Proceedings of the Edinburgh Mathematical Society (2016) **59**, 1037–1064
DOI:10.1017/S0013091515000498

APPLICATION OF BRAIDING SEQUENCES. II. POLYNOMIAL INVARIANTS OF POSITIVE KNOTS

A. STOIMENOW

School of General Studies, Gwangju Institute of Science and Technology, GIST College, 123 Cheomdan-gwagiro, Gwangju 500-712, Republic of Korea (stoimeno@stoimenov.net)

(Received 14 April 2012)

Abstract We apply the concept of braiding sequences to link polynomials to show polynomial growth bounds on the derivatives of the Jones polynomial evaluated on S^1 and of the Brandt–Lickorish–Millett–Ho polynomial evaluated on $[-2,2]$ on alternating and positive knots of given genus. For positive links, boundedness criteria for the coefficients of the Jones, HOMFLY and Kauffman polynomials are derived. (This is a continuation of the paper 'Applications of braiding sequences. I': *Commun. Contemp. Math.* **12**(5) (2010), 681–726.)

Keywords: alternating knot; positive knot; braid index; genus; link polynomial; Casson–Walker invariant

2010 *Mathematics subject classification:* Primary 57M25; 57M27
Secondary 13F20; 05A15

1. Motivation and overview of results

The notion of alternating knots has been around since knot theory began more than 100 years ago with attempts to list the few simplest knots and to prove that they are distinct. The concept acquired major importance in the field, a highlight being the solution of Tait's conjectures a century after they were formulated [**20**, **25**, **32**, **52**]. Positive knots have gradually become relevant, apparently not so much because of the combinatorial property that describes them, but because they were found to be related to a series of different subjects, including dynamical systems [**5**], algebraic curves [**35**, **36**] and singularity theory [**6**]. Positive knots also play some role in four-dimensional quantum field theories [**22**], and in relation to the recent link homological concordance invariants. Let us note that the intersection of both classes are the special alternating knots studied extensively by Murasugi (see, for example, [**31**] or also [**33**, **44**]).

The concept of braiding sequences [**53**] was originally used in relation to Vassiliev (finite degree) invariants [**1**, **2**, **4**, **55**, **56**]. Braiding sequences were later related to positive and alternating knots [**39**, **50**] by means of the fact that the set of positive (respectively, alternating) knot diagrams on which the Seifert algorithm gives a surface of given genus decomposes into finitely many such sequences. Our objective here will be to derive further consequences of this circumstance for the properties of positive and alternating links. This paper is a continuation of the first part [**46**] that focused on the study of braiding sequences with regard to Vassiliev invariants. It extended the proof (originally given by Bar-Natan and independently by Stanford) of a conjecture of Lin and Wang [**24**] on the polynomial growth (in the crossing number) of Vassiliev invariants of knots to links, tangles and graphs (see Theorem 2.10).

In §§ 3 and 4 we prove growth estimates on alternating knots of given genus [**39**] for special evaluations of the derivatives of the Brandt–Lickorish–Millett–Ho Q [**7**, **15**] and Jones V [**17**] polynomial (Theorems 3.1 and 4.2). They belong to a class of invariants, extending those of finite degree from the braiding sequence point of view (in a sense that we make precise below; see Definition 3.5). Such invariants grow (in the norm) polynomially on a fixed braiding sequence, and some of them behave polynomially or periodically polynomially on them. Some similar estimates are then derived for positive knots. Some of the results on Vassiliev invariants obtained via braiding sequences carry over to the larger class, as was discussed in [**46**].

The evaluations of the Jones polynomial we consider are interesting in several ways. The values of V at roots of unity were studied by Jones in [**18**] and have special features in the C^*-algebra approach to the definition of V. When Vassiliev invariants became popular, the values $V^{(n)}(1)$ were recognized as instances of such. Later, $V^{(n)}(-1)$ received some attention due to a certain geometric significance in relation to knot sliceness. The value $V'(-1)$ occurs most prominently in Mullins's formula for the Casson–Walker invariant $\lambda_2(K)$ of the 2-fold branched cover of S^3 over a knot K [**28**]. As a small application, an estimate of λ_2 for positive knots is derived in terms of their genus and crossing number (see Proposition 4.7).

As a special case of our estimates we obtain the following finiteness property for the Jones, HOMFLY (or skein) P [**13**, **23**] and Kauffman F [**21**] polynomials (see Corollary 5.3 and Theorem 5.11).

Theorem 1.1. *All coefficients (for fixed degree in all variables) of V, P and F are bounded (that is, admit only finitely many values) on positive links.*

For the Jones and HOMFLY polynomial this statement in fact holds for a somewhat larger class (described by a relation between Bennequin numbers and the genus; see Theorem 5.1). These results are proved in § 5.

In contrast to the other polynomials, the coefficients of the Conway polynomial grow unboundedly on positive links. This topic will be discussed in a separate part of the work, which will also combine with Theorem 1.1 to give more applications to geometric invariants.

We will mostly concentrate on knots, only occasionally remarking how to modify the arguments for links (as in Remark 5.13).

2. Preliminaries, notation and conventions

2.1. Generalities

The symbols $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the integer, natural, rational, real and complex numbers, respectively. We will also write $\mathrm{i} = \sqrt{-1}$ for the imaginary unit, in situations where no confusion (with the usage as index) arises. For a set S the expression $|S|$ denotes the cardinality of S. In what follows the symbol $\subset$ denotes a not necessarily proper inclusion.

We next need some notation related to polynomials, which are understood in the broader sense as Laurent polynomials (i.e. variables are allowed to occur with negative exponents). Let $[X]_{t^a} = [X]_a$ be the coefficient of t^a in a polynomial $X \in \mathbb{Z}[t^{\pm 1}]$. For $X \neq 0$ let $\mathcal{C}_X = \{a \in \mathbb{Z} \colon [X]_a \neq 0\}$ and let

$$\min\deg X = \min \mathcal{C}_X, \quad \max\deg X = \max \mathcal{C}_X \quad \text{and} \quad \operatorname{span} X = \max\deg X - \min\deg X$$

be the minimal degree, maximal degree and span (or breadth) of X, respectively. Similarly, one defines for $X \in \mathbb{Z}[x_1, \dots, x_n]$ the coefficient $[X]_A$ for some monomial A in the x_i, and $\min\deg_{x_i} X$, etc.

We use the following abbreviations: 'w.l.o.g.' for 'without loss of generality', 'r.h.s.' for 'right-hand side', 'l.h.s.' for 'left-hand side' and 'w.r.t.' for 'with respect to'. Further notation will be introduced when appropriate.

2.2. Links and diagrams

We say that a link diagram D is l-*almost positive* if it has exactly l negative crossings, that is, in the notation of § 2.4, $w(D) = c(D) - 2l$. A knot is l-almost positive if it has an l-almost positive diagram, but no $(l-1)$-almost positive one. In what follows, we will abbreviate '0-almost positive' to '*positive*' and '1-almost positive' to '*almost positive*'. (This applies to both knots and diagrams.) The procedure of changing the crossings in a diagram so that they become positive is called *positification*; a diagram thus obtained is *positivized*. *Negative* links and diagrams are defined as mirror images of positive ones.

Remark. Unlike the majority of publications, for example, [**9, 11, 33, 36, 43, 51, 57**], some authors (for example, [**26, 54**]) confusingly call 'positive knots' the (narrower) class of knots with positive braid representations.

A crossing in a diagram D is *reducible* if it is transversely intersected by a simple closed curve not meeting D anywhere else. If D has a reducible crossing, it is called reducible, otherwise it is *reduced*. To avoid confusion, unless otherwise stated, in what follows all diagrams are assumed to be reduced.

2.3. Genera

In the following we denote by $g(D)$ the *genus* of a diagram D, this being the genus of the surface coming from the Seifert algorithm applied on this diagram. More conveniently, if D is a link diagram, we use instead of $g(D)$ the notation $\chi(D)$ for the *Euler characteristic* of the Seifert surface given by the Seifert algorithm.

By $g(L)$ we denote the genus and by $\chi(L)$ the Euler characteristic of a link L, which are the minimal genus and the maximal Euler characteristic, respectively, of an orientable spanning (i.e. Seifert) surface for L. By $g_{\mathrm{c}}(L)$ we denote the *canonical genus* of L, which is the minimal genus $g(D)$ of some diagram D of L. Similarly, $\chi_{\mathrm{c}}(L)$, the *canonical Euler characteristic* of L, is the maximal $\chi(D)$ for all diagrams D of a link L.

Theorem 2.1 (Cromwell [9], Crowell [12] and Murasugi [29, 30]). *The Seifert algorithm applied on an alternating or positive diagram gives a minimal genus surface.*

Thus, the genus $g(L)$ of an alternating/positive link L coincides with the genus $g(D)$ of an alternating/positive diagram D of L, given by

$$g(D) = \frac{c(D) - s(D) + 2 - n(D)}{2}, \tag{2.1}$$

with $c(D)$, $s(D)$ and $n(D) = n(L)$ being the number of crossings, Seifert circles and components of D, respectively (see § 2.4.1). The preceding theorem implies that for alternating/positive links, $g = g_{\mathrm{c}}$.

Definition 2.2. The notation below is important and will be used throughout the rest of the paper. Let

$$\mathcal{A}_{n,g} := \{K \text{ an alternating knot}\colon c(K) \leqslant n,\ g(K) = g\},$$
$$\mathcal{P}_{n,g} := \{K \text{ a positive knot}\colon c(K) \leqslant n,\ g(K) = g\}.$$

We recall two main ways of estimating from below genera of arbitrary knots. One comes from the signature (see (4.5)). The other, here more important, source of estimate comes from Bennequin's inequality [**3**, Theorem 3] and its subsequent improvements.

We define the *Bennequin number* $r(D)$ of a diagram D of a link L to be

$$r(D) := \tfrac{1}{2}(w(D) - s(D) + 1). \tag{2.2}$$

Then it is known (see [**36**]) that

$$1 - \chi(L) \geqslant 2r(D), \tag{2.3}$$

which is called *Bennequin's inequality*.

A consequence is the following. Let D be an l-almost positive diagram of a knot K. By comparison of (2.1) (with $n(D) = 1$) and (2.2), we then have

$$r(D) = g(D) - l. \tag{2.4}$$

Bennequin's inequality (2.3) becomes

$$g(D) - l = r(D) \leqslant g(K) \leqslant g_{\mathrm{c}}(K) \leqslant g(D). \tag{2.5}$$

In particular, for positive diagrams ($l = 0$) all inequalities become equalities. This leads to a part of Theorem 2.1.

https://doi.org/10.1017/S0013091515000498 Published online by Cambridge University Press

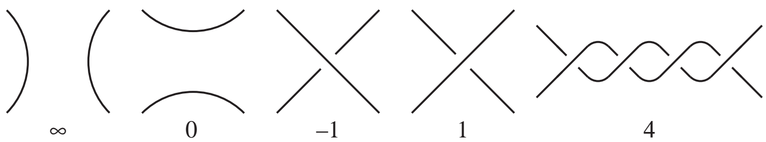


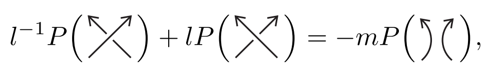
Figure 1. The Conway tangles.

2.4. Link polynomial invariants

As for polynomial invariants, our notation is fairly standard: V is the Jones polynomial [**17**], P is the HOMFLY (or skein) polynomial [**13**,**23**], F is the Kauffman polynomial [**21**] and Q is the Brandt–Lickorish–Millett–Ho polynomial [**7**, **15**].

2.4.1. *Skein (HOMFLY) polynomial*

The *skein (HOMFLY) polynomial* P is a Laurent polynomial in two variables l and m of oriented knots and links, and can be defined by the skein relation



(2.6)

and normalized so as to be 1 on the unknot. With this relation we use the convention for P of [**23**], but with l and l^{-1} interchanged.

A *skein triple* D_+, D_-, D_0 is a triple of diagrams, or of their corresponding links L_+, L_-, L_0, equal except near one crossing, where they appear as in (2.6) (from left to right). The replacement $L_\pm \to L_0$ is called *smoothing (out)* the crossing in $L_\pm$. The crossing in D_+ is called *positive*, the one in D_- *negative*. The sum of the signs of all crossings of D is called the *writhe* of D and will be written $w(D)$.

Let D be an oriented knot or link diagram. We denote by $c(D)$ the *crossing number* of D. We use $n(D) = n(L)$ to denote the *number of components* of D or its link L. We write $s(D)$ for the *number of Seifert circles* of a diagram D (the loops obtained by smoothing out all the crossings of D).

2.4.2. *Brandt–Lickorish–Millett–Ho and Kauffman polynomial*

Recall that the Brandt–Lickorish–Millett–Ho Q *polynomial* is a Laurent polynomial in one variable z for links without orientation, defined by being 1 on the unknot and the relation

$$Q(L_1) + Q(L_{-1}) = z(Q(L_0) + Q(L_\infty)), \tag{2.7}$$

where L_i are unoriented links with equal diagrams except a spot with an i-tangle (as in Figure 1). Note that Q is sometimes called 'absolute polynomial'. This name will not be used in this paper.

There are a few special values of the Q polynomial, already observed in [**7**]. We shall use two of them.

We have $Q_L(1) = 1$ for any link L. To justify this claim, observe that for $z = 1$, setting all Q polynomials in (2.7) to 1 would satisfy the relation. And since this relation uniquely determines Q (and $Q(1)$), the observed solution (for $z = 1$) is the only one.

https://doi.org/10.1017/S0013091515000498 Published online by Cambridge University Press

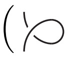
In a similar way one can argue that $Q_L(-2) = 2^{n(L)-1}$. Notice that exactly one of the two links on the r.h.s. of (2.7) has the same number of components as the links on the l.h.s. For the other link on the r.h.s. the number of components differs by ± 1.

The *Kauffman polynomial* [**21**] F is usually defined via the regular isotopy invariant $\Lambda(a,z)$ of unoriented links. For F we use the convention of [**21**], but with a and a^{-1} interchanged. In particular, we have for a link diagram D the relation

$$F(D)(a,z) = a^{w(D)}\Lambda(D)(a,z). \tag{2.8}$$

The writhe-unnormalized version Λ of F is given in our convention by the properties

$$\Lambda(L_1) + \Lambda(L_{-1}) = z(\Lambda(L_0) + \Lambda(L_\infty)), \tag{2.9}$$

$\Lambda\Big($  $\Big) = a^{-1}\Lambda\Big(\,\Big|\,\Big),\qquad \Lambda\Big($  $\Big) = a\Lambda\Big(\,\Big|\,\Big),\qquad \Lambda\Big(\bigcirc\Big) = 1.$

Thus, the positive (right-hand) trefoil has $\min\deg_l P = \min\deg_a F = 2$.

The Q polynomial is a substitution of the Kauffman polynomial: $Q(z) = F(1,z)$.

Note that for P and F there are several other variable conventions, differing from each other by possible inversion and/or multiplication of some variable by some fourth root of unity.

2.4.3. *Jones polynomial*

The *Jones polynomial* V is a Laurent polynomial in one variable t of oriented knots and links, and can be defined by being 1 on the unknot and the relation

$$t^{-1}V(D_+) - tV(D_-) = (t^{1/2} - t^{-1/2})V(D_0) \tag{2.10}$$

for a skein triple of diagrams D_+, D_- and D_0.

The Jones polynomial is obtained from P and F (with our conventions) by the substitutions (with i being the complex unit; see [**23**] or [**21**, § III])

$$V(t) = P(\mathrm{i}t, \mathrm{i}(t^{1/2} - t^{-1/2})) = F(-t^{3/4}, t^{1/4} + t^{-1/4}). \tag{2.11}$$

The Jones polynomial also has a few special values. Some discussion can be found in [**18**, § 12]. We have

$$V_L(\mathrm{e}^{\pm 2\pi\mathrm{i}/3}) = 1 \tag{2.12}$$

for all L, and

$$V_L(1) = (-2)^{n(L)-1}. \tag{2.13}$$

In particular, for knots $V_L(1) = 1$. Moreover, in this case $V_L'(1) = 0$.

Let $c(L)$, the *crossing number* of a link L, be the minimal crossing number $c(D)$ over all diagrams D of L. One main application of the Jones polynomial is the following solution to an old conjecture of Tait.

https://doi.org/10.1017/S0013091515000498 Published online by Cambridge University Press

Theorem 2.3 (Kauffman [20], Murasugi [32] and Thistlethwaite [52]). *If D is a reduced alternating diagram of a link L, then $c(D) = c(L)$.*

Additionally, Kauffman and Thistlethwaite show that for an alternating diagram D the coefficients of V alternate in sign.

The situation for positive links is somewhat different. The 4-crossing diagram of the trefoil already shows that a positive reduced diagram need not have minimal crossing number. In fact, there may be no minimal crossing positive diagram at all [**41**]. Thus, determining the crossing number of a positive knot remains a non-trivial problem. However, the following was proved in [**49**] using the Kauffman polynomial.

Theorem 2.4. *If D is a reduced positive diagram of a link L, then $c(L) \geqslant c(D)+\chi(D)$.*

This will enable us to work with the crossing number of a positive link, even without being able to calculate it exactly.

2.5. Generating functions and asymptotics

Here we recall a few elementary facts about generating functions, which we will use (but skip elaborating on) later.

Given a sequence $(a_n)_{n=0}^{\infty}$ of real numbers, one can build the generating series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

When the series converges (in a neighbourhood of 0), $f(x)$ is called the *generating function* of a_n.

In our case, f will always be a rational function, i.e. the quotient of two polynomials: $f(x) = p(x)/q(x)$. It is very well known that the sequences whose generating function is rational are precisely those satisfying linear recurrences:

$$a_n = c_0 + \sum_{i=1}^{k} c_i a_{n-i}$$

(when $n \geqslant k'$ for some $k' \geqslant k$, and with $k, k' \in \mathbb{N}$ and $c_i \in \mathbb{R}$ independent of n).

Whenever $x_i \in \mathbb{C}$, $i = 1, \dots, s$, are the zeros of q, of multiplicity $m_i > 0$, there is a partial fraction decomposition

$$\frac{p(x)}{q(x)} = \tilde{p}(x) + \sum_{i=1}^{s} \sum_{j=1}^{m_i} \frac{c_{i,j}}{(x - x_i)^j} \tag{2.14}$$

for a polynomial $\tilde{p}$ and (complex) constants $c_{i,j}$ (with $c_{i,m_i} \neq 0$).

https://doi.org/10.1017/S0013091515000498 Published online by Cambridge University Press

For two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ (of real or complex numbers, as appropriate), let us in the following say that

$$\begin{array}{ll} a_n \simeq b_n & \text{if } \lim_{n\to\infty} \dfrac{a_n}{b_n} \text{ exists (in } \mathbb{C}), \\ a_n \sim b_n \quad (a_n, b_n \in \mathbb{R}) & \text{if } 0 < \liminf_{n\to\infty} \dfrac{a_n}{b_n} \leqslant \limsup_{n\to\infty} \dfrac{a_n}{b_n} < \infty, \\ a_n \lesssim b_n, \quad b_n \gtrsim a_n \quad (a_n, b_n \in \mathbb{R}) & \text{if } \limsup_{n\to\infty} \dfrac{a_n}{b_n} < \infty, \\ a_n = O(b_n), \quad \text{`}a_n \text{ is (of asymptotics) } O(b_n)\text{'} & \text{if } \limsup_{n\to\infty} \left|\dfrac{a_n}{b_n}\right| < \infty, \text{ i.e. } |a_n| \lesssim |b_n|. \end{array}$$

In certain cases, to avoid confusion, we will index the asymptotic relation symbol, respectively, write $O_n(b_n)$ to indicate the variable going to ∞ (while all other variables are kept fixed).

An expression of the form $a_n \to \infty$ abbreviates $\lim_{n\to\infty} a_n = \infty$. Analogously, $a_{n_m} \to \infty$ means the limit for $m \to \infty$, etc.

The form of (2.14) determines the asymptotic behaviour of a_n for $n \to \infty$. Note that

$$\frac{1}{(x - x_i)^j} = \sum_{n=0}^{\infty} b_n x^n,$$

where $b_n \simeq_n x_i^{-n} n^{j-1}$.

The case we will be mostly interested in is when all (complex) roots x_i of $q(x)$ are roots of unity. This means that $q(x)$ divides (and can w.l.o.g. be assumed equal to) a power of a cyclotomic polynomial $x^c - 1$. The sequences (a_n) occurring in this case have the following feature.

Call (a_n) *almost periodically polynomial* (a.p.p.) if there are a c (period), a d (initial number of exceptions), and polynomials $P_1, \dots, P_c \in \mathbb{R}[n]$ with

$$a_n = P_{n \bmod c}(n) \quad \text{for } n \geqslant d. \tag{2.15}$$

If $d = 0$, call a_n *periodically polynomial* (p.p.).

As indicated, it is not too hard to observe that

$$(a_n) \text{ a.p.p.} \iff \sum_n a_n x^n = \frac{R(x)}{(x^c - 1)^k}, \quad R \in \mathbb{R}[x],$$

where c is a period of a_n, as in (2.15). Moreover (still referring to (2.15)), we have $d = 0$ (i.e. a_n is p.p.; the case we will almost exclusively deal with) if $\max\deg R < c \cdot k$, and $d \leqslant 1 - c\,k + \max\deg R$ otherwise. Also, $k \geqslant 1 + \max_i \max\deg P_i$.

We mention as an illustration the following example. It will not be used later, but is directly related to some other considerations that follow (around Theorem 2.9). Recall Definition 2.2.

Theorem 2.5 (Stoimenow [39]). *For fixed g, the sequence $a_n = |\mathcal{A}_{n,g}|$ is a.p.p.*

https://doi.org/10.1017/S0013091515000498 Published online by Cambridge University Press

Similar arguments apply (and will be used) for multi-indexed sequences $a_{m,n}$, $a_{l,m,n}$, etc. Then every running index will have a corresponding variable in the generating function.

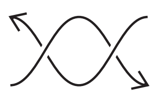
Additionally, we will work with a_n and $a_{n,\ldots}$, which are polynomials. In this case attention must be paid to which variables of the generating function are inherited from the polynomials (*polynomial variables*), and which reflect a (and which) running index (*running variables*).

One goal we will repeatedly have is to specify values for the polynomial variables so that the resulting denominator of the generating function will have only roots (in the running variable) that are roots of unity.

2.6. Braiding sequences and genus generators

Now, let us recall from [**39**, **45**] some basic facts concerning knot generators of given genus. We will set up some notation and conventions used below. This is discussed in much more detail in [**47**]. Cromwell offers in his recent book [**10**, § 5.3] an introductory exposition on the subject.

We start by defining $\sim$-equivalence of crossings. A *reverse clasp* is, up to crossing changes, a tangle like



If exactly one strand is reversed, we have a *parallel clasp*. We call a clasp *trivial* if its two crossings have opposite signs. Such a clasp can be eliminated by a Reidemeister II move.

Definition 2.6. Let D be a link diagram and let p and q be crossings. We call p and q $\sim$-*equivalent* and write $p \sim q$ if smoothing out one renders nugatory the other. We write $t(D)$ for the number of $\sim$-equivalence classes of crossings of D.

Another (and more commonly used elsewhere) way of expressing $p \sim q$ is to say that p and q can be made to form a reverse clasp after flypes. A minor argument will convince one that this is indeed an equivalence relation.

Definition 2.7. A $\sim$-equivalence class consisting of one crossing is called *trivial*, a class of more than one crossing is called *non-trivial*. A $\sim$-equivalence class is *reduced* if it has at most two crossings, otherwise it is *non-reduced*. A diagram is called *generating*, or a *generator*, if all its $\sim$-equivalence classes are reduced.

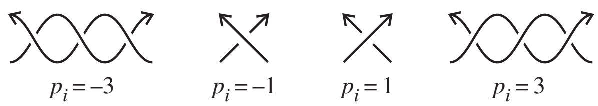
Let D be an oriented link diagram and let P be a set of crossings in D, which we call marked and number $c_1, \ldots, c_n$. We explain now, following [**38**], how to define a family of diagrams $\mathcal{D} = \mathcal{B}(D, P)$, called a *braiding sequence* (or *series*).

Consider the family of diagrams

$$\mathcal{D} = \{D(p_1, \ldots, p_n) \colon p_1, \ldots, p_n \in \mathbb{Z} \text{ odd}\}. \tag{2.16}$$

Figure 2. Two ways to do a twist at a crossing: the replacement of the crossing by the first tangle is called a $\bar{t}_2$ twist; by the second one a $\bar{t}'_2$ twist.

Herein, the diagram $D(p_1, \ldots, p_n)$ is obtained from D by replacing the crossing c_i by a tangle consisting of $|p_i|$ reverse half-twists of sign $\mathrm{sgn}(p_i)$:



(2.17)

Following [**38**] we will call $\mathcal{D}$ the *(reverse) braiding sequence* associated with (D, P) and denote it by $\mathcal{B}(D, P)$. If P is omitted, the set of all crossings of D is used by default.

Note that $\mathcal{B}(D, P)$ does not in fact depend on how crossings in P are switched. In particular, when considering sequences $\mathcal{B}(D)$ we can assume w.l.o.g. that D is alternating. Sometimes it is better to work with the *positive generating diagrams*, i.e. the positifications of D.

We call a (*positive*, respectively, *negative*) $\bar{t}'_2$ *twist* (or *reverse twist*) the replacement of the tangle for $p_i = \pm 1$ in (2.17) by the one for $p_i = \pm 3$ (with the same sign). This move does not change the canonical genus: when D' is obtained from D by a $\bar{t}'_2$ twist, then $g(D') = g(D)$. Thus, $g(D') = g(D)$ is constant for all $D' \in \mathcal{B}(D)$. As it turns out, some kind of converse of this property is true for fixed $g(D)$, up to finite indeterminacy.

Theorem 2.8 (Stoimenow [39, 47]). *The set of knot diagrams on which the Seifert algorithm gives a surface of given genus, regarded up to crossing changes and flypes, decomposes into a finite number of reverse braiding sequences $\mathcal{B}_i = \mathcal{B}(D_i)$ for generators D_i. The same is true for link diagrams of fixed number of components.*

We will be particularly interested in the case of knots. We have then for a knot diagram D of genus g,

$$t(D) \leqslant d_g := 6g - 3. \tag{2.18}$$

The notation d_g will be used often below. (In [**50**], Stoimenow and Vdovina established that this inequality is the best possible.) Note that thus for a generator, $c(D) \leqslant 2d_g = 12g - 6$ (the exact bound is $10g - 7$ for $g > 1$ [**47, 50**]).

Similarly to $\bar{t}'_2$, one can consider a $\bar{t}_2$-*twist* (*parallel twist*), creating a parallel clasp near a crossing. The difference is shown in Figure 2. Similarly, there is an unoriented twist in unoriented diagrams. One then has braiding sequences of parallel (or mixed parallel and reverse) or unoriented twists (see [**38**]).

An application of Theorem 2.8 is the following asymptotic statement.

Theorem 2.9. *With (2.18) and Definition 2.2 we have, for fixed g as $n \to \infty$,*

$$|\mathcal{A}_{n,g}| \simeq_n |\mathcal{P}_{n,g}| \simeq_n n^{d_g}. \tag{2.19}$$

For alternating knots the theorem follows easily from a more detailed property of $|\mathcal{A}_{n,g} \setminus \mathcal{A}_{n-1,g}|$, proved in [**39**] (up to the subsequent identification (2.18) of the value of d_g, which first occurred in [**50**]). The statement was later extended to positive knots in [**49**].

A fundamental feature, studied in [**38**], of any Vassiliev invariant v is that for each braiding sequence $\mathcal{D}$ as in (2.16) (with parallel or antiparallel braidings), the map

$$(x_1, \dots, x_n) \mapsto v(D(x_1, \dots, x_n)) \tag{2.20}$$

is a polynomial function in x_i; we call this the *braiding polynomial* of v on $\mathcal{D}$. The degree of v (as Vassiliev invariant) is equal to the maximum of the degrees of all its braiding polynomials. Herein, degree is taken w.r.t. all variables together, i.e. according to $\deg \prod_{i=1}^n x_i^{p_i} = \sum_{i=1}^n p_i$.

In close relation to this property is the polynomial growth of Vassiliev invariants, a proof of a conjecture of Lin and Wang (see [**24, 46**]).

Theorem 2.10. *If v is a Vassiliev invariant of degree k, then*

$$\max\{|v(D)| \colon c(D) \leqslant n\} \lesssim_n n^k.$$

3. Some evaluations of the Q polynomial

3.1. Bounds for evaluations of the Q polynomial

The aim of the following sections is to extend the polynomial growth bounds for Vassiliev invariants from [**46**] in the special case of positive and alternating knots of given genus to a larger class of invariants. We first study the Q polynomial. We continue to use the quantity d_g of (2.18).

Theorem 3.1. *For $k, g \in \mathbb{N}$ and $z \in [-2, 2]$ there are constants $C_{k,g,z}$ depending on k, g, z, but not on n, such that*

$$\max_{K \in \mathcal{A}_{n,g}} |Q_K^{(k)}(z)| \leqslant C_{k,g,z} n^{d_g q_z(k)} \quad \textit{and} \quad \max_{K \in \mathcal{P}_{n,g}} |Q_K^{(k)}(z)| \leqslant C_{k,g,z} (n + 2g - 1)^{d_g q_z(k)}$$

with

$$q_z(k) = \begin{cases} 2k+2, & z = 2, \\ 2k, & z = -2, \\ 2k+1, & z = 0, \\ \max(0, k-1), & z = 1, \\ k, & z \in (-2, 2) \setminus \{0, 1\}. \end{cases}$$

Remark 3.2. We reiterate (see § 2.4.3) that for a positive knot K the crossing number $c(K)$ is not necessarily attained in a positive diagram. Theorem 2.4, however, allows us to circumvent this obstacle.

We start with a formula allowing us to calculate Q on knots with many twists from those with few twists. Before stating the formula we fix the following notation, which will be valid until the end of § 3. We denote by L_i ($i \in \mathbb{Z} \cup \{\infty\}$) links that possess diagrams equal except in one room, where an i-tangle, in the Conway sense, is inserted; see Figure 1. The appearance of the diagrams outside this room is kept fixed.

We remark then that the defining relation (2.7) for Q can be written as

$$Q(L_1) + Q(L_{-1}) = z(Q(L_0) + Q(L_\infty)). \tag{3.1}$$

Lemma 3.3. *With the above notation we have*

$$Q(L_n) = (z^2 - 1)(Q(L_{n-2}) - Q(L_{n-4})) + Q(L_{n-6}). \tag{3.2}$$

Proof. This is straightforward from (3.1). For an explicit derivation of the result, see [**45**, Lemma 8.1]. □

3.2. An extension of Vassiliev invariants

To put our observations into a cleaner language, we establish some more terminology.

Definition 3.4. A function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is called *periodically polynomial* with period $d \in \mathbb{N}$ if there are polynomials $P_{d_1,\dots,d_n} \in \mathbb{Q}[x_1,\dots,x_n]$ for $0 \leqslant d_i < d$ such that

$$f(x_1,\dots,x_n) = P_{x_1 \bmod d,\dots,x_n \bmod d}(x_1,\dots,x_n)$$

for all $x_i \in \mathbb{Z}$. The *degree* $\deg f$ of f is given by the maximal degree of all $P_{d_1,\dots,d_n}$.

Definition 3.5.

(1) Call a knot invariant v *extended Vassiliev* of degree less than or equal to k if it induces on any braiding sequence $\mathcal{B}(D,P)$ via (2.20) a polynomial whose degree in any single variable (but *not* necessarily in *all* variables altogether) is at most k. That is, all monomials (without coefficients) of this polynomial should divide $\prod_{i=1}^n x_i^k$.

(2) Call the invariant *periodically extended Vassiliev* if the function (2.20) is a periodically polynomial function.

(3) Call the invariant *polynomially bounded* if and only if its norm has an upper bound, which is a polynomial in the parameters of the braiding sequence.

Example 3.6. The Alexander polynomial Δ is an extended Vassiliev invariant of degree 1 when one considers braiding sequences of $\bar{t}_2'$ twists only. For braiding sequences of both $\bar{t}_2$ and $\bar{t}_2'$ twists, it is well known that the *determinant* $\Delta(-1)$ behaves linearly, but only up to sign. Unfortunately, it is not obvious how to fix the sign so as to make it an extended Vassiliev invariant of degree 1, but certainly one can consider the square $(\Delta(-1))^2$, which is then an extended Vassiliev invariant of degree 2.

Theorem 3.7. *The invariant $Q^{(k)}(z)$ is polynomially bounded for $z \in [-2, 2]$. If $z = 2\cos 2\pi v$ for a rational v, then it is periodically extended Vassiliev. It is an extended Vassiliev invariant of degree less than or equal to $2k$ if $z = -2$, and of degree less than or equal to $2k+2$ if $z = 2$.*

Proof. Taking the kth derivative w.r.t. z in (3.2) we obtain, setting $Q_n^{(k)} := Q^{(k)}(L_n)$,

$$Q_n^{(k)} = (z^2-1)(Q_{n-2}^{(k)} - Q_{n-4}^{(k)}) + 2kz(Q_{n-2}^{(k-1)} - Q_{n-4}^{(k-1)}) + k(k-1)(Q_{n-2}^{(k-2)} - Q_{n-4}^{(k-2)}) + Q_{n-6}^{(k)}.$$

Hence, considering for fixed z the generating series

$$f_k = f_k(z, x) := \sum_{n=0}^{\infty} Q_{2n}^{(k)}(z) x^n,$$

we find

$$f_k = ((z^2-1)x(1-x) + x^3) f_k + 2kzx(1-x) f_{k-1} + k(k-1)x(1-x) f_{k-2} + A_k + B_k x + C_k x^2.$$

The extra term $A_k := Q_0^{(k)}(z)$, and B_k and C_k can be expressed as certain linear combinations (with coefficients in $\mathbb{Z}[z]$) of $Q_{2i}^{(k-j)}(z)$ for $0 \leqslant i, j \leqslant 2$. Rewriting this, we obtain

$$f_k = \frac{2kzx(1-x) f_{k-1} + k(k-1)x(1-x) f_{k-2} + A_k + B_k x + C_k x^2}{1 - ((z^2-1)x(1-x) + x^3)} \quad \text{for } k \geqslant 0,$$

with $f_{-1} = f_{-2} = 0$.

If all the zeros of the denominator (regarded as a polynomial in x) have norm 1, by partial fraction decomposition (based on the principle of [**37**, p. 14, Theorem 1.12]), we obtain that $Q_{2n}^{(k)}$ is a polynomial in n whose maximal degree is one less than the highest multiplicity of a (complex) zero in the denominator. (Also, the formula for f_k preserves the radius $\rho = 1$ of convergence of the series representation of f_k.) If some zero had norm smaller than 1, then we would only have an exponential bound for $Q_{2n}^{(k)}$ in n.

Now, $Y(x) = 1 - ((z^2-1)x(1-x) + x^3)$ always has the zero $x = 1$, and dividing by $1 - x$ we obtain $Z = 1 + (2 - z^2)x + x^2$. We are interested only in the situation that the zeros x_0 and x_1 of Z satisfy $|x_0|, |x_1| \geqslant 1$. (If $\min |x_0|, |x_1| < 1$, then we cannot have a polynomial bound.) Since we have $x_0 x_1 = 1$, we obtain that $x_0, x_1 \in S^1$, and they are (complex) conjugates. Hence, $2 - z^2 = -x_0 - x_1 \in [-2, 2]$, and so $z \in [-2, 2]$. The polynomial Z, and equivalently Y, has a multiple zero $x = -1$ if $z = 0$, and $x = 1$ if $z = \pm 2$. We shall first discuss these cases separately.

Case 1 ($z = -2$). From the relation for the Q polynomial, we have for knots $Q(-2) \equiv 1$ (cf. § 2.4.2). So $f_0 = 1/(1-x)$. By induction it follows that

$$f_k = \frac{P_k(x)}{(1-x)^{2k+1}}$$

with $\deg_x P_k(x) \leqslant 2k$, and hence $Q_{2n}^{(k)}(-2) = O(n^{2k})$.

Case 2 ($z = 2$). By [**7**], we have $Q(2) = (\varDelta(-1))^2$ (cf. Example 3.6), which cannot be hoped to grow less than $O(n^2)$. Hence,

$$f_0 = \frac{A_0 + B_0 x + C_0 x^2}{(1-x)^3}$$

and, inductively,

$$f_k = \frac{P_k(x)}{(1-x)^{2k+3}}.$$

Therefore, $Q^{(k)}_{2n}(2) = O(n^{2k+2})$.

Case 3 ($z = 0$). Then

$$f_0 = \frac{A_0 + B_0 x + C_0 x^2}{(x+1)^2(x-1)}$$

and, inductively,

$$f_k = \frac{P_k(x)}{(x+1)^{2k+2}(x-1)},$$

hence $Q^{(k)}_{2n}(0) = O(n^{2k+1})$.

Case 4 ($z \in [-2, 2]$ outside of $\{0, -2, 2\}$). For $z \in [-2, 2]$ outside of $\{0, -2, 2\}$ we have distinct zeros, and so we obtain

$$f_k = \frac{P_k(x)}{(x-1)(x-x_0)^{k+1}(x-x_1)^{k+1}}.$$

Hence, $Q^{(k)}_{2n}(z) = O(n^k)$. But at least for $z = 1$ the order can be reduced by one.

Case 5 ($z = 1$). By the relation for the Q polynomial, we have, as explained, $Q(1) \equiv 1$. Hence, $f_0 = 1/(1-x)$, and thus the factors $x - \mathrm{e}^{\pm 2\pi \mathrm{i}/3}$ do not occur in the denominator of f_0, and hence always appear with one power less than for general z. Thus, $Q^{(k)}_{2n}(1) = O(n^{\max(0,k-1)})$.

This argument shows the fact for one variable and for positive n, but the same argument works for negative n and for many variables, replacing the step from k to $k+1$ by a nested induction on the variables. Moreover, if $z = 2\cos 2\pi v$ for a rational v, then the formula $\cos 2z = 2\cos^2 z - 1$ shows that $2 - z^2$ is of the same form. Then x_0 and x_1 are roots of unity and $Q^{(k)}_{2n}(z)$ are periodically polynomial in n, the same being true for the multivariable case. Then the same polynomial expression holds for negative n: just prove it the same way for $n \leqslant 2$ and use that the values at 0, 1, 2 uniquely determine the extensions. The trivial periodicity for $z = \pm 2$ follows because then the denominators of f_k are powers of $1 - x$. □

Proof of Theorem 3.1. Theorem 3.1 is directly linked to Theorem 3.7.

First, we have Theorem 2.1, which says that for alternating or positive knots of genus g we need to look only at (alternating or positive) diagrams of genus g. Next, Theorem 2.8 allows us to restrict ourselves (up to some finite indeterminacy, which does not affect the existence of constants) to a single (reverse) braiding sequence. Thus, we return to the situation examined in the proof of Theorem 3.7.

Next, we use Theorems 2.3 and 2.4, which for alternating (respectively, positive) knots of crossing number n restrict us to working with diagrams of n (respectively, $n, \dots, n + 2g - 1$) crossings.

In the proof of Theorem 3.7 we studied in detail the growth of $|Q_K^{(k)}(z)|$ on a braiding sequence with one parameter, and established that

$$|Q_K^{(k)}(z)| = O(n^{q_z(k)}).$$

As remarked there, on a braiding sequence with d parameters $n_1, \dots, n_d$ the same considerations apply to each parameter.

Thus, on a braiding sequence of diagrams of genus g with d parameters,

$$\max_{K \in \mathcal{P}_{n,g}} |Q_K^{(k)}(z)|^{1/q_z(k)} = O(n_1 \cdot \dots \cdot n_d),$$

where n_i indicate the number of twists corresponding to parameter i. Therefore,

$$\max_{c(D)=n} |Q_D^{(k)}(z)|^{1/q_z(k)} = O(n^d)$$

for diagrams D on this braiding sequence. Letting $d := d_g$, that is, taking the maximal d over all braiding sequences of diagrams of genus g (see (2.18)), we obtain the statements in the theorem. □

4. Some evaluations of the Jones polynomial

4.1. Bounds for evaluations of the Jones polynomial

For the Jones polynomial, orientation of the components (for links) is relevant, so consider just reverse braiding sequences (of $\bar{t}'_2$ twists). Accordingly we make the following definition.

Definition 4.1. Call a knot invariant extended Vassiliev of degree less than or equal to k, periodically extended Vassiliev of degree less than or equal to k or polynomially bounded under $\bar{t}_2$ (or $\bar{t}'_2$) twists, if it is so in the sense of Definition 3.5, but just on braiding sequences associated with the diagram by performing $\bar{t}_2$ (or $\bar{t}'_2$) twists at the crossings.

We then have the following theorem. Here and below S^1 will denote the set of complex numbers of unit norm.

https://doi.org/10.1017/S0013091515000498 Published online by Cambridge University Press

Theorem 4.2. *The invariant $V^{(k)}(t)$ for $t \in S^1$ is polynomially bounded under $\bar{t}'_2$ twists. If t is a root of unity, it is periodically extended Vassiliev under $\bar{t}'_2$ twists. Moreover, the following inequalities hold (for values of t where the r.h.s. is defined):*

$$\max_{K \in \mathcal{A}_{n,g}} |V_K^{(k)}(t)| \leqslant C_{k,g,t} n^{d_g p_t(k)} \quad \textit{and} \quad \max_{K \in \mathcal{P}_{n,g}} |V_K^{(k)}(t)| \leqslant C_{k,g,t} (n + 2g - 1)^{d_g \tilde{p}_t(k)}$$

with

$$p_t(k) = \begin{cases} k+1, & t = -1, \\ \max(0, k-1), & t = \mathrm{e}^{\pm 2\pi \mathrm{i}/3}, \\ k, & t \in S^1 \setminus \{-1, \mathrm{e}^{\pm 2\pi \mathrm{i}/3}\}, \end{cases}$$

and

$$\tilde{p}_t(k) = \begin{cases} p_t(k), & t \in S^1, \\ 0, & |t| < 1. \end{cases}$$

Remark 4.3. The condition of alternation or positivity (in addition to fixing the genus) is necessary, because one can show, for example, that the Whitehead doubles of n-fold iterated connected sums of trefoils have unit norm values of V growing exponentially in n. In all likelihood, similar examples can be also given for $V^{(k)}$ and Q.

As was done previously for the Q polynomial, we start with a simple but helpful formula. Denote by $V_n^{(k)}$ the kth derivative of the Jones polynomials of links with diagrams equal except in one room. In that room a tangle of n half-twists (with antiparallel orientation, as rendered by $\bar{t}'_2$ moves) is inserted. Write V_n for $V_n^{(0)}$. We then have the following lemma.

Lemma 4.4.

$$V_n = (t^2 + 1)V_{n-2} - t^2 V_{n-4} = t^2(V_{n-2} - V_{n-4}) + V_{n-2}. \tag{4.1}$$

Proof. It is straightforward from (2.10). □

Remark 4.5. Formulae (3.2) and (4.1) came up in a practical context: the quest for interesting examples as presented in [**45**, §§ 9 and 10] required us to calculate V and Q on braiding sequences of diagrams of genus 2 from some initial data. For V as initial data it suffices to take the polynomials of all crossing changed versions of the generating diagram, while for Q at any crossing one twist additionally needs to be made.

Proof of Theorem 4.2. Differentiating the identity (4.1) k times w.r.t. t, we obtain

$$V_n^{(k)} = t^2(V_{n-2}^{(k)} - V_{n-4}^{(k)}) + 2kt(V_{n-2}^{(k-1)} - V_{n-4}^{(k-1)}) + k(k-1)(V_{n-2}^{(k-2)} - V_{n-4}^{(k-2)}) + V_{n-2}^{(k)}.$$

So, setting, for fixed t,

$$f_k = f_k(x,t) := \sum_{n=0}^{\infty} V_{2n}^{(k)}(t) x^n,$$

we have

$$f_k(x,t) = t^2x(1-x)f_k + 2ktx(1-x)f_{k-1} + k(k-1)x(1-x)f_{k-2} + xf_k + A_k + B_kx,$$

with A_k and B_k being certain constants. (Specifically, $A_k = V_0^{(k)}(t)$ and $B_k = V_2^{(k)}(t) - (t^2+1)A_k - 2ktA_{k-1} - k(k-1)A_{k-2}$, but these expressions will not be relevant.) Hence,

$$f_k = \frac{2ktx(1-x)f_{k-1} + k(k-1)x(1-x)f_{k-2} + A_k + B_kx}{1-(t^2+1)x + t^2x^2} \quad \text{for } k \geqslant 0, \tag{4.2}$$

with $f_{-1} = f_{-2} = 0$.

Assume first that $t \neq 0$. Then the zeros of the denominator are 1 and $1/t^2$. This time having $|1/t^2| > 1$ prevents us from extending the series to negative twists (because changing the sign of the crossings basically takes t to $1/t$). Hence, we consider $|t| < 1$ only for positive knots (but not for alternating, where negative twists may occur).

For alternating knots the zeros are simple unless $t = \pm 1$.

Case 1 ($t = 1$). One has the well-known Vassiliev invariants of the Jones polynomial. The conclusion $V_{2n}^{(k)}(1) = O(n^k)$ follows already by Theorem 2.10. But it can be also deduced by using the property $V(1) \equiv 1$ (see below (2.13)), which gives $f_0 = 1/(1-x)$, and inductively showing that $f_k = P_k(x)/(1-x)^{k+1}$.

Case 2 ($t = -1$). One has $f_0 = (A_0 + B_0x)/(1-x)^2$, as $V(-1) = \Delta(-1)$ is the determinant (in Example 3.6), which does not grow slower than $O(n)$. Hence, by induction, $f_k = P_k(x)/(1-x)^{k+2}$ and $V_{2n}^{(k)}(-1) = O(n^{k+1})$.

Case 3 (the rest of S^1). One has distinct zeros, and $V_{2n}^{(k)}(t) = O(n^k)$ as before.

Case 4 ($t = \mathrm{e}^{\pm 2\pi \mathrm{i}/3}$). The identity (2.12) divides out one of the powers of $x - \mathrm{e}^{\pm 2\pi \mathrm{i}/3}$ in the denominator of f_0. Therefore, $V_{2n}^{(k)}(\mathrm{e}^{\pm 2\pi \mathrm{i}/3}) = O(n^{\max(0,k-1)})$.

For $t \in S^1$ (and both alternating knots and positive knots) this proves the assertions. For positive knots it remains to consider the cases in which $|t| < 1$, i.e. $|1/t^2| > 1$. If $t \neq 0$, then from (4.2) we see inductively over k that the partial fraction decomposition of f_k yields terms of the form $P_k/(1-x)^l$ only for $l = 1$. The series coefficients of the other partial fraction decomposition terms are quotients of polynomials in n divided by an exponential in n, and hence $V_{2n}^{(k)}(t)$ are bounded when $n \to \infty$. If $t = 0$, then (4.2) turns into

$$f_k = k(k-1)xf_{k-2} + \frac{A_k + B_kx}{1-x},$$

so by induction $f_k = P_k(x)/(1-x)$. □

Remark 4.6. If needed, $C_{k,g,t}$ (and similarly $C_{k,g,z}$ in Theorem 3.1) can be estimated explicitly. For this one can use the inequalities in [**42**].

4.2. An application of Mullins's formula

Now consider the Casson–Walker invariant $\lambda_2(K)$ of the 2-fold branched cover of S^3 over a knot K. Mullins [**28**] proved that

$$\lambda_2(K) = -\frac{V_K'(-1)}{6V_K(-1)} + \frac{\sigma(K)}{4}, \tag{4.3}$$

where σ is the signature [**31**]. A consequence of this formula and our previous inequalities is an estimate for $\lambda_2(K)$ of a positive knot K. (The proof is constructive, so that the numbers C_1 and C_2 occurring in (4.4) can be made explicit with some work, if needed.)

Proposition 4.7. *There are constants $C_1, C_2 > 0$ such that for any positive knot K of genus $g(K)$ and crossing number $c(K)$ we have*

$$|\lambda_2(K)| \leqslant (C_1 c(K))^{C_2 g(K)}. \tag{4.4}$$

Proof. The σ term in (4.3) is not problematic: due to Murasugi [**31**], we have

$$|\sigma(K)| \leqslant 2g(K) \leqslant c(K) \tag{4.5}$$

for every knot K. Thus, there is no difficulty in incorporating the signature term into an estimate of the type of (4.4).

The point of the proposition is the estimate of the first term on the right of (4.3). The denominator is the determinant of K (from Example 3.6), which is an odd integer and can again be ignored.

For $V'(-1)$ we use Theorem 4.2. Note that now $t = -1$ and $k = 1$ are fixed, as is $\tilde{p}_t(k) = 2$.

We can obtain (4.4) by proving that

$$C_{1,g,-1} = O(C_3^g)$$

for a constant C_3, and that, with $n = c(K)$,

$$(n + 2g - 1)^{2d_g} = O((C_4 n)^{C_5 g}).$$

Now, $2g(K) \leqslant c(K)$, and thus $n + 2g - 1 \leqslant 2n$, which yields $C_4 = 2$. The property that d_g is linear in g (see (2.18)) yields C_5.

To show that C_3 exists, we must show that $C_{1,g,-1}$ in Theorem 4.2 is exponentially bounded in g. Now, $C_{1,g,-1}$ is a finite maximum (for each braiding sequence of genus g) over some linear combination of terms $V(-1)$ and $V'(-1)$ evaluated on diagrams $\tilde{D}$ of genus g with, say, at most four crossings in a $\sim$-equivalence class.

For a one-parameter braiding sequence, the values $V_{\tilde{D}}(-1)$, $V'_{\tilde{D}}(-1)$ were designated as A_0, B_0, A_1 and B_1 in the proof of Theorem 4.2. The coefficients of their linear combination are fixed for one parameter, thus they multiplicatively accumulate at most to an exponential in the number of parameters. This number is at most d_g, which is linearly bounded in g. The coefficients are thus exponentially bounded in g, and it is enough to look at the values $V_{\tilde{D}}(-1)$, $V'_{\tilde{D}}(-1)$ themselves.

The diagrams $\tilde{D}$ have at most four crossings in a $\sim$-equivalence class, and the number of such classes is at most d_g. Thus, $c(\tilde{D})$ is linearly bounded in g. Next, $V_{\tilde{D}}(-1)$, $V'_{\tilde{D}}(-1)$ can be estimated from the coefficients of $V(\tilde{D})$ (and its degrees, which are linearly bounded in $c(\tilde{D})$). To conclude, it is thus enough to notice that every coefficient of V is exponentially bounded in $c(\tilde{D})$. This follows from rather standard arguments (see, for example, [**42**]). □

Remark 4.8. For alternating knots, the same argument clearly applies. However, from the fact that the coefficients of V alternate (see the remark below Theorem 2.3), we have that

$$|V_K'(-1)| \leqslant \max(|\max\deg V_K|, |\min\deg V_K|)|V_K(-1)|,$$

leading to a much better estimate. (Of course, for positive K, similar properties of V do not hold in general.)

5. Finiteness properties for link polynomial coefficients

In this section we record some consequences of the previous theorems for positive knots.

5.1. The coefficients of the Jones and HOMFLY polynomial

The value $t = 0$ in Theorem 4.2 is of special interest, since it gives the coefficients of the Jones polynomial themselves. Using our work we can prove that any coefficient on the Jones polynomial takes a finite number of values on knot diagrams D whose Bennequin number $r(D)$ (recall § 2.3) is not too small in comparison to their genus $g(D)$.

Theorem 5.1. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function with*

$$\liminf_{n\to\infty} \left(f(n) - \frac{n}{2} \right) = \infty. \tag{5.1}$$

Let

$$\mathcal{D}_f := \{D\colon r(D) \geqslant f(g(D))\} \quad \textit{and} \quad \mathcal{D}_{f,c} := \{D\colon r(D) \geqslant f(g(D)),\ c(D) = c\}. \tag{5.2}$$

Define, for some $k \in \mathbb{Z}$,

$$\mathcal{V}_{k,f} := \{[V_D(t)]_{t^k}\colon D \in \mathcal{D}_f\} \quad \textit{and} \quad \mathcal{V}_{k,f,c} := \{[V_D(t)]_{t^k}\colon D \in \mathcal{D}_{f,c}\}. \tag{5.3}$$

Then all $\mathcal{V}_{k,f}$ are finite. More precisely, $\mathcal{V}_{k,f,c}$ stabilizes in $c \bmod 2$ when $c \to \infty$. That is, there is a $c_0 = c_0(k, f)$ such that $\mathcal{V}_{k,f,c+2} = \mathcal{V}_{k,f,c}$ for $c \geqslant c_0$.

Both statements remain true if we replace $[V]_{t^k}$ by $[P]_{l^j m^k}$ for any $j, k \in \mathbb{Z}$, and condition (5.1) by $\liminf_{n\to\infty} f(n) = \infty$. Also, the stability statement (for both polynomials) holds if one replaces '$r(D) \geqslant f(g(D))$' in (5.2) by '$r(D) = f(g(D))$'.

Remark 5.2. If we count coefficients relative to the degree, i.e. for V replace 'k' by '$k + \min\deg V$', the theorem is surely already false for positive D and $k = 1$ (in [**48**] the meaning of this coefficient is explained), and nothing similar makes sense for P or F.

Corollary 5.3. *Any coefficient of the Jones and skein polynomial takes only finitely many values on positive knots.*

Proof. Indeed, for a positive diagram D we have $w(D) = c(D)$, and thus $f(n) = n$ in (5.1) will do. □

We will later show that this corollary also holds for the Kauffman polynomial.

For the proof of Theorem 5.1 we recall a lemma from [**44**].

https://doi.org/10.1017/S0013091515000498 Published online by Cambridge University Press

Lemma 5.4 (Stoimenow [44]). *For any $l \geqslant 0$ and any l-almost positive link diagram D we have*

$$\min\deg V(D) \geqslant \frac{1-\chi(D)}{2} - 2l + \operatorname{sgn}(l). \tag{5.4}$$

(For $l = 0$ equality holds.)

We will use the following simple technical argument, which will be needed repeatedly, and thus is separated. We consider an infinite sequence of diagrams

$$D_i = D(p_{1,i}, \dots, p_{n,i}) \tag{5.5}$$

in one braiding sequence $\mathcal{D} = \mathcal{B}(D)$. By a *subsequence* we mean $\{D_{i_j}\}$ for an increasing sequence $\{i_j\}$. By *reordering indices* in (5.5) we mean the application of a fixed permutation $\rho \in S_n$ to the first subscripts:

$$D(p_{\rho(1),i}, \dots, p_{\rho(n),i}).$$

Lemma 5.5. *For (5.5), we may w.l.o.g., by reordering indices and going over to an (infinite) subsequence, assume that there is a d satisfying $1 \leqslant d \leqslant n$ such that $p_{d+1,i}, \dots, p_{n,i}$ are constant (in i), and*

$$p_{k,i+1} > p_{k,i} \tag{5.6}$$

for all $i > 0$ and $k = 1, \dots, d$.

Proof. Assume first that for some $1 \leqslant k \leqslant n$ the set $\{p_{k,i}\}_{i=1}^{\infty}$ is finite. Then reorder indices so that $k = n$ and fix a value of $p_{n,i}$ for which infinitely many D_i occur. Repeat this argument for $1 \leqslant k \leqslant n-1$, and we can fix $p_{n-1,i}$, etc. Thus, for some d we can choose D_{i_j} so that p_{k,i_j} is constant in i_j for $k = d+1, \dots, n$, while $\{p_{k,i_j}\}_{j=1}^{\infty}$ is infinite for $1 \leqslant k \leqslant d$. Clearly, $d \geqslant 1$. Replace $\{D_i\}$ by $\{D_{i_j}\}$.

When for fixed $1 \leqslant k \leqslant d$ there are infinitely many values $\{p_{k,i}\}_i$, up to going over to a subsequence, we may assume that $p_{k,i}$ is strictly increasing in i. By doing this d times for all $1 \leqslant k \leqslant d$, we obtain (5.6) for all $k = 1, \dots, d$. □

Proof of Theorem 5.1. First consider the Jones polynomial.

By assumption, we deal with diagrams D for which $r(D) - g(D)/2$ becomes (arbitrarily) large when $g(D)$ is (sufficiently) large. Let l be the number of negative crossings in D. Recall from (2.4) that $r(D) = g(D) - l$. Thus, $g(D)/2 - l$ becomes large when $g(D)$ is large. Now, for knots in (5.4) we have $(1 - \chi(D))/2 = g(D)$. Thus, we obtain that for $g(D)$ large, $\min\deg V(D)$ also becomes large.

Let us now, for the rest of the proof, fix the degree k in which we are interested in the Jones polynomial coefficient $[V(D)]_{t^k}$. The argument we made on $\min\deg V(D)$ means that, in order to evaluate $[V(D)]_{t^k}$ (to something different from 0), it is enough to consider diagrams D of bounded (from above) genus $g(D)$. Then the number l of negative crossings of D is also bounded, since $r(D)$ in (2.4) is bounded from below.

Now it follows from (5.4) that $\min\deg V(D)$ has a lower bound; call it $s = s(k)$, independent of the crossing number of D. It is thus enough to study one (reverse) braiding sequence $\mathcal{D} = \mathcal{B}(\hat{D}, P)$ with positive $\bar{t}_2'$ twists. Let $p_1, \ldots, p_n$ be the parameters of $\mathcal{D}$. We assume $n = |P|$ and p_i odd. From now on, k, s and $\mathcal{D}$ are fixed.

The proof of Theorem 5.1 will centre around the following claim.

Claim 5.6. *For each $1 \leqslant d \leqslant n$ and fixed $p_{d+1}, \ldots, p_n$ there is a number $\omega = \omega(p_{d+1}, \ldots, p_n) \leqslant k+1$ such that when $p_1, \ldots, p_d \geqslant \omega$, for all $k \geqslant k' \geqslant s$ the coefficient*

$$V_{k'}(\hat{D}(p_1, \ldots, p_d, p_{d+1}, \ldots, p_n)) := [V(\hat{D}(p_1, \ldots, p_d, p_{d+1}, \ldots, p_n))]_{k'}$$

does not depend on the values of $p_1, \ldots, p_d$.

Let us first clarify how this claim will prove Theorem 5.1. The theorem states two properties; the stability assertion ($\mathcal{V}_{k,f,c}$ stabilizes modulo 2 when $c \to \infty$) implies the finiteness assertion ($\mathcal{V}_{k,f} = \bigcup_c \mathcal{V}_{k,f,c}$ is finite). We thus consider only the stronger statement and prove it by contradiction, assuming that it is false.

If the stability assertion of the theorem is false, we have $\mathcal{V}_{k,f,c_i} \neq \mathcal{V}_{k,f,c_i+2}$ for some sequence of $c_i \to \infty$. For

$$D_i = \hat{D}(p_{1,i}, \ldots, p_{n,i}) \in \mathcal{D}_{f,c_i}$$

write

$$D_i^{j,\pm} := \hat{D}(p_{1,i}, \ldots, p_{j-1,i}, p_{j,i} \pm 2, p_{j+1,i}, \ldots, p_{n,i}) \in \mathcal{D}_{f,c_i \pm 2}.$$

(The inclusion for the negative sign holds when $p_{j,i} \geqslant 3$.) If

$$\mathcal{V}_{k,f,c_i} \setminus \mathcal{V}_{k,f,c_i+2} \neq \varnothing, \tag{5.7}$$

choose $D_i \in \mathcal{D}_{f,c_i}$ so that $V_k(D_i) \notin \mathcal{V}_{k,f,c_i+2}$. Since $D_i^{j,+} \in \mathcal{D}_{f,c_i+2}$ for all $1 \leqslant j \leqslant n$, we have in particular that

$$V_k(D_i) \neq V_k(D_i^{j,+}). \tag{5.8}$$

Otherwise, if not (5.7), we must have $\mathcal{V}_{k,f,c_i+2} \setminus \mathcal{V}_{k,f,c_i} \neq \varnothing$, and we choose $D_i \in \mathcal{D}_{f,c_i+2}$ so that $V_k(D_i) \notin \mathcal{V}_{k,f,c_i}$. Then in particular, for all $1 \leqslant j \leqslant n$ (with $p_{j,i} \geqslant 3$),

$$V_k(D_i) \neq V_k(D_i^{j,-}). \tag{5.9}$$

By using a subsequence, we can fix the generator $D = \hat{D}$ of all D_i to indeed be the same. Thus, we have a sequence of diagrams D_i as in (5.5) for which either (5.8) holds for all $1 \leqslant j \leqslant n$ or (5.9) holds for all $1 \leqslant j \leqslant n$. Let us argue using (5.9); for property (5.8) the reasoning is analogous.

Now, by Lemma 5.5, we may assume, after reordering indices, that $p_{d+1,i}, \ldots, p_{n,i}$ are fixed, while $p_{1,i}, \ldots, p_{d,i}$ are increasing in i. We compare with property (5.9), which is enough to use only for $j = 1$. But by Claim 5.6, after excluding finitely many $p_{1,i}, \ldots, p_{d,i}$ (and i), we have that V_k does not change for $p_{1,i}$ large enough, which contradicts both (5.8) and (5.9) (for $j = 1$). This shows that Claim 5.6 proves Theorem 5.1.

We will prove Claim 5.6 by induction on d. We will first do the induction step, assuming the basis of induction ($d = 1$), which we will justify later.

Assume thus that Claim 5.6 holds for $d-1 \geqslant 1$. We consider

$$\hat{D}(p_1, \ldots, p_d, p_{d+1}, \ldots, p_n)$$

for $p_{d+1}, \ldots, p_n$ fixed and $p_1, \ldots, p_d$ increasing.

We use (4.1) with $n = p_d$. (Here the assumption enters that we perform only positive $\bar{t}'_2$ twists.) One consequence of this identity is that for $s \leqslant k' \leqslant k$,

$$V_{k'}(\hat{D}(p_1, \ldots, p_d, p_{d+1}, \ldots, p_n)) \tag{5.10}$$

is determined by

$$V_{k''}(\hat{D}(p_1, \ldots, p_{d-1}, 1, p_{d+1}, \ldots, p_n)) \quad \text{and} \quad V_{k''}(\hat{D}(p_1, \ldots, p_{d-1}, 3, p_{d+1}, \ldots, p_n)) \tag{5.11}$$

for $s \leqslant k'' \leqslant k'$; cf. (5.14).

Now by the induction assumption there are numbers $\omega_i = \omega(i, p_{d+1}, \ldots, p_n)$ for $i = 1, 3$. Then for $\hat{\omega} = \max(\omega_1, \omega_3)$ we have that the expressions in (5.11) do not depend on $p_1, \ldots, p_{d-1} \geqslant \hat{\omega}$. This means that neither does (5.10).

We still have to remove the dependence of (5.10) on p_d. Again by the basis of induction, we can consider

$$\hat{D}(p) := \hat{D}(p_1, \ldots, p_{d-1}, p, p_{d+1}, \ldots, p_n), \tag{5.12}$$

where $p_{d+1}, \ldots, p_n$ had been fixed *a priori*, and now $p_1, \ldots, p_{d-1}$ are fixed to some (immaterial) values equal to or above $\hat{\omega}$. The assertion for $d = 1$ will yield some stabilization limit

$$\omega_0 = \omega(p_1, \ldots, p_{d-1}, p_{d+1}, \ldots, p_n) = \omega(\hat{\omega}, \ldots, \hat{\omega}, p_{d+1}, \ldots, p_n) \tag{5.13}$$

for $V_{k'}(\hat{D}(p))$ when $p \geqslant \omega_0$. Precisely speaking, the expression (5.13) for ω_0 is only valid after renumbering the dth parameter to become the first. Thus, in fact, the induction for Claim 5.6 should be done over the assertion that allows reordering indices. (This assertion is formally wider, but in fact equivalent, and we chose not to roll it out in this form to save notation overhead.)

With this justification of (5.13), we can now take

$$\omega(p_{d+1}, \ldots, p_n) := \max(\omega_0, \hat{\omega}) \leqslant k + 1$$

and complete the induction step needed for Claim 5.6.

It remains to prove the basis of induction, i.e. the case that $d = 1$. That is, we must prove for a one-parameter (reverse) braiding sequence $\hat{D}(p)$ as in (5.12) that there is an ω such that for $p \geqslant \omega$ and all $s \leqslant k' \leqslant k$ the coefficient $V_{k'}(\hat{D}(p))$ does not depend on p.

By induction over $k' \geqslant s$, we see that it is enough to prove this stability for $[(t^2-1)V[p]]_{k'}$, where we abbreviate $V[p] := V(\hat{D}(p))$ and $p = p_1$. But a consequence of (4.1) is

$$V[2k+1] = t^{2k}V[1] + \frac{t^{2k}-1}{t^2-1}(V[3] - t^2V[1]) \tag{5.14}$$

(cf., for example, the proof of Theorem 9.3 in [**45**]). This implies the desired stability easily. By Lemma 5.4, we have $\min\deg V[p] = g(\hat{D}) > 0$, and we see that

$\omega = \omega(p_2, \ldots, p_n) \leqslant k+1$ is sufficient for $p = p_1$ odd. This completes the proof of Theorem 5.1 for the Jones polynomial.

For the skein polynomial a similar argument applies, only that instead of Lemma 5.4 one uses the inequality for $\min\deg_l P$ of Morton [**27**]. One also needs to develop an appropriate version of Claim 5.6, but everything is completely analogous. For example, (5.14) becomes (with suggestive notation)

$$P[2k+1] = (-l^2)^k P[1] - \frac{(-l^2)^k - 1}{l^2+1}(P[3] + l^2 P[1]), \tag{5.15}$$

and again $\omega \leqslant k+1$ will suffice. □

Remark 5.7. Theorem 5.1 allows several modifications. It remains true if we consider diagrams D of fixed genus $g(D)$ in (5.3). Furthermore, a result of Gabai (see [**14**, Corollary 2.4] and also [**45**, Remark 11.1]) shows that the genus $g(K)$ of the *knot* K represented by D (and not only of the diagram D itself) is constant with at most one exception on any one-parameter $\bar{t}'_2$ twist sequence. Then, similarly to [**45**], one can also place the condition 'K is of genus $g(K) = g(D)$' in (5.3). One can also fix D to be alternating (in which case we always have $g(K) = g(D)$; see [**9**]).

Corollary 5.8. *Fix $k, l \in \mathbb{N}$ and consider*

$$\mathcal{P}_c := \{K\colon K \text{ has an } l\text{-almost positive diagram of } c \text{ crossings}\}$$

and

$$\mathcal{V}_c := \{[V_K(t)]_{t^k}\colon K \in \mathcal{P}_c\}.$$

Then $\mathcal{V}_c$ is almost everywhere 2-periodic in c, that is, there is a $c_0 = c_0(k,l) \in \mathbb{N}$ with $\mathcal{V}_{c+2} = \mathcal{V}_c$ for all $c \geqslant c_0$.

The same result holds if we modify the definition of $\mathcal{P}_c$ by only considering l-almost positive alternating diagrams of c crossings, and/or considering knot diagrams of given genus.

In particular, the corollary says: if some (value of a) coefficient of V occurs for infinitely many positive knots, then the set of all knots realizing this coefficient (value) has positive diagrams of almost all even and/or odd crossing numbers.

Remark 5.9. Note that reduced alternating almost positive diagrams do not exist. (We use this fact in [**44**] to show that alternating almost positive knots do not exist.)

Corollary 5.10. *The set $\{[V_K(t)]_{t^k}\colon K$ an l-almost positive knot$\}$ is finite for any k and l.*

Both Corollaries 5.8 and 5.10 of course also hold (in appropriate form) for the skein polynomial.

5.2. The coefficients of the Kauffman polynomial on positive knots

In this section we give a proof of the following theorem, which extends the previously exhibited coefficient finiteness property also to the Kauffman polynomial. For its proof we require the work of Yokota [**57**].

Theorem 5.11. *Each coefficient $[F(K)]_{z^k a^l}$ (for fixed k and l) of the Kauffman polynomial admits only finitely many values on positive knots K.*

It is also possible to prove some related results in the spirit of the previous paragraphs (and also possibly to prove the theorem for l-almost positive knots, by extending Yokota's results to such knots), but we shall not repeat the arguments here.

Again we consider the generating series associated with the polynomials of a certain (one-parameter) braiding sequence and find that the local relation for F makes it into a rational function. This time, however, as $F(a, z)$ (in general) contains negative powers of a, all series will be Laurent series in a.

Proof of Theorem 5.11. Recall (from § 2) our convention for F, differing from [**21**] by the interchange of a and a^{-1}. Then the relation between F and its writhe-unnormalized version $\Lambda = \Lambda(a, z)$ becomes (2.8), and from the defining relation (2.9) of Λ we have

$$\Lambda_n = z\Lambda_{n+1} + za^{-(n+1)}\Lambda_\infty - \Lambda_{n+2},$$

with $\Lambda_i := \Lambda(D_i)$ and D_i diagrams of links L_i as in (3.1). If we consider again

$$f = f(a, z, x) = \sum_{i=0}^{\infty} \Lambda_i x^i,$$

which is a Laurent series in a of bounded minimal degree, then we obtain

$$f = \frac{P(a, z, x)}{(1 - a^{-1}x)(x^2 - zx + 1)}$$

for some $P \in \mathbb{Q}[x, a, a^{-1}, z]$. This is a rational function in a, z and x divided by a power of a, and hence converges as a Laurent series in a for $|x| < |a|$, $|z| < 1$.

As we are just interested in the Taylor x-coefficients of the series for i even (for i odd we obtain certain two-component links), we build

$$\begin{aligned}\tilde{f}(a, z, x) = \sum_{i \text{ even}} F(D_i)x^i &= \frac{a^{k_1}}{2}[f(a, z, xa) + f(a, z, -xa)] \\ &= \frac{P_1(a, z, x)}{(1 - x^2)(a^2x^2 - azx + 1)(a^2x^2 + azx + 1)}\end{aligned}$$

for some $k_1 = w(D_1) \in \mathbb{Z}$ and $P_1 \in \mathbb{Q}[x, a, a^{-1}, z]$. Since $P_1(a, z, x) = P_1(a, z, -x)$, in fact we have $P_1 \in \mathbb{Q}[x^2, a^{\pm 1}, z]$. From the r.h.s. we again see that this series is rational in all variables, and hence converges as a Laurent series in a neighbourhood of the origin, in particular, for $|x|, |a|, |z| < 1$.

https://doi.org/10.1017/S0013091515000498 Published online by Cambridge University Press

Applying $\partial^k/\partial z^k|_{z=0}$ on $\tilde{f}$, we obtain

$$\frac{P_2(a,x)}{(1-x^2)(a^2x^2+1)^{k+2}}$$

for some $P_2 \in \mathbb{Q}[x^2, a^{\pm 1}]$, so that

$$\hat{f}(a,x) := \sum_{i \geqslant 0} [F(D_{2i})]_{z^k} x^i = \frac{P_3(a,x)}{(1-x)(a^2x+1)^{k+2}},$$

with $P_3(a,x) = (1/k!)P_2(a,\sqrt{x})$. Now, by Yokota's theorem [**57**] we know that (with the present convention) $\min\deg_a F(D_{2i}) > 0$ if D_{2i} are positive. Then $\hat{f}$ converges without singularity at $(0,0)$ and we can differentiate l times in a the left-hand (and therefore, also the right-hand) side, and set $a = 0$. The denominator collapses to $1-x$. To see this, consider the denominator factor $1-x$ as a constant in a and apply the quotient rule. Then again the previous arguments together with Yokota's result $\min\deg_a F(K) = 2g(K)$ for K positive complete the proof. □

Remark 5.12. It is worth remarking that Przytycki's invariance criteria [**34**] of certain evaluations of F (and analogously of the other link polynomials) under k-moves can also be deduced from these rational function expressions. It would take us aside to follow it here, but briefly, we would have to examine for which values of a and z the denominator has only zeros in x that are distinct kth roots of unity. This has been (again implicitly) investigated in [**40**] for the Q polynomial in the language of H_1 of the double branched cover.

Remark 5.13. The material from § 3 until here can be generalized straightforwardly to links. Then the genus must be replaced by the Euler characteristic χ, and one must use the fact that the Euler characteristic bounds the number of components $n \leqslant 2-\chi$. Theorems 2.1 and 2.8 hold analogously. One needs to introduce, in correspondence to Definition 2.2, sets $\mathcal{A}_{n,\chi}$ and $\mathcal{P}_{n,\chi}$ and numbers d_χ. Then again, similarly to (2.18), $d_\chi = -3\chi$ by [**47**].

6. Concluding questions and problems

We conclude this exposition with a few general remarks. One results from the desire to link the two different parts of this work.

Problem 6.1. Which extended (but not genuine) Vassiliev invariants satisfy (more) global polynomial growth bounds, and (how) can the idea of the proof of the Lin–Wang conjecture in [**46**] be adapted to show these bounds?

It is clear that not all invariants satisfy such bounds. Consider the determinant square $Q(2)$ on connected sums of trefoils: it grows exponentially.

One discrepancy between the Vassiliev and extended Vassiliev part lies in the following important difference. In the Vassiliev case, in a braiding polynomial with sufficiently many variables a top degree monomial does not contain most of the variables. This is decisively used to find recursive relations, and such an approach fails in the generalized case. One consequence is that the various dimension upper bounds for Vassiliev invariants cannot be (straightforwardly) generalized to extended Vassiliev invariants.

Problem 6.2. How can one obtain upper dimension bounds for the proposed generalizations of Vassiliev invariants of given degree?

Problem 6.3. Are $[Q(z-2)]_{z^k}$ for $k \geqslant 0$ Vassiliev invariants (where for links we consider $Q(z-2)$ as a power series in z)?

For $k=0$ this coefficient is constantly 2^{n-1} on n-component links, and for $k=1$ the problem was solved positively by Kanenobu [**19**]. The answer appears negative, though, for $k \geqslant 2$. But note that these are extended Vassiliev invariants. Thus, direct calculations as in [**16**, **53**] will not prove that they are not Vassiliev invariants. See also [**8**].

Acknowledgements. Over the very long period that parts of this work were developed, several people offered helpful remarks, discussions and references to related research. Among them are S. Baader, T. Kanenobu and C. Y. Park. The referee pointed out numerous places where my writing could be improved and recommended a substantial revision that has hopefully made the exposition more approachable.

References

1. D. Bar-Natan, On the Vassiliev knot invariants, *Topology* **34** (1995), 423–472.
2. D. Bar-Natan and A. Stoimenow, The fundamental theorem of Vassiliev invariants, in *Geometry and physics*, Lecture Notes in Pure and Applied Mathematics, Volume 184, pp. 101–134 (Dekker, New York, 1996).
3. D. Bennequin, Entrelacements et équations de Pfaff, *Astérisque* **107** (1983), 87–161.
4. J. S. Birman and X.-S. Lin, Knot polynomials and Vassiliev's invariants, *Invent. Math.* **111** (1993), 225–270.
5. J. S. Birman and R. F. Williams, Knotted periodic orbits in dynamical systems, I, Lorenz's equations, *Topology* **22**(1) (1983), 47–82.
6. M. Boileau and C. Weber, Le problème de J. Milnor sur le nombre gordien des nœuds algébriques, *Enseign. Math.* **30** (1984), 173–222.
7. R. D. Brandt, W. B. R. Lickorish and K. Millett, A polynomial invariant for unoriented knots and links, *Invent. Math.* **84** (1986), 563–573.
8. Y. Choi, M. J. Jeong and C. Y. Park, Twist of knots and the Q-polynomials, *Kyungpook Math. J.* **44**(3) (2004), 449–467.
9. P. R. Cromwell, Homogeneous links, *J. Lond. Math. Soc. (2)* **39** (1989), 535–552.
10. P. R. Cromwell, *Knots and links* (Cambridge University Press, 2004).
11. P. R. Cromwell and H. R. Morton, Positivity of knot polynomials on positive links, *J. Knot Theory Ram.* **1** (1992), 203–206.
12. R. Crowell, Genus of alternating link types, *Annals Math.* **69**(2) (1959), 258–275.
13. P. Freyd, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu and D. Yetter, A new polynomial invariant of knots and links, *Bull. Am. Math. Soc.* **12** (1985), 239–246.
14. D. Gabai, Foliations and the topology of 3-manifolds, II, *J. Diff. Geom.* **26**(3) (1987), 461–478.
15. C. F. Ho, A polynomial invariant for knots and links: preliminary report, *Abstracts Am. Math. Soc.* **6** (1985), 300.

16. G. T. Jin and J. H. Lee, Coefficients of HOMFLY polynomial and Kauffman polynomial are not finite type invariants, *J. Knot Theory Ram.* **11**(4) (2002), 545–553.
17. V. F. R. Jones, A polynomial invariant of knots and links via von Neumann algebras, *Bull. Am. Math. Soc.* **12** (1985), 103–111.
18. V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, *Annals Math.* **126** (1987), 335–388.
19. T. Kanenobu, An evaluation of the first derivative of the Q polynomial of a link, *Kobe J. Math.* **5** (1988), 179–184.
20. L. H. Kauffman, State models and the Jones polynomial, *Topology* **26** (1987), 395–407.
21. L. H. Kauffman, An invariant of regular isotopy, *Trans. Am. Math. Soc.* **318** (1990), 417–471.
22. D. Kreimer, *Knots and Feynman diagrams*, Cambridge Lecture Notes in Physics, Volume 13 (Cambridge University Press, 2000).
23. W. B. R. Lickorish and K. C. Millett, A polynomial invariant for oriented links, *Topology* **26**(1) (1987), 107–141.
24. X.-S. Lin and Z. Wang, Integral geometry of plane curves and knot invariants, *J. Diff. Geom.* **44**(1) (1996), 74–95.
25. W. W. Menasco and M. B. Thistlethwaite, The Tait flyping conjecture, *Bull. Am. Math. Soc.* **25**(2) (1991), 403–412.
26. W. W. Menasco and X. Zhang, Positive knots and knots with braid index three have property P, *J. Knot Theory Ram.* **12**(4) (2003), 427–444.
27. H. R. Morton, Seifert circles and knot polynomials, *Proc. Camb. Phil. Soc.* **99** (1986), 107–109.
28. D. Mullins, The generalized Casson invariant for 2-fold branched covers of S^3 and the Jones polynomial, *Topology* **32**(2) (1993), 419–438.
29. K. Murasugi, On the genus of the alternating knot, I, *J. Math. Soc. Jpn* **10** (1958), 94–105.
30. K. Murasugi, On the genus of the alternating knot, II, *J. Math. Soc. Jpn* **10** (1958), 235–248.
31. K. Murasugi, On a certain numerical invariant of link types, *Trans. Am. Math. Soc.* **117** (1965), 387–422.
32. K. Murasugi, Jones polynomial and classical conjectures in knot theory, *Topology* **26** (1987), 187–194.
33. T. Nakamura, Positive alternating links are positively alternating, *J. Knot Theory Ram.* **9**(1) (2000), 107–112.
34. J. Przytycki, t_k moves on links, in *Braids* (ed. J. S. Birman and A. L. Libgober), Contemporary Mathematics, Volume 78, pp. 615–656 (American Mathematical Society, Providence, RI, 1988).
35. L. Rudolph, Quasipositivity as an obstruction to sliceness, *Bull. Am. Math. Soc.* **29** (1993), 51–59.
36. L. Rudolph, Positive links are strongly quasipositive, in *Proceedings of the Kirbyfest*, Geometry and Topology Monographs, Volume 2, pp. 555–562 (Mathematical Sciences Publishers, Berkeley, CA, 1999).
37. I. Stewart, *Galois theory* (Chapman & Hall, London, 1973).
38. A. Stoimenow, Gauss sum invariants, Vassiliev invariants and braiding sequences, *J. Knot Theory Ram.* **9**(2) (2000), 221–269.
39. A. Stoimenow, Knots of genus one, *Proc. Am. Math. Soc.* **129**(7) (2001), 2141–2156.
40. A. Stoimenow, Branched cover homology and Q evaluations, *Osaka J. Math.* **39**(1) (2002), 13–21.
41. A. Stoimenow, On the crossing number of positive knots and braids and braid index criteria of Jones and Morton–Williams–Franks, *Trans. Am. Math. Soc.* **354**(10) (2002), 3927–3954.

42. A. Stoimenow, On the coefficients of the link polynomials, *Manuscr. Math.* **110**(2) (2003), 203–236.
43. A. Stoimenow, Positive knots, closed braids, and the Jones polynomial, *Annali Scuola Norm. Sup. Pisa* **2**(2) (2003), 237–285.
44. A. Stoimenow, On some restrictions to the values of the Jones polynomial, *Indiana Univ. Math. J.* **54**(2) (2005), 557–574.
45. A. Stoimenow, Knots of (canonical) genus two, *Fund. Math.* **200**(1) (2008), 1–67.
46. A. Stoimenow, Application of braiding sequences, I, On the characterization of Vassiliev and polynomial link invariants, *Commun. Contemp. Math.* **12**(5) (2010), 681–726.
47. A. Stoimenow, *Diagram genus, generators and applications*, Monographs and Research Notes in Mathematics (Chapman and Hall/CRC, 2016).
48. A. Stoimenow, Coefficients and non-triviality of the Jones polynomial, *J. Reine Angew. Math.* **657** (2011), 1–55.
49. A. Stoimenow, On the crossing number of semiadequate links, *Forum Math.* **26**(4) (2014), 1187–1246.
50. A. Stoimenow and A. Vdovina, Counting alternating knots by genus, *Math. Annalen* **333** (2005), 1–27.
51. T. Tanaka, Maximal Bennequin numbers and Kauffman polynomials of positive links, *Proc. Am. Math. Soc.* **127**(11) (1999), 3427–3432.
52. M. B. Thistlethwaite, A spanning tree expansion for the Jones polynomial, *Topology* **26** (1987), 297–309.
53. R. Trapp, Twist sequences and Vassiliev invariants, *J. Knot Theory Ram.* **3**(3) (1994), 391–405.
54. J. van Buskirk, *Positive links have positive Conway polynomial*, Springer Lecture Notes in Mathematics, Volume 1144, pp. 146–159 (Springer, 1983).
55. V. A. Vassiliev, Cohomology of knot spaces, in *Theory of singularities and its applications* (ed. V. I. Arnol′d), Advances in Soviet Mathematics (American Mathematical Society, Providence, RI, 1990).
56. P. Vogel, Algebraic structures on modules of diagrams, Preprint, Université Paris VII (available at www.math.jussieu.fr/~vogel/; 1996).
57. Y. Yokota, Polynomial invariants of positive links, *Topology* **31**(4) (1992), 805–811.