Probability in the Engineering and Informational Sciences, **33**, 2019, 172–185.
doi:10.1017/S0269964818000086

ANALYSIS OF THE NETWORK WITH MULTIPLE CLASSES OF POSITIVE AND NEGATIVE CUSTOMERS AT A TRANSIENT REGIME

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This paper is devoted to the investigation of the G-network with multiple classes of positive and negative customers. The purpose of the investigation is to analyze such a network at a transient regime, to find the state probabilities of the network that depend on time. In the first part, a description of the functioning of G-networks with positive and negative customers is provided, when a negative customer when arriving to the system destroys a positive customer of its class. Streams of positive and negative customers arriving at each of the network systems are independent. Services of positive customers of all types occur in accordance with a random selection of them for service. For nonstationary probabilities of network states, a system of Kolmogorov's difference-differential equations (DDE) has been derived. A method for their finding is proposed. It is based on the use of a modified method of successive approximations, combined with the method of series. It is proved that successive approximations converge with time to a stationary probability distribution, the form of which is indicated in this paper, and the sequence of approximations converges to the unique solution of the DDE system. Any successive approximation is representable in the form of a convergent power series with an infinite radius of convergence, the coefficients of which satisfy recurrence relations, which is convenient for computer calculations. A model example illustrating the determination of the time-dependent probabilities of network states using this technique has been calculated. The obtained results can be applied in modeling the behavior of computer viruses and attacks in information and telecommunication systems and networks, for example, as a model of the impact of some file viruses on server resources. variable.

Keywords: combined with the method of series, G-network, method of successive approximations, multiple classes of positive and negative customers, transient regime

1. NETWORK DESCRIPTION. FORMULATION OF THE PROBLEM

Consider an open G-network of queueing with n single-line queueing systems (QS), in which arrive positive and negative customers of r classes. In i-th QS from the external environment arrives a simple stream of ordinary customers (positive) with the rate λ^+ and an additional flow of negative customers, which is also the simplest with the rate λ^-, $i = \overline{1, n}$. All arriving streams are independent. Each positive customer of the input stream independently of other customers is sent to the i-th QS as a customer of class c with probability p^+_{0ic},

$\sum_{i=1}^{n}\sum_{c=1}^{r}p_{0ic}^{+}=1$. Duration of service of positive customers in the i-th QS c-class are distributed according to the exponential distribution with parameter $\mu_{ic}, i=\overline{1,n}, c=\overline{1,r}$.

The network circulates not only positive customers but also negative ones. Each negative customer of the input stream, independently of other negative customers, is sent to the i-th QS as a negative customer of class c with probability $p_{0ic}^{-}, \sum_{i=1}^{n}\sum_{c=1}^{r}p_{0ic}^{-}=1$ and destroys one positive customer of the same class. After the end of the service of the positive customer of class c in the i-th QS, it is sent to the j-th QS with probability p_{icjs}^{+} again as a positive customer of class s, and with probability p_{icjs}^{-} as a negative customer of class s, and with probability $p_{ic0}=1-\sum_{j=1}^{n}\sum_{s=1}^{r}\left(p_{icjs}^{+}+p_{icjs}^{-}\right)$ leaves the network, $i,j=\overline{1,n}$. We assume that customers to the service are randomly selected, i.e. if in the i-th QS there are k_{is} customers of class s, then the probability that to the service in it will be a customer of class c is $\frac{k_{ic}}{\sum_{s=1}^{r}k_{is}}, i=\overline{1,n}, c=\overline{1,r}$.

Under the network state at time t we mean the vector $(\vec{k},t)=(k_{11},k_{12},\ldots,k_{1r},k_{21},k_{22},\ldots,k_{2r},\ldots,k_{n1},k_{n2},\ldots,k_{nr},t)$, where $k_{ic,}$ – the number of positive customers of class c in the i-th QS which forms a homogeneous random Markov chain with continuous time and a countable number of states. It is required to find the state probabilities of the network at a transient regime, depending on the time.

It should be noted that in [2–4] the basic G-network with positive and negative customers of the same class [1] was generalized to the case of several classes of positive and negative customers under the assumption that the number of classes of both types of customers is the same. It is shown that the stationary probabilities of the states of such a network have a multiplicative form (product form). In each of these studies, various options are considered, it is established how the effect of negative customers with their classes is correlated. Thus, in [2] it is assumed that negative customers of a fixed class affect only positive customers of the same class. In [3] is used a random selection of positive customers, i.e. if a negative customer arrives at the i-th QS in which there are $k_i>0$ of positive customers (without taking into account their class), then with the probability (k_{ic}/k_i) a positive customer of class c will be destroyed. In such a G-network, we will explore in this paper, the transitional regime. In [4] the G-network with a variety of disciplines is considered: FIFO – service in the order of arrival, PS – processor separation and LIFO/PR – inversion order of service with service interruption. A positive customer is destroyed in accordance with the service discipline established in the QS, while in the i-th SMO, a negative customer of class s can destroy a positive customer of class claim with a probability K_{isic}. In [5], instead of negative customers of multiple classes in the G-network, signals of multiple classes were considered.

It should also be noted that the finding of nonstationary probabilities of states of a Markov G-network with signals and the group removal of customers by the method of successive approximations combined with the method of series has been presented in [6]; stationary state probabilities of such network in the product form has been found in [7], and for a network with triggers – in [8].

2. A SYSTEM OF KOLMOGOROV DIFFERENCE-DIFFERENTIAL EQUATIONS (DDE) FOR THE NETWORK STATES PROBABILITIES

Let us consider I_{ic} – a vector of dimension $n\times r$, consisting of zeros, with the exception of the components with the number $r(i-1)+c$, which is equal to unity, I_{00} $-n\times r$ a vector consisting of zeros, $P(\vec{k},t)$ – state probability $\vec{k}(t)$; $u(x)=\begin{cases}1, & x>0\\ 0, & x\le 0\end{cases}$ – Heaviside function.

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The following transitions of our Markov chain to the state $(\vec{k}, t+\Delta t)$ in time Δt are possible:

(1) from the state $(\vec{k}-I_{js}, t)$, in this case in the j-th QS in time, Δt a positive customer of class s will arrive with a probability $\lambda^+ p^+_{0js} u(k_{js})\Delta t + o(\Delta t), j=\overline{1,n}, s=\overline{1,r}$;

(2) from the state $(\vec{k}+I_{ic}, t)$, in this case in the i-th QS in time, Δt a negative customer of class c will arrive or after the end of the service, a positive customer of class c leaves the network or transfer to the j-th QS as a negative customer of class s, but does not find there positive customers of this class with probability $\left(\lambda^- p^-_{0ic} + \mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r} k_{is}+1} p_{ic0} + \mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r} k_{is}+1} p^-_{icjs}(1-u(k_{js}))\right)\Delta t + o(\Delta t), i=\overline{1,n}, c=\overline{1,r}$;

(3) from the state $(\vec{k}+I_{ic}-I_{js}, t)$, in this case from the i-th in time Δt a positive customer of class $c$after servicing transfer to the j-th QS as a positive customer of class s with probability $\mu_{ic}\frac{k_{ic}+1}{\sum_{s=1}^{r} k_{is}+1} p^+_{icjs} u(k_{js})\Delta t + o(\Delta t),\ i,j=\overline{1,n}, s,\ c=\overline{1,r}$;

(4) from the state $(\vec{k}+I_{ic}+I_{js}, t)$, in this case from the i-th in time, Δt a positive customer of class c after servicing transfer to the j-th QS as a negative customer of class $s$having destroyed a positive claim of this class with probability $\mu_{ic}\frac{k_{ic}+1}{\sum_{s=1}^{r} k_{is}+1} p^-_{icjs}\Delta t + o(\Delta t), i,j=\overline{1,n}, s,c=\overline{1,r}$;

(5) from the state $(\vec{k}, t)$, in this case for a period of time, Δt the network state did not change with probability

$$1-\left[\lambda^+ + \lambda^- + \sum_{i=1}^{n}\sum_{c=1}^{r}\mu_{ic}u(k_{ic})\right]\Delta t + o(\Delta t),\ i=\overline{1,n}, c=\overline{1,r};$$

(6) from other states with probability $o(\Delta t)$.

Using the full probability formula, we can write:

$$\begin{aligned}
P(\vec{k}, t+\Delta t) = &\left(1-\left[\lambda^+ + \lambda^- + \sum_{i=1}^{n}\sum_{c=1}^{r}\mu_{ic}u(k_{ic})\right]\right) P(\vec{k}, t)\Delta t \\
&+ \sum_{j=1}^{n}\sum_{s=1}^{r}\lambda^+ p^+_{0js} u\left(k_{js}\right) P(\vec{k}-I_{js}, t)\,\Delta t \\
&+ \sum_{i=1}^{n}\sum_{c=1}^{r}\left(\lambda^- p^-_{0ic} + \mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r} k_{is}+1} p_{ic0} + \mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r} k_{is}+1} p^-_{icjs}(1-u\left(k_{ic}\right))\right) \\
&P(\vec{k}+I_{ic}, t)\Delta t + \sum_{i,j=1}^{n}\sum_{s,c=1}^{r}\mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r} k_{is}+1} p^-_{icjs} P(\vec{k}+I_{ic}+I_{js}, t)\Delta t \\
&+ \sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\left(\mu_{ic}p^+_{icjs}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r} k_{is}+1} u(k_{js})\right) P(\vec{k}+I_{ic}-I_{js}, t)\Delta t + o\left(\Delta t\right).
\end{aligned}$$

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Dividing both sides of this relation by Δt and passing to the limit with $\Delta t \to 0$, we obtain that the nonstationary state probabilities of the considered network in this case satisfy the following DDE system:

$$\begin{aligned}
\frac{dP(\vec{k},t)}{dt} = &-\left[\lambda^+ + \lambda^- + \sum_{i=1}^{n}\sum_{c=1}^{r}\mu_{ic}u(k_{ic})\right]P(\vec{k},t) \\
&+ \sum_{j=1}^{n}\sum_{s=1}^{r}\lambda^+ p_{0js}^+ u\left(k_{js}\right)P(\vec{k}-I_{js},t) \\
&+ \sum_{i=1}^{n}\sum_{c=1}^{r}\left(\lambda^- p_{0ic}^- + \mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r}k_{is}+1}p_{ic0} + \mu_{ic}p_{icjs}^-\left(1-u\left(k_{ic}\right)\right)\right)P(\vec{k}+I_{ic},t) \\
&+ \sum_{i,j=1}^{n}\sum_{s,c=1}^{r}\mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r}k_{is}+1}p_{icjs}^- P(\vec{k}+I_{ic}+I_{js},t) \\
&+ \sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\left(\mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r}k_{is}+1}p_{icjs}^+ u\left(k_{js}\right)\right)P(\vec{k}+I_{ic}-I_{js},t). \qquad \mathbf{(2.1)}
\end{aligned}$$

3. FINDING THE STATE PROBABILITIES OF G-NETWORK BY THE METHOD OF SUCCESSIVE APPROXIMATIONS

The DDE system (2.1) can be represented as:

$$\begin{aligned}
\frac{dP(\vec{k},t)}{dt} = &-\Lambda(\vec{k})P(\vec{k},t) + \sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P(\vec{k}+I_{ic}-I_{js},t) \\
&+ \sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P(\vec{k}+I_{ic}+I_{js},t), \qquad \mathbf{(3.1)}
\end{aligned}$$

where

$$\Lambda(\vec{k}) = \lambda^+ + \lambda^- + \sum_{i=1}^{n}\sum_{c=1}^{r}\mu_{ic}u(k_{ic}),\ \Phi_{icjs}^{++}(\vec{k}) = \mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r}k_{is}+1}p_{icjs}^-,$$

$$\begin{aligned}
\Phi_{icjs}^{+-}(\vec{k}) = &\ \delta_{0j}\delta_{s0}\left(\lambda^- p_{0ic}^- + \mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r}k_{is}+1}p_{ic0}\right) + \left(\mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r}k_{is}+1}p_{icjs}^-\left(1-u\left(k_{ic}\right)\right)\right) \\
&+ \delta_{0i}\delta_{c1}\lambda^+ p_{0js}^+ u(k_{js}) + \delta_{0i}\delta_{c0}\lambda^+ p_{0js}^+ u\left(k_{js}\right) \\
&+ \mu_{ic}\frac{k_{ic}+1}{\sum\limits_{s=1}^{r}k_{is}+1}p_{icjs}\left(k_{js}\right), \quad \delta_{ij} = \begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}.
\end{aligned}$$

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From (3.1) it follows that

$$P(\vec{k},t)=e^{-\Lambda(\vec{k})t}\left(P(\vec{k},0)+\int\limits_0^t e^{\Lambda(\vec{k})x}\left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P(\vec{k}+I_{ic}-I_{js},x)\right.\right.$$
$$\left.\left.+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P(\vec{k}+I_{ic}+I_{js},x)\right)dx\right). \quad \mathbf{(3.2)}$$

Lets $P_q(\vec{k},t)$ – an approximation $P(\vec{k},t)$ at the q-th iteration, $P_{q+1}(k,t)$ – solution of the system (3.1) obtained by the method of successive approximations. Then it follows from (3.2) that

$$P_{q+1}(\vec{k},t)=e^{-\Lambda(\vec{k})t}\left(P_q(\vec{k},0)+\int\limits_0^t e^{\Lambda(\vec{k})\,x}\left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P_q(\vec{k}+I_{ic}-I_{js},x)\right.\right.$$
$$\left.\left.+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P_q(\vec{k}+I_{ic}+I_{js},x)\right)dx\right). \quad \mathbf{(3.3)}$$

As an initial approximation, we take the stationary distribution $P_0(k,t)=P(k)=\lim\limits_{t\to\infty}P(k,t)$, which satisfies the relation

$$\Lambda(\vec{k})P(\vec{k})=\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P(\vec{k}+I_{ic}-I_{js})+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P(\vec{k}+I_{ic}+I_{js}). \quad \mathbf{(3.4)}$$

For successive approximations, the following assertions are true.

THEOREM 1: *Successive approximations $P_q(\vec{k},t), q=0,1,2,\ldots,$ converge $t\to\infty$ to a stationary solution of the system (3.1).*

THEOREM 2: *Sequence $\left\{P_q(\vec{k},t)\right\}, q=0,1,2,\ldots,$ built according to the scheme (3.4), for any bounded by t zero approximation $P_0(\vec{k},t), 0\le P_0(\vec{k},t)\le 1,$ converges $m\to\infty$ to the unique solution of the system (3.1).*

THEOREM 3: *Any approximation $P_q(\vec{k},t), q\ge 1$ can be represented as a convergent power series*

$$P_q(\vec{k},t)=\sum_{l=0}^{\infty}d_{ql}^{+-}(\vec{k})t^l, \quad \mathbf{(3.5)}$$

whose coefficients satisfy the recurrence relations:

$$d_{q+1l}^{+-}(\vec{k})=\frac{-\Lambda(\vec{k})^{\,l}}{l!}\left\{P\left(k,0\right)+\sum_{u=0}^{l-1}\frac{(-1)^{u+1}\,u!}{\Lambda(\vec{k})^{u+1}}D_{qu}^{+-}(\vec{k})\right\}, l\ge 0,$$

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$$d_{q0}^{+-}(k) = P(\vec{k},0), d_{0l}^{+}(k) = P(\vec{k},0)\delta_{l0}, \tag{3.6}$$

$$D_{ql}^{+-}(\vec{k}) = \sum_{i,j=1}^{n}\left[\sum_{s,c=1}^{r} \Phi_{icjs}^{+-}(\vec{k})d_{ql}^{+-}(\vec{k}+I_{ic}-I_{js}) + \Phi_{icjs}^{++}(\vec{k})d_{ql}^{+-}(\vec{k}+I_{ic}+I_{js})\right].$$

4. MODEL EXAMPLE

Consider a network consisting of $n = 6$ QS, in which positive and negative customers of three types arrive. Let arriving probabilities of customers to i-th QS be, respectively, equal

$$p_{0i1}^{+} = 0{,}06; p_{0i2}^{+} = 0{,}06; p_{0i3}^{+} = 0{,}08; i = \overline{2{,}5}; p_{011}^{+} = 0{,}03;$$
$$p_{012}^{+} = 0{,}03; p_{013}^{+} = 0{,}04; p_{061}^{+} = 0{,}03;$$

$p_{062}^{+} = 0{,}03$; $p_{063}^{+} = 0{,}04$; $p_{032}^{+} = p_{042}^{+} = p_{052}^{+} = 0{,}06$; $p_{033}^{+} = p_{043}^{+} = p_{053}^{+} = 0{,}04$; $p_{0i1}^{-} = p_{0i2}^{-} = p_{063}^{-} = 0{,}05; i = \overline{1{,}6}$; $p_{0i3}^{-} = 0{,}07, i = \overline{1{,}5}$; and $\sum_{i=1}^{6}\sum_{c=1}^{3} = 1$; $\sum_{i=1}^{6}\sum_{c=1}^{3} p_{0ic}^{-} = 1$. Let the arriving rates of positive and negative customers be respectively equal $\lambda^{+} = 100$ И $\lambda^{-} = 90$.

Suppose, that service rates of customers in QS are equal:

$$\mu_{11} = 50; \mu_{12} = 30; \mu_{13} = 20; \quad \mu_{21} = 50; \mu_{22} = 40; \mu_{23} = 20; \mu_{31} = 50; \mu_{32} = 40; \mu_{33} = 20;$$
$$\mu_{41} = 50; \mu_{42} = 40; \mu_{43} = 20, \mu_{51} = 50; \mu_{52} = 30; \quad \mu_{53} = 20; \mu_{61} = 45; \mu_{62} = 45; \mu_{63} = 30.$$

Let transition probabilities of positive and negative customers between QS be equal:

$$p_{i111}^{+} = 0{,}04; p_{i112}^{+} = 0{,}03; p_{i113}^{+} = 0{,}03; p_{i121}^{+} = 0{,}08;$$
$$p_{i122}^{+} = 0{,}06; p_{i123}^{+} = 0{,}06; p_{i131}^{+} = 0{,}1 p_{i132}^{+} = 0{,}05; p_{i133}^{+} = 0{,}05;$$
$$p_{i141}^{+} = 0{,}1; p_{i142}^{+} = 0{,}07; p_{i143}^{+} = 0{,}03; p_{i151}^{+} = 0{,}12;$$
$$p_{i152}^{+} = 0{,}04; p_{i153}^{+} = 0{,}04; p_{i161}^{+} = 0{,}06; p_{i162}^{+} = 0{,}02; p_{i163}^{+} = 0{,}02;$$
$$p_{i211}^{+} = 0{,}06; p_{i212}^{+} = 0{,}02; p_{i213}^{+} = 0{,}02; p_{i221}^{+} = 0{,}1;$$
$$p_{i222}^{+} = 0{,}05; p_{i223}^{+} = 0{,}05; p_{i231}^{+} = 0{,}08\, p_{i232}^{+} = 0{,}08; p_{i233}^{+} = 0{,}04;$$
$$p_{i241}^{+} = 0{,}12; p_{i242}^{+} = 0{,}04; p_{i243}^{+} = 0{,}04; p_{i251}^{+} = 0{,}12;$$
$$p_{i252}^{+} = 0{,}04; p_{i253}^{+} = 0{,}04; p_{i261}^{+} = 0{,}04; p_{i262}^{+} = 0{,}03; p_{i263}^{+} = 0{,}03;$$
$$p_{i311}^{+} = 0{,}04; p_{i312}^{+} = 0{,}04; p_{i313}^{+} = 0{,}02; p_{i321}^{+} = 0{,}1;$$
$$p_{i322}^{+} = 0{,}07; p_{i323}^{+} = 0{,}03; p_{i331}^{+} = 0{,}12\, p_{i332}^{+} = 0{,}04; p_{i333}^{+} = 0{,}04;$$
$$p_{i341}^{+} = 0{,}1; p_{i342}^{+} = 0{,}05; p_{i343}^{+} = 0{,}05; p_{i351}^{+} = 0{,}12;$$
$$p_{i352}^{+} = 0{,}04; p_{i353}^{+} = 0{,}04; p_{i361}^{+} = 0{,}04; p_{i362}^{+} = 0{,}04; p_{i363}^{+} = 0{,}02;$$

Let we need to find, for example, state probability $P(\vec{k},t), \vec{k} = (1,1,1,2,2,2,3,3,3,2,2,1,2,2,3,3,2,1,t)$, on the condition that state $(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$ was the initial. Solving the problem by using the C# programming language, at $\varepsilon = 10^{-6}$, we obtain the dependence shown in Figure 1.

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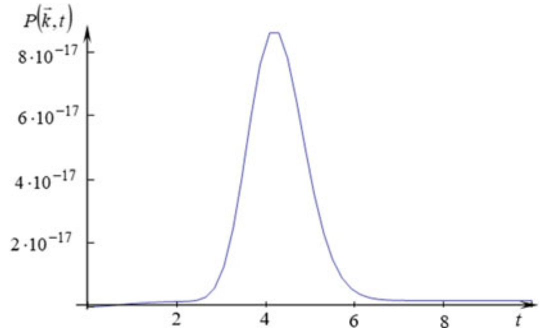

FIGURE **1.** State probability $(1,1,1,2,2,2,3,3,3,2,2,1,2,2,3,3,2,1,t)$ on the time interval $[0;8]$.

The number of terms of the series, calculated according to the formula (3.5) was found using relations $\left|d_{ql}^{+-}(k^*)\right| \le \varepsilon$, where $k^* : d_{ql}^{+-}(k^*) = \max_{\vec{k}} d_{ql}^{+-}(\vec{k})$, and number of iterations is q, using inequality $|P_{q+1}(1,1,1,2,2,2,3,3.3,2,2,1,2,2,3,3,2,1,t) - P_q(1,1,1,2,2,2,3,3.3,2,2,1,2,2,3,3,2,1,t)| \le \varepsilon$. We have obtained that the number of iterations $q^* = 70$, and the terms of the series $l^* = 63$.

5. CONCLUSIONS

In this paper, an investigation of the Markovian G-network with multiple classes of positive and negative customers has been conducted in the case when a negative customer can destroy one positive customer of its class. For this network, non-stationary state probabilities were found by the method of successive approximations, combined with the method of series.

Further studies in this direction may be related to the finding with the help of this method of nonstationary probabilities of the states of queueing networks used in solving practical problems, for example, for the network described in [9], as well as finding expected revenues in various networks with revenues and customers with multiple classes.

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APPENDIX A: PROOF OF THEOREM 1

PROOF: Spend a proof by induction. We write the expression for the first approximation

$$P_1(\vec{k},t) = e^{-\Lambda(\vec{k})\,t}\left(P_0(\vec{k},0) + \int\limits_0^t e^{\Lambda(\vec{k})x}\left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P_0(\vec{k}+I_{ic}-I_{js},x)\right.\right.$$
$$\left.\left.+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P_0(\vec{k}+I_{ic}+I_{js},x)\right)dx\right)$$
$$= e^{-\Lambda(\vec{k})\,t}\left(P_0(\vec{k},0) + \int\limits_0^t e^{\Lambda(\vec{k})x}\left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P(\vec{k}+I_{ic}-I_{js})\right.\right.$$
$$\left.\left.+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P(\vec{k}+I_{ic}+I_{js})\right)dx\right)$$
$$= e^{-\Lambda(\vec{k})\,t}\left(P_0(\vec{k},0) + \left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P(\vec{k}+I_{ic}-I_{js})\right.\right.$$
$$\left.\left.+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P(\vec{k}+I_{ic}+I_{js})\right)\int\limits_0^t e^{\Lambda(\vec{k})x}dx\right)$$
$$= e^{-\Lambda(\vec{k})\,t}\left(P_0(\vec{k},0) + \frac{1}{\Lambda(\vec{k})}\left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P(\vec{k}+I_{ic}-I_{js})\right.\right.$$
$$\left.\left.+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P(\vec{k}+I_{ic}+I_{js})\right)\left(e^{\Lambda(\vec{k})t}-1\right)\right)$$
$$\xrightarrow[t\to\infty]{}\frac{1}{\Lambda(\vec{k})}\left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P(\vec{k}+I_{ic}-I_{js})\right.$$
$$\left.+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}\left(\vec{k}\right)P(\vec{k}+I_{ic}+I_{js})\right).$$

It follows that when $q=1$ theorem holds. Suppose that the theorem is true until q-th iteration. Then from (3.3), (3.4) and L'Hospital rule, we shall have:

$$\lim_{t\to\infty}P_{q+1}(k,t) = \lim_{t\to\infty}\frac{P_q(\vec{k},0)+\int\limits_0^t e^{\Lambda(\vec{k})x}\left(\sum\limits_{i,j=1}^{n}\sum\limits_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P_q(\vec{k}+I_{ic}-I_{js},x)\right.}{e^{\Lambda(\vec{k})\,t}}$$
$$+\frac{\left.\sum\limits_{i,j=1}^{n}\sum\limits_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P_q(\vec{k}+I_{ic}+I_{js},x)\right)dx}{e^{\Lambda(\vec{k})t}}$$

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$$= \lim_{t\to\infty} \frac{e^{\Lambda(\vec{k})\,t}\left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P_q(\vec{k}+I_{ic}-I_{js},t)+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P_q(\vec{k}+I_{ic}+I_{js},t)\right)}{\Lambda(\vec{k})\,e^{\Lambda(\vec{k})\,t}}$$

$$= \lim_{t\to\infty}\frac{1}{\Lambda(\vec{k})}\left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P_q(\vec{k}+I_{ic}-I_{js},t)+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P_q(\vec{k}+I_{ic}+I_{js},t)\right)$$

$$= P(\vec{k}).$$

■

Thus the theorem is valid for $q+1$. Therefore, using the method of mathematical induction, we obtain the theorem.

APPENDIX B: PROOF OF THEOREM 2

PROOF: Because $P_0(\vec{k},t)$ limited in t function, then by virtue of (3.3) $P_1(\vec{k},t)$ is also limited, so

$$\left|P_1(\vec{k},t)-P_0(\vec{k},t)\right|\le C(\vec{k}). \quad \textbf{(B.1)}$$

We shall show that following inequality holds

$$\left|P_q(\vec{k},t)-P_{q-1}(\vec{k},t)\right|\le C^*\left(\alpha_1^*+\alpha_2^*\right)^{q-1}\frac{t^{q-1}}{(q-1)!}, \quad \textbf{(B.2)}$$

where

$$\max_{\vec{k}}\alpha_1(\vec{k})=\alpha_1^*,\ \max_{\vec{k}}\alpha_2(\vec{k})=\alpha_2^*,$$
$$\alpha_1(\vec{k})=\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\mathit{\Phi}_{icjs}^{+-}(\vec{k}),\ \alpha_2(\vec{k})=\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\mathit{\Phi}_{icjs}^{++}(\vec{k}),$$
$$\max_{\vec{k}}C(\vec{k})=C^*. \quad \textbf{(B.3)}$$

As it was shown previously, a series $\varphi_1(\vec{k})$ converges. Under $q=1$, according to (B.1) this inequality is satisfied. Assume that it is performed when $q=N$, and we will show, using (3.3), it holds when $q=N+1$. We have:

$$\left|P_{N+1}(\vec{k},t)-P_N(\vec{k},t)\right|$$
$$=\left|e^{-\Lambda(\vec{k})t}\left(P_N(\vec{k},0)+\int_0^t e^{\Lambda(\vec{k})\,x}\left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P_N(\vec{k}+I_{ic}-I_{js},x)\right.\right.\right.$$
$$\left.\left.\left|+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P_N(\vec{k}+I_{ic}+I_{js},x)\right)dx\right)\right.$$

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$$
- e^{-\Lambda(\vec{k})\,t}\left(P_{N-1}(\vec{k},0)+\int_0^t e^{\Lambda(\vec{k})\,x}\left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})P_{N-1}(\vec{k}+I_{ic}-I_{js},x)\right.\right.
$$

$$
\left.\left.+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})P_{N-1}(\vec{k}+I_{ic}+I_{js},x)\right)dx\right)\Bigg|
$$

$$
\leq\left|e^{-\Lambda(\vec{k})t}\left(\int_0^t e^{\Lambda(\vec{k})\,x}\left(\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})\left|P_N(\vec{k}+I_{ic}-I_{js},x)\right.\right.\right.\right.\ -P_{N-1}(\vec{k}+I_{ic}-I_{js},x)\Big|
$$

$$
\leq\left|e^{-\Lambda(\vec{k})\,t}\left(\int_0^t e^{\Lambda(\vec{k})x}\left(\alpha_1^*C^*\left(\alpha_1^*+\alpha_2^*\right)^{N-1}\frac{t^{N-1}}{(N-1)!}\ +\ \alpha_2^*C^*\left(\alpha_1^*+\alpha_2^*\right)^{N-1}\frac{t^{N-1}}{(N-1)!}\right)\right)\right|
$$

$$
\leq\left|e^{\,-\Lambda(\vec{k})\,t}\left(\int_0^t e^{\Lambda(\vec{k})x}C^*\left(\alpha_1^*+\alpha_2^*\right)^{N}\frac{x^{N-1}}{(N-1)!}dx\right)\right|.
$$

Because $e^{-\Lambda(\vec{k})t}e^{\Lambda(\vec{k})\,x}\leq 1$ at $x\in[0,t]$, the

$$
e^{-\Lambda(\vec{k})t}\int_0^t e^{\Lambda(\vec{k})x}\frac{x^{N-1}}{(N-1)!}dx\leq\int_0^t\frac{x^{N-1}}{(N-1)!}dx=\frac{t^N}{N!} \tag{B.4}
$$

From (B.3) follows that the inequality (B.1) occurs.

Because

$$
\lim_{q\to\infty}P_q(\vec{k},t)=\lim_{q\to\infty}\left(P_0(\vec{k},t)+\sum_{n=0}^{m-1}\left(P_{q+1}(\vec{k},t)-P_q(\vec{k},t)\right)\right)
$$

$$
=P_0(\vec{k},t)+\sum_{q=0}^{\infty}\left(P_{q+1}(\vec{k},t)-P_q(\vec{k},t)\right)
$$

$$
\leq P_0(\vec{k},t)+C^*\sum_{q=0}^{\infty}\frac{(\alpha_1^*t+\alpha_2^*t)^{\,q}}{q!}=P_0(k,t)+C^*e^{\alpha_1^*t+\alpha_2^*t},
$$

i.e. the limit of the sequence $\left\{P_q(\vec{k},t)\right\}$, $q=0,1,2,\ldots$, exists, denote it by $P_\infty(\vec{k},t)$. If we substitute $P_\infty(\vec{k},t)$ to (3.3) instead of $P(\vec{k},t)$, we see that $P_\infty(\vec{k},t)$ is a solution of equation (2.1), satisfying the initial conditions $P_\infty(\vec{k},0)=P(\vec{k},0)$ according to the conditions of the previous theorem.

Let's prove the uniqueness of the solution. Assume that there is another solution $P^*(k,t)$, then we have (3.2), if we replace in it $P(\vec{k},t),P(\vec{k},0)$, $P(\vec{k}+I_{ic}-I_{jc})$, $P(\vec{k}+I_{ic}+I_{jc})$, respectively by $P^*(\vec{k},t),P^*(\vec{k},0),P^*(\vec{k}+I_{ic}-I_{js}),P^*(\vec{k}+I_{ic}+I_{js})$. Therefore, using (3.3) we obtain:

$$
\left|P_q(\vec{k},t)-P^*(k,t)\right|\leq e^{-\Lambda(k)\,t}\left|P_q(\vec{k},t)-P^*(\vec{k},t)\right|+e^{-\Lambda(\vec{k})\,t}\int_0^t e^{\Lambda(\vec{k})x}
$$

$$
\times\int_0^t e^{\Lambda(\vec{k})x}\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})\left|P_q(\vec{k}+I_{ic}-I_{js},x)-P^*(\vec{k}+I_{ic}-I_{js},x)\right|
$$

$$
+\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})\left|P_q(\vec{k}+I_{ic}+I_{js},x)-P^*(\vec{k}+I_{ic}-I_{js},x)\right|dx.
$$

Similarly, as in the proof of inequality (B.2), we can show that $\left|P_q(\vec{k},t) - P^*(\vec{k},t)\right| \leq M(\alpha_1^* + \alpha_2^*)^q (t^q/q!)$. The right side of this inequality tends to zero as the general term of a convergent series $\sum_{q=0}^{\infty} M(\alpha_1^* + \alpha_2^*)^q \frac{t^q}{q!} = M(\vec{k})\, e^{(\alpha_1^*+\alpha_2^*)t}$, Therefore $\lim_{q\to\infty} P_q(\vec{k},t) = P^*(\vec{k},t)$. Previously we received that $\lim_{q\to\infty} P_q(\vec{k},t) = P(\vec{k},t)$, it means $P^*(\vec{k},t) = P(\vec{k},t)$, which proves the uniqueness. ■

APPENDIX C: PROOF OF THEOREM 3

Proof: Let us prove that the coefficients of the power series (3.5) satisfy the recurrence relations (3.6). We substitute successive approximations (3.5) in (3.3). Then with

$$e^{-\Lambda\left(\vec{k}\right)t} \int_0^t e^{\Lambda(\vec{k})x} x^l\, dx = \left[\frac{1}{\Lambda(\vec{k})}\right]^{l+1} l\,! \sum_{j=l+1}^{\infty} \frac{\left[-\Lambda(\vec{k})\right]^l}{j!}, l = 0,1,2,\ldots,$$

we obtain

$$\sum_{l=0}^{\infty} d_{ql}^{+-}(\vec{k})t^l = e^{-\Lambda(\vec{k})\,t} P(\vec{k},0) + \sum_{l=0}^{\infty}\sum_{i,j=1}^{n}\left[\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})\, d_{ql}^{+-}(\vec{k}+I_{ic}-I_{js}) + \Phi_{icjs}^{++}(\vec{k}) d_{ql}^{+-}(\vec{k}+I_{ic}+I_{js})\right].$$

Using the notation (3.6), this series can be written as

$$\sum_{l=0}^{\infty} d_{ql}^{+-}(\vec{k})t^l = e^{-\Lambda(\vec{k})\,t} P(\vec{k},0) + \sum_{l=0}^{\infty} D_{ql}^{+-}(\vec{k})\left[\frac{1}{\Lambda(\vec{k})}\right]^{l+1} l! \sum_{u=l+1}^{\infty}\frac{\left[-\Lambda(\vec{k})\right]^u}{u!}t^u.$$

Interchanging summation indices and expanding $e^{-\Lambda(\vec{k})t}$ to series in powers t, we get

$$\sum_{l=0}^{\infty} d_{ql}^{+-}(\vec{k})t^l = \sum_{l=0}^{\infty}\frac{\left[-\Lambda(\vec{k})\right]^l}{l!}\left\{P(\vec{k},0) + \sum_{u=0}^{l-1}\frac{(-1)^{u+1}\,u!}{\left[\Lambda(\vec{k})\right]^{u+1}} D_{qu}^{+-}(\vec{k})\right\} t^l. \quad \textbf{(C.1)}$$

Equating the left and right side of the expression (C.1) the coefficients of t^l, we obtain (3.6) for the coefficients of the series (3.5).

To find the radius of convergence of the power series (3.11) we use the Cauchy-Hadamard formula $(1/(R(\vec{k})) = \lim_{l\to\infty}\sqrt[l]{\left|d_{ql}(\vec{k})\right|}$.

From (C.1) it follows that $\left|d_{q+1l}^{+-}(\vec{k})\right| = \frac{\Lambda(\vec{k})^{\,l}}{l!}\left|P(\vec{k},0) + \sum_{u=0}^{l-1}\frac{(-1)^{u+1}u!}{\Lambda(\vec{k})^{u+1}} D_{qu}^{+-}(\vec{k})\right|$, $l \geq\ 0$.

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We shall show that $\left|D_{qu}^{+-}(\vec{k})\right|$, $q \geq 1$, $u = \overline{0, l-1}$, limited to a finite value $C_1(\vec{k})$. From the boundedness of $P(\vec{k}, 0)$ and determination of $D_{qu}^{+-}(\vec{k})$ follows

$$D_{00}^{+-}(\vec{k}) = \sum_{i,j=1}^{n} \left[\sum_{c,s=1}^{r} \varPhi_{icjs}^{+-}(\vec{k}) d_{00}^{+-}\left(k + I_{ic} - I_{js}, t\right) + \varPhi_{icjs}^{++}(\vec{k}) d_{00}^{+-}\left(k + I_{ic} + I_{js}, t\right) \right],$$
$$\leq C_{00}^{+-}(\vec{k}),$$

where $C_{00}^{+-}(\vec{k})$ – some bounded value, and all $D_{0l}^{+-}(\vec{k}) = 0$, $l = 1, 2, \ldots$ Because the $D_{q-10}^{+-}(\vec{k}) = D_{q-20}^{+-}(\vec{k}) = \ldots = D_{10}^{+-}(\vec{k}) = D_{00}^{+-}(\vec{k})$ then from (3.6) follows $D_{q-10}^{+-}(\vec{k}) < C_{00}^{+-}(\vec{k}), q \geq 1$.

By induction, we show that

$$\left|D_{q-1l}^{+-}(\vec{k})\right| \leq \frac{C_{q-1l}^{+-}(\vec{k})}{l!}, l = 1, 2, \ldots \tag{C.2}$$

For $l = 1$ we have:

$$D_{q-11}^{+-}(k) = \sum_{i,j=1}^{n} \left[\sum_{c,s=1}^{r} \varPhi_{icjs}^{+-}(\vec{k})\, d_{q-11}^{+-}(\vec{k} + I_{ic} - I_{js}) + \varPhi_{icjs}^{++}(\vec{k})\, d_{q-11}^{+-}(\vec{k} + I_{ic} + I_{js}) \right]$$
$$= \sum_{i,j=1}^{n} \left[\sum_{c,s=1}^{r} \varPhi_{icjs}^{+-}(\vec{k})\, (-\Lambda(\vec{k} + I_{ic} - I_{js})\, P(\vec{k} + I_{ic} - I_{js}, 0) \right.$$
$$+ D_{q-10}^{+-}(\vec{k} + I_{ic} - I_{js})) + \sum_{i,j=1}^{n} \sum_{c,s=1}^{r} \varPhi_{icjs}^{++}(\vec{k})$$
$$\times (-\Lambda(\vec{k} + I_{ic} + I_{js})\, P(\vec{k} + I_{ic} + I_{js}, 0) + D_{q-10}^{+-}(\vec{k} + I_{ic} + I_{js}))$$
$$\leq \frac{C_{q-11}^{+-}(\vec{k})}{1!},$$

where $C_{q-11}^{+-}(\vec{k})$ – a certain bounded value. Suppose that (C.2) holds for l-1, i.e.

$$\left|D_{q-1l-1}^{+-}(\vec{k})\right| \leq \frac{C_{q-1l-1}^{+-}(\vec{k})}{(l-1)\,!}, l = 1, 2, \ldots \tag{C.3}$$

Let us prove the validity of inequality (C.2) for l. Using (C.1), we get

$$D_{q-1l}^{+-}(\vec{k}) = \sum_{i,j=1}^{n} \left[\sum_{c,s=1}^{r} \varPhi_{icjs}^{+-}(\vec{k})\, d_{q-1l}^{+-}(\vec{k} + I_{ic} - I_{jc}) + \varPhi_{icjs}^{++}(\vec{k})\, d_{q-1l}^{+-}(\vec{k} + I_{ic} + I_{jc}) \right]$$
$$= \sum_{i,j=1}^{n} \left[\sum_{c,s=1}^{r} \varPhi_{icjs}^{+-}(\vec{k}) \times \frac{\left[-\Lambda(\vec{k} + I_{ic} - I_{js})\right]^{l}}{l!} \right.$$
$$\times \left(P(\vec{k}, 0) + \sum_{u=0}^{l-1} \frac{u!\,(-1)^{u+1}}{\left[\Lambda(\vec{k} + I_{ic} - I_{js})\right]^{u+1}} D_{q-1l-1}^{+-}(\vec{k} + I_{ic} - I_{js}) \right)$$

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$$+\sum_{i,j=1}^{n}\left[\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})\frac{\left[-\Lambda(\vec{k}+I_{ic}+I_{js})\right]^{l}}{l!}\right.$$
$$\left.\times\left(P(\vec{k}+I_{ic}+I_{js},0)+\sum_{u=0}^{l-1}\frac{u!\,(-1)^{u+1}}{\left[\Lambda(\vec{k}+I_{ic}+I_{js})\right]^{u+1}}D_{q-1l-1}^{+-}(\vec{k}+I_{ic}+I_{js})\right)\right].$$

Let's $C_1(\vec{k})=\max_{q,l}C_{ql}^{+-}(\vec{k})$, then

$$D_{q-1l}^{+-}(\vec{k})\le\sum_{i,j=1}^{n}\left[\sum_{c,s=1}^{r}\Phi_{icjs}^{+-}(\vec{k})\times\frac{\left[-\Lambda(\vec{k}+I_{ic}-I_{js})\right]^{l}}{l!}\right.$$
$$\left.\times\left(P(\vec{k},0)+\sum_{u=0}^{l-1}\frac{u!}{\left[\Lambda(\vec{k}+I_{ic}-I_{js})\right]^{u+1}}\frac{C_1(\vec{k}+I_{ic}-I_{js})}{u!}\right)\right]$$
$$+\sum_{i,j=1}^{n}\left[\sum_{c,s=1}^{r}\Phi_{icjs}^{++}(\vec{k})\frac{\left[-\Lambda(\vec{k}+I_{ic}+I_{js})\right]^{l}}{l!}\right.$$
$$\left.\times\left(P\left(\vec{k}+I_{ic}+I_{js},0\right)+\sum_{u=0}^{l-1}\frac{u!}{\left[\Lambda(\vec{k}+I_{ic}+I_{js})\right]^{u+1}}\frac{C_1(\vec{k}+I_{ic}+I_{js})}{u!}\right)\right]$$
$$\le\frac{1}{l!}\sum_{i,j=1}^{n}\left[\sum_{c,s=1}^{r}\sum_{\eta,\nu=0}^{1}\Phi_{icjs}(\vec{k})\left[-\Lambda(\vec{k})\right]^{l}\left(P(\vec{k},0)+C_1(\vec{k}+I_{ic}-I_{js})\sum_{u=0}^{l-1}\frac{1}{\left[\Lambda(\vec{k})\right]^{u+1}}\right)\right]$$
$$\le\frac{C_{q-1l}^{+-}(\vec{k})}{l!},$$

where $C_{q-1l}^{+-}(\vec{k})$ – the expression in the curly brackets, i.e., inequality (C.3) holds.

Consider the expression

$$\frac{1}{R(\vec{k})}=\lim_{l\to\infty}\sqrt[l]{\frac{\Lambda(\vec{k})^{l}}{l!}\left|P(\vec{k},0)+\sum_{u=0}^{l-1}\frac{(-1)^{u+1}u!}{\Lambda(\vec{k})^{u+1}}D_{qu}^{+-}(\vec{k})\right|}$$
$$\le\Lambda(\vec{k})\lim_{l\to\infty}\sqrt[l]{\frac{1}{l!}\left|P(\vec{k},0)+\sum_{u=0}^{l-1}\frac{(-1)^{u+1}u!}{\Lambda(\vec{k})^{u+1}}\frac{C_{q-1u}^{+-}(\vec{k})}{u!}\right|}$$
$$\le\Lambda(\vec{k})\lim_{l\to\infty}\sqrt[l]{\frac{1}{l!}\left|P(\vec{k},0)+C_1(\vec{k})\sum_{u=0}^{l-1}\frac{(-1)^{u+1}}{\Lambda(\vec{k})^{u+1}}\right|}$$
$$\le\Lambda(\vec{k})\lim_{l\to\infty}\sqrt[l]{\frac{1}{(l-1)!}}\lim_{l\to\infty}\sqrt[l]{\left(\frac{P(\vec{k},0)}{l}+\frac{C_1(\vec{k})}{l}\sum_{u=0}^{l-1}\frac{1}{\Lambda(\vec{k})^{u+1}}\right)}.\qquad\textbf{(C.4)}$$

Because the

$$S_{l-1}(\vec{k})=\sum_{u=0}^{l-1}\frac{1}{\left[\Lambda(\vec{k})\right]^{u+1}}=\begin{cases}\dfrac{1-\left[\Lambda(\vec{k})\right]^{-l}}{(\Lambda(\vec{k})-1)},\ \Lambda(\vec{k})\ne1,\\ l,\Lambda(\vec{k})=1\end{cases}$$

To

$$\lim_{l\to\infty}\frac{S_{l-1}}{l}=\begin{cases}0,\ \Lambda(\vec{k})>1\\ 1,\Lambda(\vec{k})=1\end{cases}. \tag{C.5}$$

It was shown in [6] that $\lim\limits_{l\to\infty}\sqrt[l]{\frac{1}{(l-1)!}}=0$.

Under $0<\Lambda(\vec{k})<1$, using (C.5) on the right side (C.4), we have: $\sum\limits_{u=0}^{l-1}\frac{1}{\left[\Lambda(\vec{k})\right]^{u+1}}=\frac{1-\left[\Lambda(\vec{k})\right]^{-l}}{\left(\Lambda(\vec{k})-1\right)}=\frac{a^l-1}{1-a}$, where $a=(1/\Lambda(\vec{k}))>1$, therefore,

$$\lim_{l\to\infty}\sqrt[l]{\frac{1}{l!}}\lim_{l\to\infty}\sqrt[l]{P(\vec{k},0)+C_1(\vec{k})\sum_{u=0}^{l-1}\frac{1}{\left[\Lambda(\vec{k})\right]^{u+1}}}=\lim_{l\to\infty}\sqrt[l]{\frac{1}{l!}}\lim_{l\to\infty}\sqrt[l]{P(\vec{k},0)+\frac{C_1(\vec{k})}{1-\Lambda(\vec{k})}\left(a^l-1\right)},$$

But $\sqrt[l]{b+ca^l}\leq\sqrt[l]{b}+\sqrt[l]{ca^l}=\sqrt[l]{b}+a\sqrt[l]{c}$, therefore $\lim\limits_{l\to\infty}\sqrt[l]{b+ca^l}\leq\lim\limits_{l\to\infty}b^{\frac{1}{l}}+a\lim\limits_{l\to\infty}c^{\frac{1}{l}}=1+a$ – a bounded quantity. Then, considering that $\lim\limits_{l\to\infty}\sqrt[l]{\frac{1}{l!}}=0$, we obtain that $(1/R(\vec{k}))=0$.

Therefore, the radius of convergence of the power series (3.5) is equal to $+\infty$. ■

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