Revisiting Leighton's theorem with the Haar measure

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Abstract

Leighton's graph covering theorem states that a pair of finite graphs with isomorphic universal covers have a common finite cover. We provide a new proof of Leighton's theorem that allows generalisations; we prove the corresponding result for graphs with fins. As a corollary we obtain pattern rigidity for free groups with line patterns, building on the work of Cashen–Macura and Hagen–Touikan. To illustrate the potential for future applications, we give a quasi-isometric rigidity result for a family of cyclic doubles of free groups.

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Leighton's graph covering theorem states that if X_1 and X_2 are finite graphs with isomorphic universal covers, then X_1 and X_2 have isomorphic finite covers. The case of k-regular graphs was first proved by Angluin and Gardener [1] and was soon followed by Leighton's proof of the general case [9]. Subsequently, Bass and Kulkarni gave another proof in the context of studying lattices in the automorphism groups of trees [3]. Walter Neumann revisited both proofs and proved a generalisation for coloured graphs and partial results for what he called symmetry restricted graphs [13]. Given the ubiquity of group actions on trees, it is unsurprising that Leighton's theorem has seen a number of applications [4, 7, 10].

Let Y be a finite graph. Let S be a graph given by a circle subdivided into ℓ edges $e_0, \ldots, e_{\ell-1}$. Let $\gamma : S \to Y$ be a morphism of graphs such that $\gamma(e_i)$ is not equal to $\gamma(e_{i+1})$ with reversed orientation (subscripts considered mod ℓ). Such a map γ is a *closed combinatorial path with no backtracking*, or alternatively an example of an *immersion of graphs*. Note that γ is the unique miminal length combinatorial representative of its homotopy class $[\gamma]$, up to reparameterisation. A *graph with fins* is the compact non-positively curved square complex X obtained by taking a finite collection of closed combinatorial paths with no backtracking $\{\gamma_i\}_{i=1}^n$, and taking the mapping cylinder of the map $f : \bigsqcup_{i=1}^n S_i \to Y$, where

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f restricts to γ_i on S_i. Note that there is the natural retraction $X \to Y$. We will prove the following generalisation of Leighton's theorem:

THEOREM 0·1. *Let X_1 and X_2 be compact graphs with fins such that $\widetilde{X}_1 \cong \widetilde{X}_2$. Then there exists isomorphic, finite index common covers of X_1 and X_2.*

If our graph has no fins then we recover Leighton's original theorem. Spiritually, we will prove Theorem 0·1 by walking the same path that Leighton took in the eighties. The principal obstacle to proving Leighton style theorems is the absence of a suitable fiber product if both graphs don't cover a common base. Beneath all the arithmetic employed in Leighton's original proof is a desire to construct something like a fiber product. The key idea in this paper is that the numerology can be ditched and replaced with arguments involving the Haar measure. Because the Haar measure exists for all second countable, locally compact groups, the arguments are extremely flexible and welcoming to generalisation.

0·1. *Applications to rigid line patterns in free groups*

Let X be a Gromov hyperbolic space. A *line pattern in X* is a set of equivalence classes of biinfinite quasi-geodesics up to bounded distance. A *quasi-isometry respecting line patterns* $\phi : (X, \mathcal{L}) \to (X', \mathcal{L}')$ is quasi-isometry $X \to X'$ that induces a bijection between the equivalence classes of quasi-geodesics $\mathcal{L} \to \mathcal{L}'$.

Quasi-isometries respecting line patterns were first considered by Schwartz [15] in the context of line patterns on $\mathbb{H}^n$. The notion of *pattern rigidity* was introduced in [12], in order to consider equivariant collections of subspaces up to quasi-isometry. Generalisations of Schwartz's result have been given in [5, 11] and Cashen and Macura considered line patterns in free groups [6], studying the problem in analogy to Whitehead's algorithm [17].

Let F be a free group and $S = \{g_1, \ldots, g_n\} \subseteq F$ is finite set of elements of F such that $g_i^p \neq h g_j^q h^{-1}$ for $1 \leq i < j \leq n$, $h \in F$, and $p, q \in \mathbb{Z} - \{0\}$. That is to say that the elements in S are *weakly incommensurable*. The *line pattern $\mathcal{L}_S$ generated by S* is the equivalence classes given by the following set of quasi-geodesics:

$$\left\{ h \cdot \langle g_i \rangle \mid h \in F, \ 1 \leq i \leq n \right\}.$$

The line pattern $\mathcal{L}_S$ is *rigid* if F admits no cyclic splitting relative to S. See [6] for full details. If $(F, \mathcal{L}_S)$ is a free group with line pattern, and $F' \leqslant F$ is a finite index subgroup, then F' inherits the line pattern $\mathcal{L}_S$ from F.

The following application, explained to the author by Hagen and Touikan, combines Theorem 0·1 with previous results to prove quasi-isometric rigidity for free groups with rigid line patterns:

THEOREM 0·2. *Let $(F_1, \mathcal{L}_{S_1})$ and $(F_2, \mathcal{L}_{S_2})$ be free groups with rigid line patterns. Suppose that $\phi : (F_1, \mathcal{L}_{S_1}) \to (F_2, \mathcal{L}_{S_2})$ is quasi-isometry respecting the line patterns. Then there exist finite index subgroups $F_i' \leqslant F_i$ such that there is an isomorphism that respects the line patterns $\phi' : (F_1', \mathcal{L}_{S_1}) \to (F_2', \mathcal{L}_{S_2})$.*

Proof. In [6] Cashen and Macura construct a hyperbolic CAT(0) cube complex X with line pattern $\mathcal{L}$ such that there is an action of F_i on $(X, \mathcal{L})$ that is quasi-conjugate to the action of F_i on $(F_i, \mathcal{L}_{S_i})$ for $i = 1, 2$. In general, they remark that X is not necessarily a tree, which would be the preferable state of affairs. In [8] Hagen and Touikan apply a technique called *panel collapses* to show X can be replaced with a tree T. Alternatively, this can be

achieved using Mosher, Sageev and Whyte's result about cobounded quasi-actions on bushy trees [**12**].

For each equivalence class in $\mathcal{L}$ we attach one side of the strip $[0, 1] \times \mathbb{R}$ along the unique geodesic line in the class to obtain a space $\widetilde{X}$. Thus the action of F_i on $(T, \mathcal{L})$ becomes an action of F_i on $\widetilde{X}$ for $i = 1, 2$. The actions are free and cocompact since they are quasi-conjugate to the original left action on $(F_i, \mathcal{L}_{S_i})$. The quotient $X_i = \widetilde{X}/F_i$ is a graph with fins. As X_1 and X_2 are graphs with fins with isomorphic universal covers we may apply Theorem 0·1. This common finite cover corresponds to finite index subgroups $F_i' \leqslant F_i$ identified by an isomorphism that respects the line patterns.

0·2. *Applications to quasi-isometric rigidity*

The following example, suggested to the author by Emily Stark, illustrates how Theorem 0·1 can be used to obtain rigidity results.

THEOREM 0·3. *Let F_i be a finitely generated, non-abelian free group for $i = 1, 2$. Let $w_i \in F_i$ be an element such that $\mathcal{L}_{\{w_i\}}$ is a rigid line pattern in F_i. Let $G_i = F_i *_{\langle w_i \rangle} F_i$, the amalgamated double of F_i over $\langle w_i \rangle$. If G_1 is quasi-isometric to G_2, then G_1 is commensurable with G_2.*

Proof. As the cyclic splitting given in the statement is the JSJ-decomposition for G_i, [**14**, Theorem 7.1] tells us that there is a quasi-isometry respecting line patterns

$$\phi : (F_1, \mathcal{L}_{\{w_1\}}) \longrightarrow (F_2, \mathcal{L}_{\{w_2\}}).$$

Hence, as in the proof of Theorem 0·2 both F_1 and F_2 are the fundamental groups of graphs with fins X_1 and X_2 with isomorphic universal covers $\widetilde{X}_1 \cong \widetilde{X}_2$, such that the preimages of the fins correspond to the line patterns $\mathcal{L}_{S_i}$. Let $\widehat{X}$ be the common finite cover of X_1 and X_2 given by Theorem 0·1. We can decompose X_i as a mapping cylinder of single circle mapping into a graph $\gamma_i : S_i \to Y_i$ so

$$X_i = Y_i \sqcup S_i \times [0, 1] \Big/ \{(x, 0) \sim \gamma_i(x)\}.$$

Similarly, $\widehat{X}$ can be decomposed as a mapping cylinder of a finite set of circles mapping into a graph $\hat{\gamma} : \bigsqcup_{j=1}^n \widehat{S}_j \to \widehat{Y}$, where $\widehat{Y}$ is a finite common cover of Y_1 and Y_2. So

$$\widehat{X} = \widehat{Y} \bigsqcup_{j=1}^n \widehat{S}_j \times [0, 1] \Big/ \{(\hat{x}, 0) \sim \hat{\gamma}(\hat{x})\}.$$

A graph of spaces Z_i can be constructed for G_i by taking two copies of X_i and identifying the ends of the cylinder $S_i \times \{1\} \subseteq X_i$ and obtaining the double, so G_i can be identified with $\pi_1 X_i$. A common finite cover $\widehat{Z}$ of Z_1 and Z_2 is constructed by similarly taking two copies of $\widehat{X}$ and identifying the ends of the cylinders $\bigsqcup_{j=1}^n \widehat{S}_j \times \{1\} \subseteq \widehat{X}$ to again obtain the double. Since $\widehat{X}$ covers both X_1 and X_2, it is immediate that $\widehat{Z}$ covers both Z_1 and Z_2. Therefore $\pi_1 \widehat{Z}$ is a common finite index subgroup for G_1 and G_2.

Remark 0·4. Taam and Touikan have proven have proved that a word hyperbolic group G with JSJ-decomposition containing exclusively rigid vertex groups isomorphic to closed surface groups is quasi-isometrically rigid [**16**]. Their result holds for a much wider collection of groups. In particular it can be applied to all groups in the quasi-isometry class.
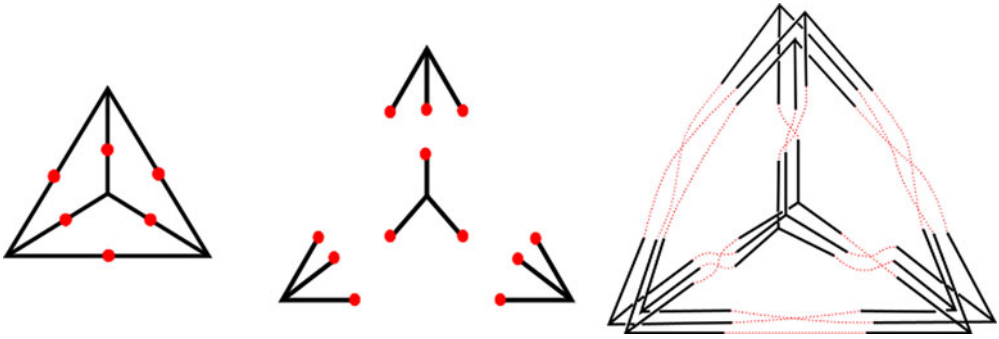
Fig. 1. From left to right, a graph with hyperplanes highlighted, the polyhedrons for the graph, and 3 copies of each polyhedron used to construct a degree three cover of the original graph.

We expect that Theorem 0·3 should similarly extend to quasi-isometric rigidity of hyperbolic groups G with JSJ-decomposition containing exclusively rigid vertex groups that are finitely generated, non-abelian free groups.

1. *Polyhedron and finite index covers*

Let X be a graph with fins. Let Y be the graph underlying X and assume we have fixed an orientation of each edge in Y. Let $\mathcal{H}$ be the set of *vertical hyperplanes* in X, that is to say the hyperplanes dual to the edges in Y. Let $\dot{X}$ denote the square complex obtained by subdividing along the vertical hyperplanes.

A *star* is a square complex P with a distinguished 0-cube p such that P is the cubical neighbourhood of p. We can obtain stars by taking the closures of the complementary components $X - \mathcal{H}$. The resulting stars are subcomplexes of $\dot{X}$.

A *polyhedron* is a star P with an isometric embedding $\phi : P \to \dot{X}$ such that $\phi(P)$ is the cubical neighborhood in $\dot{X}$ of a 0-cube in Y. A *face* (F, ψ) is a finite tree F that maps isomorphically to a vertical hyperplane in X. We say that (F, ψ) is a *face of* (P, ϕ) if (F, ψ) is isomorphic to the restriction of (P, ϕ) to a subcomplex $P \cap \phi^{-1}(\Lambda)$ where $\Lambda \in \mathcal{H}$. Up to isomorphism, there are only finitely many polyhedrons and faces. Each face is the face of precisely two polyhedrons, one of the left and one on the right. The polyhedron on the *left* [*resp. right*] is the polyhedron whose image contains the origin vertex [*resp. terminal vertex*] of the edge dual to the image of the face. If (P, ϕ) and (P', ϕ') are polyhedrons on the left and right of the face (F, ψ), then the polyhedrons can be glued together along the subcomplexes corresponding to F to obtain a new complex $P \cup P'$ that maps into X via ϕ and ϕ'.

An n-sheeted cover of X can be constructed by taking n copies of each polyhedron and face and then choosing a correspondence between the n-copies of a polyhedron on the left of a face and the n-copies on the right. The polyhedron can then be glued together to obtain an n-sheeted cover of $\dot{X}$. Reversing the subdivisions along the vertical hyperplanes gives the cover of X. See Figure 1.

The idea behind the proof of Theorem 0·1 is to replicate this construction on two graphs with fins simultaneously. This requires first defining an appropriate notion of being a polyhedron for two spaces simultaneously and understanding how we might glue them together appropriately. In this setting we obtain a set of gluing equations that we must solve in order to construct our cover. We will utilise the Haar measure to find solutions to these equations.

2. *Proof of main theorem*

Let X_1 and X_2 be graphs with fins such that $\widetilde{X}_1 \cong \widetilde{X}_2$. Let $Y_i \subseteq X_i$ be the underlying graph and $r : X_i \to Y_i$ be the natural retraction. Identify $\widetilde{X}_1 \cong \widetilde{X}_2 =: \widetilde{X}$ and $\widetilde{Y}_1 \cong \widetilde{Y}_2 =: \widetilde{Y}$. Let $p_i : \widetilde{X} \to X_i$ be the covering map. Let $\Gamma_i = \pi_1(X_i)$. Let $\mathcal{H}_i$ be the set of *vertical hyperplanes* in X_i.

Let $G = \operatorname{Aut}(\widetilde{X})$. Then there is an embedding $\Gamma_i \leqslant G$ as the group of deck transformations. Note that Γ_i is a uniform free lattice in G. We will assume that G does not invert hyperplanes. This can be achieved by either subdividing the vertical hyperplanes, or applying [**2**, Proposition 6.3] to pass to index 2 subgroups of G, Γ_1, Γ_2. Let $\underline{X} = \widetilde{X}/G$. There are natural quotient maps $q_i : X_i \to \underline{X}$. As G does not invert hyperplanes, we can assign G-invariant orientations to all the hyperplanes in $\widetilde{Y}$, which gives orientations of the vertical hyperplane $\mathcal{H}_i$. If S is a subset of $\widetilde{X}$, then G_S denotes the *setwise stabiliser* of S, and $G_{(S)}$ denotes the *pointwise stabiliser* of S.

Remark 2·1. The space $\underline{X}$ encodes the information that Leighton refers to as the *degree refinement*. Our use more closely resembles Neumann's use in [**13**], as the *colouring graph*. By giving each vertex, edge and square a unique colour we obtain colourings of X_1, X_2 and $\widetilde{X}$. The colouring of $\widetilde{X}$ is G-equivariant. In fact, everything in this section works if G is the full colour preserving automorphism group of a coloured graph with fins with free uniform lattices Γ_1 and Γ_2.

2·1. *Polyhedral pairs*

A *polyhedral pair* is a triple (P, ϕ_1, ϕ_2) where each pair P, ϕ_i is a polyhedron for X_i and the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\phi_1} & X_1 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle q_1} \\ X_2 & \xrightarrow{q_2} & \underline{X}. \end{array}$$

We will denote a polyhedral pair with boldface: $\mathbf{P} = (P, \phi_1, \phi_2)$. There are only finitely many polyhedral pairs for X_1 and X_2 up to isomorphism. By forgetting either ϕ_1 or ϕ_2 we obtain a polyhedron for either X_1 or X_2. We say that we obtain such polyhedron by *restricting* $\mathbf{P}$ *to either X_1 or X_2*. See Figure 2 for an illustrated example.

Let $\mathbf{P}$ be a polyhedral pair. Let $\tilde{\phi}_i : P \to \widetilde{X}$ be some choice of lift. Note that $\tilde{\phi}_i$ is unique up to post composition by $g_i \in \Gamma_i$. Then a polyhedral pair $\mathbf{P}$ is *admissible* if there exist a $g \in G$ such that $g \circ \tilde{\phi}_1 = \tilde{\phi}_2$. Note that this does not depend on the initial choice of lifts. Indeed, any alternative choice of lift is of the form $\gamma_i \circ \phi_1$ where $\gamma_i \in \Gamma_i$, so $(\gamma_2 g \gamma_1^{-1}) \circ (\gamma_1 \tilde{\phi}_1) = \gamma_2 \tilde{\phi}_2$. Let $\mathcal{P}$ denote the (finite) set of all admissible polyhedral pairs. All polyhedral pairs used in this paper are admissible.

Remark 2·2. If $\widetilde{X}$ is a tree, then all polyhedral pair are admissible. In general, this is not true.

We can classify admissible polyhedral pairs as follows. Let (P_1, ϕ_1) and (P_2, ϕ_2) be polyhedron for X_1 and X_2 such that $q_1 \circ \phi_1(P_1) = q_2 \circ \phi_2(P_2)$. Let $\tilde{\phi}_i : P_i \to \widetilde{X}$ be a choice of lift for each $i = 1, 2$. Let v_i be the 0-cube in $\widetilde{P}_i := \tilde{\phi}_i(P_i)$. Let $g \in G$ such that $g\widetilde{P}_1 = \widetilde{P}_2$. Such a
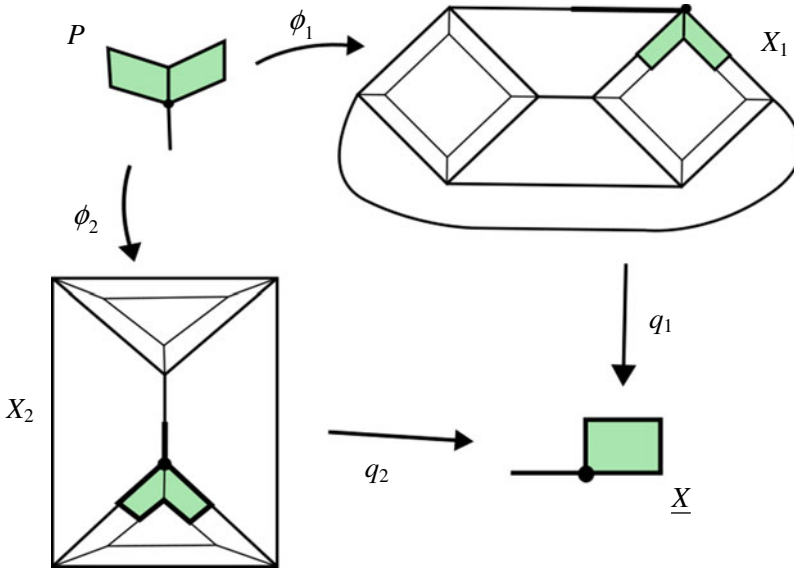
Fig. 2. A polyhedral pair illustrated.

g exists since $q_1 \circ p_1(v_1) = q_2 \circ p_2(v_2)$. Then the admissible polyhedral pairs which restrict to the polyhedron (P_i, ϕ_i) are in one to one correspondence with the elements of $G_{v_1}/G_{(\widetilde{P}_1)}$. Indeed, for each $g' \in G_{v_1}$ we obtain an isomorphism

$$\tilde{\phi}_2^{-1} \circ gg' \circ \tilde{\phi}_1 : P_1 \longrightarrow P_2$$

that allows us to identify P_1 and P_2 so that we obtain a polyhedral pair (P, ϕ_1, ϕ_2). Note that all g' in the same $G_{(\widetilde{P}_1)}$-coset give the same polyhedral pair since they correspond to identical identifications of P_1 and P_2.

A *face pair* $\mathbf{F} = (F, \psi_1, \psi_2)$ is a tuple such that each (F, ψ_i) is a face for X_i, and the following diagram commutes:

$$
\begin{array}{ccc}
F & \xrightarrow{\psi_1} & X_1 \\
\downarrow{\scriptstyle \psi_2} & & \downarrow \\
X_2 & \longrightarrow & \underline{X}.
\end{array}
$$

Let $\tilde{\psi}_i : F \to \widetilde{X}$ be a lift of ψ_i. A face pair is *admissible* if there exists $g \in G$ such that $g \circ \tilde{\psi}_1 = \tilde{\psi}_2$. As with polyhedral pairs, admissibility of face pairs does not depend on the choice of lifts.

We say that $\mathbf{F}$ *is a face of* $\mathbf{P}$ if (F, ϕ_i) is a face of the polyhedron (P, ϕ_i) for $i = 1, 2$. A polyhedral pair $\mathbf{P}$ lies on the left [*resp. right*] of $\mathbf{F}$ if the polyhedron (P, ϕ_i) lie on the left [*resp right*] of (F, ψ_i). Note that if (P, ϕ_1) lies on the left [*resp. right*] of (F, ψ_1), then (P, ϕ_2) lies on the left [*resp. right*] of (F, ψ_2), since the orientations of the edge in Y_i were obtained from G-equivariant orientations of the edges of $\widetilde{Y}$. Let $\mathcal{F}$ denote the set of all admissible face pairs.

Unlike faces of polyhedron, face pairs can have multiple polyhedral pairs on the left (or right). Let $\overleftarrow{\mathbf{F}}$ and $\overrightarrow{\mathbf{F}}$ denote the sets of polyhedral pairs on the left and right of $\mathbf{F}$. Given $\mathbf{P} \in$

$\overleftarrow{\mathbf{F}}$ and $\mathbf{P}' \in \overrightarrow{\mathbf{F}}$ we can glue $\mathbf{P}$ and $\mathbf{P}'$ together along $\mathbf{F}$. That is to say, by identifying F with the corresponding subcomplexes of P and P', we can glue P and P' together, respecting the maps ϕ_i and ϕ_i' to obtain the following commutative diagram:

$$
\begin{array}{ccc}
P \cup P' & \xrightarrow{\phi_1 \cup \phi_1'} & X_1 \\
{\scriptstyle \phi_2 \cup \phi_2'} \downarrow & & \downarrow \\
X_2 & \longrightarrow & \underline{X}.
\end{array}
$$

Performing the construction in Section 1 over both X_1 and X_2 simultaneously – thus obtaining a finite common cover – requires taking a suitable number of copies of each admissible polyhedral pair, such that the number that lie on the left of each face pair is the same as the number that lie on the right. Then we may choose a pairing between the polyhedral pairs on the left and right of each face pair and glue the the correponding faces together. The resulting graph with fins will have a well defined map into both X_1 and X_2 that will be a local isomorphism, and thus a covering map.

2·2. *Gluing equations*

We define a system of linear gluing equations. Let $\omega : \mathcal{P} \to \mathbb{R}_{>0}$ denote a weight function on the set of admissible polyhedral pairs. For each face $\mathbf{F}$ we have the following *gluing equation*:

$$
\sum_{\mathbf{P} \in \overleftarrow{\mathbf{F}}} \omega(\mathbf{P}) = \sum_{\mathbf{P} \in \overrightarrow{\mathbf{F}}} \omega(\mathbf{P}). \tag{2·1}
$$

Note that since this gives a finite set of linear equations with integer coefficients, finding a set of positive real weights satisfying all gluing equations gives a set of positive integer weights satisfying all the gluing equations.

2·3. *Solving the equation*

Let μ be the Haar measure for G. As G contains a lattice – Γ_1 for example – G is unimodular and μ is both left and right G-invariant. We recall that μ is positive on every open set and finite on every compact set.

For each polyhedral pair $\mathbf{P} = (P, \phi_1, \phi_2) \in \mathcal{P}$ recall that we let $\tilde{\phi}_i : P \to \widetilde{X}$ be a lift of ϕ_i and $\widetilde{P}_i = \tilde{\phi}_i(P)$ for $i = 1, 2$. Then let

$$
\omega(\mathbf{P}) = \mu\left(G_{(\widetilde{P}_1)} \right).
$$

As μ is left and right invariant we deduce that $\omega(\mathbf{P})$ does not depend on the choice of lifts $\tilde{\phi}_i$. Observe that $\omega(\mathbf{P})$ is a positive real number since it is a compact neighbourhood. Moreover, after scaling μ we can assume that $\omega(\mathbf{P}) \in \mathbb{Z}_{>0}$ since all stabilizers of finite sets in $\widetilde{X}$ have commensurate measures.

PROPOSITION 2·3. *The weight function ω as defined, satisfy the gluing equations* (2·1).

Proof. Given an admissible face $\mathbf{F}$ we need to enumerate the polyhedral pairs on the left of $\mathbf{F}$ (and right). Let $\tilde{\psi}_i : F \to \widetilde{X}$ be a lift of ψ_i. Let $\widetilde{F}_i = \tilde{\psi}_i(F)$. Let $g \in G$ be such that $g \circ \tilde{\psi}_1 = \tilde{\psi}_2$.

Let (P_i, ϕ_i) be the polyhedron on the left of (F, ψ_i). Let $\tilde{\phi}_i : P_i \to X$ be the lift of ϕ_i such that $\widetilde{P}_i := \tilde{\phi}_i(P_i)$ contains $\widetilde{F}_i$. Then each polyhedral pair to the left of $\mathbf{P}$ is obtained by taking some $g' \in G_{(\widetilde{F}_i)}$ and letting $\tilde{\phi}_2^{-1} \circ gg' \circ \tilde{\phi}_1$ identify P_1 and P_2. All g' in the same $G_{(\widetilde{P}_1)}$ coset define the same polyhedral pair. Therefore we can compute that

$$\sum_{\mathbf{P} \in \overleftarrow{\mathbf{F}}} \omega(\mathbf{P}) = \sum_{G_{(\widetilde{F}_1)}/G_{(\widetilde{P}_1)}} \mu(G_{(\widetilde{P}_1)}) = \mu\left(G_{(\widetilde{F}_1)}\right).$$

A similar equality holds on the right, so the proposition holds.

2·4. *Constructing a common cover*

Assuming that our weight function gives positive integer values, take $\omega(\mathbf{P})$ copies of each $\mathbf{P} \in \mathcal{P}$. Let $\mathcal{P}_\omega$ denote this multiset. Let $\overleftarrow{\mathbf{F}}_\omega$ and $\overrightarrow{\mathbf{F}}_\omega$ denote the polyhedrons in $\mathcal{P}_\omega$ on the left and right of $\mathbf{F}$ respectively. Since ω satisfies the gluing equations, for each admissible face $\mathbf{F}$ we may fix a bijection $\overleftarrow{\mathbf{F}}_\omega \to \overrightarrow{\mathbf{F}}_\omega$. By gluing the corresponding admissible faces together we obtain a graph with fins:

$$\widehat{X} = \bigsqcup_{\mathbf{P} \in \mathcal{P}_\omega} P \Big/ \sim$$

Then there is a covering map $\Phi : \widehat{X} \to X_i$ where $\Phi_i\big|_P = \phi_i$ for each $\mathbf{P} = (P, \phi_1, \phi_2) \in \mathcal{P}_\omega$. The proof of Theorem 0·1 is complete.

REFERENCES

[1] D. Angluin and A. Gardiner. Finite common coverings of pairs of regular graphs. *J. Combin. Theory Ser. B*, **30**(2) (1981), 184–187.
[2] H. Bass. Covering theory for graphs of groups. *J. Pure Appl. Algebra* **89**(1-2) (1993), 3–47.
[3] H. Bass and R. Kulkarni. Uniform tree lattices. *J. Amer. Math. Soc.* **3**(4) (1990), 843–902.
[4] J. A. Behrstock and W. D. Neumann. Quasi-isometric classification of non-geometric 3-manifold groups. *J. Reine Angew. Math.* **669** (2012), 101–120.
[5] K. Biswas and M. Mj. Pattern rigidity in hyperbolic spaces: duality and PD subgroups. *Groups Geom. Dyn.* **6**(1) (2012), 97–123.
[6] C. H. Cashen and N. Macura. Line patterns in free groups. *Geom. Topol.* **15**(3) (2011), 1419–1475.
[7] C. H. Cashen and A. Martin. Quasi-isometries between groups with two-ended splittings. *Math. Proc. Camb. Phil. Soc.* **162**(2) (2017), 249–291.
[8] M. F. Hagen and N. W. M. Touikan. Panel collapse and its applications. To appear in *Groups, Geometry and Dynamics*, https://arxiv.org/abs/1712.06553.
[9] F. T. Leighton. Finite common coverings of graphs. *J. Combin. Theory Ser. B* **33**(3) (1982), 231–238.
[10] G. Levitt. Generalised Baumslag–Solitar groups: rank and finite index subgroups. *Ann. Inst. Fourier (Grenoble)*, **65**(2) (2015), 725–762.
[11] M. Mj. Pattern rigidity and the Hilbert-Smith conjecture. *Geom. Topol.* **16**(2) (2012), 1205–1246.
[12] L. Mosher, M. Sageev and K. Whyte. Quasi-actions on trees II: Finite depth Bass–Serre trees. *Mem. Amer. Math. Soc.* **214**(1008) (2011), vi+105.
[13] W. D. Neumann. On Leighton's graph covering theorem. *Groups Geom. Dyn.* **4**(4) (2010), 863–872.
[14] P. Papasoglu. Quasi-isometry invariance of group splittings. *Ann. of Math. (2)* **161**(2) (2005), 759–830.

[15] R. E. SCHWARTZ. Symmetric patterns of geodesics and automorphisms of surface groups. *Invent. Math.* **128**(1) (1997), 177–199.

[16] A. TAAM and N. W. M. TOUIKAN. On the quasi-isometric rigidity of graphs of surface groups. Preprint, https://arxiv.org/abs/1904.10482.

[17] J. H. C. WHITEHEAD. On equivalent sets of elements in a free group. *Ann. of Math.* (2) **37**(4) (1936), 782–800.