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Krivine's classical realisability from a categorical perspective

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In a sequence of papers (Krivine 2001; Krivine 2003; Krivine 2009), J.-L. Krivine introduced his notion of *classical realisability* for classical second-order logic and Zermelo–Fraenkel set theory. Moreover, in more recent work (Krivine 2008), he has considered forcing constructions on top of it with the ultimate aim of providing a realisability interpretation for the axiom of choice.

The aim of the current paper is to show how Krivine's classical realisability can be understood as an instance of the categorical approach to realisability as started by Martin Hyland in Hyland (1982) and described in detail in van Oosten (2008). Moreover, we will give an intuitive explanation of the iteration of realisability as described in Krivine (2008).

1. Introduction

The interpretation of intuitionistic second-order logic in a realisability model based on closed λ-terms is reasonably straightforward. This was studied in detail by J.-L. Krivine and M. Parigot in the late 1980s (Krivine and Parigot 1990; Krivine 1990a). Around 1990, following the seminal paper Griffin (1990), many researchers worked out how to give a proof term assignment for classical logic using a λ-calculus with control operators that serve as realisers for classical principles like *reductio ad absurdum* or *Peirce's law* – see, for example, Streicher and Reus (1998). Krivine was one of the first to take up Griffin's suggestion in his work on so-called 'storage operators' (Krivine 1990b). Then, in a sequence of papers (Krivine 2001; Krivine 2003; Krivine 2009), beginning with his address to the Logic Colloquium 2000 in Paris, Krivine developed his theory of *classical realisability* for extensions of classical second-order logic and Zermelo–Fraenkel set theory. In more recent and still unpublished work, Krivine has embarked on the long-term project of providing a realisability interpretation for full ZFC, that is, Zermelo–Fraenkel set theory with the full Axiom of Choice (Krivine 2008). The aim is to do this by considering forcing interpretations within classical realisability models. Krivine (2008) has shown how to contract this two-step model construction into a single step.

Reading through Krivine's papers introducing classical realisability gives the impression that his account is highly original (which it definitely is!), and it is not clear how it may fit into the structural semantic approach to realisability as initiated by M. Hyland in Hyland (1982) and described in detail in the monograph van Oosten (2008). In particular, it is most puzzling that Krivine considers his classical realisability as 'generalised forcing' since toposes of the form $\mathsf{Sh}(B)$ for a complete boolean algebra B are cocomplete, whereas cocomplete realisability toposes are necessarily equivalent to **Set**. In order to clear up this confusion, we introduce the notion of an 'abstract Krivine structure' (aks) and show how to construct a classical realisability model for each such aks. Moreover, we characterise those aks's that correspond to forcing over meet semilattices with a distinguished set of truth values. We then show how any aks $\mathbb{A}$ gives rise to an *order combinatory algebra* (oca) with a filter of distinguished truth values that induces a tripos (see van Oosten (2008) and Hofstra (2006) for explanations of these notions), which also gives rise to a model of ZF.

A pleasing aspect of triposes is that they give rise to a conceptually clear account of iteration of model constructions, which is also explained in van Oosten (2008). We use this framework in explaining the iterated model construction of Krivine (2008).

2. A recap of Krivine's classical realisability

In classical realisability, as described, for example, in Krivine (2009), we consider as realisers certain closed terms in an extension of the untyped λ-calculus. In order to realise classical logic, we need at least the control operator cc.

Possibly open terms of this kind are given by the grammar

$$t ::= x \mid \lambda x.t \mid ts \mid \mathsf{cc}\, t \mid \mathsf{k}_\pi$$

where π ranges over lists or stacks (*pile* in french) of terms and k is a constant turning stacks π into terms k_π. We write Λ for the set of closed terms and Π for the set of stacks of closed terms. A *process* is a pair $t \star \pi$ of a term and a stack. We write $\Lambda \star \Pi$ for the set of processes. The relation $\succ$ of *head reduction* on processes is defined inductively by

$$\begin{array}{llcl}
\text{(pop)} & \lambda x.t \star s.\pi & \succ & t[s/x] \star \pi \\
\text{(push)} & ts \star \pi & \succ & t \star s.\pi \\
\text{(store)} & \mathsf{cc}\, t \star \pi & \succ & t \star \mathsf{k}_\pi.\pi \\
\text{(restore)} & \mathsf{k}_\pi \star t.\pi' & \succ & t \star \pi\ .
\end{array}$$

We write $\succeq$ for the reflexive transitive closure of $\succ$. The first two clauses allow us to compute weak head normal forms of λ-terms, and they constitute the core of Krivine's abstract machine (see Streicher and Reus (1998) for background information). The remaining rules tell us how to evaluate calls of the control operator cc and terms of the form k_π. It is obvious that cc is the control operator 'call with current continuation' since in order to evaluate $\mathsf{cc}\, t$ we apply t to k_π (the current continuation turned into a term using k) keeping the continuation π. When applying k_π to an argument t in context π', we evaluate t with respect to the restored context π and discard the current context π'. Motivation and further explanations can be found in Streicher and Reus (1998), which, however, is based on the alternative control operator $\mathcal{C}$ whose meaning is given by the rule

$$\mathcal{C}\, t \succ tt \star \mathsf{k}_\pi$$

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where t is applied to the 'current continuation' k_π, but now in the empty context instead of the current context π.

All this is not just a purely formal game since the above language can be interpreted in the recursively defined domain

$$D \cong \Sigma^{\mathsf{List}(D)} \cong \prod_{n\in\omega} \Sigma^{D^n}$$

where Σ is the two-point lattice $\bot \sqsubset \top$. It can be shown that $D \cong \Sigma \times D^D$, that is, D^D is a retract of D. By analogy with Streicher and Reus (1998)†, the semantic clauses are

$$\begin{aligned}
[\![\lambda x.t]\!]_\rho \langle\rangle &= \top \\
[\![\lambda x.t]\!]_\rho \langle d, k\rangle &= [\![t]\!]_{\rho[d/x]} k \\
[\![ts]\!]_\rho k &= [\![t]\!]_\rho \langle [\![s]\!]_\rho, k\rangle \\
[\![\mathsf{cc}\, t]\!]_\rho k &= [\![t]\!]_\rho \langle \mathsf{ret}(k), k\rangle \\
[\![\mathsf{k}_\pi]\!]_\rho &= \mathsf{ret}([\![\pi]\!]_\rho)
\end{aligned}$$

where

$$\begin{aligned}
\mathsf{ret}(k)\langle\rangle &= \top \\
\mathsf{ret}(k)\langle d, k'\rangle &= d(k)
\end{aligned}$$

and

$$\begin{aligned}
[\![\langle\rangle]\!]_\rho &= \langle\rangle \\
[\![t.\pi]\!]_\rho &= \langle [\![t]\!]_\rho, [\![\pi]\!]_\rho\rangle .
\end{aligned}$$

It is tempting to define a relation $\perp\!\!\!\perp \subseteq D \times \mathsf{List}(D)$ by

$$d \perp\!\!\!\perp k \qquad \text{iff} \qquad d(k) = \top,$$

which can be lifted to syntax by putting $t \perp\!\!\!\perp \pi$ if and only if $[\![t]\!] \perp\!\!\!\perp [\![\pi]\!]$. Thus $\perp\!\!\!\perp$ is a set of processes that is *saturated* in the sense that

$$p \succ q \in \perp\!\!\!\perp \quad \text{implies} \qquad p \in \perp\!\!\!\perp,$$

that is, it is closed under head expansion‡.

Saturated sets of processes are an essential ingredient for defining the classical realisability interpretation for second-order logic as in Krivine (2009). For a saturated set $\perp\!\!\!\perp$ and subsets X and Y of Π and Λ, respectively, we define

$$\begin{aligned}
X^{\perp\!\!\!\perp} &= \{t \in \Lambda \mid \forall \pi \in X.\, t \perp\!\!\!\perp \pi\} \\
Y^{\perp\!\!\!\perp} &= \{\pi \in \Pi \mid \forall t \in Y.\, t \perp\!\!\!\perp \pi\}
\end{aligned}$$

† Streicher and Reus (1998) employed the recursively defined domain $D \cong \Sigma^{D^\omega}$, which is isomorphic to Σ_∞, and thus validates $D \cong D^D$.

‡ In fact, the relation $\perp\!\!\!\perp$ under consideration is also closed under head reduction and even semantic equality.

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and say a set S (of terms or stacks) is *biorthogonally closed* if and only if $S^{\perp\!\!\perp\perp\!\!\perp} = S$. We write $\mathcal{P}_{\perp\!\!\perp}(\Lambda)$ and $\mathcal{P}_{\perp\!\!\perp}(\Pi)$ for the collections of biorthogonally closed sets of terms and stacks, respectively. In realisability models induced by $\perp\!\!\!\perp$, propositions A will be interpreted as $|A| \in \mathcal{P}_{\perp\!\!\perp}(\Lambda)$. However, it will be convenient to represent $|A|$ using a set $||A||$ of stacks with $|A| = ||A||^{\perp\!\!\perp}$, though, in general, it will be different from $|A|^{\perp\!\!\perp}$.

For a saturated set $\perp\!\!\!\perp$ of processes, second-order logic over a (typically countable) set M of individuals is interpreted as follows: n-ary predicate variables range over functions $M^n \to \mathcal{P}(\Pi)$, and formulas A are interpreted as $||A||_\rho \subseteq \Pi$ according to the clauses

$$\begin{aligned}
||X(t_1,\ldots,t_n)||_\rho &= \rho(X)([\![t_1]\!]_\rho,\ldots,[\![t_1]\!]_\rho) \\
||A{\to}B||_\rho &= |A|_\rho.||B||_\rho \\
||\forall x A(x)|| &= \bigcup_{a\in M} ||A||_{\rho[a/x]} \\
||\forall X A[X]||_\rho &= \bigcup_{R\in\mathcal{P}(\Pi)^{M^n}} ||A||_{\rho[R/X]}
\end{aligned}$$

where ρ is a valuation sending individual variables to elements of M and n-ary predicate variables to elements of $\mathcal{P}(\Pi)^{M^n}$ and $|A|_\rho = ||A||_\rho^{\perp\!\!\perp}$. If A is closed, we simply write $|A|$ and $||A||$ instead of $|A|_\rho$ and $||A||_\rho$, respectively, since the interpretation of A does not depend on ρ.

Note that we have

$$\begin{aligned}
|\forall x A| &= \bigcap_{a\in M} |A[a/x]| \\
|\forall X A| &= \bigcap_{R\in\mathcal{P}(\Pi)^{M^n}} |A[R/X]|
\end{aligned}$$

since we have

$$\left(\bigcup_{i\in I} X_i\right)^{\perp\!\!\perp} = \bigcap_{i\in I} X_i^{\perp\!\!\perp}$$

for arbitrary families $X : I \to \mathcal{P}(\Pi)$.

In general, $|A{\to}B|$ is a *proper* subset of

$$|A|{\to}|B| = \{t{\in}\Lambda \mid \forall s{\in}|A|\ ts \in |B|\}$$

since, in general,

$$ts * \pi \in \perp\!\!\!\perp \quad \not\Rightarrow \quad t * s.\pi \in \perp\!\!\!\perp,$$

but it is easy to check that for every $t \in |A|{\to}|B|$, the η-expansion $\lambda x.tx \in |A{\to}B|$. But, of course, we have $|A{\to}B| = |A|{\to}|B|$ whenever $\perp\!\!\!\perp$ is also *closed under head reduction*, that is, $\perp\!\!\!\perp \ni p \succ q$ implies $q \in \perp\!\!\!\perp$.

Krivine (1990a) gives a proof term assigment for intuitionistic second-order logic, which we reproduce in Figure 1 for convenience, where $A(F(\vec{x}))$ stands for the formula obtained from $A(X)$ by replacing every subformula of the form $X(\vec{t})$ by $F(\vec{t})$.

As proved in Krivine (2009), for example, the following soundness result holds: if $x_1{:}A_1,\ldots,x_k{:}A_k \vdash u : B$ and $v_i \in |A_i|$ for $i = 1,\ldots,k$ are derivable, then $u[\vec{v}/\vec{x}] \in |B|$, that

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$$\frac{}{\Gamma, x{:}A, \Delta \vdash x : A}$$

$$\frac{\Gamma, x{:}A \vdash u : B}{\Gamma \vdash \lambda x.u : A \to B} \qquad \frac{\Gamma \vdash u : A \to B \quad \Gamma \vdash v : A}{\Gamma \vdash uv : B}$$

$$\frac{\Gamma \vdash u : A(x)}{\Gamma \vdash u : \forall x A(x)} \,(x \text{ not free in } \Gamma) \qquad \frac{\Gamma \vdash u : \forall x A(x)}{\Gamma \vdash u : A(t)}$$

$$\frac{\Gamma \vdash u : A(X)}{\Gamma \vdash u : \forall X A(X)} \,(X \text{ not free in } \Gamma) \qquad \frac{\Gamma \vdash u : \forall X A(X)}{\Gamma \vdash u : A(F(\vec{x}))}$$

Fig. 1. Typing rules for second-order intuitionistic logic.

is, proof terms are realisers. But, of course, there may be realisers that do not come from proofs in intuitionistic second-order logic. For example, $\lambda x.\mathsf{cc}\, x$ realises Peirce's law

$$((A \to B) \to A) \to A,$$

which can be seen as follows. Suppose

$$t \in |(A \to B) \to A|$$
$$\pi \in ||A||.$$

We have to show that

$$\lambda x.\mathsf{cc}\, x \star t.\pi \in \perp\!\!\!\perp,$$

but since

$$\lambda x.\mathsf{cc}\, x \star t.\pi \succ \mathsf{cc}\, t \star \pi \succ t \star \mathsf{k}_\pi.\pi,$$

we just need to show that $\mathsf{k}_\pi \in |A \to B|$. Suppose $s \in |A|$ and $\pi' \in ||B||$. Then

$$\mathsf{k}_\pi \star s.\pi' \succ s \star \pi \in \perp\!\!\!\perp,$$

so $\mathsf{k}_\pi \star s.\pi' \in \perp\!\!\!\perp$. In particular, the term $\lambda x.\mathsf{cc}\, x$ realises $(\neg A \to A) \to A$ where $\neg A \equiv A \to \bot$ with $\bot \equiv \forall X\, X$. Accordingly, the term $\lambda f.(\lambda x.\mathsf{cc}\, x)(\lambda y.fy)$ realises $\neg\neg A \to A$ since $\lambda y.fy$ realises $\neg A \to A$ whenever f realises $\neg A \to \bot$. Thus, untyped λ-calculus extended by cc allows us to represent proofs of classical second-order logic as terms.

Note that if $\perp\!\!\!\perp$ is empty for every proposition A, the set $|A|$ is either empty (if $||A||$ is non-empty) or equals Λ (if $||A||$ is empty). Thus, in this case, the notion of a model coincides with the naive two-valued one. However, if $\perp\!\!\!\perp$ is non-empty, that is, it contains an element $t \star \pi$, then $\mathsf{k}_\pi t \in |A|$ for all propositions A since for all $\pi' \in ||A||$ we have $\mathsf{k}_\pi t \star \pi' \succ t \star \pi \in \perp\!\!\!\perp$, so $\mathsf{k}_\pi t \star \pi' \in \perp\!\!\!\perp$. This was observed in Krivine (2009), though the way to overcome the obvious problem that all propositions are realisable by some element of Λ was not discussed explicitly. However, it is implicit in most of Krivine's writings, and stated explicitly in Krivine (2010), that a proposition A has to be considered as true in a model induced by a pole $\perp\!\!\!\perp$ if $t \in |A|$ for some $t \in \Lambda$ not containing the constant k. Such

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terms are called *quasi-proofs*, and we denote the corresponding set by QP. Of course, in order to ensure consistency, the pole $\perp\!\!\!\perp$ has to be chosen in such a way that for every $t \in \mathsf{QP}$, there is a $\pi \in \Pi$ with $t \star \pi \notin \perp\!\!\!\perp$.

However, in order to realise non-logical axioms beyond classical second-order arithmetic by quasi-proofs, we may have to consider extensions with additional constants. For example, in Krivine (2003), in order to realise the axiom of *countable choice*, Krivine added a constant χ^* together with the reduction rule

$$\chi^* \star t.\pi \;\succ\; t \star n_t.\pi$$

where n_t is the Church numeral representation of a Gödel number for t[†]. This is an instance of Krivine's general point of view that new programming concepts should be motivated by their need to realise important non-logical axioms. In Krivine (2008), for example, (one-cell) memory was motivated by the need to realise Cohen forcing.

3. Abstract Krivine Structures

We saw at the end of the previous section that we cannot work with a single language. For this reason, we need to axiomatise the kind of structure needed for performing Krivine's classical realisability interpretation. Such structures were axiomatised in Krivine (2008), including a form of λ-abstraction that is technically a bit cumbersome. Instead, we will introduce a version based on combinators, which we call an *abstract Krivine structure* (aks), and which is inspired by the notion of partial combinatory algebra (pca), on which ordinary realisability is based, as explained in detail in van Oosten (2008).

Definition 3.1 (abstract Krivine structures). An *abstract Krivine structure* (aks) is given by:

— a set Λ of 'terms' together with a binary application operation (written as juxtaposition) and distinguished elements $\mathsf{K}, \mathsf{S}, \mathsf{cc} \in \Lambda$;
— a subset QP of Λ that is closed under application and contains the elements K, S and cc as elements – the elements of QP are called 'quasi-proofs';
— a set Π of 'stacks' together with a push operation (push) from $\Lambda \times \Pi$ to Π (written $t.\pi$) and a unary operation $\mathsf{k} : \Pi \to \Lambda$ (written as k_π);
— a saturated subset $\perp\!\!\!\perp$ of $\Lambda \times \Pi$.

Here *saturated* means that $\perp\!\!\!\perp$ satisfies the following closure conditions:

(S1)	$ts \star \pi \in \perp\!\!\!\perp$	whenever	$t \star s.\pi \in \perp\!\!\!\perp$
(S2)	$\mathsf{K} \star t.s.\pi \in \perp\!\!\!\perp$	whenever	$t \star \pi \in \perp\!\!\!\perp$
(S3)	$\mathsf{S} \star t.s.u.\pi \in \perp\!\!\!\perp$	whenever	$tu(su) \star \pi \in \perp\!\!\!\perp$
(S4)	$\mathsf{cc} \star t.\pi \in \perp\!\!\!\perp$	whenever	$t \star \mathsf{k}_\pi.\pi \in \perp\!\!\!\perp$

[†] The assignment $t \mapsto n_t$ could be considered to be a kind of quote construct as found in LISP. Thus, χ^* may be understood as the program $\lambda x.\, x(\mathsf{quote}(x))$.

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(S5) $\mathsf{k}_\pi \star t.\pi' \in \perp\!\!\!\perp$ whenever $t \star \pi \in \perp\!\!\!\perp$

A strong abstract Krivine structure (saks) is an aks where (S1) can be strengthened to:

(SS1) $ts \star \pi \in \perp\!\!\!\perp$ iff $t \star s.\pi \in \perp\!\!\!\perp$

Recall that a combinatory algebra is a set A with a binary application operation (denoted by juxtaposition) and distinguished elements k and s of A satisfying the equations

$$\mathsf{k}xy = x$$
$$\mathsf{s}xyz = xz(yz).$$

Note that an aks is not equationally defined but instead the axioms (S1-5) state that $\perp\!\!\!\perp$ is 'closed under head expansion'. In other words, the notion of an abstract Krivine structure is free from an equality given in advance. However, we could define a notion of observational equivalence $t \sim s$ on Λ by

$$\forall \pi \in \Pi.\ t \star \pi \in \perp\!\!\!\perp \Leftrightarrow s \star \pi \in \perp\!\!\!\perp .$$

We will show in Section 5.1 that any aks can be organised into a so-called *order combinatory algebra* (oca). A further difference compared with combinatory algebras is that there is a distinguished subset of so-called 'quasi-proofs'. Terms that are not quasi-proofs only have an auxiliary status in the sense that they are needed for formulating the operational semantics of cc via conditions (S4) and (S5). There is always a minimal choice of QP, but we have to admit more comprehensive choices of QP since we may want to realise axioms beyond classical second-order arithmetic using elements of QP[†].

We will now show how any aks gives rise to a model of classical second-order logic in a way analogous to Section 2[‡]. Again, a proposition A will be interpreted as a subset $||A||$ of Π. The elements of

$$|A| = ||A||^{\perp\!\!\!\perp} = \{t \in \Lambda \mid \forall \pi \in ||A||.\ t \star \pi \in \perp\!\!\!\perp\}$$

[†] For the domain $D \cong \Sigma^{\mathsf{List}(D)}$ with pole $\perp\!\!\!\perp = \{\langle d, k\rangle \mid d(k) = \top\}$, a natural choice for QP is the unique Scott closed subset F of D with

$$d \in F \quad \text{iff} \quad \forall k \in \mathsf{List}(F).d(k) = \bot,$$

which intuitively consists of the 'error-free' elements of D that raise an error $\top$ only if the input is not error-free. The uniqueness and existence of F follows from a well-known theorem due to A. Pitts on recursively defined predicates on recursive domains. This also extends to other kinds of domains like Girard's coherence spaces or observably sequential algorithms. In the latter case, QP is the set of stategies in D that do not contain a $\top$, that is, that are error-free.

[‡] Note that our choice of combinators does not allow us to implement functional abstraction in such a way that β-reduction holds in the sense of weak head reduction. However, this has been achieved in recent papers by Krivine (Krivine 2010; Krivine 2011) using a different, and more complicated, choice of combinators, which are actually closer to Curry's original choice. Thus, we cannot interpret implication introduction directly using λ-abstraction, but rather have to axiomatise it (*à la* Hilbert) through axiom schemes realised by K and S, respectively. However, this does not affect validity, which is our main concern in this paper.

are called 'potential' realisers of A. The actual realisers of A are the elements of $|A| \cap \mathsf{QP}$. The interpretation of formulas is given by the clauses

$$||R(\vec{t})|| = R\big([\![\vec{t}]\!]\big)$$

$$||A \to B|| = |A|.||B|| = \{t.\pi \mid t \in |A|, \pi \in ||B||\}$$

$$||\forall x A(x)|| = \bigcup_{a \in M} ||A(a)||$$

$$||\forall X A(X)|| = \bigcup_{R \in \mathcal{P}(\Pi)^{M^n}} ||A(R)||,$$

where M is the underlying set of the model and formulas are closed but may contain (constants for) elements of M or $\mathcal{P}(\Pi)^{M^n}$, respectively.

We could define propositions more restrictively by

$$\mathcal{P}_{\perp\!\!\perp}(\Pi) = \left\{X \in \mathcal{P}(\Pi) \mid X = X^{\perp\!\!\perp\perp\!\!\perp}\right\},$$

and this would not change the meaning of $|A|$ for closed formulas, though it would change the meaning of $||A||$. But as in Section 2, it turns out to be convenient to postpone the biorthogonal closure. Note that $\mathcal{P}_{\perp\!\!\perp}(\Pi)$ is in 1–1-correspondence with

$$\mathcal{P}_{\perp\!\!\perp}(\Lambda) = \left\{X \in \mathcal{P}(\Lambda) \mid X = X^{\perp\!\!\perp\perp\!\!\perp}\right\}$$

via $(-)^{\perp\!\!\perp}$. Then, if the aks under consideration is strong, we have

$$|R(\vec{t})| = R\big([\![\vec{t}]\!]\big)$$

$$|A \to B| = |A| \to |B| = \{t \in \Lambda \mid \forall s \in |A|.\ ts \in |B|\}$$

$$|\forall x A(x)| = \bigcap_{a \in M} |A(a)|$$

$$|\forall X A(X)| = \bigcap_{R \in \mathcal{P}_{\perp\!\!\perp}(\Lambda)^{M^n}} |A(R)|,$$

which allows us to redefine the realisability interpretation according to a more traditional pattern.

Again, if the aks under consideration is not strong, then, in general, we only have

$$|A \to B| \subseteq |A| \to |B| = \{t \in \Lambda \mid \forall s \in |A|.\ ts \in |B|\},$$

but elements of $|A| \to |B|$ can be uniformly transformed into elements of $|A \to B|$ using the combinator $\mathsf{E} = \mathsf{S}(\mathsf{KI})$ where $\mathsf{I} = \mathsf{SKK}$.

Lemma 3.2. If $t \in |A| \to |B|$, then $\mathsf{E}t \in |A \to B|$.

Proof. It is easy to check that

$$\mathsf{I} \star t.\pi \in \perp\!\!\!\perp \;\Leftarrow\; t \star \pi \in \perp\!\!\!\perp,$$

so we have

$$\mathsf{E}t \star s.\pi \in \perp\!\!\!\perp \;\Leftarrow\; \mathsf{KI}s(ts) \star \pi \in \perp\!\!\!\perp \;\Leftarrow\; \mathsf{I} \star ts.\pi \in \perp\!\!\!\perp \;\Leftarrow\; ts \star \pi \in \perp\!\!\!\perp.$$

Then for $s \in |A|$, $\pi \in ||B||$, we have $\mathsf{E}t \star s.\pi \in \perp\!\!\!\perp$ because $ts * \pi \in \perp\!\!\!\perp$ since $t \in |A| \to |B|$. Thus $\mathsf{E}t \in |A \to B|$ as desired. □

Thus $\mathsf{E}t$ is a combinator version of the η-expansion $\lambda x.tx$, that is, E corresponds to the λ-term $\lambda y.\lambda x.yx$.

4. Cohen forcing as an instance of abstract Krivine structures

Already in Krivine (2001), Krivine emphasises that he considers classical realisability to be a generalisation of Cohen's forcing. We will make this precise by showing that Cohen forcing is the *commutative* case of classical realisability. Note that in the case of realisability induced by a partial combinatory algebra $\mathbb{A}$, this does not make sense since if application is commutative and associative in $\mathbb{A}$, we have $x = kxy = kyx = y$, and thus $\mathbb{A}$ is trivial.

A notion of forcing is usually given by a conditional meet-semilattice, that is, a poset with a greatest element 1 such that the infimum xy of x and y exists provided they have a lower bound. For our purposes, we consider what at first sight appears to be the more general situation of a meet-semilattice $\mathbb{P}$ together with a downward closed subset $\mathcal{D}$. Such a situation induces an aks as follows.

Lemma 4.1. Let $\mathbb{P}$ be a meet-semilattice and $\mathcal{D}$ be a downward closed subset of $\mathbb{P}$. This induces a saks where $\Lambda = \Pi = \mathbb{P}$, $\mathsf{QP} = \{1\}$, application and the push operation are given by the meet operation of $\mathbb{P}$, the constants are interpreted as 1 and

$$\perp\!\!\!\perp = \{(p,q) \in \mathbb{P} \times \mathbb{P} \mid pq \in \mathcal{D}\}.$$

Now for such an aks, the set $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ of propositions coincides with the set of all subsets of $\mathbb{P}$ of the form

$$X^{\perp\!\!\!\perp} = \{p \in \mathbb{P} \mid \forall q \in X.\, pq \in \mathcal{D}\}$$

for some $X \subseteq \mathbb{P}$. Note that sets of the form $X^{\perp\!\!\!\perp}$ are always downward closed and contain $\mathcal{D}$ as a subset. If $X \subseteq \mathbb{P}$ is already downward closed, $X^{\perp\!\!\!\perp}$ can be computed in the following way, which is familiar from forcing.

Lemma 4.2. If $X \subseteq \mathbb{P}$ is downward closed,

$$X^{\perp\!\!\!\perp} = \{p \in \mathbb{P} \mid \forall q \leqslant p\, (q \in X \Rightarrow q \in \mathcal{D})\}.$$

Proof. Suppose $p \in X^{\perp\!\!\!\perp}$ and $q \in X$ with $q \leqslant p$. Then $q = pq \in \mathcal{D}$. For the converse direction, suppose $p \in \mathbb{P}$ with $\forall q \leqslant p\, (q \in X \Rightarrow q \in \mathcal{D})$. Then for $q \in X$, we have $pq \in X$ since X is downward closed, so $pq \in \mathcal{D}$ by assumption on p. □

It is also an easy exercise to prove the following lemma.

Lemma 4.3. For downward closed $X, Y \subseteq \mathbb{P}$, we have

$$X \to Y = \{p \in \mathbb{P} \mid \forall q \in X.\, pq \in Y\} = \{p \in \mathbb{P} \mid \forall q \leqslant p\, (q \in X \Rightarrow q \in Y)\},$$

and thus $Z \subseteq X \to Y$ if and only if $Z \cap X \subseteq Y$ for downward closed $Z \subseteq \mathbb{P}$.

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Using Lemma 4.2, it is easy to see that for downward closed $X \subseteq \mathbb{P}$, we have $X = X^{\perp\!\!\perp}$ if and only if $\mathcal{D} \subseteq X$ and $p \in X \setminus \mathcal{D}$ whenever for all $q \leqslant p$ with $q \notin \mathcal{D}$ there exists $r \leqslant q$ with $r \in X \setminus \mathcal{D}$. Thus $\mathcal{P}_{\perp\!\!\perp}(\Pi)$ is, via $(-) \setminus \mathcal{D}$, in 1–1-correspondence with those subsets A of the poset $\mathbb{P}_{\uparrow} = \mathbb{P} \setminus \mathcal{D}$ that are *regular* in the sense that $p \in A$ whenever $\forall q{\leqslant}p\ \exists r{\leqslant}q\ r \in A$. Lemmas 4.2 and 4.3 say that under this correspondence, negation and implication are constructed as in Cohen forcing (or Kripke models).

It is immediate from Lemma 4.3 that $X \to Y$ contains a quasi-proof (that is, 1) if and only if $X \subseteq Y$.

We can now characterise those aks's that arise from Cohen forcing.

Theorem 4.4. An aks arises, up to isomorphism, from a downward closed subset of a meet-semilattice if and only if it is strong and satisfies the following requirements:

(1) $\mathsf{k} : \Pi \to \Lambda$ is a bijection.
(2) The application operation endows Λ with the structure of a commutative idempotent monoid where $\mathsf{QP} = \{1\}$.
(3) Application coincides with the push operation when identifying Λ and Π via k.

Proof. It is clear that all these conditions are necessary, so we suppose we are given a saks satisfying the above conditions. By condition (2), application endows the set Λ with the structure of a meet-semilattice, which we call $\mathbb{P}$. For $\mathcal{D}$, we take the subset $\{t \in \Lambda \mid (t, 1) \in \perp\!\!\perp\}$ of $\mathbb{P} = \Lambda$. Note that $\mathcal{D}$ is downward closed due to condition (3). Since the aks is strong by assumption, we have

$$ts \in \mathcal{D} \quad \text{iff} \quad (ts, 1) \in \perp\!\!\perp \quad \text{iff} \quad (t, s1) \in \perp\!\!\perp \quad \text{iff} \quad (t, s) \in \perp\!\!\perp,$$

which completes the argument. □

This explains the sense in which Krivine considers forcing to be 'commutative realisability'.

5. Classical realisability tripos and topos

In this section we will show that we can associate with any aks a tripos, the so-called *Krivine tripos*, that gives rise to a model of higher-order classical logic extending the model of second-order classical logic of Section 3.

5.1. *Abstract Krivine structures as order combinatory algebras*

Hofstra and van Oosten's notion of an order partial combinatory algebra (opca) (Hofstra and van Oosten 2003) generalises both pca's and complete Heyting algebras (cHa's) as explained in van Oosten (2008). For our purposes we just need the following non-partial version, which also covers the case of complete Heyting algebras.

Definition 5.1 (order combinatory algebra with a filter). An *order combinatory algebra* (oca) is a triple $(\mathbb{A}, \leqslant, \bullet)$ where $\leqslant$ is a partial order on $\mathbb{A}$ and $\bullet$ is a binary monotone

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operation on $\mathbb{A}$ such that there exist $k, s \in \mathbb{A}$ with

$$k \bullet a \bullet b \leqslant a$$
$$s \bullet a \bullet b \bullet c \leqslant a \bullet c \bullet (b \bullet c)$$

for all $a, b, c \in \mathbb{A}$.

A *filter* on an oca $(\mathbb{A}, \leqslant, \bullet)$ is a subset Φ of $\mathbb{A}$ that is closed under $\bullet$ and contains (some choice of) k and s (for $\mathbb{A}$).

With every aks, we associate an oca with a filter as follows. The underlying set is $\mathcal{P}_{\perp\!\!\perp}(\Pi)$, on which we define a partial order by $a \leqslant b$ if and only if $a \supseteq b$. Application is defined by

$$a \bullet b = \{\pi \in \Pi \mid \forall t \in |a|, s \in |b|.\ t * s.\pi \in \perp\!\!\!\perp\}^{\perp\!\!\perp\perp\!\!\perp}$$

where $|a| = a^{\perp\!\!\perp}$, and similarly for b. It is obvious that $a \leqslant b$ if and only if $|a| \subseteq |b|$. Note that if the aks under consideration is strong, we have

$$|a \bullet b| = \{ts \mid t \in |a|, s \in |b|\}^{\perp\!\!\perp\perp\!\!\perp},$$

which explains how we arrived at the definition of $\bullet$. The filter is defined by

$$\Phi = \{a \in \mathcal{P}_{\perp\!\!\perp}(\Pi) \mid |a| \cap \mathsf{QP} \neq \varnothing\},$$

that is, a is in the filter if and only if $|a|$ contains a quasi-proof.

In order to show that $(\mathcal{P}_{\perp\!\!\perp}(\Pi), \leqslant, \bullet)$ is actually an oca, we have to identify appropriate $k, s \in \mathcal{P}_{\perp\!\!\perp}(\Lambda)$ satisfying the following conditions for all $x, y, z \in \mathcal{P}_{\perp\!\!\perp}(\Pi)$:

(1) $k \bullet x \bullet y \leqslant x$
(2) $s \bullet x \bullet y \bullet z \leqslant x \bullet z \bullet (y \bullet z)$.

The most obvious choices for k and s are $\{\mathsf{K}\}^{\perp\!\!\perp}$ and $\{\mathsf{S}\}^{\perp\!\!\perp}$, respectively, because then $|k| = \{\mathsf{K}\}^{\perp\!\!\perp\perp\!\!\perp}$ and $|s| = \{\mathsf{S}\}^{\perp\!\!\perp\perp\!\!\perp}$.

One could show by brute force that these choices of k and s validate the conditions (1) and (2), but instead we will give a more elegant argument, which was suggested to us by Benno van den Berg. First we define

$$x \to y = \{t.\pi \mid t \in |x|, \pi \in y\}^{\perp\!\!\perp\perp\!\!\perp}$$

for $x, y \in \mathcal{P}_{\perp\!\!\perp}(\Pi)$ and observe the following result.

Lemma 5.2. From $x \leqslant y \to z$, it follows that $x \bullet y \leqslant z$.

Proof. Suppose $x \leqslant y \to z$. Then we have

$$\forall u \in |x|. \forall v \in |y|. \forall \pi \in z.\ u \star v.\pi \in \perp\!\!\!\perp,$$

from which it follows that $z \subseteq x \bullet y$. Thus $x \bullet y \leqslant z$ as desired. □

Moreover, we also have the following result.

Lemma 5.3. If $u \in |x|$ and $v \in |y|$, then $uv \in |x \bullet y|$.

Proof. Suppose $u \in |x|$ and $v \in |y|$. Let $\pi \in x \bullet y$. Then $u \star v.\pi \in \perp\!\!\!\perp$, so $uv * \pi \in \perp\!\!\!\perp$ by property (S1) of $\perp\!\!\!\perp$ □

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Later in the paper we will use the fact that the converse of the implication of Lemma 5.2 holds in the following restricted sense.

Lemma 5.4. If $x \bullet y \leqslant z$, then for all $t \in |x|$, we have $\mathsf{E}t \in |y \to z|$.

Proof. Suppose $x \bullet y \leqslant z$, that is,

$$\forall t \in |x \bullet y|. \forall \pi \in z.\, t \star \pi \in \perp\!\!\!\perp .$$

Thus, by Lemma 5.3, we have

$$\forall u \in |x|. \forall v \in |y|. \forall \pi \in z.\, uv \star \pi \in \perp\!\!\!\perp .$$

Since $uv \star \pi \in \perp\!\!\!\perp$ implies $\mathsf{E}u \star v.\pi \in \perp\!\!\!\perp$, it follows that

$$\forall u \in |x|. \forall v \in |y|. \forall \pi \in z.\, \mathsf{E}u \star v.\pi \in \perp\!\!\!\perp .$$

Thus

$$\forall t \in |x|.\, \mathsf{E}t \in |y \to z|$$

as desired. □

We are now ready to show that (1) and (2) hold for $k = \{\mathsf{K}\}^{\perp\!\!\!\perp}$ and $s = \{\mathsf{S}\}^{\perp\!\!\!\perp}$.

(1) To show that $k \bullet x \bullet y \leqslant x$, it suffices, by (two applications of) Lemma 5.2, to show that $k \leqslant x \to y \to x$. But, it is obvious that $\mathsf{K} \in |x \to y \to x|$, so

$$k = \{\mathsf{K}\}^{\perp\!\!\!\perp\perp\!\!\!\perp} \subseteq |x \to y \to x|.$$

(2) To show that

$$s \bullet x \bullet y \bullet z \leqslant x \bullet z \bullet (y \bullet z),$$

it suffices, by (multiple applications of) Lemma 5.2, to show that

$$s \leqslant x \to y \to z \to (x \bullet z \bullet (y \bullet z)).$$

So we just need to show that

$$\mathsf{S} \in |x \to y \to z \to (x \bullet z \bullet (y \bullet z))|.$$

To do this, we suppose $u \in |x|$, $v \in |y|$, $w \in |z|$ and $\pi \in x \bullet z \bullet (y \bullet z)$. Applying Lemma 5.3 iteratively, we get $uw(vw) \in |x \bullet z \bullet (y \bullet z)|$, and thus $uw(vw) \star \pi \in \perp\!\!\!\perp$. By property (S3) of $\perp\!\!\!\perp$, it follows that $\mathsf{S} * u.v.w.\pi \in \perp\!\!\!\perp$ as desired.

We still need to show that

$$\Phi = \{a \in \mathcal{P}_{\perp\!\!\!\perp}(\Pi) \mid |a| \cap \mathsf{QP} \neq \varnothing\}$$

is actually a filter on $(\mathcal{P}_{\perp\!\!\!\perp}(\Pi), \leqslant, \bullet)$. Suppose a and b are in Φ. Then there exist $u \in |a| \cap \mathsf{QP}$ and $v \in |b| \cap \mathsf{QP}$. By Lemma 5.3, we have $uv \in |a \bullet b|$. Since QP is closed under application, we have $uv \in \mathsf{QP}$. Thus $a \bullet b \in \Phi$. Since $\mathsf{S}, \mathsf{K} \in \mathsf{QP}$ and

$$\mathsf{K} \in \{\mathsf{K}\}^{\perp\!\!\!\perp\perp\!\!\!\perp} = |k|$$
$$\mathsf{S} \in \{\mathsf{S}\}^{\perp\!\!\!\perp\perp\!\!\!\perp} = |s|,$$

it follows that $k, s \in \Phi$.

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We will now collect together a few facts about an oca $\mathbb{A}$ endowed with a filter Φ from van Oosten (2008) and Hofstra (2006), which we will need for verifying the construction of the Krivine tripos in Section 5.2. For convenience, we will often write xy instead of $x \bullet y$ for $x, y \in \mathbb{A}$. A *polynomial* over $\mathbb{A}$ is a term built from elements of $\mathbb{A}$ and a (countable) set of variables using the application operation $\bullet$.

If $\mathbb{A}$ is an oca, then for every polynomial $t[\vec{x}, x]$, there exists a polynomial $\lambda^* x.t$ whose free variables are included in the list $\vec{x}$ such that

$$(\lambda^* x.t)a \leqslant t[\vec{x}, a]$$

for all $a \in \mathbb{A}$. Moreover, if all constants of t are in Φ, then $\lambda^* x.t \in \Phi$ provided all items of $\vec{x}$ are in Φ. For example, $k' = \lambda^* x.\lambda^* y.y \in \Phi$.

Using these facts, we can define the following pairing and projection operations in every oca $\mathbb{A}$

$$p = \lambda^* x.\lambda^* y.\lambda^* z.zxy$$
$$p_1 = \lambda^* z.zk$$
$$p_2 = \lambda^* z.zk',$$

which are elements of Φ and validate the laws

$$p_1(pxy) \leqslant x$$
$$p_2(pxy) \leqslant y.$$

5.2. *The Krivine tripos*

Given an oca $\mathbb{A} = (\mathbb{A}, \leqslant, \bullet)$ and a filter Φ on it, we may associate with it the following **Set**-indexed preorder $[-, \mathbb{A}]_\Phi$:

— $[I, \mathbb{A}]_\Phi = \mathbb{A}^I$ is the set of all functions from set I to $\mathbb{A}$.
— This set is endowed with the *entailment* relation

$$\phi \vdash_I \psi \quad \text{iff} \quad \exists a \in \Phi \forall i \in I.\ a \bullet \phi_i \leqslant \psi_i.$$

— For $u : J \to I$, the *reindexing map* $[u, \mathbb{A}]_\Phi = u^* : \mathbb{A}^I \to \mathbb{A}^J$ sends ϕ to $u^*\phi = (\phi_{u(j)})_{j \in J}$.

It is easy to see that $\vdash_I$ actually defines a preorder on $\mathbb{A}^I$. Let $e = \lambda^* x.x \in \Phi$. Then for all $\varphi \in \mathbb{A}^I$, we have $\forall i \in I.\ e \bullet \varphi_i \leqslant \varphi_i$ and thus $\varphi \vdash_I \varphi$. Suppose $\varphi \vdash_I \psi$ and $\psi \vdash_I \theta$. Then there exists $a, b \in \Phi$ such that $a \bullet \varphi_i \leqslant \psi_i$ and $b \bullet \psi_i \leqslant \theta_i$ for all $i \in I$. Then for

$$c = \lambda^* x.b \bullet (a \bullet x) \in \Phi,$$

we have

$$c \bullet \varphi_i \leqslant b \bullet (a \bullet \varphi_i) \leqslant b \bullet \psi_i \leqslant \theta_i$$

for all $i \in I$. Thus $\varphi \vdash_I \theta$.

Suppose $u : J \to I$ is a map in **Set** and $\varphi \vdash_I \psi$. Then there exists $a \in \Phi$ with $\forall i \in I.\ a \bullet \varphi_i \leqslant \psi_i$. Thus, *a fortiori*, we have

$$\forall j \in J.\ a \bullet \varphi_{u(j)} \leqslant \psi_{u(j)},$$

that is, $u^*\varphi \vdash_J u^*\psi$. Thus u^* preserves entailment.

From now on we will assume that $\mathbb{A}$ and the filter Φ on it is induced by an aks, as described in Section 5.1. Under this assumption, we can give the following characterisation of entailment, which will turn out as crucial for proving that $[-,\mathbb{A}]_\Phi$ is indeed a tripos.

Lemma 5.5. For all sets I, we have

$$\varphi \vdash_I \psi \quad \text{iff} \quad \exists t \in \mathsf{QP}.\forall i \in I.\, t \in |\varphi_i| {\rightarrow} |\psi_i| \quad \text{iff} \quad \exists t \in \mathsf{QP}.\forall i \in I.\, t \in |\varphi_i \to \psi_i|$$

for all $\varphi, \psi \in [I,\mathbb{A}]_\Phi$.

Proof. Suppose $\varphi \vdash_I \psi$. Then there exists $a \in \Phi$ such that $\forall i \in I.\ a \bullet \varphi_i \leqslant \psi_i$. By Lemma 5.3, for all $i \in I$, $t \in |a|$ and $s \in |\varphi_i|$, we have $ts \in |a \bullet \phi_i| \subseteq |\psi_i|$. Let $t \in |a| \cap \mathsf{QP}$. Then for all $i \in I$, we have $t \in |\varphi_i| \to |\psi_i|$.

Suppose, for some $t \in \mathsf{QP}$, we have $t \in |\varphi_i| \to |\psi_i|$ for all $i \in I$. Then, by Lemma 5.4, we have $\mathsf{E}t \in |\varphi_i \to \psi_i|$ for all $i \in I$ and $\mathsf{E}t \in \mathsf{QP}$ since QP is closed under application and contains K and S as elements.

Suppose there exists a $t \in \mathsf{QP}$ such that $\forall i \in I.\ t \in |\varphi_i \to \psi_i|$. Then we have

$$\forall i \in I.\ \{t\}^{\perp\!\!\!\perp\perp\!\!\!\perp} \subseteq |\varphi_i \to \psi_i|,$$

so for $a = \{t\}^{\perp\!\!\!\perp}$, we have

$$\forall i \in I.\forall u \in |a|.\forall v \in |\varphi_i|.\forall \pi \in \psi_i.\ u \star v.\pi \in \perp\!\!\!\perp,$$

from which it follows that $\forall i \in I.\ a \bullet \varphi_i \leqslant \psi_i$ and thus $\varphi \vdash_I \psi$ since $a = \{t\}^{\perp\!\!\!\perp} \in \Phi$ (because $t \in \mathsf{QP}$ and $t \in \{t\}^{\perp\!\!\!\perp\perp\!\!\!\perp} = |a|$). □

The following lemma will be useful in the proof of Theorem 5.9.

Lemma 5.6. Let I be a set and $\varphi, \psi, \theta \in [I,\mathbb{A}]_\Phi$. We write $\varphi{\rightarrow}\psi$ for the family $(\varphi_i{\rightarrow}\psi_i)_{i\in I}$. Then $\theta \vdash_I \varphi{\rightarrow}\psi$ if and only if there exists an $a \in \Phi$ such that $\forall i \in I.\, a \bullet \theta_i \bullet \varphi_i \leqslant \psi_i$.

Proof. Suppose $\theta \vdash_I \varphi \to \psi$. Then there is an $a \in \Phi$ such that

$$\forall i \in I.\, a \bullet \theta_i \leqslant \varphi_i{\rightarrow}\psi_i.$$

By Lemma 5.2, it follows that

$$\forall i \in I.\, a \bullet \theta_i \bullet \varphi_i \leqslant \psi_i.$$

For the converse direction, we suppose $a \in \Phi$ with

$$\forall i \in I.\, a \bullet \theta_i \bullet \varphi_i \leqslant \psi_i.$$

Then, by Lemma 5.3, we have

$$\forall i \in I.\forall t \in |a \bullet \theta_i|.\forall s \in |\varphi_i|.\, ts \in |a \bullet \theta_i \bullet \varphi_i| \subseteq |\psi_i|,$$

and thus

$$\forall i \in I.\forall t \in |a \bullet \theta_i|.\, \mathsf{E}t \in |\varphi_i{\rightarrow}\psi_i|$$

by Lemma 5.4. Since $\mathsf{E} \in \mathsf{QP}$, by Lemma 5.5 there is a $b \in \Phi$ with

$$\forall i \in I.\, b \bullet (a \bullet \theta_i) \leqslant \varphi_i{\rightarrow}\psi_i.$$

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Let $c \in \Phi$ with $c \bullet x \leqslant b \bullet (a \bullet x)$ for all $x \in \mathbb{A}$. Thus for all $i \in I$, we have

$$c \bullet \theta_i \leqslant b \bullet (a \bullet \theta_i) \leqslant \varphi_i \rightarrow \psi_i,$$

from which it follows that $\theta \vdash_I \varphi \rightarrow \psi$ as desired. □

Furthermore, for every set I, we will need an 'equality predicate' $\mathrm{eq}_I : I \times I \rightarrow \mathbb{A}$ on I defined by

$$\mathrm{eq}_I(i,j) = \begin{cases} \{\mathsf{I}\}^{\perp\!\!\perp} & \text{if } i = j \\ \Pi & \text{otherwise.} \end{cases}$$

Note that $\mathrm{eq}_I(i,i) \in \Phi$ since $\mathsf{I} \in \mathsf{QP}$. The equality predicate has the following remarkable properties.

Lemma 5.7. For every $i \in I$, we have $\mathsf{I} \in |\mathrm{eq}_I(i,i)|$ and $eq_I(i,i) \bullet a \leqslant a$ for all $a \in \mathbb{A}$. If $i, j \in I$ with $i \neq j$, then $\mathrm{eq}_I(i,j) \bullet a \leqslant b$ for all $a, b \in \mathbb{A}$.

Proof. It is obvious that $|\mathrm{eq}_I(i,i)| = \{\mathsf{I}\}^{\perp\!\!\perp\perp\!\!\perp}$. Thus $\mathsf{I} \in |\mathrm{eq}_I(i,i)|$ since $\mathsf{I} \in \{\mathsf{I}\} \subseteq \{\mathsf{I}\}^{\perp\!\!\perp\perp\!\!\perp}$. Let $a \in \mathbb{A}$. By Lemma 5.2, to show $\mathrm{eq}_I(i,i) \bullet a \leqslant a$, it suffices to show that $\mathrm{eq}_I(i,i) \leqslant a \rightarrow a$, which holds since $\mathsf{I} \in |a \rightarrow a|$ and thus $\{\mathsf{I}\}^{\perp\!\!\perp\perp\!\!\perp} \subseteq |a \rightarrow a|$.

Suppose $i, j \in I$ with $i \neq j$ and $a, b \in \mathbb{A}$. Then

$$\mathrm{eq}_I(i,j) = \Pi \supseteq a \rightarrow b,$$

from which it follows that $\mathrm{eq}_I(i,j) \leqslant a \rightarrow b$ and thus $\mathrm{eq}_I(i,j) \bullet a \leqslant b$ by Lemma 5.2. □

Lemma 5.8. Let $a \in \mathbb{A}$ and $t \in \left|\{\mathsf{I}\}^{\perp\!\!\perp} \rightarrow a\right|$. Then:

(i) $\mathsf{SI}t \in \left|\{\mathsf{I}\}^{\perp\!\!\perp} \rightarrow a\right|$; and
(ii) $\mathsf{SI}t \in |\Pi \rightarrow b|$ for all $b \in \mathbb{A}$.

Proof. Suppose $t \in \left|\{\mathsf{I}\}^{\perp\!\!\perp} \rightarrow a\right|$.

(i) Suppose $s \in \{\mathsf{I}\}^{\perp\!\!\perp\perp\!\!\perp}$ and $\pi \in a$. We have to show that $\mathsf{SI}t \star s.\pi \in \perp\!\!\perp$. Since $t \star s.\pi \in \perp\!\!\perp$, we have $ts \star \pi \in \perp\!\!\perp$. Thus $\mathsf{I} \star ts.\pi \in \perp\!\!\perp$ and, accordingly, $ts.\pi \in \{\mathsf{I}\}^{\perp\!\!\perp}$. Thus $s \star ts.\pi \in \perp\!\!\perp$, so $\mathsf{I}s(ts) \star \pi \in \perp\!\!\perp$ also. Hence, by property (S3) of $\perp\!\!\perp$, we have $\mathsf{S} \star \mathsf{I}.t.s.\pi \in \perp\!\!\perp$ and thus, as desired, $\mathsf{SI}t \star s.\pi$ also.

(ii) Suppose $s \in \Pi^{\perp\!\!\perp}$ and $\pi \in b$. We have to show that $\mathsf{SI}t \star s.\pi \in \perp\!\!\perp$. Since $s \in \Pi^{\perp\!\!\perp}$, we have $s \star ts.\pi \in \perp\!\!\perp$. Thus $\mathsf{I}s(ts) \star \pi \in \perp\!\!\perp$ also. By property (S3) of $\perp\!\!\perp$, we also have $\mathsf{S} \star \mathsf{I}.t.s \in \perp\!\!\perp$, from which it follows that $\mathsf{SI}t \star s.\pi \in \perp\!\!\perp$ as desired. □

We are now ready to prove the main result of this section.

Theorem 5.9. If $\mathbb{A}$ and Φ arise from an aks, the indexed preorder $[-, \mathbb{A}]_\Phi$ is a *tripos*, that is, we have:

— All $[I, \mathbb{A}]_\Phi$ are pre-Heyting-algebras whose structures are preserved by reindexing.
— For every $u : J \rightarrow I$ in **Set**, the reindexing map u^* has a left adjoint $\exists_u$ and a right adjoint $\forall_u$ satisfying the (Beck–)Chevalley condition.
— There is a *generic predicate* $\mathsf{T} \in [\Sigma, \mathbb{A}]_\Phi$ such that all other predicates can be obtained from T by appropriate reindexing.

Moreover, it is boolean in the sense that all $[I, \mathbb{A}]_\Phi$ are pre-boolean-algebras.

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Proof. Recall that we often denote application in the oca $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ by juxtaposition.

We will first show that $[I, \mathbb{A}]_\Phi$ has finite infima. Let

$$\top = \{\pi \in \Pi \mid \forall t \in \Lambda.\, t \star \pi \in \perp\!\!\!\perp\}.$$

This is obviously an element in $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ and satisfies $a \leqslant \top$ for all $a \in \mathbb{A}$. Let $\top_I$ be the constant family in $[I, \mathbb{A}]_\Phi$ with value $\top$. If $\varphi \in [I, \mathbb{A}]_\Phi$, then for all $i \in I$, we have $(\lambda^* x.x)\varphi_i \leqslant |\top|$. Since $\lambda^* x.x \in \Phi$, we have $\varphi \vdash_I \top_I$. Thus $\top_I$ is a greatest element in $[I, \mathbb{A}]_\Phi$. To show that $[I, \mathbb{A}]_\Phi$ has binary infima, suppose $\varphi, \psi \in \mathbb{A}^I$. Let $\varphi \wedge \psi \in \mathbb{A}^I$ with $(\varphi \wedge \psi)_i = p\varphi_i\psi_i$ for all $i \in I$. Since $\forall i \in I.\, p_1(p\varphi_i\psi_i) \leqslant \varphi_i$, we have $\varphi \wedge \psi \vdash_I \varphi$, and since $\forall i \in I.\, p_2(p\varphi_i\psi_i) \leqslant \psi_i$, we have $\varphi \wedge \psi \vdash_I \psi$. Suppose $\theta \vdash_I \varphi, \psi$. Then there exist $a, b \in \Phi$ such that for all $i \in I$, we have $a\theta_i \leqslant \varphi_i$ and $b\theta_i \leqslant \psi_i$. For $c = \lambda^* x.p(ax)(bx) \in \Phi$, we have for all $i \in I$ that

$$c\theta_i \leqslant p(a\theta_i)(b\theta_i) \leqslant p\varphi_i\psi_i = (\varphi \wedge \psi)_i,$$

and thus $\theta \vdash_I \varphi \wedge \psi$ as desired. It is obvious that every reindexing u^* preserves $\top$ and $\wedge$.

Next we will show that all $[I, \mathbb{A}]_\Phi$ have implication. Suppose $\varphi, \psi \in [I, \mathbb{A}]_\Phi$. We define $\varphi{\to}\psi$ as $(\varphi{\to}\psi)_i = \varphi_i{\to}\psi_i$ for $i \in I$. Suppose $\theta \vdash_I \varphi{\to}\psi$. Then there exists $a \in \Phi$ with $a\theta_i \leqslant \varphi_i{\to}\psi_i$ for all $i \in I$. Then, by Lemma 5.2, we have $a\theta_i\varphi_i \leqslant \psi_i$ for all $i \in I$. Thus we have

$$a(p_1(p\theta_i\varphi_i))(p_2(p\theta_i\varphi_i)) \leqslant a\theta_i\varphi_i \leqslant \psi_i$$

for all $i \in I$. Thus, for $f = \lambda^* x.a(p_1 x)(p_2 x) \in \Phi$, we have

$$f(\theta_i \wedge \varphi_i) \leqslant \psi_i$$

for all $i \in I$, that is, $\theta \wedge \varphi \vdash_I \psi$. For the converse direction, suppose $\theta \wedge \varphi \vdash_I \psi$. Hence there is an $a \in \Phi$ with $a(p\theta_i\varphi_i) \leqslant \psi_i$ for all $i \in I$, and for $f = \lambda^* x.\lambda^* y.a(pxy) \in \Phi$, we have

$$f\theta_i\varphi_i \leqslant \psi_i$$

for all $i \in I$. Thus, by Lemma 5.6, it follows that $\theta \vdash_I \varphi{\to}\psi$. Thus we have shown that $\varphi{\to}\psi$ is actually the exponential in $[I, \mathbb{A}]_\Phi$. It follows from $\varphi{\to}\psi \vdash_I \varphi{\to}\psi$ that $(\varphi{\to}\psi) \wedge \varphi \vdash_I \psi$. Since for $u : J \to I$ we have $u^*(\varphi{\to}\psi) = u^*\varphi{\to}u^*\psi$ and u^* preserves $\wedge$, it follows that

$$(u^*\varphi{\to}u^*\psi) \wedge u^*\varphi = u^*((\varphi{\to}\psi) \wedge \varphi) \vdash_J u^*\psi.$$

Thus reindexing preserves implication.

Next we show that $[-, \mathbb{A}]_\Phi$ has universal quantification. For $\alpha : J \to I$ and $\varphi \in [J, \mathbb{A}]_\Phi$, we define $\forall_u(\varphi)$ in $[I, \mathbb{A}]_\Phi$ by

$$\forall_\alpha(\varphi)_i = \left(\bigcup_{j \in J} \mathrm{eq}_I(\alpha(j), i){\to}\varphi_j\right)^{\perp\!\!\!\perp\perp\!\!\!\perp}$$

for all $i \in I$. Note that

$$|\forall_\alpha(\varphi)_i| = \left(\bigcup_{j \in J} \mathrm{eq}_I(\alpha(j), i){\to}\varphi_j\right)^{\perp\!\!\!\perp} = \bigcap_{j \in J} |\mathrm{eq}_I(\alpha(j), i){\to}\varphi_j|.$$

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Suppose $\psi \in [I, \mathbb{A}]_\Phi$. We have to show that

$$\alpha^*\psi \vdash_J \varphi \qquad \text{iff} \qquad \psi \vdash_I \forall_\alpha(\varphi).$$

For the if direction, suppose $\psi \vdash_I \forall_\alpha(\varphi)$. Then there is a $c \in \Phi$ with

$$c\psi_i \leqslant \mathrm{eq}_I(\alpha(j), i) \rightarrow \varphi_j$$

for all $i \in I$ and $j \in J$. Thus, in particular, we have

$$c\psi_{\alpha(j)} \leqslant \mathrm{eq}_I(\alpha(j), \alpha(j)) \rightarrow \varphi_j$$

for all $j \in J$. Since $c \in \Phi$, we have

$$\psi_{\alpha(j)} \vdash_{j \in J} \mathrm{eq}_I(\alpha(j), \alpha(j)) \rightarrow \varphi_j$$

and, accordingly,

$$\mathrm{eq}_I(\alpha(j), \alpha(j)) \vdash_{j \in J} \psi_{\alpha(j)} \rightarrow \varphi_j$$

by that part of propositional logic we have already established for $[I, \mathbb{A}]_\Phi$. Thus, by Lemma 5.5, there is a $t \in \mathsf{QP}$ such that

$$\forall j \in J. \forall s \in |\mathrm{eq}_I(\alpha(j), \alpha(j))|.\, ts \in |\psi_{\alpha(j)} \rightarrow \varphi_j|,$$

from which it follows that

$$\forall j \in J.\, t\mathsf{I} \in |\psi_{\alpha(j)} \rightarrow \varphi_j|$$

since, by Lemma 5.7, we have $\mathsf{I} \in |\mathrm{eq}_I(\alpha(j), \alpha(j))|$ for all $j \in J$. Thus, we have

$$\forall j \in J. \forall s \in |\psi_{\alpha(j)}|.\, t\mathsf{I}s \in |\psi_{\alpha(j)}| \rightarrow |\varphi_j|,$$

from which it follows by Lemma 5.5, since $t\mathsf{I} \in \mathsf{QP}$, that $\alpha^*\psi \vdash_J \varphi$ as desired.

For the reverse direction, suppose $\alpha^*\psi \vdash_J \varphi$. Hence, there exists an $a \in \Phi$ such that $\forall j \in J.\, a\psi_{\alpha(j)} \leqslant \varphi_j$. Then $b = \lambda^* x.\lambda^* y.y(ax) \in \Phi$. Suppose $i \in I$ and $j \in J$. If $\alpha(j) = i$, then, by Lemma 5.7,

$$b\psi_i \mathrm{eq}_I(\alpha(j), i) \leqslant \mathrm{eq}_I(\alpha(j), i)(a\psi_i) \leqslant a\psi_i \leqslant a\psi_{\alpha(j)} \leqslant \varphi_j.$$

Otherwise, again by Lemma 5.7, we have

$$b\psi_i \mathrm{eq}_I(\alpha(j), i) \leqslant \mathrm{eq}_I(\alpha(j), i)(a\psi_i) \leqslant \varphi_j.$$

Thus we have shown that

$$\forall i \in I, j \in J.\, b\psi_i \mathrm{eq}_I(\alpha(j), i) \leqslant \varphi_i,$$

from which it follows by Lemma 5.6 that there is a $c \in \Phi$ with

$$\forall i \in I, j \in J.\, c\psi_i \leqslant \mathrm{eq}_I(\alpha(j), i) \rightarrow \varphi_j.$$

Thus we have

$$\forall i \in I, j \in J.\, |c\psi_i| \subseteq |\mathrm{eq}_I(\alpha(j), i) \rightarrow \varphi_j|,$$

from which it follows that

$$\forall i \in I.\, |c\psi_i| \subseteq \bigcap_{j \in J} |\mathrm{eq}_I(\alpha(j), i) \rightarrow \varphi_j| = |\forall_\alpha(\varphi)_i|.$$

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Thus we have

$$\forall i \in I.\, c\psi_i \leqslant \forall_\alpha(\varphi)_i,$$

and since $c \in \Phi$, it follows that $\psi \vdash_I \forall_\alpha(\varphi)$ as desired.

To show that $\forall$ satisfies the (Beck–)Chevalley condition, we suppose

$$\begin{array}{ccc} P & \xrightarrow{\;q\;} & J \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \alpha} \\ K & \xrightarrow[\;\beta\;]{} & I \end{array}$$

is a pullback in **Set** and $\varphi \in [J, \mathbb{A}]_\Phi$. We have to show that $\beta^*\forall_\alpha\varphi \cong \forall_p q^*\varphi$. Note that by abstract nonsense, $\beta^*\forall_\alpha\varphi \vdash_K \forall_p q^*\varphi$ does hold anyway. Thus, it suffices to show that $\forall_p q^*\varphi \vdash_K \beta^*\forall_\alpha\varphi$. To do this, by Lemma 5.5, it suffices to show that for every $k \in K$, the term $\mathsf{SI} \in \mathsf{QP}$ sends elements of $|(\forall_p q^*\varphi)_k|$ to elements of $|(\beta^*\forall_\alpha\varphi)_k|$. Suppose $k \in K$. We have

$$|(\forall_p q^*\varphi)_k| = \bigcap_{z\in P} |\mathrm{eq}_K(p(z), k) \to \varphi_{q(z)}|$$

and

$$|(\beta^*\forall_\alpha\varphi)_k| = \bigcap_{j\in J} |\mathrm{eq}_I(\alpha(j), \beta(k)) \to \varphi_j|\,.$$

Suppose

$$t \in \bigcap_{z\in P} |(\mathrm{eq}_K(p(z), k) \to \varphi_{q(z)}|$$

and $j \in J$. Suppose $\alpha(j) = \beta(k)$. Then there is a $z \in P$ with $p(z) = k$ and $q(z) = j$. By assumption on t, we have

$$t \in |(\mathrm{eq}_K(p(z), k) \to \varphi_{q(z)}|\,,$$

so

$$t \in |\mathrm{eq}_I(\alpha(j), \beta(k)) \to \varphi_j|$$

since

$$\mathrm{eq}_K(p(z), k) = \mathrm{eq}_I(\alpha(j), \beta(k)).$$

Thus, by Lemma 5.8 (i), we have

$$\mathsf{SI}t \in |\mathrm{eq}_I(\alpha(j), \beta(k)) \to \varphi_j|$$

since

$$\mathrm{eq}_I(\alpha(j), \beta(k)) = \{\mathsf{I}\}^\perp.$$

Otherwise, if $\alpha(j) \neq \beta(k)$, then

$$\mathsf{SI}t \in |\mathrm{eq}_I(\alpha(j), \beta(k)) \to \varphi_j|$$

by Lemma 5.8 (ii) since

$$\mathrm{eq}_I(\alpha(j), \beta(k)) = \Pi.$$

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Thus, in either case,

$$\mathsf{SI}t \in |\mathrm{eq}_I(\alpha(j), \beta(k)) \to \varphi_j| \, .$$

Next we show that there exists a generic predicate T. Let $\Sigma = \mathbb{A}$ and $\mathsf{T} = \mathrm{id}_{\mathbb{A}} \in [\mathbb{A}, \mathbb{A}]_\Phi$. Then for $\varphi \in [I, \mathbb{A}]_\Phi$, we have $\varphi = \varphi^* \mathsf{T}$ as desired.

It is well known that the remaining logical structure can be obtained from the already established one by second-order encoding *à la* Russell–Prawitz.

Since $\mathsf{cc} \in \mathsf{QP}$ realises *reductio ad absurdum*, it follows by Lemma 5.5 that all $[I, \mathbb{A}]_\Phi$ are actually pre-boolean-algebras. Thus the tripos $[-, \mathbb{A}]_\Phi$ is boolean. □

For every tripos, the equality predicate on I is given by $\exists_{\delta_I}(\top_I)$ where $\delta_I = \langle \mathrm{id}_I, \mathrm{id}_I \rangle$ is the diagonal on I and $\exists_{\delta_I} \dashv \delta_I^*$. We observe that this notion of equality on I coincides with the one given by eq_I.

Lemma 5.10. For every set I and $\rho \in [I \times I, \mathbb{A}]_\Phi$, we have

$$\mathrm{eq}_I \vdash_{I \times I} \rho \quad \text{iff} \quad \top_I \vdash_I \delta_I^* \rho,$$

and thus $\exists_{\delta_I}(\top_I) \cong \mathrm{eq}_I$.

Proof. Suppose $\mathrm{eq}_I \vdash_{I \times I} \delta_I^* \rho$. Then, by Lemma 5.5, there is a $t \in \mathsf{QP}$ such that

$$\forall i, j \in I. \forall s \in |\mathrm{eq}_I(i, j)|. \, ts \in \rho(i, j).$$

Then for all $i \in I$, the term $\mathsf{K}(t\mathsf{I}) \in \mathsf{QP}$ sends elements of $|\top|$ to elements of $|\rho(i, i)|$. Thus $\top_I \vdash_I \delta_I^* \rho$ by Lemma 5.5.

For the converse direction, suppose $\top_I \vdash_I \delta_I^* \rho$. Then there exists $a \in \Phi$ such that $a\top \leqslant \rho(i, i)$ for all $i \in I$. Thus, by Lemma 5.7, we have $\mathrm{eq}_I(i, j)(a\top) \leqslant \rho(i, j)$ for all $i, j \in I$. Let $b \in \Phi$ with $bxy \leqslant yx$ for all $x, y \in \mathbb{A}$. Then we have

$$b(a\top)\mathrm{eq}_I(i, j) \leqslant eq_I(i, j)(a\top) \leqslant \rho(i, j)$$

for all $i, j \in I$. Accordingly, since $b(a\top) \in \Phi$, it follows by Lemma 5.5 that $\mathrm{eq}_I \vdash_{I \times I} \rho$ as desired. □

As described in van Oosten (2008), the boolean tripos $[-, \mathbb{A}]_\Phi$ induces a boolean topos $\mathbf{Set}\big[[-, \mathbb{A}]_\Phi\big]$, which we may call the *classical realisability topos* induced by the abstract Krivine structure under consideration, or simply the *Krivine topos.*

Also as described in van Oosten (2008), for any tripos P over a topos $\mathcal{S}$, there is a 'constant objects' functor ∇_{P} from $\mathcal{S}$ to the topos $\mathcal{S}[\mathsf{P}]$ induced by P. This functor sends $I \in \mathcal{S}$ to the object $\big(I, \exists_{\delta_I}(\top_I)\big)$. By Lemma 5.10, this gives rise to an embedding ∇ of **Set** into the classical realisability topos sending a set I to (I, eq_I).

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6. Forcing within classical realisability

Let P be a meet-semilattice. We write pq as shorthand for $p \wedge q$. Let C be an upward closed subset of P. With every $X \subseteq P$, we associate[†]

$$|X| = \{p \in P \mid \forall q. (\mathsf{C}(pq) \to X(q))\}.$$

Such subsets of P are called propositions. We say

$$p \text{ forces } X \qquad \text{iff} \qquad p \in |X|$$

and want

$$\begin{array}{rll} p \text{ forces } X \to Y & \text{iff} & \forall q. (|X|(q) \to |Y|(pq)) \\ p \text{ forces } \forall i \in I.X_i & \text{iff} & \forall i \in I.\, p \text{ forces } X_i \end{array}$$

to hold. Obviously, we have

$$\begin{array}{rll} p \text{ forces } X \to Y & \text{iff} & \forall q. (|X|(q) \to |Y|(pq)) \\ & \text{iff} & \forall q. (|X|(q) \to \forall r. (\mathsf{C}(pqr) \to Y(r))) \\ & \text{iff} & \forall q, r.\, (\mathsf{C}(pqr) \to |X|(q) \to Y(r)) \\ & \text{iff} & p \in |\{qr \mid |X|(q) \to Y(r)\}| \end{array}$$

and

$$p \text{ forces } \forall i \in I.X_i \qquad \text{iff} \qquad p \in |\bigcap_{i \in I} X_i|.$$

As in Krivine (2008), we want to consider this construction inside a classical realisability topos. That this gives a topos again follows from Pitts' *iteration theorem*, as explained in van Oosten (2008) and Hofstra (2008). This theorem says that for any tripos P over a topos $\mathcal{S}$ and any tripos Q over $\mathcal{S}[\mathsf{P}]$, the resulting topos $\mathcal{S}[\mathsf{P}][\mathsf{Q}]$ is again induced by a tripos, provided the functor $\nabla_{\mathsf{Q}} : \mathcal{S}[\mathsf{P}] \to \mathcal{S}[\mathsf{P}][\mathsf{Q}]$ preserves epis, namely by the tripos $(\nabla_{\mathsf{Q}}\nabla_{\mathsf{P}})^*\mathsf{Sub}_{\mathcal{S}[\mathsf{P}][\mathsf{Q}]}$. The requirement on ∇_{Q} is certainly satisfied in our case because Q is localic over $\mathcal{S}[\mathsf{P}]$. Alas, it is not obvious by general reasoning that the tripos $(\nabla_{\mathsf{Q}}\nabla_{\mathsf{P}})^*\mathsf{Sub}_{\mathcal{S}[\mathsf{P}][\mathsf{Q}]}$ is induced by an appropriate aks. Nevertheless, the fact that this is the case was shown in Krivine (2008). Our aim now is to explain and reveal the intuition behind his construction.

In fact, in most cases, P will not be a meet-semilattice inside a classical realisability topos, *but* it will be one 'from the point of view' of $\mathsf{C} \subseteq P$. This means that, as in Krivine (2008), we are given an external[‡] set P, a distinguished element $1 \in P$, a binary operation on P (denoted by juxtaposition) and a predicate[§] $\mathsf{C} : P \to \mathcal{P}_{\perp\!\!\!\perp}(\Lambda)$ such that the

[†] Traditionally, we would associate with X the set $X^{\perp} = \{p \in P \mid \forall q \in X. \neg\mathsf{C}(pq)\}$, but classically we have $|X| = (P \setminus X)^{\perp}$.

[‡] In other words, P is an object of **Set**.

[§] This predicate induces a predicate $\mathsf{C}^{\perp\perp}$on P in the classical realisability topos.

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following conditions hold in the classical realisability topos:

$$\mathsf{C}(p(qr)) \leftrightarrow \mathsf{C}((pq)r)$$
$$\mathsf{C}(pq) \leftrightarrow \mathsf{C}(qp)$$
$$\mathsf{C}(p) \leftrightarrow \mathsf{C}(pp)$$
$$\mathsf{C}(1p) \leftrightarrow \mathsf{C}(p)$$
$$\big(\mathsf{C}(p) \leftrightarrow \mathsf{C}(q)\big) \to \big(\mathsf{C}(pr) \leftrightarrow \mathsf{C}(qr)\big),$$

together with

$$\mathsf{C}(pq) \to \mathsf{C}(p)$$

expressing the requirement that C be upward closed. We may define a congruence on P by

$$p \simeq q \equiv \forall r. (C(rp) \leftrightarrow C(rq)),$$

with respect to which P is a commutative idempotent monoid, that is, a meet-semilattice, inside the classical realisability topos of which C is an upward closed subset whose complement contains at most one element.

A term t realises p forces $X \to Y$ if and only if

$$\forall q, r. \forall u {\in} \mathsf{C}(p(qr)). \forall s {\in} |X|(q). \forall \pi {\in} Y(r).\ t * u.s.\pi \in \perp\!\!\!\perp .$$

Thus, we may want to define a notion of a pair (t, p) realising $X \to Y$. To do this, we have to find a new aks whose term and stack part are $\Lambda \times P$ and $\Pi \times P$, respectively. The quasi-proofs of the new structure are the pairs of the form $(t, 1)$ with $t \in \mathsf{QP}$. The pole $\perp\!\!\!\perp \subseteq (\Lambda \times P) * (\Pi \times P)$ on the new structure is given by

$$(t, p) * (\pi, q) \in \perp\!\!\!\perp \ \text{ iff } \ \forall u \in \mathsf{C}(pq)\ t * \pi^u \in \perp\!\!\!\perp$$

where π^u is obtained from π by inserting u at its bottom. The push operation on the new structure is given quite straightforwardly by $(t, p).(\pi, q) = (t.\pi, pq)$, whereas application is a bit more intricate, which is why we will postpone its definition.

Propositions with respect to this new aks are now subsets of $\Pi \times P$ understood as functions from $P \to \mathcal{P}(\Pi)$. Now, given such propositions X and Y, we have

$$\begin{aligned}(t, p) \in |X \to Y| \quad &\text{iff} \quad \forall (s, q) \in |X|. \forall (r, \pi) \in Y.\ (t, p) * (s, q).(\pi, r) \in \perp\!\!\!\perp \\ &\text{iff} \quad \forall (s, q) \in |X|. \forall (r, \pi) \in Y. \forall u \in \mathsf{C}(p(qr)).\ t * s.\pi^u \in \perp\!\!\!\perp\end{aligned}$$

in accordance with the above explicitation of t realises p forces $X \to Y$. The only difference is that the realiser u of $\mathsf{C}(p(qr))$ is now placed at a distinguished position, namely the bottom of the stack.

In order to jump back and forth between

$$t \text{ realises } p \text{ forces } A$$

and

$$(t', p) \in |A|,$$

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Krivine (2008) introduced 'read' and 'write' constructs in the original aks, namely commands χ and χ' whose operational semantics is given by

$$\text{(read)} \qquad \chi * t.\pi^s \in \perp\!\!\!\perp \qquad \text{whenever} \qquad t * s.\pi \in \perp\!\!\!\perp$$

$$\text{(write)} \qquad \chi' * t.s.\pi \in \perp\!\!\!\perp \qquad \text{whenever} \qquad t * \pi^s \in \perp\!\!\!\perp .$$

Using these, we can transform t into t' and *vice versa*. Krivine concludes from this that to *realise forcing we need global memory*.

Moreover, these two new commands allow us to give a correct definition of application. Let α be a uniform realiser of $\mathsf{C}((pq)r) \to \mathsf{C}(p(qr))$ and $\underline{\alpha}$ be a quasi-proof with

$$\underline{\alpha} * t.\pi^u \in \perp\!\!\!\perp \text{ whenever } t * \pi^{\alpha u} \in \perp\!\!\!\perp,$$

which may be taken as $\lambda^* x.\chi(\lambda^* y.\chi' x(\alpha y))$. We now define application in the new aks as

$$(t, p)(s, q) \equiv (\underline{\alpha}(ts), pq)$$

for which we have

$$\begin{aligned} (t,p)(s,q) * (\pi, r) \in \perp\!\!\!\perp\!\!\!\perp \quad & \text{iff} \quad \forall u \in \mathsf{C}((pq)r)\, \underline{\alpha}(ts) * \pi^u \in \perp\!\!\!\perp \\ & \text{if} \quad \forall u \in \mathsf{C}((pq)r)\, ts * \pi^{\alpha u} \in \perp\!\!\!\perp \\ & \text{if} \quad \forall u \in \mathsf{C}((pq)r)\, t * s.\pi^{\alpha u} \in \perp\!\!\!\perp \\ & \text{if} \quad \forall u \in \mathsf{C}((p(qr))\, t * s.\pi^u \in \perp\!\!\!\perp \\ & \text{iff} \quad (t,p) * (s,q).(\pi, r) \in \perp\!\!\!\perp\!\!\!\perp, \end{aligned}$$

as required by condition (S1).

7. Conclusions

We have identified a notion of an abstract Krivine structure as an axiomatic account of Krivine's classical realisability. An important aspect of this notion is the explanation of the role of the distinguished set QP of 'quasi-proofs' without which all models with a non-empty pole $\perp\!\!\!\perp$ would be inconsistent. This point has not been emphasised in most of Krivine's writings, though a notable exception is the recent Krivine (2010).

Based on this notion of an abstract Krivine structure, we have shown the precise sense in which Cohen forcing is the commutative case of classical realisability.

We have also shown how Krivine's work on classical realisability can be seen as an instance of the categorical approach to realisability initiated by Martin Hyland. This has been achieved by associating with every abstract Krivine structure an order pca $\mathbb{A}$ of propositions together with a filter Φ of those propositions that we want to regard as 'true'. From $\mathbb{A}$ and Φ, we have constructed a boolean tripos giving rise to a categorical model of classical higher-order logic. This tripos gives rise to the ensuing classical realisability topos. This view has been helpful for us in getting a more structural understanding of forcing within classical realisability using Pitts' Iteration Theorem.

We leave as an open question whether techniques of Algebraic Set Theory (see, for example, van den Berg and Moerdijk (2009)) can be used to show that every abstract Krivine structure gives rise to a model for ZF.

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References

van den Berg, B. and Moerdijk, I. (2009) A unified approach to algebraic set theory. In: Cooper, S. B., Geuvers, H., Pillay, A. and Väänänen, J. (eds.) *Logic Colloquium 2006*, Lecture Notes in Logic **32**, Cambridge University Press 18–37.

Griffin, T. (1990) A formulae-as-types notion of control. In: *POPL '90: Proceedings of the 17th ACM SIGPLAN–SIGACT symposium on Principles of Programming Languages*, ACM 47–58.

Hofstra, P. and van Oosten, J. (2003) Ordered partial combinatory algebras. *Mathematical Proceedings of the Cambridge Philosophical Society* **134** (3) 445–463.

Hofstra, P. (2006) All realizability is relative. *Mathematical Proceedings of the Cambridge Philosophical Society* **141** (2) 239–264.

Hofstra, P. (2008) Iterated realizability as a comma construction. *Mathematical Proceedings of the Cambridge Philosophical Society* **144** (1) 39–51.

Hyland, J. M. E. (1982) The effective topos. In: Troelstra, A. S. and van Dalen, D. (eds.) *Proceedings of the The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout 1981)*, Studies in Logic and the Foundations of Mathematics **110**, North-Holland 165–216.

Krivine, J.-L. and Parigot, M. (1990) Programming with proofs. *Journal of Information Processing and Cybernetics/Elektronische Informationsverarbeitung und Kybernetik* **26** (3) 149–167.

Krivine, J.-L. (1990a) *Lambda-calcul, types et modèles*, Masson.

Krivine, J.-L. (1990b) Opérateurs de mise en mémoire et traduction de Gödel. *Archive for Mathematical Logic* **30** (4) 241–267.

Krivine, J.-L. (2001) Types lambda-calculus in classical Zermelo–Fraenkel set theory. *Archive for Mathematical Logic* **40** (3) 189–205.

Krivine, J.-L. (2003) Dependent choice, 'quote' and the clock. *Theoretical Computer Science* **308** 259–276.

Krivine, J.-L. (2008) Structures de réalisabilité, RAM et ultrafiltre sur *n*. Avaialable from `http://www.pps.jussieu.fr/~krivine/Ultrafiltre.pdf`.

Krivine, J.-L. (2009) Realizability in classical logic. In: *Interactive models of computation and program behaviour*, Panoramas et synthèses **27**, SMF.

Krivine, J.-L. (2011) Realizability algebras : a program to well order $\mathbb{R}$. *Logical Methods in Computer Science* **7** 1–47.

Krivine, J.-L. (2010) Realizability algebras II: new models for ZF + DC. Avaialable from `http://www.pps.jussieu.fr/~krivine/articles/R_ZF.pdf`.

van Oosten, J. (2008) *Realizability: An Introduction to its Categorical Side*, Elsevier.

Streicher, T. and Reus, B. (1998) Classical logic, continuation semantics and abstract machines. *Journal of Functional Programming* **8** (6) 543–572.