded modal logic*.

DEFINITION 2.13. *The minimal graded modal logic* $\mathtt{K_g}$ *consists of the following axiom schemata and inference rules*:

(*Ax*1) *all instances of propositional tautologies*
(*Ax*2) $\langle a\rangle_0\varphi \leftrightarrow \top$
(*Ax*3) $\langle a\rangle_n\bot \leftrightarrow \bot\ (n > 0)$
(*Ax*4) $\langle a\rangle_{n+1}\varphi \rightarrow \langle a\rangle_n\varphi$
(*Ax*5) $[a](\varphi \rightarrow \psi) \rightarrow (\langle a\rangle_n\varphi \rightarrow \langle a\rangle_n\psi)$
(*Ax*6) $\neg\langle a\rangle(\varphi \wedge \psi) \wedge \langle a\rangle_{!m}\varphi \wedge \langle a\rangle_{!n}\psi \rightarrow \langle a\rangle_{!(m+n)}(\varphi \vee \psi)$
(MP) *from* φ *and* $\varphi \rightarrow \psi$ *infer* ψ
(Gen) *from* φ *infer* $[a]\varphi$.

Let $\mathtt{Thm}(\mathtt{K_g})$ *denote the set of all theorems in the system* $\mathtt{K_g}$.

REMARK 2.14. *Let* $n > 0$. *The operator* $\langle a\rangle_n$ *is normal, i.e., it admits the axiom* (*Ax3*). *It is also clear that* $\langle a\rangle_n$ *is monotone*: *from* $\varphi \rightarrow \psi$ *one can get* $\langle a\rangle_n\varphi \rightarrow \langle a\rangle_n\psi$. *Similarly, the dual operator* $[a]_n$ *is monotone. However,* $\langle a\rangle_n$ *is not additive because* $\langle a\rangle_n(\varphi \vee \psi) \rightarrow (\langle a\rangle_n\varphi \vee \langle a\rangle_n\psi)$ *is not valid. Moreover, one can easily verify that the* (*multi*)*modal logic* $\mathtt{K}$ *is a sublogic of* $\mathtt{K_g}$. *Note that the formulae* $\langle a\rangle(\varphi \vee \psi) \leftrightarrow \langle a\rangle\varphi \vee \langle a\rangle\psi$ *and* $[a](\varphi \wedge \psi) \leftrightarrow [a]\varphi \wedge [a]\psi$ *are theorems of* $\mathtt{K_g}$.

A *graded epistemic logic* is a set Λ of $\mathcal{L}^g_{\mathrm{EL}}$-formulae such that (i) $\mathtt{Thm}(\mathtt{K_g}) \subseteq \Lambda$ and (ii) Λ is closed under the rules (MP) and (Gen). By $\vdash_\Lambda \varphi$ we mean that φ is a theorem of Λ. The completeness of $\mathtt{K_g}$ for the Kripke semantics has been shown in de Caro (1988) and Fattorosi-Barnaba & de Caro (1985).

THEOREM 2.15 (Completeness of $\mathtt{K_g}$ for Kripke models, de Caro (1988)). *For any* $\varphi \in \mathcal{L}^g_{EL}$, $\vdash_{\mathtt{K_g}} \varphi$ *if and only if* $\mathcal{F} \Vdash_K \varphi$ *for any Kripke frame* $\mathcal{F}$.

The completeness for the semantics on graded models is a straightforward corollary.

THEOREM 2.16 (Completeness of $\mathtt{K_g}$ for graded models). *For any* $\varphi \in \mathcal{L}^g_{EL}$, $\vdash_{\mathtt{K_g}} \varphi$ *if and only if* $\mathfrak{F} \Vdash_g \varphi$ *for any graded frame* $\mathfrak{F}$.

Proof. The soundness is shown easily. To prove the completeness, assume $\nvdash_{\mathtt{K_g}} \varphi$. Then $\mathcal{F}_{\mathtt{K_g}} \nVdash_K \varphi$ where $\mathcal{F}_{\mathtt{K_g}}$ is the canonical model for $\mathtt{K_g}$ defined in de Caro (1988) and Fattorosi-Barnaba & de Caro (1985). By Proposition 2.5(4), $\mathcal{F}^{\circ}_{\mathtt{K_g}} \nVdash_g \varphi$. □

The completeness can also be directly shown by a canonical model construction using the semantics on graded models. This construction will be used in the next section to prove completeness for extensions of graded epistemic logic on frame classes satisfying

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particular frame properties. We therefore give the construction in detail. The alternative completeness proof is found in the appendix §7. The result that is relevant to show completeness for particular frame classes is Proposition 7.7, page 682.

§3. Graded logics of knowledge and of belief. In this section, we first consider a scala of extensions of the minimal graded modal logic $\mathtt{K_g}$, including their corresponding frame properties, after which we explain how the most relevant cases $\mathtt{S5_g}$ and $\mathtt{KD45_g}$ can be seen as graded versions of, respectively, the standard logics of $\mathtt{S5}$ knowledge and $\mathtt{KD45}$ belief (also known as consistent/introspective belief). Additionally we illustrate, just as in the motivating example in the introductory setting, how in those settings belief in a proposition can be modelled as higher *confidence* in its truth than in its falsity, a rather different usage of graded modalities than the abovementioned $\mathtt{KD45}$ belief.

Table 1 shows the axioms and their correspondents in the weak second-order language. For any graded frame $\mathfrak{F} = (W, \{\sigma_a\}_{a\in\mathcal{A}})$, we use lower letters x, y, z etc. to denote variables ranging over W, and capital letters X, Y, Z etc. to denote variables ranging over $\mathcal{P}^+(W)$. The quantifiers can bind first-order and second-order variables.

PROPOSITION 3.1. *Let* $\mathfrak{F} = (W, \{\sigma_a\}_{a\in\mathcal{A}})$, $a \in \mathcal{A}$ *and* $m, n > 0$. *Then*:

— $\mathfrak{F} \Vdash_g \mathrm{D}_n$ *iff* $\mathfrak{F} \models \forall x\exists Y(\sigma_a(x)(Y) \geq n)$.
— $\mathfrak{F} \Vdash_g \mathrm{T}_n$ *iff* $\mathfrak{F} \models \forall x(\sigma_a(x)(x) \geq n)$.
— $\mathfrak{F} \Vdash_g 4_{mn}$ *iff* $\mathfrak{F} \models \forall xyZ(\sigma_a(x)(y) \geq 1 \;\&\; \sigma_a(y)(Z) \geq m \rightarrow \sigma_a(x)(Z) \geq n)$.
— $\mathfrak{F} \Vdash_g \mathrm{B}_{mn}$ *iff* $\mathfrak{F} \models \forall xy(\sigma_a(x)(y) \geq m \rightarrow \sigma_a(y)(x) \geq n)$.
— $\mathfrak{F} \Vdash_g 5_{mn}$ *iff* $\mathfrak{F} \models \forall xYz(\sigma_a(x)(Y) \geq m \;\&\; \sigma_a(x)(z) \geq 1 \rightarrow \sigma_a(z)(Y) \geq n)$.

Proof. By straightforward verification. □

For any subset $\Gamma \subseteq \{\mathrm{D}_n, \mathrm{T}_n, 4_{mn}, 5_{mn} \mid m, n > 0\}$, let $\mathtt{K_g}\Gamma$ be the graded epistemic logic generated by Γ, i.e., the system obtained from $\mathtt{K_g}$ by adding all substitution instances of formulae in Γ as new axioms.

THEOREM 3.2. *For any* $\Gamma \subseteq \{D_n, T_n, 4_{mn}, 5_{mn} \mid m, n > 0\}$, *the graded epistemic logic* $\mathtt{K_g}\Gamma$ *is sound and complete with respect to the class of all graded frames satisfying all frame conditions corresponding to axioms in* Γ.

Proof. It suffices to show that the canonical frame for $\Lambda = \mathtt{K_g}\Gamma$ is a graded frame for $\mathtt{K_g}\Gamma$. In each of the following cases, assume the axiom belongs to Λ.

— (D_n) By $\langle a\rangle_n\top \in u \in W^\Lambda$, we have $\mathfrak{M}^\Lambda, u \Vdash_g \langle a\rangle_n\top$. Hence there exists $Y \subseteq W^\Lambda$ such that $\sigma_a^\Lambda(x)(Y) \geq n$.
— (T_n) Let $u \in W^\Lambda$ and $\varphi \in u$. Then $\langle a\rangle_n\varphi \in u$. Then $\sigma_a^\Lambda(u)(u) \geq n$.
— (4_{mn}) Assume $\sigma_a^\Lambda(u)(v) \geq 1$ and $\sigma_a^\Lambda(v)(Z) \geq m$. Assume $\varphi \in \bigcap Z$. Then $\langle a\rangle_m\varphi \in v$. Then $\langle a\rangle\langle a\rangle_m\varphi \in u$. By axiom 4_{mn}, $\langle a\rangle_n\varphi \in u$. By Proposition 7.7, $\sigma_a^\Lambda(u)(Z) \geq n$.

Table 1. *Axioms and their names, and corresponding frame properties* ($m, n > 0$)

D_n	$\langle a\rangle_n\top$	$\forall x\exists Y(\sigma_a(x)(Y) \geq n)$
T_n	$\varphi \rightarrow \langle a\rangle_n\varphi$	$\forall x(\sigma_a(x)(x) \geq n)$
4_{mn}	$\langle a\rangle\langle a\rangle_m\varphi \rightarrow \langle a\rangle_n\varphi$	$\forall xyZ(\sigma_a(x)(y) \geq 1 \;\&\; \sigma_a(y)(Z) \geq m \rightarrow \sigma_a(x)(Z) \geq n)$
B_{mn}	$\varphi \rightarrow [a]_m\langle a\rangle_n\varphi$	$\forall xy(\sigma_a(x)(y) \geq m \rightarrow \sigma_a(y)(x) \geq n)$
5_{mn}	$\langle a\rangle_m\varphi \rightarrow [a]\langle a\rangle_n\varphi$	$\forall xYz(\sigma_a(x)(Y) \geq m \;\&\; \sigma_a(x)(z) \geq 1 \rightarrow \sigma_a(z)(Y) \geq n)$

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— (5_{mn}) Assume $\sigma_a^\Lambda(u)(Y) \geq m$ and $\sigma_a^\Lambda(u)(z) \geq 1$. Suppose $\sigma_a^\Lambda(z)(Y) < n$. By Proposition 7.7, there exists $\varphi \in \bigcap Y$ such that $\langle a\rangle_n\varphi \notin v$. By the assumption, $\langle a\rangle_m\varphi \in u$. Hence $[a]\langle a\rangle_n\varphi \in u$. By $\sigma_a^\Lambda(u)(z) \geq 1$, $\langle a\rangle_n\varphi \in v$, a contradiction. □

The graded epistemic logics $\mathtt{KD45_g}$ and $\mathtt{S5_g}$, that will continue to play an important role in this article, are defined as follows:

$$\begin{array}{lcl} \mathtt{KD45_g} & = & \mathrm{K_g}\{\mathrm{D}_1, 4_{nn}, 5_{nn} \mid n > 0\} \\ \mathtt{S5_g} & = & \mathrm{K_g}\{\mathrm{T}_1, 4_{nn}, 5_{nn} \mid n > 0\}. \end{array}$$

As a matter of minor interest, we note that axiom B_{n1} is derivable from T_1 and 5_{nn} in $\mathtt{S5_g}$, and that dually, 5_{nn} is derivable from 4_{nn} and B_{11} in the axiomatization consisting of $\mathtt{S5_g}$ plus B_{11} and minus 5_{nn}. We therefore did not include B_{11} as a case in Theorem 3.2. The logic $\mathtt{KD45_g}$ can be viewed as the graded version of the standard logic of belief $\mathtt{KD45}$, and the logic $\mathtt{S5_g}$ can be viewed as the graded version of the standard logic of knowledge $\mathtt{S5}$. We can make this correspondence clear in different ways. Firstly, consider graded models where all grades are either 0 or 1. Then $D_1 = D$, $T_1 = T$, $4_{11} = 4$, $B_{11} = B$, and $5_{11} = 5$ are the standard modal logical axioms characterizing the frame properties of, respectively, seriality, reflexivity, transitivity, symmetry, and euclidicity. Secondly, consider truth in all accessible worlds. This is definable as $\neg\langle a\rangle_1\neg\varphi$. We can thus define knowledge in $\mathtt{S5_g}$ as

$$K_a\varphi := \neg\langle a\rangle_1\neg\varphi.$$

Clearly, this also defines belief as *conviction* in $\mathtt{KD45}$, in the irrevocable sense of Lenzen (1978) and Segerberg (1998). We resist the temptation to write $B_a\varphi$ for that, and simply write (as Segerberg) $K_a\varphi$ for both *K*nowledge and *K*onviction.

To see why this temptation must be resisted, let us return to our original motivation that we can measure the certainty in a proposition φ as the number of worlds in which it is true. A way to define belief in φ in a graded model is when the certainty of φ is (strictly) larger than the certainty in $\neg\varphi$. This can be a primitive binary operator in a logical language. An *infinitary* way to define belief by abbreviation in our language is

$$B_a\varphi := \bigvee_{n\in\mathbb{N}} (\langle a\rangle_n\varphi \wedge \neg\langle a\rangle_n\neg\varphi).$$

This may not be a formula. But on finite models, a finite subset of $\mathbb{N}$ will suffice and this finite disjunction will then be a formula in the language. We recall that the idea of belief as a majority of φ worlds was mentioned in the introduction as motivating our investigation (van der Hoek, 1992). Similar ideas have been pursued for a long time by, for example, Ghosh & de Jongh (2013), Lenzen (2003), Pacuit & Salame (2004), and Segerberg (1971). It also relates to probabilistic approaches. Pacuit & Salame (2004) is an interesting case as it proposes 'majority spaces' to allow for the definition of belief on infinite domains, and gives a complete axiomatization for a graded modal logic in that setting (although otherwise very different from ours).

In graded models for the logics $\mathtt{KD45_g}$ and $\mathtt{S5_g}$, instead of associating a degree n with a pair of worlds (w, v) such that $\sigma_a(w)(v) = n$, we can associate that degree with the second world v of that pair. It is easy to see that the frame axioms enforce that, in case $\sigma_a(w)(v) = n$, then for any x and m with $\sigma_a(x)(v) = m$, $m = n$ (all arrows pointing to a world have the same weight). In such cases, a simpler visualization suffices than for graded models in general.

For example, the 'S5-like' graded model on the left can be pictured as the one on the right, wherein worlds in the same epistemic equivalence class are linked (and reflexivity,

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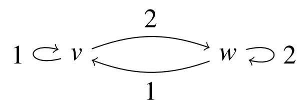
symmetry, and transitivity are thus assumed). In the continuation, we will use this visualization.



§4. Graded public announcement logic. In this section and in the next section, we consider dynamic extensions of graded epistemic logics. In this section, we first discuss the public announcement extension, followed by a motivating example. In the next section, we present the extension with graded event models and their corresponding modalities, and subsequently examples of such complex dynamics.

The language of the public announcement logic $\mathcal{L}^g_{\text{PA}}$ is an extension of $\mathcal{L}^g_{\text{EL}}$ by adding a clause for public announcement formulae of the form $\langle\varphi\rangle\psi$ to the inductive language definition, and where $[\varphi]\psi$ is defined by abbreviation as $\neg\langle\varphi\rangle\neg\psi$.

DEFINITION 4.1. *Given a graded model $\mathfrak{M} = (W, \{\sigma_a\}_{a\in\mathcal{A}}, V)$ and a formula $\varphi \in \mathcal{L}^g_{PA}$ such that $[\![\varphi]\!]_{\mathfrak{M}} \neq \emptyset$, define the updated model of $\mathfrak{M}$ by φ as $\mathfrak{M}^\varphi = (W^\varphi, \{\sigma^\varphi_a\}_{a\in\mathcal{A}}, V^\varphi)$ where*

$W^\varphi = [\![\varphi]\!]_{\mathfrak{M}}$;
for all $w, u \in W^\varphi$, $\sigma^\varphi_a(w)(u) = \sigma_a(w)(u)$;
$V^\varphi(p) = V(p) \cap W^\varphi$, for each $p \in$ Prop.

The truth of a public announcement formula $\langle\varphi\rangle\psi$ is defined as follows:

$$\mathfrak{M}, w \Vdash_g \langle\varphi\rangle\psi \text{ iff } \mathfrak{M}, w \Vdash_g \varphi \text{ and } \mathfrak{M}^\varphi, w \Vdash_g \psi.$$

The public announcements respect graded bisimulation over graded models, namely, we have the following model-theoretic result:

PROPOSITION 4.2. *Let $\mathfrak{M} = (W, \{\sigma_a\}_{a\in\mathcal{A}}, V)$ and $\mathfrak{M}' = (W', \{\sigma'_a\}_{a\in\mathcal{A}}, V')$ be graded models. For every formula $\varphi \in \mathcal{L}^g_{PA}$ such that $[\![\varphi]\!]_{\mathfrak{M}} \neq \emptyset$, if $Z : \mathfrak{M} \rightleftharpoons_g \mathfrak{M}'$, then $Z^\varphi : \mathfrak{M}^\varphi \rightleftharpoons_g \mathfrak{M}'^\varphi$, where $Z^\varphi = Z \cap ([\![\varphi]\!]_{\mathfrak{M}} \times [\![\varphi]\!]_{\mathfrak{M}'})$.*

Proof. Assume $Z : \mathfrak{M} \rightleftharpoons_g \mathfrak{M}'$. Let $wZ^\varphi w'$. Then wZw'. Hence the atomic condition is satisfied. For the forth condition, assume that $\sigma^\varphi_a(w)(X) = i > 0$ and $\sigma^\varphi_a(w)(u) > 0$ for all $u \in X$. Then $\sigma_a(w)(X) = i > 0$ and $\sigma_a(w)(u) > 0$ for all $u \in X$. Then there exists $X' \in \mathcal{P}(W')$ such that $\sigma'_a(w')(X') = i > 0$ and XZX'. Since $X \subseteq W^\varphi$, one can easily show that $X' \subseteq W'^\varphi$. Hence $\sigma'^\varphi_a(w')(X') \geq n$ and $XZ^\varphi X'$. The back condition is similar. □

Let $\texttt{PAL}_\texttt{g}$ be the proof system consisting of $\texttt{K}_\texttt{g}$ plus the set of reduction axioms $\texttt{RA}_\texttt{PAL}$ listed in Table 2. We call this *graded public announcement logic*.

THEOREM 4.3. *Graded public announcement logic $\texttt{PAL}_\texttt{g}$ is sound and complete with respect to the class of graded models.*

Table 2. *Reduction axioms* $\texttt{RA}_\texttt{PAL}$

(RAt)	$\langle\varphi\rangle p \leftrightarrow (\varphi \wedge p)$
(R¬)	$\langle\varphi\rangle\neg\psi \leftrightarrow \varphi \wedge \neg\langle\varphi\rangle\psi$
(R∧)	$\langle\varphi\rangle(\psi \wedge \chi) \leftrightarrow \langle\varphi\rangle\psi \wedge \langle\varphi\rangle\chi$
(R◇)	$\langle\varphi\rangle\langle a\rangle_n\psi \leftrightarrow (\varphi \wedge \langle a\rangle_n\langle\varphi\rangle\psi)$
(RComp)	$\langle\varphi\rangle\langle\psi\rangle\chi \leftrightarrow \langle\langle\varphi\rangle\psi\rangle\chi$

Proof. The completeness is reduced to the completeness of $\mathtt{K_g}$ by reduction axioms. The soundness can be checked routinely. Here we check only the validity of $(\mathrm{R}\langle a\rangle_n)$. If $n = 0$, it is valid obviously. Suppose $n > 0$. Assume $\mathfrak{M}, w \Vdash_g \langle\varphi\rangle\langle a\rangle_n\psi$. Therefore, $\mathfrak{M}, w \Vdash_g \varphi$ and $\mathfrak{M}^\varphi, w \Vdash_g \langle a\rangle_n\psi$. Then there is a finite subset $X \subseteq W^\varphi$ such that $\sigma_a^\varphi(w)(X) \geq n$ and $\mathfrak{M}^\varphi, u \Vdash_g \psi$ for every $u \in X$. Therefore, also $\sigma_a(w)(X) \geq n$. Let $u \in X$. Clearly, $\mathfrak{M}, u \Vdash_g \varphi$. Therefore, $\mathfrak{M}, u \Vdash_g \langle\varphi\rangle\psi$. Hence, $\mathfrak{M}, w \Vdash_g \langle a\rangle_n\langle\varphi\rangle\psi$. Conversely, assume $\mathfrak{M}, w \Vdash_g \varphi \wedge \langle a\rangle_n\langle\varphi\rangle\psi$. Then there is a finite subset $X \subseteq W$ such that $\sigma_a(w)(X) \geq n$ and $\mathfrak{M}, u \Vdash_g \langle\varphi\rangle\psi$ for all $u \in X$. Let u be any state in X. Then $\mathfrak{M}, u \Vdash_g \varphi$ and $\mathfrak{M}^\varphi, u \Vdash_g \psi$. We still have that $\sigma_a^\varphi(w)(X) \geq n$. Hence, $\mathfrak{M}^\varphi, w \Vdash_g \langle a\rangle_n\psi$. Therefore we can conclude that $\mathfrak{M}, w \Vdash_g \langle\varphi\rangle\langle a\rangle_n\psi$. □

For any graded epistemic logic Λ, a graded model $\mathfrak{M}$ is called a *graded model for* Λ if $\mathfrak{M} \models \varphi$ for all $\varphi \in \Lambda$. The class of all graded models for Λ is denoted by $\mathtt{Mod}(\Lambda)$. We say that Λ *respects public announcement* if $\mathtt{Mod}(\Lambda)$ is closed under the model operation $(.)^\varphi$, i.e., $\mathfrak{M} \in \mathtt{Mod}(\Lambda)$ implies $\mathfrak{M}^\varphi \in \mathtt{Mod}(\Lambda)$, for any formula $\varphi \in \mathcal{L}^g_{\mathrm{PA}}$. The *public announcement extension* of Λ is defined as the logic $\mathtt{PAL_g}\Lambda$ obtained from Λ by adding all reduction axioms $\mathtt{RA_{PAL}}$ listed in Table 2.

THEOREM 4.4. *If a graded epistemic logic* Λ *respects public announcement, then the public announcement logic* $\mathtt{PAL_g}\Lambda$ *is sound and complete with respect to* $\mathtt{Mod}(\Lambda)$.

Proof. Directly from Theorem 4.3. □

The following rule (RE) of the replacement of equivalents is derivable in $\mathtt{PAL_g}\Lambda$ (use the method in van Ditmarsch *et al.*, (2007)):

$$\frac{\varphi \leftrightarrow \psi}{\chi \leftrightarrow \chi[\varphi/\psi]}(\mathrm{RE}),$$

where $\chi[\varphi/\psi]$ is obtained from χ by replacing one or more occurrences of φ in χ by ψ. An alternative complete axiomatization consists of $\mathtt{PAL_g}\Lambda \setminus \mathrm{RComp} \cup \mathrm{RE}$, along the lines spelled out in detail by Wang and Cao (2013).

For $\mathtt{PA_gK_g}\Gamma$ with $\Gamma = \{\mathrm{T}_1, 4_{nn}, 5_{nn} \mid n > 0\}$ we write $\mathtt{PA_gS5_g}$. This is the graded modal equivalent of the public announcement logic by Plaza (1989).

COROLLARY 4.5. $\mathtt{PA_gS5_g}$ *is sound and complete with respect to* $\mathtt{Mod}(\mathtt{PA_gS5_g})$.

Proof. We use that the weak second-order conditions for the characteristic axioms in $\mathtt{PA_gS5_g}$, by Proposition 3.1, are universal (i.e., without existential quantifiers); hence they are preserved under taking subframes. Therefore $\mathtt{S5_g}$ respects public announcement. □

It should be noted the logic $\mathtt{KD45_g}$ does *not* respect public announcements, as the D_1 axiom is an existential condition. It is well known that consistency of belief (i.e., in our setting, whether $\langle a\rangle_1\top$ is true) may not be preserved after truthful announcement (Balbiani, van Ditmarsch, Herzig, & de Lima, 2012; van Ditmarsch *et al.*, 2007).

EXAMPLE 4.6. *Consider a single-agent S5 model* $\mathfrak{M}$ *consisting of five worlds*:

$$\overline{p}q\overline{r} - \overline{p}q\overline{r} - pqr - p\overline{q}r - p\overline{q}r.$$

We assume the agent is anonymous, so the links have not been labelled. We name worlds by their valuations, where for example $\overline{p}q\overline{r}$ *stands for a world where p is false, q is true, and r is false. We can see this as a graded 'S5-like' model where the grades of all worlds are* 1 (*as explained in the previous section*). *We note that the two* $\overline{p}q\overline{r}$ *worlds are graded bisimilar, but that they cannot be identified* (*unless we were to increase the grade of that*

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single world to 2). *We have that* $\mathfrak{M} \Vdash_g B_a p$*, as p is true in three and false in two worlds.* (*And this is indeed a model validity.*)

The announcement of q will make the agent lose her belief in p, as $\mathfrak{M}^q$ *consists of three worlds only of which one satisfies p and two satisfy* $\neg p$:

$$\overline{p}q\overline{r}—\overline{p}q\overline{r}—pqr.$$

Therefore, we have that $\mathfrak{M} \Vdash_g B_a p \wedge [q]\neg B_a p$.

On the other hand, the announcement of r in $\mathfrak{M}$ *will strengthen the agent's belief in p as there are now no longer* $\neg p$ *worlds, even up to it becoming knowledge*:

$$pqr—p\overline{q}r—p\overline{q}r.$$

We now have that $\mathfrak{M} \Vdash_g \neg K_a p \wedge [r]K_a p$.

§5. Graded event model logic. Graded public announcement logic is a straightforward extension of graded epistemic logic. For the dynamics of nonpublic events, *action models*, also known as *event models*, are very appropriate (Baltag, Moss, & Solecki, 1998). An event model is a structure like a Kripke model, but with preconditions instead of valuations per domain object. Executing an action corresponds to computing a modal product of a Kripke model and an event model, thus producing a new Kripke model. The peculiarity of event model logic is that such event models also figure as syntactic primitives, i.e., as parameters of dynamic modalities. In graded modal logic, we can entirely copy this approach, with the obvious difference that the actions are now based on graded frames instead of Kripke frames.

We will first give essential definitions, the semantics, a complete axiomatization, and after that some extended examples. The axiomatization is not as straightforward as that of graded public announcement logic. The interaction between graded modalities and graded events is surprisingly straightforward, and the comparison with standard event model logic rather surprising.

DEFINITION 5.1. *A* graded event model *is a tuple* $\mathfrak{E} = (E, \{\sigma_a\}_{a\in\mathcal{A}}, Pre)$ *where E is the domain of* events *or* actions, $(E, \{\sigma_a\}_{a\in\mathcal{A}})$ *is a graded frame, and* $Pre : E \to \mathcal{L}$*, where* $\mathcal{L}$ *is a logical language, is a* precondition *function.*

In Definition 5.1, $\mathcal{L}$ can be any logical language. In this contribution, we only consider the following logical language. The logical language $\mathcal{L}^g_{\text{DEL}}$ is defined as the extension of $\mathcal{L}^g_{\text{EL}}$ with an inductive clause $\langle\mathfrak{E}, e\rangle\varphi$, where e is in the domain E of $\mathfrak{E}$, and with the restriction that E is finite. The formulas in the language $\mathcal{L}^g_{\text{DEL}}$ and the finite graded event models should be simultaneously defined. (This means that given a formula $\psi = \langle\mathfrak{E}, e\rangle\varphi$, all precondition formulas of events in $\mathfrak{E}$ are less complex than ψ.)

A public announcement is a singleton (graded) event model, with as precondition the announcement formula, and with that event graded 1 for all agents.

DEFINITION 5.2. *Given a graded model* $\mathfrak{M} = (W, \{\sigma_a\}_{a\in\mathcal{A}}, V)$ *and a graded event model* $\mathfrak{E} = (E, \{\sigma_a\}_{a\in\mathcal{A}}, Pre)$ *we define the product update of* $\mathfrak{M}$ *by* $\mathfrak{E}$ *as the graded model* $\mathfrak{M} \otimes \mathfrak{E} = (W^E, \{\sigma^E_a\}_{a\in\mathcal{A}}, V^E)$ *where*

— $W^E = \{(w, e) : \mathfrak{M}, w \Vdash_g Pre(e)\}$.
— $\sigma^E_a(w, e)(v, f) = \sigma_a(w)(v) \cdot \sigma_a(e)(f)$.
— $V^E(p) = \{(w, e) : w \in V(p)\}$*, for each* $p \in$ Prop.

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Table 3. *Reduction axioms* $\mathtt{RA_{DEL}}$ *for graded event models*

(DRAt)	$\langle\mathfrak{E}, e\rangle p \leftrightarrow Pre(e) \wedge p$
(DR¬)	$\langle\mathfrak{E}, e\rangle\neg\varphi \leftrightarrow Pre(e) \wedge \neg\langle\mathfrak{E}, e\rangle\varphi$
(DR∧)	$\langle\mathfrak{E}, e\rangle(\varphi \wedge \psi) \leftrightarrow \langle\mathfrak{E}, e\rangle\varphi \wedge \langle\mathfrak{E}, e\rangle\psi$
(DRComp)	$\langle\mathfrak{E}, e\rangle\langle\mathfrak{E}', e'\rangle\varphi \leftrightarrow \langle\mathfrak{E} \circ \mathfrak{E}', (e, e')\rangle\varphi$
(DR⊗)	$\langle\mathfrak{E}, e\rangle\langle a\rangle_m\varphi \leftrightarrow Pre(e) \wedge \bigvee_S \bigwedge_{f\in E}\langle a\rangle_{n_f}\langle\mathfrak{E}, f\rangle\varphi$
	where $m = \Sigma_{f\in E}(n_f \cdot \sigma_a(e)(f))$ and $S = \{\vec{n_f} : m = \Sigma_{f\in E}(n_f \cdot \sigma_a(e)(f))\}$

The truth of $\langle\mathfrak{E}, e\rangle\varphi$ *at a state in a graded model is defined as follows*:

$$\mathfrak{M}, w \Vdash_g \langle\mathfrak{E}, e\rangle\varphi \textit{ iff } \mathfrak{M}, w \Vdash_g Pre(e) \textit{ and } \mathfrak{M} \otimes \mathfrak{E}, (w, e) \Vdash_g \varphi.$$

Given graded model $\mathfrak{M}$ and graded event model $\mathfrak{E}$, it is easy to see that $\mathfrak{M} \otimes \mathfrak{E}$ is a graded model.

DEFINITION 5.3. *Given graded event models* $\mathfrak{E} = (E, \{\sigma_a\}_{a\in\mathcal{A}}, Pre)$ *and* $\mathfrak{E}' = (E', \{\sigma'_a\}_{a\in\mathcal{A}}, Pre')$, *their* composition $\mathfrak{E}'' = \mathfrak{E} \circ \mathfrak{E}'$ *is defined as the graded event model* $\mathfrak{E}'' = (E'', \{\sigma''_a\}_{a\in\mathcal{A}}, Pre'')$ *where*

— $E'' = E \times E'$.
— $\sigma''_a(e, e')(f, f') = \sigma_a(e)(f) \cdot \sigma'_a(e')(f')$.
— $Pre''(e, e') = Pre(e) \wedge \langle\mathfrak{E}, e\rangle Pre'(e')$.

PROPOSITION 5.4. *Schema* $\langle\mathfrak{E} \circ \mathfrak{E}', (e, e')\rangle\varphi \leftrightarrow \langle\mathfrak{E}, e\rangle\langle\mathfrak{E}', e'\rangle\varphi$ *is valid.*

Proof. Obvious. □

PROPOSITION 5.5. *Let* $\mathfrak{M} = (W, \{\sigma_a\}_{a\in\mathcal{A}}, V)$ *and* $\mathfrak{M}' = (W', \{\sigma'_a\}_{a\in\mathcal{A}}, V')$ *be graded models. For any graded event model* $\mathfrak{E} = (E, \{\sigma_a\}_{a\in\mathcal{A}}, Pre)$, *if* $Z : \mathfrak{M} \rightleftharpoons_g \mathfrak{M}'$, *then* $Z^{\mathfrak{E}} : \mathfrak{M} \otimes \mathfrak{E} \rightleftharpoons_g \mathfrak{M}' \otimes \mathfrak{E}$, *where* $Z^{\mathfrak{E}} = Z \cap (W^E \times W'^E)$.

Proof. Similar to Theorem 4.2. □

The reduction axioms $\mathtt{RA_{DEL}}$ are listed in Table 3.[1] Let $\mathtt{K_g RA_{DEL}}$ be the axiomatic system obtained from $\mathtt{K_g}$ by adding the reduction axioms in $\mathtt{RA_{DEL}}$.

THEOREM 5.6. *The dynamic graded epistemic logic* $\mathtt{K_g RA_{DEL}}$ *is sound and complete with respect to the class of all graded models.*

Proof. As in the case for the public announcement logic $\mathtt{PAL_g}$, the completeness of $\mathtt{K_g RA_{DEL}}$ is reduced to the completeness of $\mathtt{K_g}$ by reduction axioms. The soundness can be checked routinely, with the exception of the axiom DR⊗.

We now prove that DR⊗ is valid. Let $\mathfrak{M} = (W, \{\sigma_a\}_{a\in A}, V)$ and $w \in W$ be given.

(⇒) Let $\mathfrak{M}, w \Vdash_g \langle\mathfrak{E}, e\rangle\langle a\rangle_m\varphi$. By definition, $\mathfrak{M}, w \Vdash_g Pre(e)$ and $\mathfrak{M} \otimes \mathfrak{E}, (w, e) \Vdash_g \langle a\rangle_m\varphi$, i.e., there is X such that $\sigma_a(w, e)(X) \geq m$ and for all $(v, f) \in X$, $\mathfrak{M} \otimes \mathfrak{E}, (v, f) \Vdash_g \varphi$. If $m = 0$, take $X = \emptyset$ and all $n_f = 0$ and we are done. So let $m > 0$.

Let $F = \{f \in E : \exists v \in W, (v, f) \in X\}$, let for all $f \in F$, $V_f = \{v \in W : (v, f) \in X\}$ and $\max_f := \sigma_a(w)(V_f)$, and for all $f \notin F$, $\max_f := 0$. The set V_f consists of all worlds (occurring in pairs of X) wherein f can be executed. First observe that:

[1] The (correct version of) axiom (DR⊗) was suggested by an anonymous reviewer of the journal.

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$$\begin{array}{rcl}\sigma_a(w,e)(X) & = & \Sigma_{(v,f)\in X}\sigma_a(w,e)(v,f)\\ & = & \Sigma_{(v,f)\in X}(\sigma_a(w)(v)\cdot\sigma_a(e)(f))\\ & = & \Sigma_{f\in F}(\Sigma_{v\in V_f}(\sigma_a(w)(v)\cdot\sigma_a(e)(f)))\\ & = & \Sigma_{f\in F}(\sigma_a(w)(V_f)\cdot\sigma_a(e)(f))\\ & = & \Sigma_{f\in F}(\max_f\cdot\sigma_a(e)(f))\\ & = & \Sigma_{f\in F}(\max_f\cdot\sigma_a(e)(f))+\Sigma_{f\notin F}(0\cdot\sigma_a(e)(f))\\ & = & \Sigma_{f\in E}(\max_f\cdot\sigma_a(e)(f)).\end{array}$$

For all $f \in F$, choose $n_f \le \max_f$ such that $m = \Sigma_{f\in E}(n_f \cdot \sigma_a(e)(f))$. This choice can be made, because $\Sigma_{f\in E}(n_f \cdot \sigma_a(e)(f)) \le \Sigma_{f\in E}(\max_f \cdot\sigma_a(e)(f)) = \sigma_a(w,e)(X)$, and our assumption was that $m \le \sigma_a(w,e)(X)$.

We can now prove our claim that for all $f \in E$, $\mathfrak{M}, w \Vdash_g \langle a\rangle_{n_f}\langle\mathfrak{E}, f\rangle\varphi$. By the semantic definition this is equivalent to: there is a Y such that $\sigma_a(w,Y) \ge n_f$ and $\mathfrak{M}, v \Vdash_g \langle\mathfrak{E}, f\rangle\varphi$ for all $v \in Y$. For $f \notin F$, $n_f \le \max_f = 0$, so this is satisfied for choice $Y = \emptyset$ (we recall that $\langle a\rangle_0\psi$ is a validity for any ψ). For $f \in E$, Choose $Y = V_f$. We now use the assumption that $\sigma_a(w,e)(X) \ge m$, from which, given our choice of n_f as above, it follows that $\sigma_a(w,Y) = \sigma_a(w,V_f) = \max_f \ge n_f$. We further need to establish $\mathfrak{M}, v \Vdash_g \langle\mathfrak{E}, f\rangle\varphi$, i.e., $\mathfrak{M}, v \Vdash_g Pre(f)$ and $\mathfrak{M}\otimes\mathfrak{E}, (v,f) \Vdash_g \varphi$. Both follow from the observation that $(v,f)\in X$: we recall for all $(v,f)\in X$, $\mathfrak{M}\otimes\mathfrak{E}$, $(v,f) \Vdash_g \varphi$.

($\Leftarrow$)

The other direction follows more directly, by taking the n_f given in the assumption.

Given that soundness is established, the completeness of $\mathtt{K_gRA_{DEL}}$ is reduced to the completeness of $\mathtt{K_g}$ by showing that all formulae are provably equivalent to formulae without graded event models. □

Also, as for public announcement (graded) modal logic $\mathtt{PAL_g}$, we can extend the logic with axioms for frame properties, and thus (e.g.,) show that graded dynamic epistemic logic with additionally axioms $\{T_n, 4_{mn}, 5_{mn} \mid m,n > 0\}$ is sound and complete on class $\mathtt{S5_g}$. Furthermore, we similarly have the choice in the axiomatization between the composition of event models axiom DRComp or the derivation rule RE of the replacement of equivalents "From $\varphi \leftrightarrow \psi$, infer $\chi \leftrightarrow \chi[\varphi/\psi]$." Both axiomatizations are complete.

Let us explain the shapes of the axioms to reduce a graded modality after an announcement, and to reduce a graded modality after a graded event, in relation to each other and to their classical counterparts for Kripke semantics $R\Diamond_K$ (Plaza, 1989) and $DR\otimes_K$ (Baltag *et al.*, 1998). We recall that they are as follows, where in $DR\otimes$ set S consists of all lists of grades n_f for which $m = \Sigma_{f\in E}(n_f\cdot\sigma_a(e)(f))$.

$$\begin{array}{l|lllll} R\Diamond_K & \langle\psi\rangle\langle a\rangle\varphi & \leftrightarrow & \psi & \wedge & \langle a\rangle\langle\psi\rangle\varphi\\ DR\otimes_K & \langle\mathfrak{E},e\rangle\langle a\rangle\varphi & \leftrightarrow & Pre(e) & \wedge & \bigvee_{f\in R_a(e)}\langle a\rangle\langle\mathfrak{E},f\rangle\varphi\\ R\Diamond & \langle\psi\rangle\langle a\rangle_n\varphi & \leftrightarrow & \psi & \wedge & \langle a\rangle_n\langle\psi\rangle\varphi\\ DR\otimes & \langle\mathfrak{E},e\rangle\langle a\rangle_m\varphi & \leftrightarrow & Pre(e) & \wedge & \bigvee_S\bigwedge_{f\in E}\langle a\rangle_{n_f}\langle\mathfrak{E},f\rangle\varphi.\end{array}$$

To see why $R\Diamond$ is a special case of $DR\otimes$, consider the following rephrasings of the axiom $DR\otimes$ for the case of a graded event model for public announcement.

$$\begin{array}{lllllll} \text{(i)} & \langle\mathfrak{E},e\rangle\langle a\rangle_m\varphi & \leftrightarrow & Pre(e) & \wedge & \bigvee_S\bigwedge_{f\in E}\langle a\rangle_{n_f}\langle\mathfrak{E},f\rangle\varphi\\ \text{(ii)} & \langle\mathfrak{E},e\rangle\langle a\rangle_m\varphi & \leftrightarrow & Pre(e) & \wedge & \bigvee_S\langle a\rangle_{n_e}\langle\mathfrak{E},e\rangle\varphi\\ \text{(iii)} & \langle\mathfrak{E},e\rangle\langle a\rangle_{n_e}\varphi & \leftrightarrow & Pre(e) & \wedge & \langle a\rangle_{n_e}\langle\mathfrak{E},e\rangle\varphi\\ \text{(iv)} & \langle\psi\rangle\langle a\rangle_{n_e}\varphi & \leftrightarrow & \psi & \wedge & \langle a\rangle_{n_e}\langle\psi\rangle\varphi\end{array}$$

We can identify (i) and (ii) because the graded event model for a public announcement is a singleton $E = \{e\}$. We can identify (ii) and (iii) because the set S consists of the one-item list n_e only, so that $m = n_e \cdot \sigma_a(e)(e) = n_e \cdot 1 = n_e$. We can identify (iii) and (iv) because $Pre(e) = \psi$. This is straightforward.

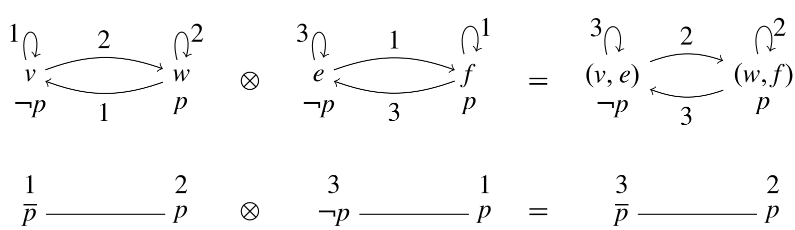
How to see DR$\otimes_K$ as a special case of DR$\otimes$ is, we think, rather interesting. First, note that in the summation $m = \Sigma_{f\in E}(n_f \cdot \sigma_a(e)(f))$ we can restrict the set E of all events to the set $F = \{f \in E : \sigma_a(e)(f) > 0\}$ of all events with positive grade from e's perspective for agent a. This set F is of course the same as the set $R_a(e) = \{f \in E : (e,f) \in R_a\}$ in Kripke semantics, of events f that are accessible from e by a. Therefore, $m = \Sigma_{f\in R_a(e)}(n_f \cdot \sigma_a(e)(f))$.

Next, note that what only counts in Kripke semantics is that the disjunction over all S is such that their grades add up *in some way*: it only matters that $m > 0$. So from the set S consisting of all lists of grades n_f for which $m = \Sigma_{f\in R_a(e)}(n_f \cdot \sigma_a(e)(f))$ we can choose any member that makes m positive. For this it suffices that for any of the $f \in R_a(e)$, n_f is positive. In other words, we can equate $\bigvee_S \bigwedge_{f\in R_a(e)} \langle a\rangle_{n_f} \langle \mathfrak{E}, f\rangle\varphi$ with $\bigvee_{f\in R_a(e)} \langle a\rangle_{n_f} \langle \mathfrak{E}, f\rangle\varphi$. There is no need to count as long as we have 1. The last identification we need is

$$\begin{array}{lcl} \langle \mathfrak{E}, e\rangle\langle a\rangle_m\varphi & \leftrightarrow & Pre(e) \wedge \bigvee_{f\in R_a(e)} \langle a\rangle_{n_f}\langle \mathfrak{E}, f\rangle\varphi \\ \langle \mathfrak{E}, e\rangle\langle a\rangle\varphi & \leftrightarrow & Pre(e) \wedge \bigvee_{f\in R_a(e)} \langle a\rangle\langle \mathfrak{E}, f\rangle\varphi. \end{array}$$

Here again, we use that it only matters that m and n_f are positive natural numbers, as in the standard modal language $\langle a\rangle$ replaces $\langle a\rangle_n$ for any positive n. This observation also explains the relation between R$\Diamond_K$ and R$\Diamond$.

We demonstrate the execution of graded event models and their usage in modelling multiagent system dynamics with a number of examples. We illustrate change of *knowledge*, namely where both the graded (static) model and the graded event model satisfy the properties characterized by T_1, 4_{nn}, and 5_{nn} for all $n > 0$, the principles of the logic $\mathsf{S5_g}$. We further recall that a simpler visualization then suffices than for graded models in general, where we only need to give weights to worlds. As an example of graded event model execution, including the simpler visualization[2] consider the following.



Observe that the execution is according to the semantics of event model execution. For example, we have that $\sigma_a(w,f)(v,e) = \sigma_a(w)(v) \cdot \sigma_a(f)(e) = 1 \cdot 3 = 3$.

The initial model represents that the agent considers p twice as likely as $\neg p$ (so is inclined to believe that p; for example, to believe that she is running a fever), the event model represents an update that is three times more likely to be with $\neg p$ than with p (for example, a partial observation of the value of p that is strongly inclined to be the observation of $\neg p$; let us say the reading of a badly visible thermometer in a nearly dark

[2] With some further simplifying assumptions already used before, such as writing $\overline{p}$ for the valuation where p is false. Still, we write $\neg p$ for the precondition *formula*, not valuation, of an event.

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sickroom by a therefore conservative estimate, strongly favouring low readings), and as a result of executing that event she changed her belief into that of $\neg p$ (she now believes that she is not running a fever).

Some other examples of graded event model execution are depicted in Figure 1. Let us explain them informally. To keep things simple, let in all cases the graded model be called $\mathfrak{M}$, where an initial p world is w and an initial $\neg p$ world is v, and let the graded event model be called $\mathfrak{E}$, where the left event is called e and the right event is called f (and in case v the top event is called g). We recall the definitions of knowledge and belief from §3: An agent believes φ if the φ worlds exceed the $\neg\varphi$ words, and an agent knows φ if the degree of $\neg\varphi$-accessible worlds is 0 (i.e., no $\neg\varphi$ worlds are accessible).

(i) We have that $\mathfrak{M}, w \Vdash_g \langle \mathfrak{E}, f\rangle\langle a\rangle_2 p$ but $\mathfrak{M}, w \Vdash_g \langle a\rangle_1\langle \mathfrak{E}, f\rangle p$. Executing $\mathfrak{E}$ in $\mathfrak{M}$ duplicates the model. Differently said: all weights double.

(ii) But it is not always the case that executing a two-event model duplicates the graded model. The typical example is when one of the events can never be executed, as here (precondition $\bot$). So we now have $\mathfrak{M}, w \Vdash_g \langle \mathfrak{E}, f\rangle\langle a\rangle_1 p$ and also $\mathfrak{M}, w \Vdash_g \langle a\rangle_1\langle \mathfrak{E}, f\rangle p$. The degrees correspond before and after.

(iii) A different way to achieve the effect of (i) is to execute a singleton event with grade 2. It is tempting to consider a notion of 'event emulation,' along the lines of van Eijck, Ruan, & Sadzik (2012), under which the events under (i) and (iii) are 'the same' (i.e., have the same update effect).

(iv) This executes a classic scenario in dynamic epistemic logic: given a situation wherein two agents a and b are uncertain about p, and where this is common knowledge, agent b receives information about p and such that agent a observes that b is informed about p without getting that information (for example, b receives a letter containing the truth about p and opens and reads the letter in the

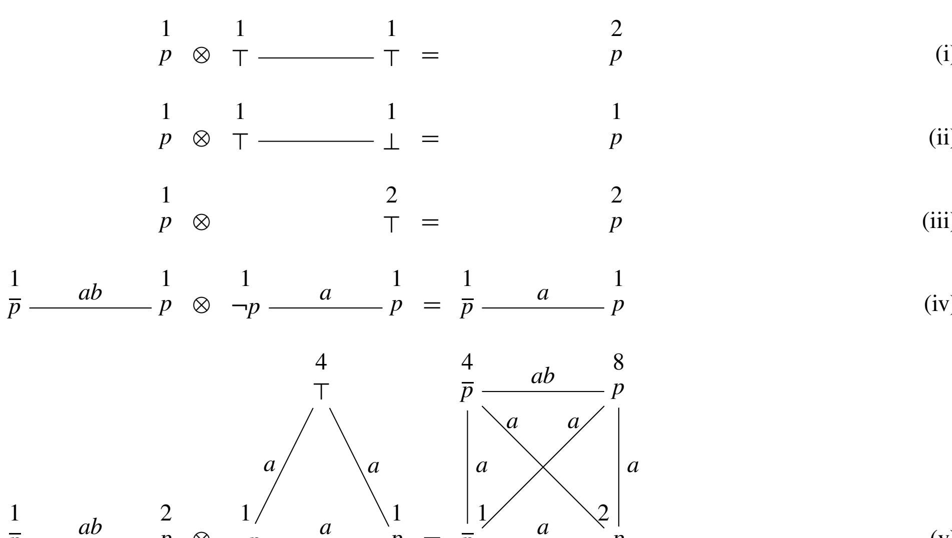

Fig. 1. Examples of graded event execution. For nonlabelled edges assume an agent a. Label ab means that the worlds are indistinguishable for a and for b (there are two edges). The depicted grades are only those for agent a. We further only assume that b has positive grade in any member of any of his equivalence classes.

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presence of a). (In the graded event model (and in the resulting graded model), the equivalence classes for b are singleton. Therefore, there is only an a link between the events/worlds.)

(v) As a more complex variation on (iv), now consider a and b being uncertain about p but with a certain bias towards p (they both believe p, because the degree of the p world is larger than the degree of the $\neg p$ world). Again, b receives a letter containing the truth about p and opens and reads the letter in the presence of a. However, a was temporarily absent (ordering cups of coffee at the counter) and considers it possible that b has not yet read the letter. She even considers that much more likely than that b read the letter (weight 4). In the resulting model $\mathfrak{M} \otimes \mathfrak{E}$, in (bottom right) state (w, f), agent a believes p (which is true but unjustified belief) and agent b knows p; and a incorrectly believes that b is ignorant about p: we have that $\mathfrak{M} \otimes \mathfrak{E}, (w, f) \Vdash_g \langle a \rangle_{10} p \wedge \langle a \rangle_5 \neg p$. Therefore $B_a p$. Also, $\mathfrak{M} \otimes \mathfrak{E}, (w, f) \Vdash_g \neg \langle b \rangle_1 \neg p$, i.e., $K_b p$. Whereas, without further complicating matters with notation: the grade of worlds where b knows whether p is 3 and the grade of worlds where b is ignorant about p is 12, so that we have $K_b p \wedge B_a \neg(K_b p \vee K_b \neg p)$, for a incorrectly believes that b is still uncertain about p.

§6. Conclusion. We proposed graded epistemic logics, interpreted on graded models that are generalizations of Kripke models. We provided axiomatizations for such logics, also with additional frame properties. Our main contribution is that we defined dynamic extensions of graded epistemic logics, namely, graded public announcements logic and graded event model logic, where we also presented complete axiomatizations for these logics. The interaction between the dynamics and the graded modality is quite different from the usual interaction in dynamic epistemic logics. We illustrate our logics with derived belief and knowledge operators.

§7. Appendix: Completeness revisited. We first recall some standard terminology. A graded epistemic logic Λ is said to be *consistent*, if $\bot \notin \Lambda$. A formula φ is a *consequence* of a set of formulae Γ in Λ, notation $\Gamma \vdash_\Lambda \varphi$, if there is a finite subset $\Delta \subseteq_{<\omega} \Gamma$ with $\bigwedge \Delta \to \varphi \in \Lambda$. We understand $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$. A set of formulae $\Gamma \subseteq \mathcal{L}^g_{\mathrm{EL}}$ is said to be Λ-*consistent*, if $\Gamma \nvdash_\Lambda \bot$. A Λ-consistent set Γ is called *maximal* Λ-*consistent*, if Γ has no proper superset which is Λ-consistent. We use u, v etc. to denote maximal consistent sets. It is easy to check that the Lindenbaum lemma holds, i.e., every Λ-consistent set of formulae can be extended to be a maximal one.

LEMMA 7.1. *Let Λ be a consistent graded epistemic logic, and u be a maximal Λ-consistent set. The following hold*:

(1) $\Lambda \subseteq u$ *and* $\bot \notin u$.
(2) $\neg\varphi \in u$ *iff* $\varphi \notin u$.
(3) $\varphi \vee \psi \in u$ *iff* $\varphi \in \Gamma$ *or* $\psi \in u$.
(4) $\varphi \wedge \psi \in u$ *iff* $\varphi \in u$ *and* $\psi \in u$.
(5) *if* $\langle a \rangle_{!n} \varphi \in u$, *then* $\langle a \rangle_{!m} \varphi \notin u$ *for any* $m \neq n$.
(6) *either* $\forall n \in \mathbb{N}(\langle a \rangle_n \varphi \in u)$, *or* $\exists n \in \mathbb{N}(\langle a \rangle_{!n} \varphi \in u)$.
(7) *if* $\varphi \to \psi \in \Lambda$ *and* $\langle a \rangle_{!n} \psi \in u$, *there exists unique* $m \leq n$ *with* $\langle a \rangle_{!m} \varphi \in u$.

Proof. Items (1)–(4) are properties that hold for every maximal consistent set.

(5) Assume $\langle a\rangle_{!n}\varphi \in u$ and $\langle a\rangle_{!m}\varphi \in u$. Assume $m < n$ without loss of generality. Then $m+1 \leq n$. By $\langle a\rangle_{!n}\varphi \in u$, we have $\langle a\rangle_{n}\varphi \in u$. Because $\langle a\rangle_{n}\varphi \to \langle a\rangle_{m+1}\varphi \in \Lambda$, we have $\langle a\rangle_{m+1}\varphi \in u$, a contradiction. Then $m = n$.

(6) Assume $\langle a\rangle_{n}\varphi \notin u$ for some $n \in \mathbb{N}$. Then $n \neq 0$. Let m be the least number such that $\langle a\rangle_{m}\varphi \notin u$. Then $\langle a\rangle_{m-1}\varphi \in u$. Hence $\langle a\rangle_{!(m-1)}\varphi \in u$.

(7) Assume $\varphi \to \psi \in \Lambda$ and $\langle a\rangle_{!n}\psi \in u$. Then $\langle a\rangle_{n}\psi \in u$ and $\langle a\rangle_{n+1}\psi \notin u$. Since $\varphi \to \psi \in \Lambda$, we have $\langle a\rangle_{n+1}\varphi \to \langle a\rangle_{n+1}\psi \in \Lambda$. Hence $\langle a\rangle_{n+1}\varphi \notin u$. By (6), there exists $m \in \mathbb{N}$ with $\langle a\rangle_{!m}\varphi \in u$. By (5), m is unique. Assume $n < m$. Then $n+1 \leq m$. Hence $\langle a\rangle_{m}\varphi \to \langle a\rangle_{n+1}\varphi \in \Lambda$. By $\langle a\rangle_{m}\varphi \in u$, we have $\langle a\rangle_{n+1}\varphi \in u$, a contradiction. ☐

DEFINITION 7.2. *For any graded epistemic logic* Λ*, the* canonical model $\mathfrak{M}^{\Lambda} = (W^{\Lambda}, \{\sigma_a^{\Lambda}\}_{a\in\mathcal{A}}, V^{\Lambda})$ *is defined as follows*:

(1) $W^{\Lambda} = \{u \mid u$ *is a maximal* Λ*-consistent set of* $\mathcal{L}_{EL}^{g}$*-formulae*$\}$.

(2) *Define* σ_a^{Λ} *as follows*:

$$\sigma_a^{\Lambda}(u)(v) = \begin{cases} \omega, & \text{if } \forall\varphi \in v \forall n \in \mathbb{N}(\langle a\rangle_{n}\varphi \in u). \\ min\{n \in \mathbb{N} \mid \langle a\rangle_{!n}\varphi \in u \;\&\; \varphi \in v\}, & \text{otherwise}. \end{cases}$$

(3) $V^{\Lambda}(p) = \{u \in W^{\Lambda} \mid p \in u\}$ *for each* $p \in$ Prop.

Note that the definition of the function σ^{Λ} *is sound by Lemma 7.1(6). We say that* $\mathfrak{F}^{\Lambda} = (W^{\Lambda}, \{\sigma_a^{\Lambda}\}_{a\in\mathcal{A}})$ *is the* canonical frame *for* Λ.

LEMMA 7.3. *For pairwise different maximal* Λ*-consistent sets* $u_0, \ldots, u_n$ $(n > 0)$*, there exist formulae* $\varphi_0, \ldots, \varphi_n$ *such that* $\varphi_i \in u_i$ *and* $\varphi_i \wedge \varphi_j \leftrightarrow \bot \in \Lambda$ *for all* $i \neq j \leq n$.

Proof. By induction on $n > 0$. For $n = 1$, let $u_0 \neq u_1$. Then there is a formula $\varphi_0 \in u_0$ such that $\varphi_0 \notin u_1$. Then $\neg\varphi_0 \in u_1$, and $\varphi_0 \wedge \neg\varphi_0 \leftrightarrow \bot \in \Lambda$. For the inductive step, let $u_0, \ldots, u_n, u_{n+1}$ be pairwise different. By induction hypothesis, $\varphi_0 \in u_0, \ldots, \varphi_n \in u_n$ such that $\varphi_i \wedge \varphi_j \leftrightarrow \bot \in \Lambda$ for $i \neq j \leq n$. Let $\psi_0, \ldots, \psi_n \in u_{n+1}$ and $\psi_i \notin u_i$ for $i \leq n$. Let $\psi = \psi_0 \wedge \cdots \wedge \psi_n$. Then $\psi \notin u_i$ and hence $\neg\psi \in u_i$ for $i \leq u$. Let $\theta_i = \varphi_i \wedge \neg\psi$ for $i \leq n$ and $\theta_{n+1} = \psi$. Obviously $\theta_i \in u_i$ and $\theta_i \wedge \theta_j \leftrightarrow \bot \in \Lambda$ for $i \neq j \leq n+1$. ☐

LEMMA 7.4. *Let* u *be a maximal* Λ*-consistent set, and* $\varphi_0, \ldots, \varphi_k$ $(k \geq 0)$ *be formulae such that* $\langle a\rangle_{!n_i}\varphi_i \in u$ *and* $\varphi_i \wedge \varphi_j \leftrightarrow \bot \in \Lambda$ *for all* $i \neq j \leq k$*. Let* $\varphi = \varphi_0 \vee \cdots \vee \varphi_k$ *and* $n = n_0 + \cdots + n_k$*. Then* $\langle a\rangle_{!n}\varphi \in u$.

Proof. By induction on k. We separately distinguish $k = 0$, that is a trivial case, and $k = 1$. For $k = 1$, let $\langle a\rangle_{!n_0}\varphi_0 \in u$, $\langle a\rangle_{!n_1}\varphi_1 \in u$, $\varphi = \varphi_0 \vee \varphi_1$, $n = n_0 + n_1$ and $\varphi_0 \wedge \varphi_1 \leftrightarrow \bot \in \Lambda$. Then we have $\neg(\varphi_0 \wedge \varphi_1) \in \Lambda$. By (Gen), we have $[a]\neg(\varphi_0 \wedge \varphi_1) \in \Lambda$. Then $\neg\langle a\rangle(\varphi_0 \wedge \varphi_1) \in \Lambda$. Since $\neg\langle a\rangle(\varphi_0 \wedge \varphi_1) \to (\langle a\rangle_{!n_0}\varphi_0 \to (\langle a\rangle_{!n_1}\varphi_1 \to \langle a\rangle_{!n}\varphi)) \in \Lambda$ by (Ax6), $\langle a\rangle_{!n_0}\varphi_0 \in u$ and $\langle a\rangle_{!n_1}\varphi_1 \in u$, we have $\langle a\rangle_{!n}\varphi \in u$. For the inductive step, let $\langle a\rangle_{!n_i}\varphi_i \in u$ and $\varphi_i \wedge \varphi_j \leftrightarrow \bot \in \Lambda$ for all $i \neq j \leq k+1$. Let $\varphi = \varphi_0 \vee \cdots \vee \varphi_k$, and $n = n_0 + \cdots + n_k$. By induction hypothesis, we have $\langle a\rangle_{!n}\varphi \in u$. Clearly $\neg\langle a\rangle(\varphi \wedge \varphi_{k+1}) \in \Lambda$ and $\langle a\rangle_{!n_{k+1}}\varphi_{k+1} \in u$. By a similar argument as for the case $k = 1$, we have $\langle a\rangle_{!(n+n_{k+1})}(\varphi \vee \varphi_{k+1}) \in u$. ☐

LEMMA 7.5. *Let* u, v *be maximal* Λ*-consistent sets. Then*

(1) $\sigma_a^{\Lambda}(u)(v) \geq n$ *iff* $\langle a\rangle_{n}\varphi \in u$ *for all* $\varphi \in v$.

(2) *if* $\langle a\rangle\varphi \in u$*, there exists* $v \in W^{\Lambda}$ *such that* $\sigma_a^{\Lambda}(u)(v) \geq 1$ *and* $\varphi \in v$.

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Proof. (1) The case for $n = 0$ is obvious. Let $n > 0$. Assume $\sigma_a^\Lambda(u)(v) \geq n$ and $\varphi \in v$ but $\langle a\rangle_n\varphi \notin u$. By Lemma 7.1(6), there exists $m \in \mathbb{N}$ such that $\langle a\rangle_{!m}\varphi \in u$. Clearly, $m < n$. Hence $\sigma_a^\Lambda(u)(v) \leq m < n$, a contradiction. Conversely, assume $\langle a\rangle_n\varphi \in u$ for all $\varphi \in v$, but $\sigma_a^\Lambda(u)(v) = k < n$. Then $\psi \in v$ and $\langle a\rangle_{!k}\psi \in u$ for some formula ψ. Hence $\langle a\rangle_{k+1}\psi \notin u$. By the assumption, for $\psi \in v$, we have $\langle a\rangle_n\psi \in u$. Since $k < n$, we have $k + 1 \leq n$ and so $\langle a\rangle_n\psi \to \langle a\rangle_{k+1}\psi \in \Lambda$. Then $\langle a\rangle_{k+1}\psi \in u$, a contradiction.

(2) Assume $\langle a\rangle\varphi \in u$. Consider the set $\Gamma = \{\varphi\} \cup \{\psi \mid [a]\psi \in u\}$. Now, we show that Γ is Λ-consistent. Suppose not. There exist $\psi_1, \ldots, \psi_n$ such that $[a]\psi_i \in u$ for $i \in \{1, \ldots, n\}$ and $\varphi_1 \wedge \cdots \wedge \psi_n \to \neg\varphi \in \Lambda$. By (Gen) and the distributivity of $[a]$ over conjunction, $[a]\varphi_1 \wedge \cdots \wedge [a]\psi_n \to \neg\langle a\rangle\varphi \in \Lambda$. Hence $\neg\langle a\rangle\varphi \in u$, i.e., $\langle a\rangle\varphi \notin u$, a contradiction. □

LEMMA 7.6 (Truth). *For every formula $\varphi \in \mathcal{L}^g_{EL}$, $\mathfrak{M}^\Lambda, u \Vdash_g \varphi$ iff $\varphi \in u$.*

Proof. By induction on the complexity of φ. The atomic and Boolean cases are easy. Let $\varphi := \langle a\rangle_n\psi$. For $n = 0$, the lemma holds obviously. For $n = 1$, the lemma holds by Lemma 7.5(2). Assume $n > 1$.

(1) Assume $\mathfrak{M}^\Lambda, u \Vdash_g \langle a\rangle_n\psi$. There is $X = \{v_0, \ldots, v_m\} \subseteq W^\Lambda$ with $\sigma_a^\Lambda(u)(X) \geq n$ and $v_i \Vdash_g \psi$ for all $i \leq m$. Let $\sigma_a^\Lambda(u)(v_i) = n_i$ for $i \leq m$. We may assume that each $n_i > 0$, and that states in X are pairwise different. There are two cases:

(1.1) $n_i \geq n$ for some $i \leq m$. By induction hypothesis and the assumption $v_i \Vdash_g \psi$, we have $\psi \in v_i$. Since $n_i \geq n$, by Lemma 7.5(1), $\langle a\rangle_n\psi \in u$.

(1.2) $0 < n_i < n$ for every $i \leq m$. Let $k = n_0 + \cdots + n_m \geq n$. Obviously, $m > 0$, otherwise, $\sigma_a^\Lambda(u)(X) = n_0 < n$. Since $\sigma_a^\Lambda(u)(v_i) = n_i$ for each $i \leq m$, there exists $\chi_i \in v_i$ such that $\langle a\rangle_{!n_i}\chi_i \in u$ for $i \leq m$. Since states in X are pairwise different, $\xi_i \in v_i$ and $\xi_i \wedge \xi_j \leftrightarrow \bot \in \Lambda$ for $i \neq j \leq m$. Let $\theta_i = \chi_i \wedge \xi_i \wedge \psi$ for $i \leq m$, and $\theta = \theta_0 \vee \cdots \vee \theta_m$. Then $\theta_i \in v_i$ and $\theta_i \wedge \theta_j \leftrightarrow \bot \in \Lambda$ for $i \neq j \leq m$.

Now we show $\langle a\rangle_{!n_i}\theta_i \in u$. Suppose $\langle a\rangle_{n_i}\theta_i \notin u$. Then there is $r < n_i$ such that $\langle a\rangle_{!r}\theta_i \in u$, a contradiction to $\sigma^\Lambda(u)(v_i) = n_i$. Suppose $\langle a\rangle_{n_i+1}\theta_i \in u$. Clearly, $\theta_i \to \chi_i \in \Lambda$. Hence $\langle a\rangle_{n_i+1}\chi_i \in u$, a contradiction.

Therefore, $\langle a\rangle_{!n_i}\theta_i \in u$ for all $i \leq m$. Finally, by Lemma 7.4, we have $\langle a\rangle_{!k}\theta \in u$. Since $k \geq n$, $\langle a\rangle_n\theta \in u$. Since $\theta \to \psi \in \Lambda$, $\langle a\rangle_n\psi \in u$.

(2) Assume $\langle a\rangle_n\psi \in u$. Since $n > 1$, we have $\langle a\rangle_n\psi \to \langle a\rangle\psi$ and so $\langle a\rangle\psi \in u$. By Lemma 7.5, there exists $v \in W^\Lambda$ such that $\sigma_a^\Lambda(u)(v) \geq 1$ and $\psi \in v$. We distinguish the following three cases.

If there are infinitely many such v, then by induction and the semantic definition of $\langle a\rangle_n\psi$ we have $\mathfrak{M}^\Lambda, u \Vdash_g \langle a\rangle_n\psi$.

If there exists $v \in W^\Lambda$ such that $\sigma_a^\Lambda(u)(v) \geq n$ and $\psi \in v$, then again by induction and the semantic definition of $\langle a\rangle_n\psi$ we have $\mathfrak{M}^\Lambda, u \Vdash_g \langle a\rangle_n\psi$.

Finally, assume that there are only finitely many pairwise different $v_0, \ldots, v_m \in W^\Lambda$ such that for each $i \leq m$, $0 < \sigma_a^\Lambda(u)(v_i)$ and $\psi \in v_i$, and suppose for each $i \leq m$ that $0 < \sigma_a^\Lambda(u)(v_i) = n_i < n$. Clearly, $\langle a\rangle_{n_i+1}\psi \in u$ since $\langle a\rangle_n\psi \in u$ and $n \geq n_i + 1$ (Axiom 4). Let $\chi_i \in v_i$, $\langle a\rangle_{!n_i}\chi_i \in u$, and $\xi_i \in v_i$, $\xi_i \wedge \xi_j \leftrightarrow \bot \in \Lambda$ for $i \neq j \leq m$. Let $k = n_0 + \cdots + n_m$. It suffices to show $k \geq n$. Let $\theta_i = \chi_i \wedge \xi_i \wedge \psi$ for $i \leq m$, and $\theta = \theta_0 \vee \cdots \vee \theta_m$. By the argument in (1), $\langle a\rangle_{!k}\theta \in u$. Let $\theta' = \neg\bigvee_{i \leq m}(\chi_i \wedge \xi_i)$. Then $\theta \wedge \theta' \leftrightarrow \bot \in \Lambda$ and $\psi \leftrightarrow \theta \vee (\theta' \wedge \psi) \in \Lambda$.

Now we show $\langle a\rangle(\theta' \wedge \psi) \notin u$. Suppose not. There is $w \in W^\Lambda$ such that $\sigma_a^\Lambda(u)(w) \geq 1$ and $\theta' \wedge \psi \in w$. Then $\psi \in w$. Hence $w = v_j$ for some $j \leq m$. Then $\theta' \in v_j$, a contradiction. Therefore, $\neg\langle a\rangle(\theta' \wedge \psi) \in u$, i.e., $\langle a\rangle_{!0}(\theta' \wedge \psi) \in u$. From that and $\langle a\rangle_{!k}\theta \in u$ follows by

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(Axiom 6) that $\langle a\rangle_{!k}(\theta \vee (\theta' \wedge \psi)) \in u$, i.e., $\langle a\rangle_{!k}\psi \in u$. Suppose $k < n$. Since $\langle a\rangle_n\psi \in u$, $\langle a\rangle_{k+1}\psi \in u$, a contradiction. Therefore, $k \geq n$. By induction hypothesis, similarly to above, again it follows that $\mathfrak{M}^\Lambda, u \Vdash_g \langle a\rangle_n\psi$. □

By Truth Lemma 7.6, one easily obtains strong completeness of $\mathrm{K_g}$ with respect to the class of all graded frames. This completes the alternative proof of Theorem 2.16.

The final result in the appendix is used to prove completeness of extensions Λ of $\mathrm{K_g}$ for frame classes with additional properties.

PROPOSITION 7.7. *For every graded epistemic logic* Λ*,* $u \in W^\Lambda$ *and* $X \in \mathcal{P}^+(W^\Lambda)$*,* $\sigma_a^\Lambda(u)(X) \geq n$ *iff* $\langle a\rangle_n\varphi \in u$ *for all* $\varphi \in \bigcap X$*.*

Proof. For the 'only if' part, assume that $\sigma_a^\Lambda(u)(X) \geq n$ and $\varphi \in \bigcap X$. If $\exists v \in X(\sigma_a^\Lambda(u)(v) = \omega)$, then obviously $\langle a\rangle_n\varphi \in u$. Assume $\forall v \in X(\sigma_a^\Lambda(u)(v) < \omega)$. Let $X = \{v_0, \dots, v_m\}$ and $\sigma_a^\Lambda(u)(v_i) = n_i < \omega$ for $i \leq m$. For all $v_i \in X$, from $\varphi \in v_i$ and Truth Lemma 7.6 it follows that $\mathcal{M}^\Lambda, v_i \Vdash_g \varphi$. From that and $n_0 + \cdots + n_m \geq n$ one gets $\mathcal{M}^\Lambda, u \Vdash_g \langle a\rangle_n\varphi$. By Lemma 7.6, one gets $\langle a\rangle_n\varphi \in u$.

For the 'if' part, assume $\langle a\rangle_n\varphi \in u$ for all $\varphi \in \bigcap X$. For a contradiction, assume $\sigma_a^\Lambda(u)(X) < n$. Let $X = \{v_0, \dots, v_m\}$ and $\sigma_a^\Lambda(u)(v_i) = n_i$ for $i \leq m$. Then $n_0 + \cdots + n_m = k < n$. Let $\chi_i \in v_i$, $\langle a\rangle_{!n_i}\chi_i \in u$, and $\xi_i \in v_i$, $\xi_i \wedge \xi_j \leftrightarrow \bot \in \Lambda$ for $i \neq j \leq m$. Let $\theta_i = \chi_i \wedge \xi_i$ for $i \leq m$, and $\theta = \theta_0 \vee \cdots \vee \theta_m$. Then $\theta_i \in v_i$ and $\theta_i \wedge \theta_j \leftrightarrow \bot \in \Lambda$ for $i \neq j \leq m$. It is easy to see that $\langle a\rangle_{!n_i}\theta_i \in u$. Hence $\langle a\rangle_{!n_i}\theta_i \in u$ for all $i \leq m$. By Lemma 7.4, $\langle a\rangle_{!k}\theta \in u$. Note that $\theta \in \bigcap X$ and $\langle a\rangle_{k+1}\theta \notin u$. By $k+1 \leq n$, $\langle a\rangle_n\theta \notin u$, a contradiction. □

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