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SPANNING TREES IN $\mathbb{Z}$-COVERS OF A FINITE GRAPH AND MAHLER MEASURES

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Abstract

Using the special value at $u = 1$ of Artin–Ihara L-functions, we associate to every $\mathbb{Z}$-cover of a finite connected graph a polynomial, which we call the *Ihara polynomial*. We show that the number of spanning trees for the finite intermediate graphs of such a cover can be expressed in terms of the Pierce–Lehmer sequence associated to a factor of the Ihara polynomial. This allows us to express the asymptotic growth of the number of spanning trees in terms of the Mahler measure of this polynomial. Specialising to the situation where the base graph is a bouquet or the dumbbell graph gives us back previous results in the literature for circulant and I-graphs (including the generalised Petersen graphs). We also express the p-adic valuation of the number of spanning trees of the finite intermediate graphs in terms of the p-adic Mahler measure of the Ihara polynomial. When applied to a particular $\mathbb{Z}$-cover, our result gives us back Lengyel's calculation of the p-adic valuations of Fibonacci numbers.

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1. Introduction

The aim of the present paper is to explain how the number of spanning trees in a $\mathbb{Z}$-cover of finite graphs evolves, by providing an explicit recipe to compute the invariants that describe this evolution in terms of a polynomial that can be associated to the cover in question.

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1.1. Historical remarks. Before describing in detail the main results of this paper, let us provide an overview of the main questions that motivated the present paper.

Iwasawa theory is concerned with the study of the evolution of certain invariants within a tower of objects (see [18] for a comprehensive survey). The first example of this is provided by the evolution, as $n \to +\infty$, of the group of $\mathbb{F}_n$-rational points of the Jacobian of a curve defined over a finite field $\mathbb{F}$, where $\mathbb{F}_n \supseteq \mathbb{F}$ is the unique extension (up to isomorphism) of $\mathbb{F}$ having degree n. This example was studied by Weil and led him to formulate his celebrated conjectures concerning the properties of the zeta functions associated to varieties defined over a finite field. Iwasawa pursued analogous investigations concerning the evolution of class groups of number fields in a tower of cyclotomic extensions, which are akin to the extensions of a function field that are obtained by increasing the field of constants, as explained in [47, page 188]. This initiated a large series of works that study the evolution of different invariants along a tower of number fields whose Galois group is a p-adic Lie group (see [25] for a survey). Moreover, Iwasawa theory has been extended to the study of the evolution of invariants of many different arithmetic objects, such as elliptic curves or even general motives (see [14] for one of the most general frameworks available at present).

In a somehow different direction, ideas from Iwasawa theory have found applications also outside number theory and algebraic geometry. More precisely, the torsion subgroups of the first homology groups of a tower of hyperbolic 3-manifolds whose base is the complement of a knot or a link have been shown to evolve according to a pattern that is very similar to the one appearing in Iwasawa theory (see [21, 24, 55]). Considering hyperbolic manifolds allows one to study towers whose groups of deck transformations are not necessarily profinite, which is not possible when one studies towers of number fields. For instance, one can consider a $\mathbb{Z}$-cover of hyperbolic manifolds. In this case, when the base of the tower consists of the complement of a knot in the three-dimensional sphere, the Alexander polynomial of the knot in question can be used to describe explicitly the growth of the torsion inside the first homology groups of the manifolds in question, as proven by Ueki [56] in the p-adic case, and by González-Acuña and Short [17] and Riley [46] in the Archimedean case. These results are particularly interesting in view of the widely explored analogy between number fields and knots (see [43] for a survey).

Finally, an analogue of Iwasawa theory has also been developed to study the evolution of the so-called Picard group of degree zero of a finite connected graph X, as one moves along a tower. This finite group, defined for instance in [6, Section 1.3], is analogous to the class group of a number field, or to the Picard group of degree zero of a curve defined over a finite field. Its cardinality, usually denoted by $\kappa(X)$, is given by the number of spanning trees of the graph in question. The evolution of this number when the finite graph in question varies along a tower has been the subject of a series of papers written by several authors in collaboration with the second author of the present paper [9, 35, 39, 40, 57]. More precisely, if $\ell \in \mathbb{N}$ is a rational prime and

$$\cdots \to X_{\ell^n} \to \cdots \to X_\ell \to X_1 = X \tag{1-1}$$

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is a tower of finite graphs, such that each X_{ℓ^n}/X is a Galois cover with Galois group $\mathbb{Z}/\ell^n\mathbb{Z}$, it was shown in [39, 40, 57] that there exist nonnegative integers μ_ℓ, λ_ℓ and an integer ν_ℓ such that

$$\mathrm{ord}_\ell(\kappa(X_{\ell^n})) = \mu_\ell \cdot \ell^n + \lambda_\ell \cdot n + \nu_\ell \tag{1-2}$$

for n large enough, where ord_ℓ denotes the usual ℓ-adic valuation on $\mathbb{Q}$. Moreover, it was shown in [35] that if p is another rational prime different than ℓ, then there exist a nonnegative integer μ_p and an integer ν_p such that

$$\mathrm{ord}_p(\kappa(X_{\ell^n})) = \mu_p \cdot \ell^n + \nu_p \tag{1-3}$$

for n large enough. Furthermore, given an integer $d \geq 1$, and a tower of finite graphs

$$\cdots \to X_{\ell^n}^{(d)} \to \cdots \to X_\ell^{(d)} \to X_1 = X,$$

such that each $X_{\ell^n}^{(d)}/X$ is a Galois cover with Galois group $(\mathbb{Z}/\ell^n\mathbb{Z})^d$, it was shown in [9] that there exists a polynomial $P \in \mathbb{Q}[t_1, t_2]$ of total degree at most d, and linear in t_2, such that

$$\mathrm{ord}_\ell(\kappa(X_{\ell^n}^{(d)})) = P(\ell^n, n) \tag{1-4}$$

for every n that is large enough. These advances in the Iwasawa theory of finite graphs can be seen as being analogous to more classical theorems and conjectures in the Iwasawa theory of number fields. More precisely, (1-2) is analogous to a classical theorem of Iwasawa [22] for $\mathbb{Z}_\ell$-extensions of number fields, whereas (1-3) is analogous to a result of Washington for the cyclotomic $\mathbb{Z}_\ell$-extension of an abelian number field, proved in [58], and (1-4) is akin to a conjecture of Greenberg, which is discussed by Cuoco and Monsky in [7, Section 7].

The results (1-2) and (1-4) were originally proven by working on the 'analytic side' of Iwasawa theory, that is, by constructing appropriate elements of the Iwasawa algebra

$$\mathbb{Z}_\ell[\![\mathbb{Z}_\ell^d]\!] \cong \mathbb{Z}_\ell[\![T_1, \ldots, T_d]\!].$$

However, Gonet [15, 16] reproved (1-2) using a module theoretical approach, which was recently shown to be closely related to the analytic approach in the work of Kleine and Müller [27]. More precisely, this work proves an analogue of the Iwasawa main conjecture in the setting of graphs, which allows Kleine and Müller to prove (1-4) in an algebraic way. Moreover, Kataoka's recent work [26] studies the Fitting ideals that appear in this setting, and Kleine and Müller's more recent work [28] shows how to adapt some of these ideas to the nonabelian setting.

To conclude this overview, let us mention that the recent work of Lei and Müller [33, 34] shows how one can obtain natural towers of finite graphs by looking at the isogeny graphs associated to elliptic curves defined over a finite field $\mathbb{F}$. More precisely, in [33], the authors consider ℓ-isogeny graphs $\tilde{G}_N^m$ of ordinary elliptic curves with a $\Gamma_1(Np^m)$-level structure, where p is the characteristic of $\mathbb{F}$, while ℓ is a prime different from p and N is a fixed integer coprime to p. In particular, they fix an ordinary

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elliptic curve E defined over $\mathbb{F}$, which also admits a nontrivial ℓ-isogeny defined over $\mathbb{F}$, and they prove that there exists an integer m_0 such that the connected components $(\tilde{\mathcal{G}}_N^m)_{m=m_0}^{+\infty}$ of the graphs $(\tilde{G}_N^m)_{m=m_0}^{+\infty}$ that contain a vertex corresponding to E give rise to a $\mathbb{Z}_p$-tower. These graphs generalise the celebrated isogeny volcanoes, which are vastly used in cryptography, and have been classified in recent work of Bambury *et al.* [1]. In a subsequent paper [34], Lei and Müller considered ℓ-isogeny graphs of elliptic curves with full $\Gamma(Np^n)$-level structures, and they showed that their ordinary connected components do not give rise to Galois covers, while their supersingular ones do, at least when $N \leq 2$ and for a positive proportion of primes p. In this case, the resulting tower has a nonabelian Galois group, isomorphic to $\mathrm{GL}_2(\mathbb{Z}_p)$, which therefore fits into the framework developed by Kleine and Müller in [28].

1.2. Main results. Inspired by the results mentioned in the previous section, we show in the present paper how the invariants appearing in (1-2) and (1-3) can be explicitly computed when (1-1) comes from a $\mathbb{Z}$-cover of finite graphs. More precisely, every Galois cover of graphs Y/X with Galois group G can be constructed from a voltage assignment, which is a function $\alpha \colon \mathbf{E}_X \to G$ such that $\alpha(\overline{e}) = \alpha(e)^{-1}$ for every $e \in \mathbf{E}_X$, where $\mathbf{E}_X$ denotes the set of directed edges of X, and $\overline{e}$ denotes the inverse of an edge (see Section 3.1 for further details). Indeed, if G is an arbitrary group and $\alpha \colon \mathbf{E}_X \to G$ is a voltage assignment, one can construct a graph $X(G, \alpha)$, introduced by Gross in [19], which generalises the usual notion of a Cayley graph, and is endowed with a canonical map $X(G, \alpha) \to X$, which is a Galois cover if and only if $X(G, \alpha)$ is connected, in which case, $\mathrm{Gal}(X(G, \alpha)/X) \cong G$. Moreover, if Y/X is a Galois cover of finite graphs, with Galois group G, there exists a voltage assignment $\alpha \colon \mathbf{E}_X \to G$ and an isomorphism of covers $Y/X \cong X(G, \alpha)/X$, as outlined in [9, Section 3].

Now, let G be an arbitrary group and $\alpha \colon \mathbf{E}_X \to G$ be a voltage assignment. Then, for every normal subgroup $H \trianglelefteq G$ that has finite index, one has a finite graph $X_H := X(G/H, \alpha_H)$, where $\alpha_H \colon \mathbf{E}_X \to G/H$ denotes the voltage assignment obtained by composing α with the natural projection map $\pi \colon G \twoheadrightarrow G/H$. If each of the finite graphs X_H is connected, then it is a Galois cover of X, whose Galois group is canonically isomorphic to G/H. In this setting, one of the main goals, which is related to the results mentioned above, is to describe how the number of spanning trees $\kappa(X_H)$ depends on H. When $G = \mathbb{Z}_\ell^d$ for some $d \geq 1$, this is the content of the results that we recalled in the previous section, that lead to (1-2), (1-3) and (1-4).

As we mentioned above, in this paper, we focus on the case $G = \mathbb{Z}$, and we provide a global analogue of the results obtained in (1-2) and (1-3). More precisely, for every finite graph X and every voltage assignment $\alpha \colon \mathbf{E}_X \to \mathbb{Z}$ such that each finite graph $X_n := X(\mathbb{Z}/n\mathbb{Z}, \alpha_n)$ is connected, where $\alpha_n := \alpha_{n\mathbb{Z}}$, we show in Theorem 3.6 that the number of spanning trees $\kappa(X_n)$ of the graph X_n is intimately related to the Pierce–Lehmer sequence $\{\Delta_n(J_\alpha)\}_{n=1}^{+\infty}$ associated to a factor $J_\alpha \in \mathbb{Z}[t]$ of the *Ihara polynomial* $\mathcal{I}_\alpha \in \mathbb{Z}[t^{\pm 1}]$, which is a Laurent polynomial that can be explicitly constructed from the voltage assignment α, as we explain in Section 3.3. The Archimedean and p-adic absolute values of the aforementioned Pierce–Lehmer sequence, introduced by

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Pierce [44] and Lehmer [32], turn out to be related to the Archimedean and p-adic Mahler measures of the polynomial $\mathcal{I}_\alpha$, as we explain in Section 2.3. In particular, these Mahler measures provide the main term that explains the order of growth of the different absolute values of the Pierce–Lehmer sequence $\{\Delta_n(J_\alpha)\}_{n=1}^{+\infty}$. This suffices to describe the asymptotic behaviour of the Archimedean (respectively p-adic) absolute value of the number of spanning trees $\kappa(X_n)$ whenever no root of $\mathcal{I}_\alpha$ lies on the unit circle of $\mathbb{C}$ (respectively $\mathbb{C}_p$), as we explain in Corollaries 3.10 and 3.15. In particular, we show in Examples 3.13 and 3.14 that our Archimedean result generalises previous work of Mednykh and Mednykh [41, 42].

One may of course wonder about the behaviour of the Archimedean or p-adic absolute value of $\kappa(X_n)$ when $\mathcal{I}_\alpha$ has some of its roots on the Archimedean (or p-adic) unit circle. In fact, this question is central to understanding the behaviour of the sequence $\kappa(X_n)$, as for almost every prime p, all the roots of $\mathcal{I}_\alpha$ will lie on the p-adic unit circle. In the case of the Archimedean absolute value, one can only get some upper and lower bounds for $\kappa(X_n)$, but not an exact asymptotic, as follows from Weyl's equidistribution theorem (see Remark 3.12). In the p-adic case, to understand the absolute value of $\kappa(X_n)$, one needs to take into account a correcting factor, which is described in Theorem 2.3. Doing so, we arrive at the following result, which we now present in a simplified version. For the precise formulation, see Theorem 3.17.

THEOREM 1.1. *Let X be a finite connected graph whose Euler characteristic $\chi(X)$ does not vanish, and $\alpha\colon \mathbf{E}_X \to \mathbb{Z}$ be a voltage assignment such that for every $n \geq 1$, the finite graph $X_n := X(\mathbb{Z}/n\mathbb{Z}, \alpha_n)$ is connected (which can be checked using Theorem 3.2). Let $\mathcal{I}_\alpha \in \mathbb{Z}[t^{\pm 1}]$ be the Ihara polynomial associated to α, and set*

$$J_\alpha(t) := t^b (t-1)^{-e} \mathcal{I}_\alpha(t),$$

where $b := -\mathrm{ord}_{t=0}(\mathcal{I}_\alpha)$, and $e := \mathrm{ord}_{t=1}(\mathcal{I}_\alpha)$. Fix a rational prime $p \in \mathbb{N}$ and let

$$\mu_p(X,\alpha) = -m_p(J_\alpha)/\log(p),$$

where $m_p(J_\alpha)$ denotes the logarithmic p-adic Mahler measure of J_α, defined as in (2-3). Then, there exist two explicit functions

$$\begin{aligned} \mathbb{N} &\to \mathbb{Z}_{\geq 0} \\ n &\mapsto \lambda_{p,n}(X,\alpha) \end{aligned} \quad \text{and} \quad \begin{aligned} \mathbb{N} &\to \mathbb{Z} \\ n &\mapsto \nu_{p,n}(X,\alpha) \end{aligned}$$

whose images are finite, and an integer $c_p(X,\alpha)$, such that

$$\mathrm{ord}_p(\kappa(X_n)) = \mu_p(X,\alpha)\cdot n + \lambda_{p,n}(X,\alpha)\cdot \mathrm{ord}_p(n) + \nu_{p,n}(X,\alpha) + c_p(X,\alpha) \tag{1-5}$$

for all $n \in \mathbb{N}$.

Specialising (1-5) to the subsequence $\{\kappa(X_{p^k})\}_{k=1}^{\infty}$ gives

$$\mathrm{ord}_p(\kappa(X_{p^k})) = \mu_p(X,\alpha)\cdot p^k + \lambda_{p,p^k}(X,\alpha)\cdot k + \nu_{p,p^k}(X,\alpha) + c_p(X,\alpha)$$

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and specialising to the subsequence $\{\kappa(X_{\ell^k})\}_{k=1}^{\infty}$, where ℓ is another rational prime different from p, gives

$$\mathrm{ord}_p(\kappa(X_{\ell^k})) = \mu_p(X,\alpha)\cdot \ell^k + \nu_{p,\ell^k}(X,\alpha) + c_p(X,\alpha).$$

After studying the dependency on k of the constants $\lambda_{p,p^k}(X,\alpha), \nu_{p,p^k}(X,\alpha)$ and $\nu_{p,\ell^k}(X,\alpha)$, one gets back (1-2) and (1-3), as we explain in Remark 3.8. Moreover, we obtain similar results by specialising Theorem 1.1 to sequences of integers divisible only by a finite number of primes, as we explain in Corollary 2.10. These identities can be seen as analogous to a result proven by Friedman [13] for cyclotomic $\mathbb{Z}_{p_1}\times\cdots\times\mathbb{Z}_{p_s}$-extensions of number fields that are abelian over $\mathbb{Q}$.

To conclude, we show in Section 3.7 that Theorem 1.1 allows one to recover a well-known formula that computes the p-adic valuations of Fibonacci numbers, which is due to Lengyel [36].

1.3. Notation and conventions. Let p be a rational prime. We let $\mathbb{C}_p$ denote a fixed completion of an algebraic closure of the p-adic rational numbers $\mathbb{Q}_p$. As usual, $|\cdot|_p$ and ord_p denote the p-adic absolute value and the p-adic valuation on $\mathbb{C}_p$, respectively. They are related via

$$\mathrm{ord}_p(x) = -\frac{\log |x|_p}{\log p}$$

for all $x \in \mathbb{C}_p$, and they are normalised so that $\mathrm{ord}_p(p) = 1$. We also denote by $\mathbb{C}$ the field of complex numbers, endowed with the usual Archimedean absolute value $|\cdot|_\infty$.

If G is an abelian group, not necessarily finite, we let $G^\vee = \mathrm{Hom}_{\mathbb{Z}}(G, W_\infty)$, where W_∞ denotes the group of roots of unity in an algebraic closure $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ of $\mathbb{Q}$. An element of $G^\vee$ will be called a character of finite order. Here, we depart from the usual notation, since $G^\vee$ is not necessarily the Pontryagin dual of G. For each rational prime p, we fix once and for all an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Via these embeddings, we view the characters in $G^\vee$ as taking values in $\mathbb{C}_p$ once a rational prime p has been fixed. If n is a positive integer, then we let W_n denote the group of n th roots of unity. The symbol $\mathbb{N} = \{1, 2, \ldots\}$ refers to the collection of all positive integers.

2. Mahler measures and Pierce–Lehmer sequences

In this section, we remind the reader about the resultant of two polynomials, which appears in Section 2.1, and about the p-adic and Archimedean Mahler measures of polynomials, which we treat in Section 2.2. Moreover, we devote Section 2.3 to collecting some results about Pierce–Lehmer sequences. In particular, Theorems 2.2 and 2.3 provide explicit formulae to compute the p-adic valuations of Pierce–Lehmer sequences.

2.1. Resultant. Let F be a field and let

$$p(t) = a_m t^m + \cdots + a_0 = a_m \prod_{i=1}^{m} (t-\alpha_i)$$

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and

$$q(t) = b_n t^n + \cdots + b_0 = b_n \prod_{j=1}^{n} (t - \beta_i)$$

be two polynomials in $F[t]$ of degrees m and n, respectively. Here, the roots α_i and β_i are assumed to be in a fixed algebraic closure of F. The resultant $\mathrm{Res}(p, q)$ of p and q is defined to be

$$\mathrm{Res}(p, q) = a_m^n b_n^m \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j) \tag{2-1}$$

and is easily seen to be an element of F. Let $r(t)$ be another polynomial with coefficients in F. From the definition in (2-1), the two properties

$$\mathrm{Res}(p, q) = (-1)^{mn} \mathrm{Res}(q, p) \quad \text{and} \quad \mathrm{Res}(p \cdot r, q) = \mathrm{Res}(p, q) \cdot \mathrm{Res}(r, q)$$

follow immediately. Furthermore, one has

$$a_m^n \prod_{i=1}^{m} q(\alpha_i) = \mathrm{Res}(p, q) = (-1)^{mn} b_n^m \prod_{j=1}^{n} p(\beta_j),$$

which can be seen as an instance of Weil's reciprocity law for the projective line over F. Finally, the resultant can also be defined as the determinant of the Sylvester matrix of p and q, as shown for instance in [5, Lemma 3.3.4]. This allows one to define the resultant $\mathrm{Res}(f, g) \in R$ of any pair of polynomials $f, g \in R[t]$ that have coefficients in an arbitrary commutative ring with unity R.

2.2. Mahler measure. Recall that if

$$f(t) = a_d t^d + \cdots + a_0 \in \mathbb{C}[t]$$

is a nonzero polynomial of degree d, which can be factorised as

$$f(t) = a_d \prod_{i=1}^{d} (t - \alpha_i)$$

for some $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$, then one defines its Archimedean Mahler measure to be

$$M_\infty(f) := |a_d|_\infty \prod_{i=1}^{d} \max\{1, |\alpha_i|_\infty\} \in \mathbb{R}_{>0}. \tag{2-2}$$

This invariant, originally studied by Lehmer [32], was generalised by Mahler [38] to polynomials with any number of variables.

Now, let p be a rational prime and let

$$g(t) = b_d t^d + \cdots + b_0 \in \mathbb{C}_p[t]$$

be a nonzero polynomial of degree d, which factors as

$$g(t) = b_d \prod_{i=1}^{d} (t - \beta_i)$$

for some $\beta_1, \ldots, \beta_d \in \mathbb{C}_p$. Following [56], we define similarly the p-adic Mahler measure of $g(t)$ to be

$$M_p(g) := |b_d|_p \prod_{i=1}^{d} \max\{1, |\beta_i|_p\} \in \mathbb{R}_{>0}.$$

This invariant and its Archimedean analogue are clearly multiplicative. Furthermore, the p-adic Mahler measure of a polynomial

$$g(t) = \sum_{i=0}^{d} b_i t^i \in \mathbb{C}_p[t]$$

can be easily computed from its coefficients, thanks to the formula

$$M_p(g) = \max\{|b_i|_p \mid i = 0, \ldots, d\},$$

which was proved by Ueki in [56, Proposition 2.7]. Finally, we introduce the logarithmic Archimedean Mahler measure

$$m_\infty(f) := \log(M_\infty(f))$$

of a polynomial $f(t) \in \mathbb{C}[t]$, and analogously the logarithmic p-adic Mahler measure

$$m_p(g) := \log(M_p(g)) \tag{2-3}$$

of a polynomial $g(t) \in \mathbb{C}_p[t]$.

REMARK 2.1. We note in passing that the logarithmic p-adic Mahler measure introduced in (2-3) does not coincide with the p-adic logarithmic Mahler measure introduced by Besser and Deninger in [3], which is a p-adic number.

2.3. Pierce–Lehmer sequences.

Let

$$f(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_0 \in \mathbb{Z}[t],$$

with $a_d \neq 0$ and write

$$f(t) = a_d \prod_{i=1}^{d} (t - \alpha_i)$$

for some $\alpha_1, \ldots, \alpha_d \in \overline{\mathbb{Q}}$. The associated Pierce–Lehmer sequence is defined to be

$$\Delta_n(f) = a_d^n \prod_{i=1}^{d} (\alpha_i^n - 1) = \operatorname{Res}(f(t), t^n - 1) \in \mathbb{Z}. \tag{2-4}$$

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Fix now a rational prime p and an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, as we did in Section 1.3, and view all the algebraic numbers $\alpha_1, \dots, \alpha_d$ as lying in $\mathbb{C}_p$ via this embedding.

THEOREM 2.2. *With the notation as above, one has*

$$|\Delta_n(f)|_p = M_p(f)^n \prod_{\substack{i=1 \\ |\alpha_i|_p=1}}^{d} |\alpha_i^n - 1|_p.$$

PROOF. Noting that for $\alpha \in \mathbb{C}_p$ and $n \in \mathbb{N}$, one has

$$|\alpha^n - 1|_p = \begin{cases} |\alpha|_p^n & \text{if } |\alpha|_p > 1, \\ 1 & \text{if } |\alpha|_p < 1, \end{cases}$$

one calculates

$$\begin{aligned} |\Delta_n(f)|_p &= |a_d|_p^n \prod_{\substack{i=1 \\ |\alpha_i|_p>1}}^{d} |\alpha_i|_p^n \prod_{\substack{i=1 \\ |\alpha_i|_p=1}}^{d} |\alpha_i^n - 1|_p \\ &= M_p(f)^n \prod_{\substack{i=1 \\ |\alpha_i|_p=1}}^{d} |\alpha_i^n - 1|_p. \end{aligned}$$

□

It follows that, to determine the p-adic valuation of the numbers $\Delta_n(f)$, one needs to understand the p-adic valuation of numbers of the form $\alpha^n - 1$, where $n \in \mathbb{N}$ and $\alpha \in \overline{\mathbb{Q}}_p \subseteq \mathbb{C}_p$ is a p-adic number such that $|\alpha|_p = 1$. The following theorem, which is inspired by [56, Lemma 2.11], provides a first step in this direction.

THEOREM 2.3. *Let $\alpha \in \overline{\mathbb{Q}}_p$ be such that $|\alpha|_p = 1$, and assume that α is not a root of unity. Let $\mathfrak{m}$ be the maximal ideal of the valuation ring $\mathcal{O}$ of $\overline{\mathbb{Q}}_p$ and let $N(\alpha)$ be the multiplicative order of α modulo $\mathfrak{m}$. Then, there exists a function $c : \mathbb{N} \to \mathbb{Q}$ such that $c(m)$ is constant for m large and for which*

$$\operatorname{ord}_p(\alpha^n - 1) = \begin{cases} 0 & \text{if } N(\alpha) \nmid n, \\ \operatorname{ord}_p(n) + c(\operatorname{ord}_p(n)) & \text{if } N(\alpha) \mid n. \end{cases} \tag{2-5}$$

PROOF. First of all, let us write

$$|\alpha^n - 1|_p = \prod_{\zeta \in W_n} |\alpha - \zeta|_p, \tag{2-6}$$

where $W_n \subseteq \mathcal{O}^\times$ denotes the group of roots of unity of order dividing n. Now, the natural embedding of the ring of Witt vectors of $\overline{\mathbb{F}}_p$ inside $\mathcal{O}$ gives rise to the Teichmüller lift

$$\tau : \overline{\mathbb{F}}_p^\times \hookrightarrow \mathcal{O}^\times,$$

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which is a morphism of groups that sends any $\beta \in \overline{\mathbb{F}}_p^\times$ to a root of unity whose order coincides with the multiplicative order of β in $\overline{\mathbb{F}}_p^\times$. Therefore, if

$$\pi : \mathcal{O} \twoheadrightarrow \mathcal{O}/\mathfrak{m} = \overline{\mathbb{F}}_p$$

is the natural projection map, the root of unity $\xi = \tau(\pi(\alpha)) \in \mathcal{O}^\times$ has order $N = N(\alpha)$, and we have that $|\alpha - \xi|_p < 1$ because τ is a section of π.

We can then use this root of unity ξ to write the following formula:

$$|\alpha^n - 1|_p = \prod_{\zeta \in W_n} |\alpha - \xi + \xi - \zeta|_p, \tag{2-7}$$

which follows from (2-6). Using this formula, we can prove immediately the first part of (2-5). Indeed, if $N \nmid n$, then for every $\zeta \in W_n$, the order of the root of unity ζ/ξ is not a power of p, because otherwise there would exist some $r \in \mathbb{N}$ such that $\zeta^{p^r} = \xi^{p^r}$, from which it would follow that $\xi^{np^r} = 1$, and thus that $N \mid np^r$. Since we know that $(N, p) = 1$, this would imply that $N \mid n$, contradicting our assumption. Therefore, [56, Lemma 2.9] implies that

$$|\xi - \zeta|_p = |1 - \zeta/\xi|_p = 1$$

for all $\zeta \in W_n$, which entails that $|\alpha - \xi + \xi - \zeta|_p = 1$ for every $\zeta \in W_n$, because $|\alpha - \xi|_p < 1$ by construction. Finally, we see thanks to (2-7) that $|\alpha^n - 1|_p = 1$, which proves the first part of the statement (2-5).

Suppose now that $N \mid n$. As before, we have that $|\xi - \zeta|_p = 1$ unless the order of $\mu := \zeta/\xi$ is a power of p. Therefore,

$$|\alpha^n - 1|_p = \prod_{\mu \in W_{p^m}} |\alpha - \xi\mu|_p = \prod_{\mu \in W_{p^m}} |\alpha - \xi + \xi(1 - \mu)|_p, \tag{2-8}$$

where $m := \mathrm{ord}_p(n)$. Moreover, if μ has order p^k, for some $k \in \mathbb{N} \cup \{0\}$ such that

$$p^{k-1}(p-1)\mathrm{ord}_p(\alpha - \xi) > 1,$$

we have that $|\xi(1 - \mu)|_p = |1 - \mu|_p > |\alpha - \xi|_p$, as follows from the classical fact that

$$\mathrm{ord}_p(1 - \mu) = \frac{1}{p^{k-1}(p-1)}$$

for every root of unity $\mu \in W_{p^m} \setminus \{1\}$ of order p^k, whose proof can be found for example in [56, Lemma 2.9].

Hence, if we define $s \in \mathbb{N} \cup \{0\}$ to be the minimal nonnegative integer such that

$$p^s(p-1)\mathrm{ord}_p(\alpha - \xi) > 1, \tag{2-9}$$

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and we set $r := \min(s, m)$, we see from (2-8) that

$$\begin{aligned}|\alpha^n - 1|_p &= \Big(\prod_{\mu \in W_{p^r}} |\alpha - \xi\mu|_p\Big)\Big(\prod_{\mu \in W_{p^m} \setminus W_{p^r}} |\alpha - \xi + \xi(1-\mu)|_p\Big) \\ &= |\alpha - \xi|_p \Big(\prod_{\mu \in W_{p^r} \setminus \{1\}} \frac{|\alpha - \xi\mu|_p}{|1-\mu|_p} |1-\mu|_p\Big)\Big(\prod_{\mu \in W_{p^m} \setminus W_{p^r}} |1-\mu|_p\Big) \\ &= |\alpha - \xi|_p \Big(\prod_{\mu \in W_{p^r} \setminus \{1\}} \frac{|\alpha - \xi\mu|_p}{|1-\mu|_p}\Big)\Big(\prod_{\mu \in W_{p^m} \setminus \{1\}} |1-\mu|_p\Big) \\ &= |\alpha - \xi|_p \Big(\prod_{\mu \in W_{p^r} \setminus \{1\}} \frac{|\alpha - \xi\mu|_p}{|1-\mu|_p}\Big)|n|_p,\end{aligned}$$

where the last equality follows from the fact that

$$\prod_{\mu \in W_{p^m} \setminus \{1\}} |1-\mu|_p = \prod_{\mu \in W_n \setminus \{1\}} |1-\mu|_p = \Big|\mathrm{Res}\Big(\frac{t^n - 1}{t-1}, t-1\Big)\Big|_p = |n|_p.$$

Therefore, we see that $\mathrm{ord}_p(\alpha^n - 1) = \mathrm{ord}_p(n) + c(\mathrm{ord}_p(n))$, where the expression

$$\begin{aligned}c(m) &:= \mathrm{ord}_p(\alpha - \xi) + \sum_{\mu \in W_{p^r} \setminus \{1\}} (\mathrm{ord}_p(\alpha - \xi\mu) - \mathrm{ord}_p(1-\mu)) \\ &= \mathrm{ord}_p(\alpha^{p^r} - \xi^{p^r}) - r\end{aligned}$$

depends only on m and α, and is evidently constant when m becomes sufficiently large. This proves the second part of (2-5), and concludes the proof. □

REMARK 2.4. Let $\alpha \in \overline{\mathbb{Q}}$. Then, there exists a finite subset $S \subseteq \mathbb{N}$ such that for every rational prime $p \in \mathbb{N} \setminus S$ and every embedding $\iota\colon \mathbb{Q}(\alpha) \hookrightarrow \overline{\mathbb{Q}}_p$, we have that $|\iota(\alpha)|_p = 1$ and

$$\mathrm{ord}_p(\iota(\alpha) - \tau(\pi(\iota(\alpha)))) \in \mathbb{N}.$$

Therefore, for every rational prime $p \in \mathbb{N} \setminus (S \cup \{2\})$, we see that $s = 0$ is the minimal nonnegative integer such that (2-9) holds true.

From Theorems 2.2 and 2.3, we obtain several corollaries. First of all, one can obtain an explicit formula for the p-adic valuation of the elements of the Pierce–Lehmer sequence associated to a polynomial $f \in \mathbb{Z}[t]$.

COROLLARY 2.5. *Let $f \in \mathbb{Z}[t] \setminus \{0\}$ be a polynomial that does not vanish at any root of unity, and fix a prime p. Let $\beta_1, \dots, \beta_d \in \overline{\mathbb{Q}}_p$ denote the p-adic roots of f, counted with multiplicities. Then, for every $n \in \mathbb{N}$, we define*

$$\mu_p(f) := -m_p(f)/\log(p), \tag{2-10}$$

$$B_{p,n}(f) := \{\beta \in \overline{\mathbb{Q}}_p : f(\beta) = 0, |\beta|_p = 1, |\beta^n - 1|_p < 1\}, \tag{2-11}$$

$$\lambda_{p,n}(f) := \#\{j \in \{1, \dots, d\} : \beta_j \in B_{p,n}(f)\} \tag{2-12}$$

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and for every $\beta \in \overline{\mathbb{Q}}_p$ *such that* $|\beta|_p = 1$,

$$s_p(\beta) := \min\{s \in \mathbb{Z}_{\geq 0} : p^s(p-1)\mathrm{ord}_p(\beta - \tau_p(\pi_p(\beta))) > 1\} \tag{2-13}$$

and we write $r_{p,n}(\beta) := \min(\mathrm{ord}_p(n), s_p(\beta))$. *Using this notation,*

$$\mathrm{ord}_p(\Delta_n(f)) = \mu_p(f) \cdot n + \lambda_{p,n}(f) \cdot \mathrm{ord}_p(n) + \nu_{p,n}(f), \tag{2-14}$$

where $\nu_{p,n}(f) := \sum_{\substack{j\in\{1,\ldots,d\}\\ \beta_j \in B_{p,n}(f)}} (\mathrm{ord}_p(\beta_j^{p^{r_{p,n}(\beta_j)}} - \tau_p(\pi_p(\beta_j))^{p^{r_{p,n}(\beta_j)}}) - r_{p,n}(\beta_j))$.

PROOF. We see from Theorem 2.2 that

$$\mathrm{ord}_p(\Delta_n(f)) = \mu_p(f) \cdot n + \sum_{\substack{j\in\{1,\ldots,d\}\\ \beta_j \in B_{p,n}(f)}} \mathrm{ord}_p(\beta_j^n - 1),$$

and Theorem 2.3 implies that

$$\mathrm{ord}_p(\beta^n - 1) = \mathrm{ord}_p(n) + \mathrm{ord}_p(\beta^{p^{r_{p,n}(\beta)}} - \tau_p(\pi_p(\beta))^{p^{r_{p,n}(\beta)}}) - r_{p,n}(\beta)$$

for every $\beta \in B_{p,n}(f)$, which allows us to conclude the proof. □

Moreover, we can use Theorems 2.2 and 2.3 to pin down the asymptotic behaviour of the p-adic valuation of the Pierce–Lehmer sequence associated to an integral polynomial that does not vanish on the p-adic unit circle or on roots of unity.

COROLLARY 2.6. *Let* $f(t) \in \mathbb{Z}[t] \setminus \{0\}$ *and assume that* $f(\alpha) \neq 0$ *for every* $\alpha \in \mathbb{C}_p$ *such that* $|\alpha|_p = 1$. *Then,*

$$|\Delta_n(f)|_p = M_p(f)^n \tag{2-15}$$

for all $n \in \mathbb{N}$. *If one only assumes that* $f(\zeta) \neq 0$ *for every* $\zeta \in W_\infty$,

$$|\Delta_n(f)|_p^{1/n} \to M_p(f)$$

as $n \to \infty$.

PROOF. This follows directly from Theorems 2.2 and 2.3. We leave the details to the reader. □

REMARK 2.7. A similar result holds true for the Archimedean Mahler measure. More precisely, we see directly from the definition given in (2-2) that for every polynomial $f \in \mathbb{Z}[t] \setminus \{0\}$ that does not vanish on the unit circle of $\mathbb{C}$, the asymptotic

$$|\Delta_n(f)|_\infty \sim M_\infty(f)^n \tag{2-16}$$

holds true as $n \to +\infty$. If one only assumes that the roots of f are not roots of unity, then one sees that

$$|\Delta_n(f)|_\infty^{1/n} \to M_\infty(f)$$

as $n \to \infty$. This follows from an inequality originally proved by Gelfand, as explained for instance in [11, Lemma 1.10]. However, if f has some root on the unit circle of $\mathbb{C}$, the

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behaviour of the absolute values $|\Delta_n(f)|_\infty$ is quite chaotic, as exemplified for instance by [11, Theorem 2.16], which shows that the sequence of ratios $|\Delta_n(f)/\Delta_{n-1}(f)|_\infty$ converges if and only if f has no roots on the unit circle of $\mathbb{C}$.

The p-adic valuation of various subsequences of a Pierce–Lehmer sequence can be understood from Theorems 2.2 and 2.3. For instance, the following corollary shows that such a p-adic valuation of the sub-sequence $\{\Delta_{p^n}(f)\}_{n=0}^{+\infty}$ associated to a polynomial $f \in \mathbb{Z}[t]$ that does not vanish at roots of unity exhibits a behaviour similar to the p-adic valuation of the class number in $\mathbb{Z}_p$-extensions of number fields, which was already studied in the seminal work of Iwasawa [22].

COROLLARY 2.8. *Let $f(t) \in \mathbb{Z}[t]$ be a polynomial that does not vanish at roots of unity, p be a rational prime and $\beta_1, \ldots, \beta_d$ denote the p-adic roots of f, counted with multiplicities. Then, there exist two constants $k_0(f,p) \in \mathbb{N}$ and $\nu_p(f) \in \mathbb{Z}$, depending on p and f, such that the following equality:*

$$\operatorname{ord}_p(\Delta_{p^k}(f)) = \mu_p(f) \cdot p^k + \lambda_p(f) \cdot k + \nu_p(f)$$

holds true for every $k \geq k_0(f,p)$, where $\mu_p(f) := -m_p(f)/\log(p)$ and

$$\lambda_p(f) := \#\{j \in \{1, \ldots, d\} : |\beta_j|_p = 1,\ |\beta_j - 1|_p < 1\}.$$

PROOF. This follows directly from Corollary 2.5. Indeed, the invariant $\mu_p(f)$ coincides with the one introduced in (2-10). Moreover, let us note that $B_{p,p^k}(f) = B_{p,1}(f)$ for every $k \geq 1$. To see this, fix any $\beta \in \overline{\mathbb{Q}}_p$ such that $|\beta|_p = 1$. Then, we have that $|\beta^{p^k} - 1|_p < 1$ if and only if the multiplicative order of $\pi_p(\beta) \in \overline{\mathbb{F}}_p^\times$ is a multiple of p^k. However, the aforementioned multiplicative order is coprime to p. Therefore, $|\beta^{p^k} - 1|_p < 1$ if and only if $\pi_p(\beta) = 1$, which is equivalent to saying that $|\beta - 1|_p < 1$. Hence, we see immediately from the definition of the sets $B_{p,n}(f)$ given in (2-11) that $B_{p,p^k}(f) = B_{p,1}(f)$ for every $k \geq 1$, as we wanted to show. This shows in particular that $\lambda_{p,p^k}(f) = \lambda_p(f)$ for every $k \geq 1$, where $\lambda_{p,p^k}(f)$ is the invariant defined in (2-12).

To conclude, it suffices to define

$$k_0(f,p) := \max\{s_p(\beta) : \beta \in \overline{\mathbb{Q}}_p, f(\beta) = 0, |\beta|_p = 1\},$$

where $s_p(\beta)$ is the invariant defined in (2-13). Then, for every $k \geq k_0(f,p)$ and every $\beta \in \overline{\mathbb{Q}}_p$, such that $|\beta|_p = 1$ and $f(\beta) = 0$, we have that $r_{p,p^k}(\beta) = s_p(\beta)$, as follows immediately from the fact that $r_{p,p^k}(\beta) = \min(k, s_p(\beta))$. Therefore, using the definition of $\nu_{p,n}(f)$ given in Corollary 2.5, we see that for every integer $k \geq k_0(f,p)$, the invariant

$$\begin{aligned}
\nu_{p,p^k}(f) &= \sum_{\substack{j \in \{1,\ldots,d\} \\ \beta_j \in B_{p,p^k}(f)}} (\operatorname{ord}_p(\beta_j^{p^{r_{p,p^k}(\beta_j)}} - \tau_p(\pi_p(\beta_j))^{p^{r_{p,p^k}(\beta_j)}}) - r_{p,p^k}(\beta_j)) \\
&= \sum_{\substack{j \in \{1,\ldots,d\} \\ \beta_j \in B_{p,1}(f)}} (\operatorname{ord}_p(\beta_j^{p^{s_p(\beta_j)}} - \tau_p(\pi_p(\beta_j))^{p^{s_p(\beta_j)}}) - s_p(\beta_j))
\end{aligned}$$

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is independent of k. This allows us to set

$$\nu_p(f) := \sum_{\substack{j\in\{1,\ldots,d\}\\ \beta_j\in B_{p,1}(f)}} (\mathrm{ord}_p(\beta_j^{p^{s_p(\beta_j)}} - \tau_p(\pi_p(\beta_j))^{p^{s_p(\beta_j)}}) - s_p(\beta_j)),$$

and to conclude our proof. □

Moreover, if $\ell, p \in \mathbb{N}$ are two distinct rational primes, the ℓ-adic valuation of the sub-sequence $\{\Delta_{p^n}(P)\}_{n=0}^{+\infty}$ exhibits a behaviour that is similar to that observed by Washington [58] in the case of $\mathbb{Z}_p$-towers of number fields.

COROLLARY 2.9. *Let $f(t) \in \mathbb{Z}[t]$ be a polynomial that does not vanish at roots of unity. Let p and ℓ be two distinct rational primes. Then, there exist two constants $k_0(f, p, \ell) \in \mathbb{N}$ and $\nu_p(f, \ell) \in \mathbb{Z}$, depending on p, ℓ and f, such that the equality*

$$\mathrm{ord}_p(\Delta_{\ell^k}(f)) = \mu_p(f) \cdot \ell^k + \nu_p(f, \ell)$$

holds true for every $k \geq k_0(f, p, \ell)$, where again $\mu_p(f) = -m_p(f)/\log(p)$.

PROOF. This follows once again from Corollary 2.5. Indeed, $\mu_p(f)$ is once again identical to the invariant defined in (2-10). Moreover, $\mathrm{ord}_p(\ell^k) = 0$ for every $k \geq 0$, which shows that the part of the equality (2-14) involving the invariant $\lambda_{p,n}(f)$ does not appear when $n = \ell^k$.

To conclude, let $\beta_1, \ldots, \beta_d \in \overline{\mathbb{Q}}_p$ be the p-adic roots of $f(t)$, counted with multiplicities, that lie on the p-adic unit circle, and $N_1, \ldots, N_d$ be the multiplicative orders of their reductions $\pi_p(\beta_1), \ldots, \pi_p(\beta_d) \in \overline{\mathbb{F}}_p^\times$. Then, for every $j \in \{1, \ldots, d\}$, we have that $|\beta_j^{\ell^k} - 1|_p < 1$ if and only if there exists some nonnegative integer $a_j \leq k$ such that $N_j = \ell^{a_j}$. Moreover,

$$r_{p,\ell^k}(\beta_j) = \min(\mathrm{ord}_p(\ell^k), s_p(\beta_j)) = 0$$

for every $j \in \{1, \ldots, d\}$ and every $k \geq 0$. Therefore, if we set

$$k_0(f, p, \ell) := \max\{\mathrm{ord}_\ell(N_j) : j \in \{1, \ldots, d\}\}$$
$$\nu_p(f, \ell) := \sum_{\substack{j\in\{1,\ldots,d\}\\ \beta_j\in B_{p,\ell^{k_0}}(f)}} \mathrm{ord}_p(\beta_j - \tau_p(\pi_p(\beta_j))),$$

where we write k_0 instead of $k_0(f, p, \ell)$, then, we see from the definition of $\nu_{p,n}(f)$ given in Corollary 2.5 that $\nu_{p,\ell^k}(f) = \nu_p(f, \ell)$ for every $k \geq k_0(f, p, \ell)$, and this allows us to conclude our proof. □

In fact, Corollaries 2.8 and 2.9 can be generalised by looking at sequences of integers that are divisible only by a finite number of primes, as done by Friedman [13] for cyclotomic $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}$-extensions of number fields that are abelian over $\mathbb{Q}$.

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COROLLARY 2.10. *Let $f(t) \in \mathbb{Z}[t]$ be a polynomial that does not vanish at roots of unity. Let $\ell_1, \dots, \ell_r$ be distinct prime numbers, and let $\mathcal{S} \subseteq \mathbb{N}$ be the set of those integers whose prime divisors are contained in $\{\ell_1, \dots, \ell_r\}$. Then:*

- *for every $j \in \{1, \dots, r\}$, there exist a nonnegative integer $\lambda_j(f)$ and an integer $\nu_j(f)$ such that for every $n = \ell_1^{k_1} \cdots \ell_j^{k_j} \cdots \ell_r^{k_r} \in \mathcal{S}$,*

$$\mathrm{ord}_{\ell_j}(\kappa(X_n)) = \mu_{\ell_j}(f) \cdot n + \lambda_j(f) \cdot k_j + \nu_j(f)$$

  as long as n is big enough, where again $\mu_{\ell_j}(f) = -m_{\ell_j}(f)/\log(\ell_j)$;
- *for every prime $p \notin \{\ell_1, \dots, \ell_r\}$, there exists an integer $\nu(f)$ such that*

$$\mathrm{ord}_p(\kappa(X_n)) = \mu_p(f) \cdot n + \nu(f)$$

  for every $n \in \mathcal{S}$ that is big enough.

PROOF. The proof is similar to the proofs of the two previous corollaries, and we leave it to the reader. □

REMARK 2.11. In the situation where the polynomial $f(t)$ is monic, the p-adic valuation of a Pierce–Lehmer sequence was also studied in [23]. In this situation, there is no p-adic Mahler measure appearing in the formulae.

REMARK 2.12. Note that the Pierce–Lehmer sequence $\{\Delta_n(f)\}_{n\in\mathbb{N}}$ associated to any polynomial $f \in \mathbb{Z}[t]$ satisfies a linear recurrence, as explained in [32, Section 8]. Therefore, studying the p-adic valuation of Pierce–Lehmer sequences can be seen as a special case of the more general problem of studying the p-adic valuation of linearly recurrent sequences, which has been the subject of great attention (see for instance [4]). We also refer the interested reader to the works [10, 12], which treat problems related to the p-adic valuation of Pierce–Lehmer sequences.

3. Graph theory

The aim of this section is to prove Theorem 3.17, which provides an explicit expression for the p-adic valuation of the number of spanning trees in a $\mathbb{Z}$-tower of graphs in terms of a polynomial naturally associated to this tower, which we call the *Ihara polynomial* and which we define in Section 3.3. In particular, Theorem 1.1 is a simplified version of Theorem 3.17, as we explain in Section 3.6. To do so, we first recall some fundamentals about graphs and their covers in Sections 3.1 and 3.2. Then, we devote Section 3.3 to the proof of Theorem 3.6, which provides an explicit formula relating the number of spanning trees of the members of a $\mathbb{Z}$-cover of graphs to the Pierce–Lehmer sequence associated to the Ihara polynomial of this tower. We provide an explicit example that verifies this relation in Section 3.4. Moreover, Section 3.5 shows how to combine Theorem 3.6 with the results proven in Section 2.3, to provide some asymptotic expressions for the growth of the number of spanning trees in a $\mathbb{Z}$-tower. In particular, this generalises two previous results of Mednykh and Mednykh [41, 42].

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3.1. Galois covers of locally finite graphs. The aim of this subsection is to formally introduce the kinds of graphs that are considered in this article and their Galois theory.

3.1.1. Locally finite graphs. Let $X = (V_X, \mathbf{E}_X)$ be a graph in the sense of Serre (see [48] and also [52]), where V_X and $\mathbf{E}_X$ are two sets, to be interpreted as the sets of vertices and (directed) edges of X. In particular, each edge $e \in \mathbf{E}_X$ has an origin $o(e) \in V_X$ and a terminus $t(e) \in V_X$, giving rise to the incidence map

$$\begin{aligned} \mathrm{inc}\colon \mathbf{E}_X &\to V_X \times V_X \\ e &\mapsto (o(e), t(e)) \end{aligned}$$

and to the inversion map $\mathbf{E}_X \to \mathbf{E}_X$, denoted by $e \mapsto \overline{e}$, such that

$$(o(\overline{e}), t(\overline{e})) = (t(e), o(e))$$

and $\overline{\overline{e}} = e \neq \overline{e}$ for every $e \in \mathbf{E}_X$.

A graph X is called finite if both V_X and $\mathbf{E}_X$ are finite sets. Moreover, a graph X is called locally finite if for each vertex $v \in V_X$, the set of edges with origin at v, defined as

$$\mathbf{E}_{X,v} = \{e \in \mathbf{E}_X : o(e) = v\}$$

is finite. In this case, one defines the valency (or degree) of a vertex $v \in V_X$ to be

$$\mathrm{val}_X(v) = |\mathbf{E}_{X,v}|.$$

Any finite graph is in particular locally finite.

ASSUMPTION 3.1. In this paper, all graphs will be locally finite.

3.1.2. Paths and loops. Let us recall that a path $c = e_1 \cdot \cdots \cdot e_m$ in a graph X consists of a sequence of directed edges $e_i \in \mathbf{E}_X$ such that $t(e_i) = o(e_{i+1})$ for each index $i \in \{1, \ldots, m-1\}$. The origin and the terminus of the path $c = e_1 \cdot \cdots \cdot e_m$ are defined as $o(c) = o(e_1)$ and $t(c) = t(e_m)$, respectively.

A graph X is called connected if given any two vertices $v_1, v_2 \in V_X$, there exists a path c in X such that $o(c) = v_1$ and $t(c) = v_2$. Finally, a loop based at a vertex $v_0 \in X$ is a path c in X such that $o(c) = t(c) = v_0$.

This allows one to define the fundamental group of X based at a vertex $v_0 \in V_X$, which is denoted by $\pi_1(X, v_0)$, as the set of loops based at v_0, considered modulo homotopy (see [52, Section 3.5] for the precise definition of this equivalence relation in the context of graphs), endowed with the group operation given by the concatenation of paths (see [52, Section 5.3] for details).

3.1.3. Galois covers of graphs. Let Y and X be two graphs. A morphism of graphs $f : Y \to X$ is called a cover (or a covering map) if the following two conditions are satisfied:

(1) $f : V_Y \to V_X$ is surjective;
(2) for all $w \in V_Y$, the restriction $f|_{\mathbf{E}_{Y,w}}$ induces a bijection

$$f|_{\mathbf{E}_{Y,w}} : \mathbf{E}_{Y,w} \xrightarrow{\approx} \mathbf{E}_{X,f(w)}.$$

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We will often refer to Y/X as a cover if the covering map is understood from the context. Given a cover $f : Y \to X$, one defines as usual $\mathrm{Aut}_f(Y/X)$ to be the subgroup of $\mathrm{Aut}(Y)$ consisting of the automorphisms $\iota \in \mathrm{Aut}(Y)$ satisfying $f \circ \iota = f$. Again, we will often write $\mathrm{Aut}(Y/X)$ instead of $\mathrm{Aut}_f(Y/X)$ if f is understood.

Let us also recall that a cover $f : Y \to X$ is called Galois if the following two conditions are satisfied:

(1) the graph Y is connected (and hence also X);
(2) the group $\mathrm{Aut}(Y/X)$ acts transitively on the fibre $f^{-1}(v)$ for all $v \in V_X$.

If Y/X is a Galois cover, we write $\mathrm{Gal}(Y/X)$ instead of $\mathrm{Aut}(Y/X)$. In this case, one has the usual Galois correspondence between subgroups of $\mathrm{Aut}(Y/X)$ and equivalence classes of intermediate covers of Y/X.

3.1.4. Voltage assignments. Let X be a graph and let G be a group. A voltage assignment on X with values in G is defined to be a function $\alpha : \mathbf{E}_X \to G$ satisfying

$$\alpha(\bar{e}) = \alpha(e)^{-1}$$

for every $e \in \mathbf{E}_X$. Each such voltage assignment can be defined starting from an orientation of X, which is a subset $S \subseteq \mathbf{E}_X$ such that for each edge $e \in \mathbf{E}_X$, either e or $\overline{e}$ belong to S, but not both. Then, to get a voltage assignment as above, it suffices to define it on any orientation S and set $\alpha(\overline{s}) := \alpha(s)^{-1}$ for every $s \in S$.

3.1.5. Covers from voltage assignments. Given a graph X, a group G and a voltage assignment

$$\alpha \colon \mathbf{E}_X \to G,$$

one can construct a new graph $X(G, \alpha)$ as follows:

- the vertices of $X(G, \alpha)$ are given by $V_X \times G$;
- the (directed) edges of $X(G, \alpha)$ are given by $\mathbf{E}_X \times G$;
- the origin, terminus and inverse maps are defined as

$$\begin{aligned} o(e, \sigma) &= (o(e), \sigma), \\ t(e, \sigma) &= (t(e), \sigma \cdot \alpha(e)), \\ \overline{(e, \sigma)} &= (\bar{e}, \sigma \cdot \alpha(e)) \end{aligned}$$

  for each edge $(e, \sigma) \in \mathbf{E}_X \times G$.

It is easy to see that if X is locally finite, then so is $X(G, \alpha)$, and that the map of graphs

$$p \colon X(G, \alpha) \to X$$

defined as $p(v, \sigma) := v$ on each vertex $(v, \sigma) \in V_X \times G$, and as $p(e, \sigma) := e$ on each edge $(e, \sigma) \in \mathbf{E}_X \times G$, is actually a covering map. Moreover, this covering map is Galois whenever $X(G, \alpha)$ is connected and, in this case, $\mathrm{Gal}(X(G, \alpha)/X) \cong G$ canonically.

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To conclude, let us observe that the construction of $X(G,\alpha)$ is functorial with respect to α. More precisely, for each morphism of groups $f\colon G \to H$, one gets a new voltage assignment $\beta := f \circ \alpha$ with values in H, and a natural map of graphs

$$f_*\colon X(G,\alpha) \to X(H,\beta), \tag{3-1}$$

which is defined on each vertex $(v,\sigma) \in V_X \times G$ as $f_*(v,\sigma) := (v, f(\sigma))$, and on each edge $(e,\sigma) \in \mathbf{E}_X \times G$ as $f_*(e,\sigma) := (e, f(\sigma))$. Finally, it is easy to see that this morphism f_* is a covering map whenever f is surjective.

3.1.6. Monodromy representations. Let X be a graph and $\alpha\colon \mathbf{E}_X \to G$ be a voltage assignment with values in a group G. Then, the monodromy representation attached to α at a vertex $v_0 \in V_X$ is given by the following map:

$$\begin{aligned} \rho_{\alpha,v_0}\colon \pi_1(X,v_0) &\to G \\ [e_1 \cdot \dots \cdot e_n] &\mapsto \alpha(e_1) \cdot \dots \cdot \alpha(e_n), \end{aligned}$$

which is easily seen to be a well-defined morphism of groups. Moreover, this map can be used to detect when the graph $X(G,\alpha)$ is connected, and thus when the natural covering map $p\colon X(G,\alpha) \to X$ is Galois, as we recall in the following theorem, which is proven in [45, Section 2.3.1].

THEOREM 3.2. *Let X be a connected graph and $\alpha\colon \mathbf{E}_X \to G$ be a voltage assignment. Then, the graph $X(G,\alpha)$ is connected if and only if the monodromy representation ρ_{α,v_0} attached to α at some (equivalently, any) vertex $v_0 \in V_X$ is surjective.*

REMARK 3.3. Let X be a connected graph, $v_0 \in V_X$ a vertex of X, and $\alpha\colon \mathbf{E}_X \to G$ a voltage assignment such that ρ_{α,v_0} is surjective. Fix moreover a universal cover $\pi\colon \widetilde{X} \twoheadrightarrow X$ and a vertex $w_0 \in \widetilde{X}$ such that $\pi(w_0) = v_0$. Thanks to the universal property of the universal cover, proved for example in [52, Theorem 5.10], it can be shown that the intermediate Galois cover of $\widetilde{X} \to X$ given by $X(G,\alpha)$ corresponds to the subgroup $\varphi_{w_0}(\ker(\rho_{\alpha,v_0})) \trianglelefteq \mathrm{Gal}(\widetilde{X}/X)$, where

$$\varphi_{w_0}\colon \pi_1(X,v_0) \xrightarrow{\sim} \mathrm{Gal}(\widetilde{X}/X)$$

is the usual group isomorphism.

3.1.7. Systems of Galois covers. Let X be a graph and $\alpha\colon \mathbf{E}_X \to G$ a voltage assignment taking values in a group G. Given a group homomorphism $f\colon G \to H$ and a vertex $v_0 \in V_X$, the monodromy representation attached at v_0 to the Galois cover f_* defined in (3-1) is given by $f \circ \rho_{\alpha,v_0}$. This shows in particular that if the graph $X(G,\alpha)$ is connected, the graph $X(H, f \circ \alpha)$ will be connected whenever f is surjective. Moreover, any morphism of groups $f\colon G \to H$ induces another morphism of groups

$$\ker(f) \to \mathrm{Aut}(X(G,\alpha)/X(H, f \circ \alpha)), \tag{3-2}$$

which sends each $\tau \in \ker(f)$ to the automorphism $\phi_\tau\colon X(G,\alpha) \to X(G,\alpha)$ defined by setting $\phi_\tau(v,\sigma) := (v, \tau \cdot \sigma)$ for each vertex $(v,\sigma) \in V_X \times G$, and $\phi_\tau(e,\sigma) := (e, \tau \cdot \sigma)$

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for each edge $(e,\sigma) \in \mathbf{E}_X \times G$. The morphism of groups (3-2) is actually an isomorphism whenever f is surjective, as follows from the unique lifting theorem [52, Theorem 5.1].

In particular, the previous discussion shows that any voltage assignment

$$\alpha\colon \mathbf{E}_X \to G$$

for which $X(G,\alpha)$ is connected induces a system of Galois covers indexed by the lattice of quotients of the group G. As we will see in the upcoming sections of this paper, it is interesting to study how various graph invariants evolve when moving across this system.

3.2. The number of spanning trees in finite abelian covers of finite graphs. One particularly interesting kind of invariant of a connected finite graph X is given by its Picard group $\mathrm{Pic}^0(X)$, also known as the Jacobian, sandpile or class group of X. Its cardinality, denoted by $\kappa(X)$, is given by the number of spanning trees of the graph X. The aim of this section is to recall, following [57, Section 3], how this number changes in an abelian cover of a finite graph, using Ihara's determinant formula.

3.2.1. Ihara zeta functions. To do so, we will make use of another invariant of a finite connected graph X, namely its Ihara zeta function, which we denote by $Z_X(u)$. This is a rational function of u, which can be explicitly computed thanks to the Ihara determinant formula, which we recall below in (3-3), and is proven in [2, 29]. More precisely, given an ordering $V_X = \{v_1,\ldots,v_g\}$ of the vertices of X, we let $A_X := (a_{i,j}) \in \mathbb{Z}^{g\times g}$ denote the adjacency matrix of X, which is defined by setting $a_{i,j} := \#\{e \in \mathbf{E}_X\colon\ o(e) = v_i,\ t(e) = v_j\}$. Moreover, we let $D_X := (d_{i,j}) \in \mathbb{Z}^{g\times g}$ denote the valency (or degree) matrix of X, which is a diagonal matrix defined by setting $d_{i,i} := \mathrm{val}_X(v_i)$ for each $i \in \{1,\ldots,g\}$. Then, we can write the Ihara zeta function $Z_X(u)$ using the following explicit formula:

$$Z_X(u) = \frac{1}{(1-u^2)^{-\chi(X)}\cdot \det(\mathrm{Id}_g - A_X u + (D_X - \mathrm{Id}_g)u^2)}, \tag{3-3}$$

where Id_g denotes the $g\times g$ identity matrix and $\chi(X) := |V_X| - |\mathbf{E}_X|/2$ is the Euler characteristic of X. In particular, we have that $Z_X(u)^{-1} = (1-u^2)^{-\chi(X)}\cdot h_X(u)$, where

$$h_X(u) := \det(\mathrm{Id}_g - A_X u + (D_X - \mathrm{Id}_g)u^2) \in \mathbb{Z}[u]$$

is a polynomial.

3.2.2. Hashimoto's formula. This explicit formula can be used to relate the Ihara zeta function to the number of spanning trees of X. More precisely, given a finite connected graph X, one has

$$h_X'(1) = -2\chi(X)\kappa(X), \tag{3-4}$$

as was proven by Hashimoto in [20, Theorem B] (see also [2, Part II, Sections 5 and 6]). Such a formula, which can be considered as an analogue of the class number formula in the context of graph theory, admits an equivariant generalisation.

3.2.3. Artin–Ihara L-functions. Given a Galois cover of finite connected graphs Y/X, one can associate to any linear complex representation $\rho\colon \mathrm{Gal}(Y/X) \to \mathrm{GL}_n(\mathbb{C})$ an Artin–Ihara L-function $L_{Y/X}(u,\rho)$. This admits an explicit description analogous to (3-3). More precisely, let $d_\rho \in \mathbb{N}$ denote the degree of the representation ρ, and fix an ordering $V_X = \{v_1, \dots, v_g\}$ of the vertices of X and a section $\iota\colon V_X \to V_Y$ of the projection $V_Y \to V_X$. Then, [53, Theorem 18.15] shows that the Artin–Ihara L-function $L_{Y/X}(u,\rho)$ can be explicitly computed thanks to the following formula:

$$L_{Y/X}(u,\rho) = \frac{1}{(1-u^2)^{-\chi(X)\cdot d_\rho} \cdot \det(\mathrm{Id}_{gd_\rho} - A_\rho u + Q_\rho u^2)},$$

where $A_\rho, Q_\rho \in \mathbb{C}^{gd_\rho \times gd_\rho}$ are two explicit matrices, whose definitions we now recall. Given $\sigma \in G$,

$$A(\sigma) := (\#\{e \in \mathbf{E}_Y \colon o(e) = \iota(v_i),\ t(e) = \sigma(\iota(v_j))\})_{i,j=1,\dots,g},$$

and we define

$$A_\rho := \sum_{\sigma\in G} A(\sigma) \otimes \rho(\sigma) \quad \text{and} \quad Q_\rho := (D_X \otimes \mathrm{Id}_{d_\rho}) - \mathrm{Id}_{g\cdot d_\rho},$$

where $\otimes$ denotes the Kronecker product of matrices. For more details, we refer the interested reader to [53, Definition 18.13]. As before, this explicit formula allows one to write $L_{Y/X}(u,\rho)^{-1} = (1-u^2)^{-\chi(X)\cdot d_\rho} \cdot h_{Y/X}(u,\rho)$, where

$$h_{Y/X}(u,\rho) := \det(\mathrm{Id}_{gd_\rho} - A_\rho u + Q_\rho u^2) \in \mathbb{C}[u]$$

is a polynomial. In particular, if $\mathrm{Gal}(Y/X)$ is abelian and ψ is a character of $\mathrm{Gal}(Y/X)$, we have that $L_{Y/X}(u,\psi)^{-1} = (1-u^2)^{-\chi(X)} \cdot h_{Y/X}(u,\psi)$, where

$$h_{Y/X}(u,\psi) := \det(\mathrm{Id}_g - A_\psi u + (D_X - \mathrm{Id}_g)u^2), \tag{3-5}$$

because $d_\psi = 1$ and $Q_\psi = D_X - \mathrm{Id}_g$, as follows easily from [53, Definition 18.13].

3.2.4. Spanning trees and abelian covers. To conclude this subsection, let us recall that the Artin–Ihara L-functions satisfy the Artin formalism (see [2, 51]). This implies that for every Galois cover of finite graphs Y/X, with Galois group $G := \mathrm{Gal}(Y/X)$, the Ihara zeta function $Z_Y(u)$ admits the following factorisation:

$$Z_Y(u) = \prod_{\rho\in\mathrm{Irr}(G)} L_{Y/X}(u,\rho)^{d_\rho},$$

where $\mathrm{Irr}(G)$ denotes the set of equivalence classes of complex irreducible representations of a finite group G (see [53, Corollary 18.11]). Therefore, we see easily that

$$h_Y(u) = \prod_{\rho\in\mathrm{Irr}(G)} h_{Y/X}(u,\rho)^{d_\rho}, \tag{3-6}$$

using the relation $\chi(Y) = |G| \cdot \chi(X)$ between the Euler characteristics of Y and X, which is explained at the end of [52, Section 5.1], and the classical identity $|G| = \sum_{\rho\in\mathrm{Irr}(G)} d_\rho^2$,

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proven for example in [49, Section 2.4, Corollary 2]. Finally, if ρ_0 denotes the trivial representation of G and $\chi(X) \neq 0$,

$$|G| \cdot \kappa(Y) = \kappa(X) \cdot \prod_{\rho \neq \rho_0} h_{Y/X}(1,\rho)^{d_\rho},$$

thanks to (3-4) and (3-6), combined with the fact that

$$h_{Y/X}(1,\rho_0) = h_X(1) = 0,$$

which holds because the Laplacian matrix $D_X - A_X$ is singular (as explained in [6, Proposition 2.8]). In particular, if Y/X is a Galois cover of finite graphs such that $\chi(X) \neq 0$ and $G := \mathrm{Gal}(Y/X)$ is abelian,

$$|G| \cdot \kappa(Y) = \kappa(X) \prod_{\psi \neq \psi_0} h_{Y/X}(1,\psi), \tag{3-7}$$

where ψ_0 denotes the trivial character of G and $\psi \in G^\vee$ runs over all nontrivial characters of G.

3.3. Exact formulae for the number of spanning trees. In this subsection, we introduce what we take the liberty to call the *Ihara polynomial* $\mathcal{I}_\alpha$ associated to a voltage assignment $\alpha : \mathbf{E}_X \to G$. When $G = \mathbb{Z}^d$ for some $d \in \mathbb{N}$, this polynomial was introduced, with a slightly different terminology, in the work of Silver and Williams [50] (see Remark 3.4 for a comparison between the two definitions). Moreover, when $G \in \{\mathbb{Z}, \mathbb{Z}_\ell\}$, for some rational prime ℓ, this polynomial was considered by Lei and the second author of the present paper [35]. In the general case, this invariant consists of an element of the group ring $\mathbb{Z}[G]$, which we write as a generalised polynomial ring $\mathbb{Z}[t^G]$ by adding a formal variable t.

3.3.1. The Ihara polynomial. More precisely, let X be a finite connected graph such that $\chi(X) \neq 0$, and let us start with a voltage assignment $\alpha : \mathbf{E}_X \to G$. As before, let us fix an ordering of the vertices of X, given by $V_X = \{v_1, \dots, v_g\}$. Then, we can define the matrix

$$A_\alpha(t) := \Bigg(\sum_{\substack{e \in \mathbf{E}_X \\ \mathrm{inc}(e)=(v_i,v_j)}} t^{\alpha(e)} \Bigg) \in \mathbb{Z}[t^G]^{g\times g},$$

which we use to introduce the *Ihara polynomial*

$$\mathcal{I}_\alpha(t) = \det(D_X - A_\alpha(t)) \in \mathbb{Z}[t^G], \tag{3-8}$$

where D_X denotes, as before, the valency (or degree) matrix of X. When no confusion seems to occur, we will just write $\mathcal{I}_\alpha$ instead of $\mathcal{I}_\alpha(t)$.

We note in particular that the Ihara polynomial is self-reciprocal. In other words, we have the following identity:

$$\mathcal{I}_\alpha\Big(\frac{1}{t}\Big) = \mathcal{I}_\alpha(t), \tag{3-9}$$

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which comes from the fact that the transpose of the matrix $A_\alpha(t)$ equals the matrix $A_\alpha(t^{-1})$ by definition. Moreover, for every morphism of groups $f\colon G \to H$, we have by definition that

$$\mathcal{I}_\beta = f_*(\mathcal{I}_\alpha), \tag{3-10}$$

where $\beta := f \circ \alpha$ and $f_*\colon \mathbb{Z}[t^G] \to \mathbb{Z}[t^H]$ denotes the morphism of rings induced by f.

REMARK 3.4. For $G = \mathbb{Z}^d$ with $d \in \mathbb{N}$, this polynomial was introduced in [50] under the name of Laplacian polynomial. In particular, an unsigned d-periodic graph $\mathfrak{X}$ in the sense of [50, Section 6] can be obtained as $\mathfrak{X} = X(\mathbb{Z}^d, \alpha)$, where X is a finite graph and $\alpha\colon \mathbf{E}_X \to \mathbb{Z}^d$ is a voltage assignment.

3.3.2. The Ihara polynomial and the number of spanning trees. Suppose now that X is a finite connected graph such that $\chi(X) \neq 0$, and fix a voltage assignment

$$\alpha\colon \mathbf{E}_X \to G$$

with values in some finite abelian group G, such that the induced graph $X(G, \alpha)$ is connected. The following result expresses how the number of spanning trees changes from X to $X(G, \alpha)$, using the Ihara polynomial $\mathcal{I}_\alpha$.

PROPOSITION 3.5. *For every finite connected graph X, such that $\chi(X) \neq 0$, and every voltage assignment $\alpha\colon \mathbf{E}_X \to G$ with values in a finite abelian group G, such that the associated graph $X(G, \alpha)$ is connected,*

$$|G| \cdot \kappa(X(G, \alpha)) = \kappa(X) \cdot \prod_{\psi \neq \psi_0} \mathcal{I}_\alpha(\psi(1)),$$

where $\mathcal{I}_\alpha(\psi(1)) := \psi(\mathcal{I}_\alpha) \in \mathbb{C}$ is obtained by applying to $\mathcal{I}_\alpha$ the natural linear extension of the character ψ to the group ring $\mathbb{Z}[t^G]$.

PROOF. First of all, observe that $\mathcal{I}_\alpha(\psi(1)) = \det(D_X - \widetilde{A}_\psi)$, where we define

$$\widetilde{A}_\psi := \Bigg(\sum_{\substack{e \in \mathbf{E}_X \\ \mathrm{inc}(e) = (v_i, v_j)}} \psi(\alpha(e)) \Bigg) \in \mathbb{C}^{g \times g}$$

for every character $\psi \in G^\vee$. In particular, one can prove that $\widetilde{A}_\psi = A_\psi$, as explained in [40, Corollary 5.3]. Therefore, we see from the definition of the polynomial $h_{Y/X}(u, \psi)$, which was given in (3-5), that $\mathcal{I}_\alpha(\psi(1)) = h_{Y/X}(1, \psi)$ for every character $\psi \in G^\vee$. To conclude the proof, it is just sufficient to substitute this equality in the explicit expression

$$|G| \cdot \kappa(Y) = \kappa(X) \prod_{\psi \neq \psi_0} h_{Y/X}(1, \psi),$$

which was recalled in (3-7). □

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3.3.3. $\mathbb{Z}$-towers of graphs and Pierce–Lehmer sequences. From now on, we will specialise to the case of $\mathbb{Z}$-towers of graphs. More precisely, we will consider a finite connected graph X such that $\chi(X) \neq 0$, endowed with a voltage assignment $\alpha\colon \mathbf{E}_X \to \mathbb{Z}$ with values in the additive group of the integers, such that the derived graph $X_\infty := X(\mathbb{Z}, \alpha)$ is connected, which is equivalent to saying that there exists a vertex $v_0 \in V_X$ such that the monodromy representation $\rho_{\alpha,v_0}\colon \pi_1(X, v_0) \to \mathbb{Z}$ is surjective, as explained in Theorem 3.2. In this case, we have a natural isomorphism $\mathbb{Z}[t^{\mathbb{Z}}] \cong \mathbb{Z}[t^{\pm 1}]$. Therefore, we see from (3-9) that the Ihara polynomial can be written as

$$\mathcal{I}_\alpha(t) = c_0 + c_1(t + t^{-1}) + \cdots + c_b(t^b + t^{-b})$$

for some $c_0, \ldots, c_b \in \mathbb{Z}$ such that $c_b \neq 0$. Clearing denominators, we can define

$$I_\alpha(t) := t^b \mathcal{I}_\alpha(t) \in \mathbb{Z}[t],$$

which is a polynomial of degree $2b$ such that $I_\alpha(0) \neq 0$. Finally, we define $e := \mathrm{ord}_{t=1}(I_\alpha)$, and we write

$$I_\alpha(t) = (t-1)^e J_\alpha(t) \tag{3-11}$$

for some polynomial $J_\alpha \in \mathbb{Z}[t]$ such that $J_\alpha(0) \cdot J_\alpha(1) \neq 0$.

Now, one can associate to the voltage assignment $\alpha\colon \mathbf{E}_X \to \mathbb{Z}$ a system of finite graphs $X_n := X(\mathbb{Z}/n\mathbb{Z}, \pi_n \circ \alpha)$, where $\pi_n\colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ is the natural quotient map. In particular, each of these graphs will be connected, because X_∞ is assumed to be connected, and the maps π_n are surjective. Moreover, the number of spanning trees of each graph X_n can be computed in terms of a Pierce–Lehmer sequence associated to the polynomial J_α, as the following result shows.

THEOREM 3.6. *Let X be a finite connected graph such that $\chi(X) \neq 0$ and fix a voltage assignment $\alpha\colon \mathbf{E}_X \to \mathbb{Z}$ such that the graph $X_\infty := X(\mathbb{Z}, \alpha)$ is connected. Moreover, for every $n \in \mathbb{N}$, we let $X_n := X(\mathbb{Z}/n\mathbb{Z}, \pi_n \circ \alpha)$, where $\pi_n\colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ is the natural quotient map. Then, the number of spanning trees of X_n can be computed as*

$$\kappa(X_n) = (-1)^{b(n-1)} \cdot \kappa(X) \cdot n^{e-1} \cdot \frac{\Delta_n(J_\alpha)}{\Delta_1(J_\alpha)}, \tag{3-12}$$

where the integers $b := -\mathrm{ord}_{t=0}(\mathcal{I}_\alpha) \geq 0$ and $e := \mathrm{ord}_{t=1}(\mathcal{I}_\alpha) \geq 1$ are defined in terms of the Ihara polynomial $\mathcal{I}_\alpha \in \mathbb{Z}[t^{\pm 1}]$, whose definition was recalled in (3-8). Moreover, $\{\Delta_n(J_\alpha)\}_{n\in\mathbb{N}}$ is the Pierce–Lehmer sequence, defined as in (2-4), which is associated to the polynomial $J_\alpha(t) := t^b \cdot (t-1)^{-e} \cdot \mathcal{I}_\alpha(t) \in \mathbb{Z}[t]$.

PROOF. First of all, observe that $e \geq 1$ because $\mathcal{I}_\alpha(1) = \det(D_X - A_X)$, where D_X and A_X are respectively the degree and adjacency matrices associated to X, whose difference $D_X - A_X$ is singular, as explained in [6, Proposition 2.8]. Now, applying Proposition 3.5 to the Galois cover X_n/X,

$$n \cdot \kappa(X_n) = \kappa(X) \cdot \prod_{\psi \neq \psi_0} \mathcal{I}_{\alpha_n}(\psi(1)), \tag{3-13}$$

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where $\alpha_n := \pi_n \circ \alpha$ for every $n \in \mathbb{N}$. Moreover, it is easy to see using (3-10) that

$$\prod_{\psi \neq \psi_0} \mathcal{I}_{\alpha_n}(\psi(1)) = \prod_{\psi \neq \psi_0} \mathcal{I}_{\alpha}(\psi(1)) = \prod_{\zeta \in W_n^*} \mathcal{I}_{\alpha}(\zeta), \tag{3-14}$$

where $W_n^* := W_n \setminus \{1\}$ denotes the set of nontrivial roots of unity whose order divides n. Now, let us observe that

$$\begin{aligned}\prod_{\zeta \in W_n^*} \mathcal{I}_{\alpha}(\zeta) &= \prod_{\zeta \in W_n^*} (\zeta^{-b} \cdot I_{\alpha}(\zeta)) = (-1)^{b(n-1)} \prod_{\zeta \in W_n^*} I_{\alpha}(\zeta) \\ &= (-1)^{b(n-1)} \mathrm{Res}\Big(I_{\alpha}(t), \frac{t^n - 1}{t - 1}\Big)\end{aligned} \tag{3-15}$$

as follows from the definition of resultant recalled in Section 2.1. Thus,

$$n \cdot \kappa(X_n) = (-1)^{b(n-1)} \cdot \kappa(X) \cdot \mathrm{Res}\Big(I_{\alpha}(t), \frac{t^n - 1}{t - 1}\Big) \tag{3-16}$$

by combining (3-13) with (3-14) and (3-15). To conclude, it suffices to observe that

$$\begin{aligned}\mathrm{Res}\Big(I_{\alpha}(t), \frac{t^n - 1}{t - 1}\Big) &= \mathrm{Res}\Big(t - 1, \frac{t^n - 1}{t - 1}\Big)^e \cdot \mathrm{Res}\Big(J_{\alpha}(t), \frac{t^n - 1}{t - 1}\Big) \\ &= n^e \cdot \mathrm{Res}\Big(J_{\alpha}(t), \frac{t^n - 1}{t - 1}\Big) = n^e \cdot \frac{\Delta_n(J_{\alpha})}{\Delta_1(J_{\alpha})},\end{aligned}$$

thanks to the multiplicative property of resultants. □

REMARK 3.7. Formulae such as (3-16) appear also in the theory of curves over finite fields and in knot theory. Indeed:

- if C is a nonsingular, geometrically irreducible projective curve over a finite field $\mathbb{F}_q$ with at least one rational point over $\mathbb{F}_q$, and J is its Jacobian variety, then [47, Corollary, page 110] implies that

$$\#J(\mathbb{F}_{q^n}) = |\mathrm{Res}(P_C(t), t^n - 1)|,$$

  where $P_C(t)$ is the Weil polynomial of C, defined as the reverse of the L-polynomial $L_C(t)$;
- if $K \subseteq S^3$ is a knot and M_n is a Galois cover of $M := S^3 \setminus K$, with Galois group $\mathbb{Z}/n\mathbb{Z}$, Fox's formula (see [59]) implies that

$$\#H_1(X_n, \mathbb{Z})_{\mathrm{tors}} = |\mathrm{Res}(A_K(t), t^n - 1)|,$$

  where $A_K(t)$ is the Alexander polynomial associated to the knot K.

REMARK 3.8. Let us note that, in the setting of Theorem 3.6, the explicit formula (3-12) implies that J_{α} does not vanish at roots of unity. Indeed, if this was the case,

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we would have $\kappa(X_n) = 0$ for some $n \geq 2$, which is absurd because X_n is not empty. Therefore, combining Theorem 3.6 with Corollaries 2.8 and 2.9, one can recover (1-2) and (1-3) for the $\mathbb{Z}_\ell$-towers $\{X_{\ell^n}\}_{n=0}^{+\infty}$ induced from a $\mathbb{Z}$-tower $\{X_n\}_{n=1}^{+\infty}$. More generally, combining Theorem 3.6 with Corollary 2.5 will allow us to prove Theorem 1.1, as we will explain in Theorem 3.17.

REMARK 3.9. Since the polynomial J_α is a reciprocal polynomial, it is known that the quantity $|\Delta_n(J_\alpha)/\Delta_1(J_\alpha)|$ is a square when n is odd and J_α is a monic polynomial, as explained for instance in [10, Section 2]. It follows that if p and ℓ are two distinct rational primes with ℓ odd and J_α is monic, then

$$\mathrm{ord}_p(\kappa(X_{\ell^k})) = \mathrm{ord}_p(\kappa(X)) + \mathrm{ord}_p(\Delta_{\ell^k}(J_\alpha)/\Delta_1(J_\alpha)),$$

and the parity of the number $\mathrm{ord}_p(\kappa(X_{\ell^k}))$, for all $k \geq 1$, depends only on the parity of $\mathrm{ord}_p(\kappa(X))$. This remark explains the parity of the p-adic valuation of the number of spanning trees in various $\mathbb{Z}_\ell$-towers appearing in [39, 40, 57] and [35], since in each case, the tower in question is constructed from a voltage assignment $\alpha\colon \mathbf{E}_X \to \mathbb{Z}_\ell$ such that $\alpha(\mathbf{E}_X) \subseteq \mathbb{Z}$. We point out as well that (3-12) is compatible with various results in the literature, such as [31, Theorem 5.5], [42, Theorem 5.5] and [41, Theorem 3].

3.4. An explicit example. Let us revisit [57, Example 2] using the results proven in the present paper. Consider the bouquet graph $X = B_2$ on two loops and pick an orientation $S = \{s_1, s_2\}$. Consider the function $\alpha : S \to \mathbb{Z}$ given by $\alpha(s_1) = 3$ and $\alpha(s_2) = 5$. Note that $\alpha(s_1^2 \cdot \bar{s}_2) = 1$, and thus Theorem 3.2 implies that $X(\mathbb{Z}, \alpha)$ is connected. Therefore, so are all the finite graphs

$$X_n := X(\mathbb{Z}/n\mathbb{Z}, \pi_n \circ \alpha),$$

where $\pi_n\colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ denotes the canonical projection map. The infinite graph $X(\mathbb{Z}, \alpha)$ is a connected 4-regular graph that is not a tree, but it is a quotient of the infinite 4-regular tree. All the finite graphs X_n are finite quotients of $X(\mathbb{Z}, \alpha)$. Moreover, one can draw each of these graphs X_n, using their definition, and we did so for $n \in \{1, \ldots, 10, 25, 27\}$ in Figure 1, where we also drew a line $X_n \to X_m$ whenever $m \mid n$.

Note in particular that the $\mathbb{Z}_2$-tower considered in [57, Example 2] corresponds to the leftmost column of the previous figure. For the $\mathbb{Z}$-tower considered in the present example, the Ihara polynomial is given by

$$\mathcal{I}_\alpha(t) = 4 - (t^3 + t^{-3}) - (t^5 + t^{-5}) = t^{-5} \cdot (t-1)^2 \cdot J_\alpha(t),$$

with $J_\alpha(t) = -(t^8 + 2t^7 + 4t^6 + 6t^5 + 8t^4 + 6t^3 + 4t^2 + 2t + 1)$. Using SAGEMATH [54], for each $n \in \{1, \ldots, 10\}$ we computed the number of spanning trees $\kappa(X_n)$, the resultants $\mathrm{Res}(I_\alpha(t), (t^n - 1)/(t-1))$ and the values of the Pierce–Lehmer sequence $\Delta_n(J_\alpha)$. Doing so, we obtained the values that are tabulated in the following table, which

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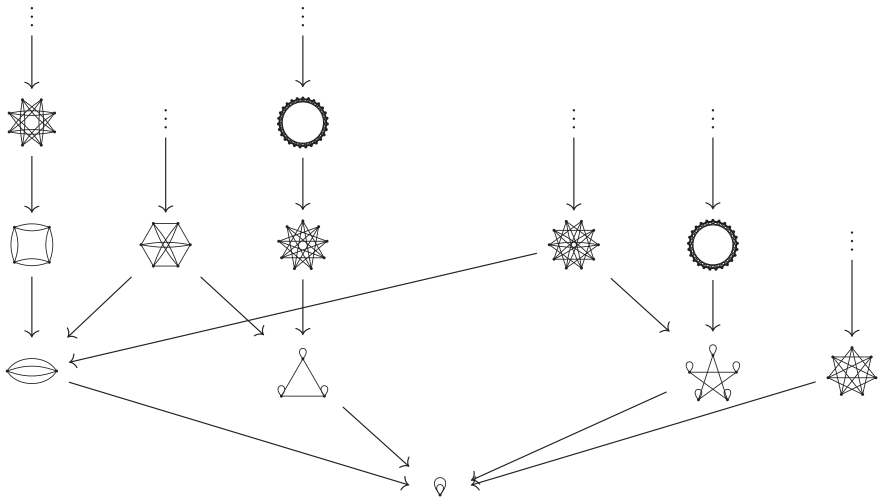


FIGURE 1. The $\mathbb{Z}$-tower of graphs described in Section 3.4.

shows in particular that the relationship between these invariants is the one predicted by (3-12):

n	1	2	3	4	5	6	7	8	9	10
$\kappa(X_n)$	1	4	3	32	5	300	1183	1024	12321	16820
$\mathrm{Res}(I_\alpha(t), \frac{t^n-1}{t-1})$	1	−8	9	−128	25	−1800	8281	−8192	110889	−168200
$\Delta_n(J_\alpha)$	−34	68	−34	272	−34	1700	−5746	4352	−46546	57188

3.5. Asymptotics for the number of spanning trees. The aim of this subsection is to obtain some asymptotic results for the number of spanning trees in a $\mathbb{Z}$-tower of graphs, using the relation between the number of spanning trees and Pierce–Lehmer sequences, provided by Theorem 3.6, in combination with the asymptotic results for Pierce–Lehmer sequences, which we explored in Section 2.3.

3.5.1. Archimedean asymptotics. We will start from the Archimedean asymptotics of the number of spanning trees, which are provided by the following corollary of Theorem 3.6.

COROLLARY 3.10. *Let X be a finite connected graph such that $\chi(X) \neq 0$ and fix a voltage assignment $\alpha\colon \mathbf{E}_X \to \mathbb{Z}$ such that $X(\mathbb{Z}, \alpha)$ is connected. Then, if the polynomial J_α defined by (3-11) does not have any root on the unit circle of $\mathbb{C}$,*

$$\kappa(X_n) \sim n^{e-1} \frac{\kappa(X)}{|\Delta_1(J_\alpha)|} M_\infty(I_\alpha)^n$$

as $n \to +\infty$.

PROOF. This follows directly by combining (3-12) with (2-16). □

REMARK 3.11. Let us note that given an Ihara polynomial $\mathcal{I}_\alpha$ associated to some voltage assignment α, either $M_\infty(\mathcal{I}_\alpha) = 1$ or $M_\infty(\mathcal{I}_\alpha) \geq 2$, as was proved in [50, Proposition 12.7].

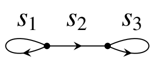
REMARK 3.12. It is reasonable to ask what happens when the Ihara polynomial $\mathcal{I}_\alpha$ has some roots on the Archimedean unit circle. In this case, it is easy to see that these roots will prevent one from getting a precise asymptotic for the growth of $\kappa(X_n)$. To see this, fix some $\alpha = e^{2\pi i\theta} \in \mathbb{C}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then, the sequence

$$|\alpha^n - 1|_\infty^2 = 2(1 - \cos(2\pi n\theta))$$

is distributed on the interval $[0, 4]$ according to the probability density function $(2/\pi)\sqrt{4x - x^2}$, thanks to Weyl's equidistribution theorem [30, Ch. 1, Example 2.1], and to the explicit computation of the probability density function of the random variable $2(1 - \cos(2\pi U))$, where U is a random variable that is uniformly distributed in the interval $[0, 1]$, which follows from the general transformation formula for probability density functions (see [8, Theorem 3.8.4]). Therefore, we see that any asymptotic expansion for the growth of $\kappa(X_n)$ would have to feature some oscillating term, which takes into account this equidistribution phenomenon.

The previous Corollary 3.10 allows us to recover the asymptotics for the number of spanning trees of two particular examples of $\mathbb{Z}$-towers, which were thoroughly studied in [41, 42].

EXAMPLE 3.13. Consider the following orientation $S = \{s_1, s_2, s_3\}$ on the dumbbell graph:



and fix a function $\alpha : S \to \mathbb{Z}$ such that $\alpha(s_2) = 0$. This defines a voltage assignment on the dumbbell graph X, with values in $\mathbb{Z}$. Moreover, the derived covers $X_n := X(\mathbb{Z}/n\mathbb{Z}, \pi_n \circ \alpha)$, where $\pi_n : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ denotes the canonical projection, are given by the I-graphs $I(n, k, l)$, where $k := \alpha(s_1)$ and $l := \alpha(s_3)$. In particular, if $k = 1$, one gets the family of generalised Petersen graphs $\mathrm{GP}(n, l)$.

Now, one sees from Theorem 3.2 that the graph $X(\mathbb{Z}, \alpha)$ is connected if and only if $(k, l) = 1$, which we will assume for the rest of this example. Then, we can compute the Ihara polynomial associated to the voltage assignment α, which is given by

$$\mathcal{I}_\alpha(t) = (3 - t^k - t^{-k})(3 - t^l - t^{-l}) - 1 = t^{-(k+l)} \cdot I_\alpha(t)$$

where $I_\alpha(t) = (3t^k - t^{2k} - 1) \cdot (3t^l - t^{2l} - 1) - t^{k+l}$. Since $I_\alpha(1) = I'_\alpha(1) = 0$, while $I''_\alpha(1) \neq 0$, we see that $e := \mathrm{ord}_{t=1}(\mathcal{I}_\alpha) = 2$, which allows us to write

$$I_\alpha(t) = (t - 1)^2 \cdot J_\alpha(t)$$

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for some $J_\alpha \in \mathbb{Z}[t]$ such that $J_\alpha(1) \neq 0$. A simple calculation shows that J_α has no roots on the unit circle, as explained for instance in [42, Lemma 5.2]. Moreover, it is easy to see that

$$|\Delta_1(J_\alpha)| = |J_\alpha(1)| = |I''_\alpha(1)|/2 = k^2 + l^2$$

and that $\kappa(X) = 1$. Therefore, Corollary 3.10 shows that

$$\kappa(I(n,k,l)) \sim \frac{n}{k^2 + l^2} \cdot M_\infty(I_\alpha)^n$$

as $n \to \infty$, which is precisely [42, Theorem 6.1].

Example 3.14. Consider the graph X consisting of a bouquet with k loops for some $k \in \mathbb{N}$, and take an orientation $S = \{s_1, \dots, s_k\}$ of X. Moreover, fix any function $\alpha : S \to \mathbb{Z}$ such that

$$1 \leq \alpha(s_1) < \alpha(s_2) < \cdots < \alpha(s_k)$$

and write $a_i := \alpha(s_i)$ for every $i \in \{1, \dots, k\}$. Then, for n large enough, the derived graph $X_n := X(\mathbb{Z}/n\mathbb{Z}, \pi_n \circ \alpha)$ is the circulant graph $C_n(a_1, \dots, a_k)$. Note in particular that the example described in Section 3.4 belongs to this more general family.

Once again, it is easy to see by Theorem 3.2 that the graph $X(\mathbb{Z}, \alpha)$ is connected if and only if $(a_1, \dots, a_k) = 1$, which we will assume for the rest of this example. Then, we can compute the Ihara polynomial associated to the voltage assignment α, and we obtain

$$\mathcal{I}_\alpha(t) = 2k - \sum_{i=1}^{k} (t^{a_i} + t^{-a_i}) = t^{-a_k} I_\alpha(t),$$

where $I_\alpha(t) := 2kt^{a_k} - \sum_{i=1}^{k} (t^{a_k + a_i} + t^{a_k - a_i}) \in \mathbb{Z}[t]$. In particular, we easily see that

$$I_\alpha(1) = I'_\alpha(1) = 0 \neq I''_\alpha(1),$$

which implies that $e := \mathrm{ord}_{t=1}(\mathcal{I}_\alpha) = 2$, and that $I_\alpha(t) = (t-1)^2 \cdot J_\alpha(t)$ for some $J_\alpha \in \mathbb{Z}[t]$ satisfying $J_\alpha(1) \neq 0$. Moreover, one can easily show that J_α does not have any root on the unit circle of $\mathbb{C}$, as explained in [41, Lemma 2]. Finally, we see that $\kappa(X) = 1$ and

$$|\Delta_1(J_\alpha)| = |J_\alpha(1)| = |I''_\alpha(1)|/2 = \sum_{i=1}^{k} a_i^2$$

which, thanks to Corollary 3.10, implies that

$$\kappa(C_n(a_1, \dots, a_k)) \sim \frac{n}{\sum_{i=1}^{k} a_i^2} \cdot M_\infty(I_\alpha)^n$$

as $n \to \infty$, which is [41, Theorem 5] in the particular case when $d = 1$.

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3.5.2. p-adic asymptotics. Let us look at the asymptotics of the p-adic valuations of the number of spanning trees in a $\mathbb{Z}$-tower, for a fixed prime p. As in the Archimedean setting, we start from the case when the Ihara polynomial $\mathcal{I}_\alpha$ does not have any nontrivial root lying on the unit circle of $\mathbb{C}_p$.

COROLLARY 3.15. *Let X be a finite connected graph such that $\chi(X) \neq 0$, and fix a voltage assignment $\alpha\colon \mathbf{E}_X \to \mathbb{Z}$ such that $X(\mathbb{Z}, \alpha)$ is connected. Fix moreover a rational prime $p \in \mathbb{N}$. Then, if the polynomial J_α defined by (3-11) has no root on the unit circle of $\mathbb{C}_p$,*

$$|\kappa(X_n)|_p = |n|_p^{e-1} \frac{|\kappa(X)|_p}{|\Delta_1(J_\alpha)|_p} M_p(I_\alpha)^n$$

for every $n \in \mathbb{N}$, where $e := \mathrm{ord}_{t=1}(\mathcal{I}_\alpha)$ is the order of vanishing at $t = 1$ of the Ihara polynomial associated to α.

PROOF. This follows immediately by combining (3-12) with (2-15). □

REMARK 3.16. Let $f(t) = \sum_{j=0}^{d} c_j t^j \in \mathbb{Z}[t]$ be any polynomial. Then, every root of f lies in the unit circle of $\mathbb{C}_p$ whenever $p \nmid c_0 \cdot c_d$. Therefore, we see that, for every given $\mathbb{Z}$-tower of graphs, Corollary 3.15 can be applied only for finitely many primes p.

3.6. Exact formulae for the *p*-adic valuation of the number of spanning trees. The previous remark prompts us to study the case when J_α has some roots on the unit circle of $\mathbb{C}_p$. In this case, we can prove the following result (which is the more precise version of Theorem 1.1 from §1), which gives a partial analogue of Iwasawa's theorem for $\mathbb{Z}$-towers.

THEOREM 3.17. *Let X be a finite connected graph whose Euler characteristic $\chi(X)$ does not vanish, and let $\alpha\colon \mathbf{E}_X \to \mathbb{Z}$ be a voltage assignment such that $X(\mathbb{Z}, \alpha)$ is connected. Let $\mathcal{I}_\alpha \in \mathbb{Z}[t]$ be the Ihara polynomial associated to α, and set*

$$J_\alpha(t) := t^b (t-1)^{-e} \mathcal{I}_\alpha(t),$$

where $b := -\mathrm{ord}_{t=0}(\mathcal{I}_\alpha)$, and $e := \mathrm{ord}_{t=1}(\mathcal{I}_\alpha)$.

Fix now a rational prime $p \in \mathbb{N}$, an algebraic closure $\overline{\mathbb{Q}}_p$ of the field of p-adic numbers, and let O_p be the ring of integers of $\overline{\mathbb{Q}}_p$. Using this notation, we can define the quantities

$$\mu_p(X, \alpha) := -m_p(J_\alpha)/\log(p),$$
$$c_p(X, \alpha) := \mathrm{ord}_p(\kappa(X)) - \mathrm{ord}_p(\Delta_1(J_\alpha)),$$

where $m_p(J_\alpha)$ denotes the logarithmic p-adic Mahler measure of J_α, defined as in (2-3).

Moreover, let $\beta_1, \ldots, \beta_d \in \overline{\mathbb{Q}}_p$ be the p-adic roots of J_α, counted with multiplicity. Then, for every $n \in \mathbb{N}$, we introduce the set

$$B_{p,n}(X, \alpha) := \{\beta \in \overline{\mathbb{Q}}_p : J_\alpha(\beta) = 0,\ |\beta|_p = 1,\ |\beta^n - 1|_p < 1\}, \tag{3-17}$$

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which can be used to define the quantity

$$\lambda_{p,n}(X,\alpha) := \#\{j \in \{1,\dots,d\} \colon \beta_j \in B_{p,n}(X,\alpha)\} + e - 1.$$

Finally, for every $\beta \in \mathcal{O}_p$ *such that* $|\beta|_p = 1$,

$$s_p(\beta) := \min\{s \in \mathbb{Z}_{\geq 0} \colon p^s(p-1)\mathrm{ord}_p(\beta - \tau_p(\pi_p(\beta))) > 1\} \tag{3-18}$$

and for every $n \in \mathbb{N}$, *we set* $r_{p,n}(\beta) := \min(\mathrm{ord}_p(n), s_p(\beta))$, *where* $\tau_p(\pi_p(\beta))$ *denotes the Teichmüller lift of the reduction* $\pi_p(\beta)$ *of* β *modulo the maximal ideal of* $\mathcal{O}_p$. *This can be used to define the quantity*

$$\nu_{p,n}(X,\alpha) := \sum_{\substack{j \in \{1,\dots,d\} \\ \beta_j \in B_{p,n}(X,\alpha)}} (\mathrm{ord}_p(\beta_j^{p^{r_{p,n}(\beta_j)}} - \tau_p(\pi_p(\beta_j))^{p^{r_{p,n}(\beta_j)}}) - r_{p,n}(\beta_j)).$$

Then,

$$\mathrm{ord}_p(\kappa(X_n)) = \mu_p(X,\alpha)\cdot n + \lambda_{p,n}(X,\alpha)\cdot \mathrm{ord}_p(n) + \nu_{p,n}(X,\alpha) + c_p(X,\alpha)$$

for every $n \in \mathbb{N}$.

PROOF. From Theorem 3.6, one has the identity

$$\mathrm{ord}_p(\kappa(X_n)) = \mathrm{ord}_p(\Delta_n(J_\alpha)) + (e-1)\cdot \mathrm{ord}_p(n) + c_p(X,\alpha). \tag{3-19}$$

Moreover, Corollary 2.5 implies that

$$\mathrm{ord}_p(\Delta_n(J_\alpha)) = \mu_p(X,\alpha)\cdot n + A_{p,n}(X,\alpha)\cdot \mathrm{ord}_p(n) + \nu_{p,n}(X,\alpha), \tag{3-20}$$

where $A_{p,n}(X,\alpha) := \#\{j \in \{1,\dots,d\} \colon \beta_j \in B_{p,n}(X,\alpha)\}$, because we have that $B_{p,n}(X,\alpha) = B_{p,n}(J_\alpha)$ and $\nu_{p,n}(X,\alpha) = \nu_{p,n}(J_\alpha)$ by definition. Therefore, we can conclude the proof by combining (3-20) with (3-19). □

Using Theorem 3.17, one can easily show that once we fix the base graph X, the voltage assignment $\alpha \colon \mathbf{E}_X \to \mathbb{Z}$ and the prime number p, we can subdivide $\mathbb{N}$ in a finite number of sequences, given by imposing certain divisibility conditions. Along each of these sequences, the invariant $\mathrm{ord}_p(\kappa(X_n))$ can be computed as a linear form in n and $\mathrm{ord}_p(n)$, as we show more precisely in the following theorem.

THEOREM 3.18. *Let* X *be a finite connected graph whose Euler characteristic* $\chi(X)$ *does not vanish, and* $\alpha \colon \mathbf{E}_X \to \mathbb{Z}$ *be a voltage assignment such that for every* $n \geq 1$, *the finite graph* $X_n := X(\mathbb{Z}/n\mathbb{Z}, \alpha_n)$ *is connected (which can be checked using Theorem 3.2). Moreover, for every prime* $p \in \mathbb{Z}$,

$$\mu_p(X,\alpha) := -m_p(\mathcal{I}_\alpha)/\log(p),$$

where $m_p(\mathcal{I}_\alpha)$ *denotes the logarithmic p-adic Mahler measure of the Ihara polynomial* $\mathcal{I}_\alpha$. *Then, for every rational prime p, there exist a finite set* $\mathcal{N}_p(X,\alpha) \subseteq \mathbb{N}$ *of integers coprime to p, and an integer* $R_p(X,\alpha) \geq 0$ *such that for every* $\mathfrak{n} \subseteq \mathcal{N}_p(X,\alpha)$ *and every*

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$r \in \{0, \dots, R_p(X, \alpha)\}$, there exist two integers $\lambda_p(X, \alpha, \mathfrak{n}) \geq 0$ and $\nu_p(X, \alpha, \mathfrak{n}, r)$ such that

$$\mathrm{ord}_p(\kappa(X_n)) = \mu_p(X, \alpha) \cdot n + \lambda_p(X, \alpha, \mathfrak{n}) \cdot \mathrm{ord}_p(n) + \nu_p(X, \alpha, \mathfrak{n}, r) \tag{3-21}$$

for every $n \in \mathcal{S}_p(X, \alpha, \mathfrak{n}, r)$, where $\mathcal{S}_p(X, \alpha, \mathfrak{n}, r)$ consists of those $n \in \mathbb{N}$ such that:

- *$N \mid n$ for each $N \in \mathfrak{n}$;*
- *$N' \nmid n$ for each $N' \in \mathcal{N}_p(X, \alpha) \setminus \mathfrak{n}$;*
- *$\mathrm{ord}_p(n) = r$ if $r < R_p(X, \alpha)$, or $\mathrm{ord}_p(n) \geq R_p(X, \alpha)$ if $r = R_p(X, \alpha)$.*

Moreover, the finite set $\mathcal{N}_p(X, \alpha)$, the integer $R_p(X, \alpha)$, and the two invariants $\lambda_p(X, \alpha, \mathfrak{n})$ and $\nu_p(X, \alpha, \mathfrak{n}, r)$ can be explicitly computed in terms of the polynomial $\mathcal{I}_\alpha$.

Proof. Fix a finite connected graph X and a voltage assignment $\alpha\colon \mathbf{E}_X \to \mathbb{Z}$. Moreover, fix a rational prime p and let $\beta_1, \dots, \beta_d$ denote the roots of J_α, counted with multiplicities, that lie on the unit circle of $\mathbb{C}_p$, and let $N_1, \dots, N_d$ denote the multiplicative orders of $\pi_p(\beta_1), \dots, \pi_p(\beta_d)$ in $\overline{\mathbb{F}}_p^\times$. Then,

$$\mathcal{N}_p(X, \alpha) := \{N_1, \dots, N_d\},$$
$$R_p(X, \alpha) := \max_{j=1,\dots,d} s_p(\beta_j),$$

where $s_p(\beta_j)$ is defined as in (2-13). Moreover, for every subset $\mathfrak{n} \subseteq \mathcal{N}_p(X, \alpha)$,

$$\lambda_p(X, \alpha, \mathfrak{n}) := \#\mathfrak{n} + e - 1,$$

as we will now show.

First of all, suppose that $\mathfrak{n} = \emptyset$. In other words, let us take an integer $n \in \mathbb{N}$ such that $N_1 \nmid n, \dots, N_d \nmid n$. Then, the set $B_{p,n}(X, \alpha)$ defined in (3-17) is empty. Therefore, Theorem 3.17 shows that for every $r \in \{0, \dots, R_p(X, \alpha)\}$,

$$\nu_p(X, \alpha, \emptyset, r) := c_p(X, \alpha)$$

and (3-21) will hold true.

Now let us suppose that $\mathfrak{n} \neq \emptyset$ and let $J \subseteq \{1, \dots, d\}$ be the unique nonempty subset such that $\mathfrak{n} = \{N_j : j \in J\}$. Then, if we suppose in addition that $p \neq 2$ and $p \nmid \mathrm{Disc}(J_\alpha)$, we have that $B_{p,n}(X, \alpha) = \{\beta_j : j \in J\}$. Moreover, the quantity $r_{p,n}(\beta)$, which was defined in (3-18), vanishes whenever $\beta \in B_{p,n}(X, \alpha)$, because $\mathrm{ord}_p(\beta - \tau_p(\pi_p(\beta))) \in \mathbb{N}$ and $p - 1 > 1$ in this case. Therefore, Theorem 3.17 shows that for every prime $p \neq 2$ such that $p \nmid \mathrm{Disc}(J_\alpha)$ and every $r \in \{0, \dots, R_p(X, \alpha)\}$,

$$\nu_p(X, \alpha, \mathfrak{n}, r) := c_p(X, \alpha) + \sum_{j \in J} \mathrm{ord}_p(\beta_j - \tau_p(\pi_p(\beta_j))),$$

and (3-21) will hold true. However, if $p = 2$ and $\mathrm{Disc}(J_\alpha)$ is odd, we still have that $B_{p,n}(X, \alpha) = \{\beta_j : j \in J\}$ and $\mathrm{ord}_p(\beta - \tau_p(\pi_p(\beta))) \in \mathbb{N}$ for every $\beta \in B_{p,n}(X, \alpha)$.

Therefore, we see that for every $\beta \in B_{p,n}(X, \alpha)$, we have $r_{p,n}(\beta) = 0$ if $2 \nmid n$, and $r_{p,n}(\beta) \leq 1$ otherwise, because $\mathrm{Disc}(J_\alpha)$ is assumed to be odd. In other words, if $r = 0$,

$$\nu_2(X, \alpha, \mathfrak{n}, 0) := c_2(X, \alpha) + \sum_{j \in J} \mathrm{ord}_2(\beta_j - \tau_2(\pi_2(\beta_j))),$$

while if $r \geq 1$,

$$\begin{aligned}\nu_2(X, \alpha, \mathfrak{n}, r) := c_2(X, \alpha) &\\ &+ \sum_{\substack{j \in J \\ \mathrm{ord}_2(\beta_j - \tau_2(\pi_2(\beta_j)))=1}} (\mathrm{ord}_2(\beta_j^2 - \tau_2(\pi_2(\beta_j))^2) - 1) \\ &+ \sum_{\substack{j \in J \\ \mathrm{ord}_2(\beta_j - \tau_2(\pi_2(\beta_j)))\geq 2}} \mathrm{ord}_2(\beta_j - \tau_2(\pi_2(\beta_j)))\end{aligned}$$

and Theorem 3.17 will ensure that (3-21) holds true when $p = 2$.

To conclude, let us assume that $p \mid \mathrm{Disc}(J_\alpha)$ and let again $\mathfrak{n} = \{N_j : j \in J\}$ for some nonempty $J \subseteq \{1, \ldots, d\}$, so that $B_{p,n}(X, \alpha) = \{\beta_j : j \in J\}$. Then, if $r = R_p(X, \alpha)$, which implies that $\mathrm{ord}_p(n) \geq R_p(X, \alpha)$,

$$r_{p,n}(\beta) = s_p(\beta) := \min\{s \in \mathbb{Z}_{\geq 0} : p^s(p - 1)\mathrm{ord}_p(\beta - \tau_p(\pi_p(\beta))) > 1\}$$

for every $\beta \in B_{p,n}(X, \alpha)$. Therefore, Theorem 3.17 guarantees that if we take $\nu_p(X, \alpha, \mathfrak{n}, R_p(X, \alpha))$ to be

$$c_p(X, \alpha) + \sum_{j \in J} (\mathrm{ord}_p(\beta_j^{p^{s_p(\beta_j)}} - \tau_p(\pi_p(\beta_j))^{p^{s_p(\beta_j)}}) - s_p(\beta_j)),$$

the identity (3-21) will hold true. Finally, if we fix $r \in \{0, \ldots, R_p(X, \alpha) - 1\}$, we can take $\nu_p(X, \alpha, \mathfrak{n}, r)$ to be

$$c_p(X, \alpha) + \sum_{j \in J} (\mathrm{ord}_p(\beta_j^{p^{\min(r, s_p(\beta_j))}} - \tau_p(\pi_p(\beta_j))^{p^{\min(r, s_p(\beta_j))}}) - \min(r, s_p(\beta_j)))$$

and Theorem 3.17 still guarantees that (3-21) holds true. □

REMARK 3.19. The previous proof shows that the p-adic valuation of the number of spanning trees is actually constant along many of the sequences $\mathcal{S}_p(X, \alpha, \mathfrak{n}, r)$. More precisely, let $\|J_\alpha\|$ be the greatest common divisor of the coefficients of J_α. Then, if p is a prime such that

$$p \nmid \mathrm{Disc}(J_\alpha) \cdot \|J_\alpha\|,$$

for every $\mathfrak{n} = \{N_j : j \in J\} \subseteq \mathcal{N}_p(X, \alpha)$,

$$\mathrm{ord}_p(\kappa(X_n)) = c_p(X, \alpha) + \sum_{j \in J} \mathrm{ord}_p(\beta_j - \tau_p(\pi_p(\beta_j)))$$

whenever $n \in \mathcal{S}_p(X, \alpha, \mathfrak{n}, 0)$.

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REMARK 3.20. It is clear from the proof of Theorem 1.1 that the multiplicative orders $N_1, \ldots, N_d$ play a crucial role in the understanding of the evolution of the p-adic valuation of the number of spanning trees along a $\mathbb{Z}$-tower. Therefore, it would be nice to know how these orders vary with the prime number p. This can be understood in terms of a far reaching generalisation of Artin's primitive root conjecture, due to Lenstra [37], which is known to hold under the assumption of the generalised Riemann hypothesis.

3.7. The Fibonacci tower. To conclude this paper, let us note how the formula provided by Theorem 3.17 generalises a famous formula for the p-adic valuation of the Fibonacci numbers, due to Lengyel [36]. More precisely, if in Example 3.14 we take the bouquet on two loops X, with an orientation $S = \{s_1, s_2\}$, and we let α be the unique voltage assignment $\alpha : \mathbf{E}_X \to \mathbb{Z}$ such that $\alpha(s_1) = 1$ and $\alpha(s_2) = 2$, then we obtain the $\mathbb{Z}$-tower portrayed in Figure 2.

It turns out that the number of spanning trees of the finite layers of this tower is intimately related to the sequence of Fibonacci numbers. To show this, let us observe that $\kappa(X) = 1$ and

$$\mathcal{I}_\alpha(t) := 4 - (t + t^{-1}) - (t^2 + t^{-2}),$$

which implies that $e = 2$ and $J_\alpha(t) = -(t^2 + 3t + 1)$. We denote by $\beta_1 = (-3 - \sqrt{5})/2$ and $\beta_2 = (-3 + \sqrt{5})/2$ the roots of J_α, and we observe that $\Delta_1(J_\alpha) = -5$.

Then, Theorem 3.6 can be combined with a simple computation to show that

$$\kappa(X_n) = n\frac{\Delta_n(J_\alpha)}{\Delta_1(J_\alpha)} = n\frac{(-1)^n(\beta_1^n - 1)(\beta_2^n - 1)}{-5} = nF_n^2,$$

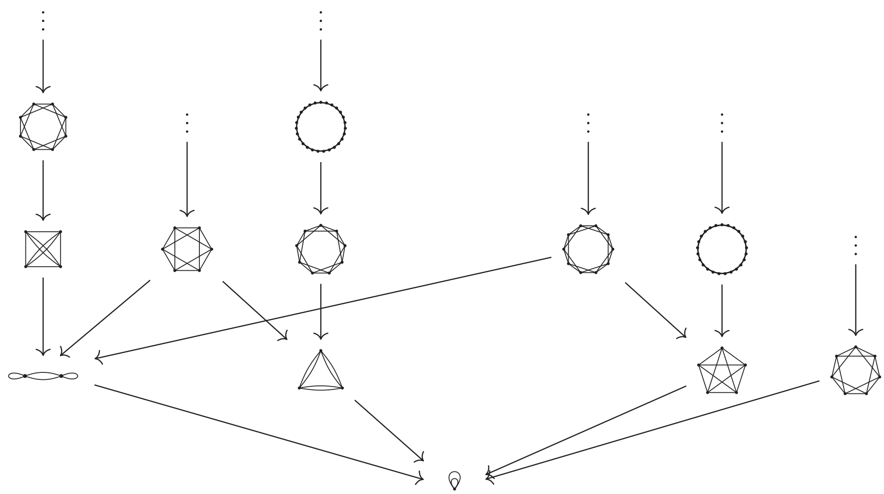

FIGURE 2. The $\mathbb{Z}$-tower of graphs described in Section 3.7.

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where F_n is the nth Fibonacci number. Therefore, Theorem 3.17 implies that

$$2 \cdot \mathrm{ord}_p(F_n) = \mathrm{ord}_p(\kappa(X_n)) - \mathrm{ord}_p(n) = \#B_{p,n}(X,\alpha) \cdot \mathrm{ord}_p(n) + \nu_{p,n}(X,\alpha)$$

for every prime $p \neq 5$ and every $n \in \mathbb{N}$, because $m_p(J_\alpha) = 0$ for every prime $p \in \mathbb{N}$.

Now, let us note that the two roots β_1 and β_2 of the polynomial J_α are both reciprocal units, which implies that the set $B_{p,n}(X,\alpha)$ is either empty or consists of the two roots $\{\beta_1, \beta_2\}$. The latter scenario occurs if and only if n is a multiple of the multiplicative order N_p of β_1 (and β_2) in $\overline{\mathbb{F}}_p^\times$. Therefore, Theorem 3.17 shows that N_p coincides with the so called rank of apparition of the prime p, that is, with the smallest index n such that $p \mid F_n$. Since there exists $k \in \mathbb{N}$ such that $N_p \mid p^k - 1$, we see that $\mathrm{ord}_p(N_p) = 0$ and thus that $\sum_{j=1}^{2} \mathrm{ord}_p(\beta_j - \tau_p(\pi_p(\beta_j))) = 2 \cdot \mathrm{ord}_p(F_{N_p})$, provided one assumes moreover that $p \neq 2$. Hence, these considerations entail that

$$\mathrm{ord}_p(F_n) = \begin{cases} \mathrm{ord}_p(n) + \mathrm{ord}_p(F_{N_p}) & \text{if } N_p \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

for every $p \neq 2, 5$, as was proven in [36, Section 3].

To conclude this example, and this paper, let us see what happens when $p = 2$ and $p = 5$. In the first case, when $p = 2$, we can observe that J_α has no roots in $\mathbb{F}_2$, which implies necessarily that $N_2 = 3$. Moreover, to compute the Teichmüller representatives $\tau_2(\pi_2(\beta_j))$, for $j \in \{1, 2\}$, we can work globally and consider the number field $K = \mathbb{Q}(\sqrt{5}, \zeta_3)$, where ζ_3 is a primitive third root of unity. A simple calculation shows that 2 decomposes as a product of two primes in K both with inertia degree 2 and ramification degree 1. Using SageMath [54], or by hand, one calculates that $(\beta_1 - \zeta_3) = \mathfrak{p}$, where $\mathfrak{p}$ is one of the two primes lying above 2. If we denote the other prime lying above 2 by $\mathfrak{q}$, then we also have $(\beta_1 - \zeta_3^2) = \mathfrak{q}$. Moreover, we have $(\beta_1^2 - \zeta_3^2) = \mathfrak{p}^3$ and $(\beta_1^2 - \zeta_3^4) = \mathfrak{q}^3$. A similar calculation can be performed for β_2. After embedding K into $\overline{\mathbb{Q}}_2$ with any embedding, these calculations show that $N_2 = 3$ and

$$\mathrm{ord}_2(\beta_j - \tau_2(\pi_2(\beta_j))) = 1, \quad \mathrm{ord}_2(\beta_j^2 - \tau_2(\pi_2(\beta_j^2))) = 3,$$

for every $j \in \{1, 2\}$. Finally, one can check easily that $\mathrm{ord}_2(F_3) = 1$. Thus, (3-18) implies that

$$\mathrm{ord}_2(F_n) = \begin{cases} 0 & \text{if } n \equiv 1, 2 \pmod{3}, \\ 1 & \text{if } n \equiv 3 \pmod{6}, \\ \mathrm{ord}_2(n) + 2 & \text{if } n \equiv 0 \pmod{6}, \end{cases}$$

which was proven in [36, Lemma 2].

Let us now suppose that $p = 5$. Since $-J_\alpha \equiv (t-1)^2 \pmod{5}$, we see that the multiplicative order of β_1 and β_2 in $\overline{\mathbb{F}}_5^\times$ is 1, which is not the rank of apparition of the prime 5. Moreover, one has that $\mathrm{ord}_5(\beta_j - \tau_5(\pi_5(\beta_j))) = 1/2$. Indeed, $\tau_5(\pi_5(\beta_j)) = \tau_5(1) = 1$. Hence, $\mathrm{ord}_5(\beta_j - \tau_5(\pi_5(\beta_j))) = \mathrm{ord}_5(\sqrt{5}) = 1/2$ for every $j \in \{1, 2\}$, as we

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wanted to show. Finally, let us observe that $s_5(\beta_j) = 0$ for every $j \in \{1, 2\}$. Combining this with the fact that $c_5(X, \alpha) = -1$,

$$\mathrm{ord}_5(F_n) = \mathrm{ord}_5(n) + \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{2} = \mathrm{ord}_5(n)$$

for every $n \in \mathbb{N}$, as was proven in [36, Lemma 1]. This shows that Theorem 3.17 can be seen as a generalisation of Lengyel's theorem to sequences that arise as the number of spanning trees in a $\mathbb{Z}$-cover of finite graphs.

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References

[1] H. Bambury, F. Campagna and F. Pazuki, 'Ordinary isogeny graphs over $\mathbb{F}_p$: the inverse volcano problem', *Ann. Sc. Norm. Super. di Pisa Cl. Sci. (5)*, to appear.
[2] H. Bass, 'The Ihara–Selberg zeta function of a tree lattice', *Internat. J. Math.* **3**(6) (1992), 717–797.
[3] A. Besser and C. Deninger, 'p-adic Mahler measures', *J. reine angew. Math.* **517** (1999), 19–50.
[4] Y. Bilu, F. Luca, J. Nieuwveld, J. Ouaknine and J. Worrell, 'On the p-adic zeros of the Tribonacci sequence', *Math. Comp.* **93** (2023), 21 pages.
[5] H. Cohen, *A Course in Computational Algebraic Number Theory*, Graduate Texts in Mathematics, 138 (Springer-Verlag, Berlin, 1993).
[6] S. Corry and D. Perkinson, *Divisors and Sandpiles: An Introduction to Chip–Firing*, AMS Non-Series Monographs, 114 (American Mathematical Society, Providence, RI, 2018).
[7] A. A. Cuoco and P. Monsky, 'Class numbers in Z_p^d-extensions', *Math. Ann.* **255**(2) (1981), 235–258.
[8] M. H. DeGroot and M. J. Schervish, *Probability and Statistics*, 4th edn (Addison-Wesley, Boston, MA, 2012).
[9] S. DuBose and D. Vallières, 'On $\mathbb{Z}_\ell^d$-towers of graphs', *Algebr. Comb.* **6**(5) (2023), 1331–1346.
[10] M. Einsiedler, G. Everest and T. Ward, 'Primes in sequences associated to polynomials (after Lehmer)', *LMS J. Comput. Math.* **3** (2000), 125–139.
[11] G. Everest and T. Ward, *Heights of Polynomials and Entropy in Algebraic Dynamics*, Universitext (Springer-Verlag, London, 1999).
[12] A. Flatters, 'Primitive divisors of some Lehmer–Pierce sequences', *J. Number Theory* **129**(1) (2009), 209–219.
[13] E. C. Friedman, 'Ideal class groups in basic $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}$-extensions of abelian number fields', *Invent. Math.* **65**(3) (1981/82), 425–440.
[14] T. Fukaya and K. Kato, *A Formulation of Conjectures on p-adic Zeta Functions in Noncommutative Iwasawa Theory* (American Mathematical Society, Providence, RI, 2006), 1–85.
[15] S. R. Gonet, 'Iwasawa theory of Jacobians of graphs', *Algebr. Comb.* **5**(5) (2022), 827–848.
[16] S. R. Gonet, 'Jacobians of finite and infinite voltage covers of graphs', PhD Thesis, The University of Vermont and State Agricultural College, 2021.
[17] F. González-Acuña and H. Short, 'Cyclic branched coverings of knots and homology spheres', *Rev. Mat. Complut.* **4**(1) (1991), 97–120.

[18] R. Greenberg, 'Iwasawa theory—past and present', in: *Class Field Theory—Its Centenary and Prospect (Tokyo, 1998)*, Advanced Studies in Pure Mathematics, 30 (ed. K. Miyake) (Mathematical Society of Japan, Tokyo, 2001), 335–385.

[19] J. L. Gross, 'Voltage graphs', *Discrete Math.* **9** (1974), 239–246.

[20] K. Hashimoto, 'On zeta and L-functions of finite graphs', *Internat. J. Math.* **1**(4) (1990), 381–396.

[21] J. Hillman, D. Matei and M. Morishita, *Pro-p Link Groups and p-Homology Groups*, Contemporary Mathematics, 416 (American Mathematical Society, Providence, RI, 2006), 121–136.

[22] K. Iwasawa, 'On Γ-extensions of algebraic number fields', *Bull. Amer. Math. Soc. (N.S.)* **65** (1959), 183–226.

[23] Q. Ji and H. Qin, 'The numerical factors of $\Delta_n(f, g)$', *Indian J. Pure Appl. Math.* **46**(5) (2015), 701–714.

[24] T. Kadokami and Y. Mizusawa, 'Iwasawa type formula for covers of a link in a rational homology sphere', *J. Knot Theory Ramifications* **17**(10) (2008), 1199–1221.

[25] M. Kakde, 'From the classical to the noncommutative Iwasawa theory (for totally real number fields)', in: *Non-abelian Fundamental Groups and Iwasawa Theory*, London Mathematical Society Lecture Note Series, 393 (eds. F. Pop, J. Coates, M. Kim, M. Saïdi, and P. Schneider) (Cambridge University Press, Cambridge, 2011), 107–131.

[26] T. Kataoka, 'Fitting ideals of Jacobian groups of graphs', *Algebr. Comb.* **7** (2024), 597–625.

[27] S. Kleine and K. Müller, 'On the growth of the Jacobian in $\mathbb{Z}_p^l$-voltage covers of graphs', *Algebr. Comb.* **7** (2024), 1011–1038.

[28] S. Kleine and K. Müller, 'On the non-commutative Iwasawa main conjecture for voltage covers of graphs', *Israel J. Math.*, to appear; arXiv:2307.15395.

[29] M. Kotani and T. Sunada, 'Zeta functions of finite graphs', *J. Math. Sci. Univ. Tokyo* **7**(1) (2000), 7–25.

[30] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Pure and Applied Mathematics (Wiley, New York–London–Sydney, 1974).

[31] Y. S. Kwon, A. D. Mednykh and I. A. Mednykh, 'On Jacobian group and complexity of the generalized Petersen graph $GP(n, k)$ through Chebyshev polynomials', *Linear Algebra Appl.* **529** (2017), 355–373.

[32] D. H. Lehmer, 'Factorization of certain cyclotomic functions', *Ann. of Math. (2)* **34**(3) (1933), 461–479.

[33] A. Lei and K. Müller, 'On ordinary isogeny graphs with level structure', *Expo. Math.* **42** (2024), 125589.

[34] A. Lei and K. Müller, 'On towers of isogeny graphs with full level structure', Preprint, 2023, arXiv:2309.00524.

[35] A. Lei and D. Vallières, 'The non-ℓ-part of the number of spanning trees in abelian ℓ-towers of multigraphs', *Res. Number Theory* **9**(1) (2023), Paper no. 18, 16 pages.

[36] T. Lengyel, 'The order of the Fibonacci and Lucas numbers', *Fibonacci Quart.* **33**(3) (1995), 234–239.

[37] H. W. Lenstra, 'On Artin's conjecture and Euclid's algorithm in global fields', *Invent. Math.* **42**(1) (1977), 201–224.

[38] K. Mahler, 'On some inequalities for polynomials in several variables', *J. Lond. Math. Soc. (2)* s1–37(1) (1962), 341–344.

[39] K. McGown and D. Vallières, 'On abelian ℓ-towers of multigraphs II', *Ann. Math. Qué.* **47**(2) (2023), 461–473.

[40] K. McGown and D. Vallières, 'On abelian ℓ-towers of multigraphs III', *Ann. Math. Qué.* **48**(1) (2024), 1–19.

[41] A. D. Mednykh and I. A. Mednykh, 'The number of spanning trees in circulant graphs, its arithmetic properties and asymptotic', *Discrete Math.* **342**(6) (2019), 1772–1781.

[42] I. A. Mednykh, 'On Jacobian group and complexity of the I-graph $I(n, k, l)$ through Chebyshev polynomials', *Ars Math. Contemp.* **15**(2) (2018), 467–485.

[43] M. Morishita, *Knots and Primes. An Introduction to Arithmetic Topology. Based on the Japanese Original (Springer, 2009)*, Universitext (Springer, Berlin, 2012).
[44] T. A. Pierce, 'The numerical factors of the arithmetic forms $\prod_{i=1}^{n}(1 \pm \alpha_i^m)$', *Ann. of Math. (2)* **18**(2) (1916), 53–64.
[45] A. Ray and D. Vallières, 'An analogue of Kida's formula in graph theory', Preprint, 2022, arXiv:2209.04890.
[46] R. Riley, 'Growth of order of homology of cyclic branched covers of knots', *Bull. Lond. Math. Soc.* **22**(3) (1990), 287–297.
[47] M. Rosen, *Number Theory in Function Fields*, Graduate Texts in Mathematics, 210 (Springer-Verlag, New York, 2002).
[48] J.-P. Serre, *Arbres, Amalgames*, SL_2, Astérisque, 46, Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass (Société Mathématique de France, Paris, 1977).
[49] J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Mathematics, 42 (Springer-Verlag, New York–Heidelberg, 1977).
[50] D. S. Silver and S. G. Williams, 'Lehmer's question, graph complexity growth and links', *New York J. Math.* **27** (2021), 981–1008.
[51] H. M. Stark and A. A. Terras, 'Zeta functions of finite graphs and coverings. II', *Adv. Math.* **154**(1) (20000), 132–195.
[52] T. Sunada, *Topological Crystallography*, Surveys and Tutorials in the Applied Mathematical Sciences, 6 (Springer, Tokyo, 2013); with a view towards discrete geometric analysis.
[53] A. Terras, *Zeta Functions of Graphs: A Stroll Through the Garden*, Cambridge Studies in Advanced Mathematics, 128 (Cambridge University Press, Cambridge, 2011).
[54] The Sage Developers, Sagemath, the Sage Mathematics Software System (Version 9.5). https://www.sagemath.org.
[55] J. Ueki, 'On the Iwasawa invariants for links and Kida's formula', *Int. J. Math. Anal. (N.S.)* **28**(6) (2017), 1750035.
[56] J. Ueki, 'p-adic Mahler measure and $\mathbb{Z}$-covers of links', *Ergodic Theory Dynam. Systems* **40**(1) (2020), 272–288.
[57] D. Vallières, 'On abelian ℓ-towers of multigraphs', *Ann. Math. Qué.* **45**(2) (2021), 433–452.
[58] L. C. Washington, 'The non-p-part of the class number in a cyclotomic Z_p-extension', *Invent. Math.* **49**(1) (1978), 87–97.
[59] C. Weber, 'Sur une formule de R. H. Fox concernant l'homologie des revêtements cycliques', *Enseign. Math. (2)* **25**(3–4) (1979), 261–272 (1980).

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