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Δ-MOVES ON LINKS AND JONES POLYNOMIAL EVALUATIONS

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ABSTRACT. We determine the effect on the Jones polynomial evaluated at $t = i$ and $t = e^{\pi i/3}$ of an oriented link whenever certain twists are performed.

1. . Let $L \subset S^3$ be any oriented link diagram. A Δ_m^j*-move* is any local change of the diagram in which m coherently oriented strands are given j half-twists, as in Figure 1. (The notation is suggested by analogy with the *fundamental m-braid* $\Delta = \sigma_1 \cdots \sigma_{m-1}\sigma_1 \cdots \sigma_{m-2} \cdots \sigma_1$.) Let $\Delta_m^j(L)$ denote any oriented link obtained by performing a Δ_m^j-move on a diagram L. In [13], [18] J. H. Przytycki observed that the Jones polynomial relation $V_{\Delta_3^2(L)}(i) = -V_L(i)$ follows abstractly from Birman and Wajnryb's study [2] of finite quotients of the braid groups B_n, and he suggested that an elementary argument should be possible. We generalize Przytycki's observation and give two very short geometric proofs.

THEOREM 1. *Let $L \subset S^3$ be any oriented link. For any nonnegative integer n,*

$$(1_n) \quad V_{\Delta^2_{2n+1}(L)}(i) = (-1)^{\lceil \frac{n}{2} \rceil} V_L(i)$$
$$(2_n) \quad V_{\Delta^4_{2n}(L)}(i) = (-1)^n V_L(i),$$

where $\lceil \frac{n}{2} \rceil$ denotes the smallest integer $\geq \frac{n}{2}$.

In [18] Przytycki also showed that $V_{\Delta_2^6(L)}(e^{\pi i/3}) = -V_L(e^{\pi i/3})$ for any oriented link L. We extend this result in the following theorem.

THEOREM 2. *Let L be any oriented link. For any nonnegative integer n,*

$$(1_n) \quad V_{\Delta^4_{2n+1}(L)}(e^{\pi i/3}) = V_L(e^{\pi i/3})$$
$$(2_n) \quad V_{\Delta^6_{2n}(L)}(e^{\pi i/3}) = (-1)^n V_L(e^{\pi i/3}).$$

REMARK. We have defined Δ_m^j*-move* using left-hand twists. Results identical to those above using right-hand twists (Δ_m^{-j}-moves) follow by applying Theorem 1 to the links $\Delta_{2n+1}^{-2}(L), \Delta_{2n}^{-4}(L)$ and Theorem 2 to the links $\Delta_{2n+1}^{-4}(L), \Delta_{2n}^{-6}(L)$.

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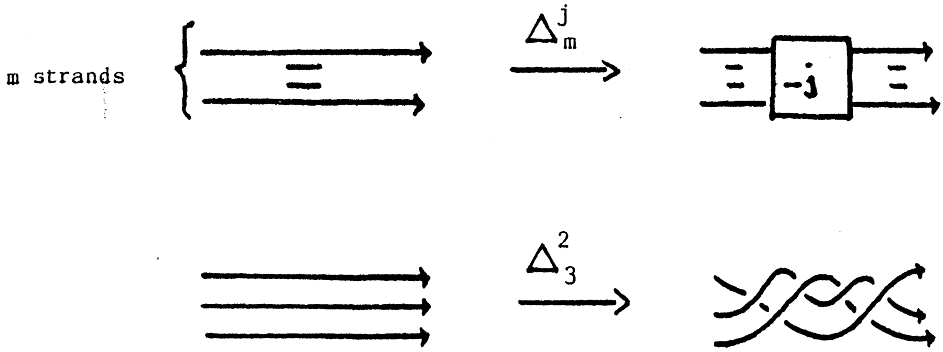

Figure 1

The author is grateful to L. Richard Hitt, Dale Rolfsen and Józef Przytycki for helpful discussions, and to University of British Columbia and University of Washington for their hospitality.

2. In [14] Morton modified techniques of Jones and Ocneanu to define a version $P_L \in Z[v^{\pm 1}, z^{\pm 1}]$ of the 2-variable polynomial for an oriented link L by the conditions $v^{-1}P_{L_+} - vP_{L_-} = zP_{L_0}$, $P_{\text{unknot}} = 1$. Here L_+, L_-, L_0 are links that differ only in a neighborhood of one crossing, as in Figure 2. (L_+, L_-, L_0 is called a *skein triple* of oriented links.) In [15] Morton and Traczyk observed that P_L actually resides in a quotient ring $\Lambda = Z[v^{\pm 1}, z, \delta]/\langle z\delta = v^{-1} - v\rangle$ which is isomorphic to a subring of $Z[v^{\pm 1}, z^{\pm 1}]$ via the assignment $\delta \mapsto z^{-1}(v^{-1} - v)$. The Jones polynomial $V_L(t)$ can be recovered as $P_L(t, t^{\frac{1}{2}} - t^{-\frac{1}{2}}, -t^{\frac{1}{2}} - t^{-\frac{1}{2}})$. The reader is cautioned that in order to compute $V_L(i)$, a consistent choice of $e^{\pi i/4}$ or $e^{5\pi i/4}$ for $i^{\frac{1}{2}}$ must be made.

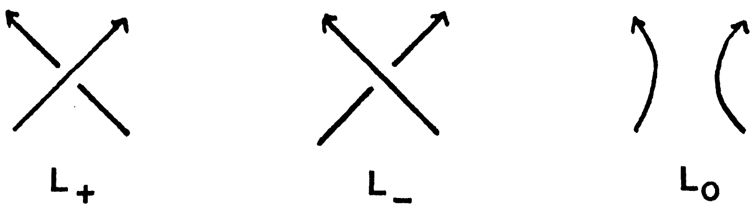

Figure 2

We briefly review the aspects of linear skein theory used in our first proof of Theorem 1. This elegant and surprisingly powerful theory was introduced by Conway [3], [4] and subsequently developed by others, including Giller [5], Lickorish and Millet [11], Morton and Traczyk [14], [15]. Most of the ideas that we borrow appear in [14], and the reader is referred to that paper for details.

Consider a *room* R_n, as in Figure 3, having *n inputs* and *n outputs.* An *oriented n-tangle* (n-tangle, for brevity) is any oriented 1-manifold T connecting all of the inputs and outputs, as in Figure 3. Geometric n-braids are a special type of n-tangle. The *closure* $T^{\wedge}$ is the oriented link obtained by joining inputs to corresponding outputs outside of R_n in the obvious way so that no new crossings are created. Let $\mathcal{T}_n$ denote the collection of all n-tangles up to isotopy of R_n fixing its boundary. Henceforth we will identify any n-tangle with its equivalence class in $\mathcal{T}_n$. Tangle composition (i.e., stacking n-tangles) induces a semigroup structure on $\mathcal{T}_n$.

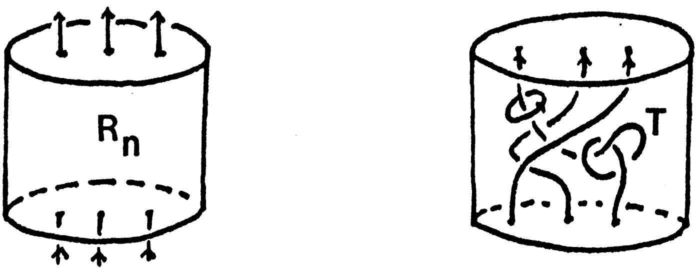

Figure 3

Let $\Lambda[\mathcal{T}_n]$ denote the free Λ-module generated by $\mathcal{T}_n$. The *linear skein* L_n is the quotient Λ-module $\Lambda[\mathcal{T}_n]/\langle v^{-1}T_+ - vT_- = zT_0, T \amalg \text{unknot} = \delta T\rangle$. Here, as for links, T_+, T_-, T_0 are n-tangles that differ only in a neighborhood of a single crossing, as in Figure 2. The correspondence $T \longmapsto P_{T^\wedge}$ induces a homomorphism $P\colon L_n \longrightarrow \Lambda$, and the pairing $(S,T) \longmapsto P_{ST^\wedge}$ induces a bilinear form $\phi\colon L_n \times L_n \longrightarrow \Lambda$. Using the invariance of P_L under *mutation* (see [11]), one can show that ϕ is symmetric; by [15, proof of Theorem 2] ϕ is nondegenerate as well.

A free Λ-basis $\{s_\pi\}$ for L_n, indexed by the elements of the symmetric group S_n, is described in [15]: s_π consists of n unknotted arcs—the first arc joining input 1 to output $\pi(1)$ and lying below all of the other arcs; the second arc joining input 2 to output $\pi(2)$ and lying below all of the remaining arcs, etc. Notice that s_π can be constructed as a geometric n-braid representing a negative word in B_n. Furthermore, when $\pi \neq 1$ we may assume: *if r is the largest index such that σ_r^{-1} occurs, then σ_r^{-1} occurs exactly once.*

We now specialize the ring Λ to $\Lambda' = Z[i^{\frac{1}{2}}]$ by setting $v = i, z = i^{\frac{1}{2}} - i^{-\frac{1}{2}}, \delta = -i^{\frac{1}{2}} - i^{-\frac{1}{2}}$. The resulting Λ'-module $L_n \otimes_\Lambda \Lambda'$ will be denoted by L'_n. Denote $s_\pi \otimes 1 \in L'_n$ by s'_π. The elements s'_π constitute a free Λ'-basis for L'_n. Suitably tensoring each of the maps P, ϕ with $1_{\Lambda'}$ produces a homomorphism $P'\colon L'_n \longrightarrow \Lambda'$ and a bilinear form $\phi'\colon L'_n \times L'_n \longrightarrow \Lambda'$. We will see that ϕ' is degenerate; in fact, it's the degeneracy that makes Theorem 1 possible.

3. **Proof of Theorem 1.** The following general observation about Δ_m^4-moves provides a basis for induction. For a proof of the lemma, see Figure 4 below.

LEMMA 1. *Any Δ_m^4-move, $m > 2$, can be accomplished using one Δ_{m-1}^4-move, two Δ_{m-1}^2-moves, one Δ_{m-2}^{-4}-move and one Δ_2^4-move.*

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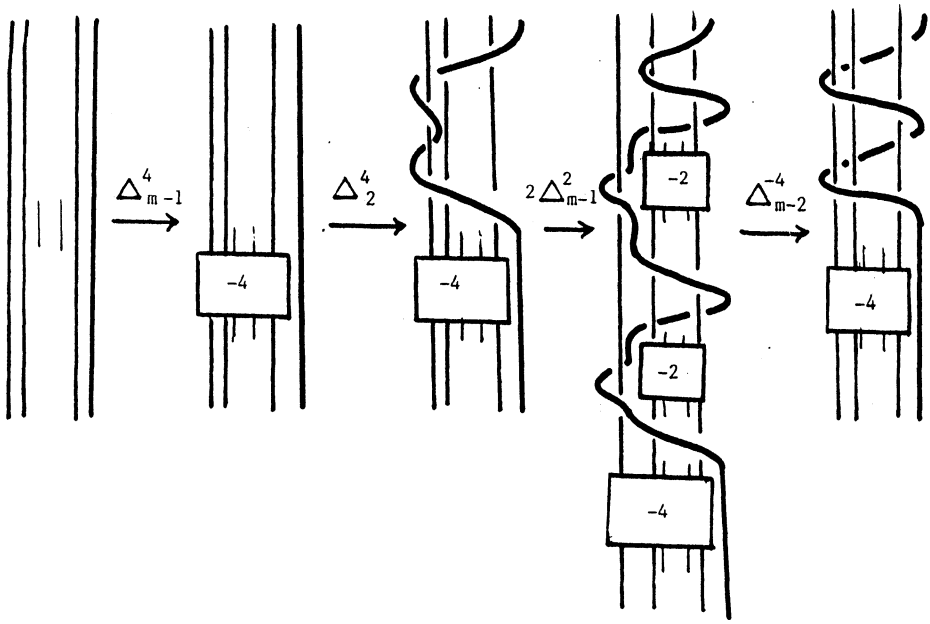

Figure 4

We prove statements (1_n) and (2_n) of Theorem 1 together, using induction on the total number m of strands being twisted.

First consider $m = 2$. Any link $\Delta_2^4(L)$ can be viewed as the closure of the tangle product $T\Delta_2^4$, where Δ_2^4 is regarded as a 2-tangle and T is the complementary 2-tangle. It's a simple matter to check that Δ_2^4 is equivalent to -1 in L_2'. Hence $V_{\Delta_2^4(L)}(i) = \phi'(T, \Delta_2^4) = -\phi'(T, 1) = -V_L(i)$.

Assume the statement of Theorem 1 for links in which $m-1$ strands are twisted. We'll prove the result for m. If m is odd, then in view of the proof for $m = 2$, it suffices to prove the result for each basis element s_π' (which we identify with its n-tangle representative constructed in § 2). In fact, since the homomorphism $P: L_m \longrightarrow \Lambda$ factors through a quotient of L_m in which s_1 is a Λ-linear combination of the elements $s_\pi, \pi \neq 1$ (Kauffman's *diagram algebra* [8], [15] is a concrete realization of this quotient) we need only prove the result for basis elements $s_\pi' \in L_m', \pi \neq 1$: replace s_π' by the m-tangle $\tilde{s}_\pi'$ obtained by permuting all trivial right-most strands to the left, as illustrated by example in Figure 5. As braids, s_π' and $\tilde{s}_\pi'$ represent conjugate elements in B_m, and since Δ_m^2 lies in the center of B_m, the closures of $s_\pi'\Delta_m^2$ and $\tilde{s}_\pi'\Delta_m^2$ are isotopic links. We'll work only with the latter.

Figure 6 below reveals that the Δ_m^2-move on the closure of $\tilde{s}_\pi'$ can be accomplished by first performing a Δ_{m-1}^4-move and then a Δ_{m-2}^{-2}-move. Letting $m = 2n+1$, any Δ_{2n+1}^2-move has the same effect on the Jones polynomial evaluated at $t = i$ as a Δ_{2n-1}^2-move when n is even; it has the opposite effect when n is odd. This assertion is equivalent to statement (1_n) of Theorem 1.

If m is even, then Lemma 1 shows that $V_{\Delta_m^4(L)}(i) = -V_{\Delta_{m-2}^4(L)}(i)$ for any oriented link

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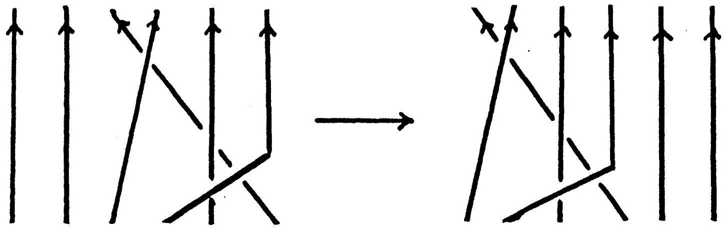

Figure 5

L. This establishes statement (2_n). ■

NOTE ADDED IN PROOF. Józef Przytycki has informed me that Hugh Morton (private correspondence) obtained the above result for $m = 3$ using these techniques.

4. **Proof of Theorem 2.** Lickorish and Millett have shown in [9] that for any oriented link L, $V_L(e^{\pi i/3}) = \sigma_L i^{(c_L-1)}(i\sqrt{3})^{d_L}$, where $\sigma_L = \pm 1, c_L$ is the number of components of L and d_L = dim $H_1(D_L; Z_3), D_L$ being the double branched cover of L. Also, Przytycki [18] has shown that $d_{\Delta^4_{2n+1}(L)} = d_L$ and $d_{\Delta^6_{2n}(L)} = d_L$ for any oriented link L. Thus in order to establish Theorem 2 it suffices to determine the effect of Δ^4_{2n+1}-moves and Δ^6_{2n}-moves on σ (the invariant σ has been studied by Lipson [12]). If L_+, L_-, L_0 is any skein triple of oriented links, we will shorten the notation $\sigma_{L_+}, \sigma_{L_-}, \sigma_{L_0}$ to $\sigma_+, \sigma_-, \sigma_0$ (similarly for c and d).

LEMMA 2. *(1) If $d_0 = d_+ + 1$ and $c_0 = c_+ + 1$, then $\sigma_0 = \sigma_+$. (2) If $d_0 = d_+ - 1$ and $c_0 = c_+ - 1$, then $\sigma_0 = \sigma_+$.*

PROOF. Using the above-mention result of [9] together with the defining relation for V_L, we obtain

$$\sigma_+ e^{\pi i/6}(i\sqrt{3})^{d_+} + \sigma_- e^{-\pi i/6}(i\sqrt{3})^{d_-} + \sigma_0 i^{(c_0-c_+)}(i\sqrt{3})^{d_0} = 0,$$

since $c_+ = c_-$ (this equation appears in [12]). When $d_0 = d_+ + 1$ and $c_0 = c_+ + 1$ we can write

$$(i\sqrt{3})^{d_- - d_+} = \sigma_-\left[\left(\frac{3}{2}\sigma_0 - \frac{1}{2}\sigma_+\right) + \frac{\sqrt{3}}{2}i(\sigma_0 - \sigma_+)\right].$$

Since $\frac{3}{2}\sigma_0 - \frac{1}{2}\sigma_+$ is never 0, $d_- - d_+$ must be even and hence $\sigma_0 - \sigma_+ = 0$. This proves statement (1). The proof of (2) is similar. ■

The link $\Delta^4_{2n+1}(L)$ can be obtained by joining L and the $(2n+1, 2(2n+1))$ torus link T with $2n+1$ bands as in Figure 7.

Since d is invariant under Δ^4_{2n+1}-moves, $d_T = 2n$. If we break $2n$ bands joining L and T, we obtain $L\#T$, and d must increase by $2n$. (Break each band by smoothing its crossing.) Breaking a single band can alter d by at most ± 1 (see [17]), so each time we break a band d must increase by 1 (and so does c). By Lemma 2 (1) $\sigma_{\Delta^4_{2n+1}(L)} = \sigma_{L\#T} = \sigma_L \sigma_T$.

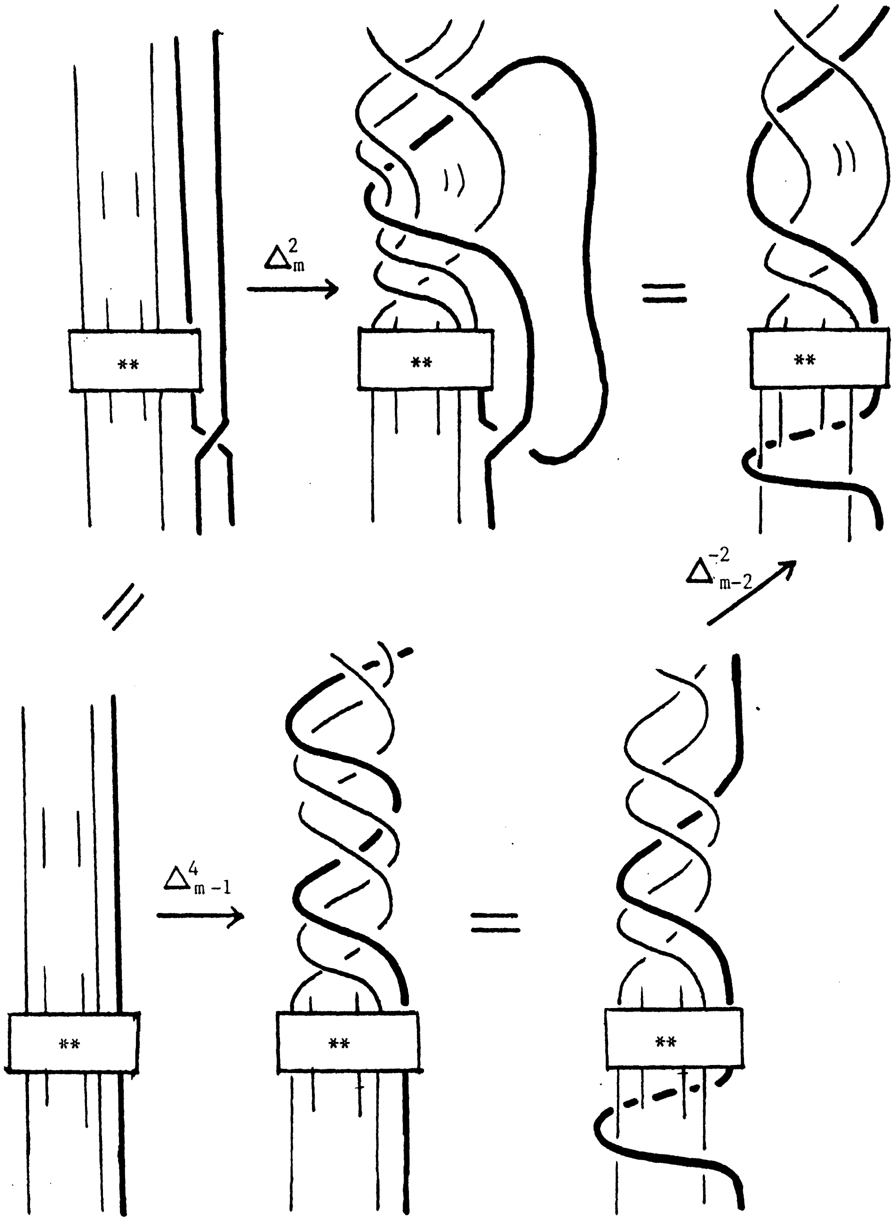

Figure 6

It will be convenient to denote σ_T also by σ_{2n+1}. In order to compute σ_{2n+1}, reverse the orientation of any component of T; call the altered link T'. By the Jones reversing result [6], [10] (See [18] for our situation) $\sigma_{T'} = \sigma_T$. Now smooth any crossing of T' that involves the reversed component to obtain the $\big(2n-1, 2(2n-1)\big)$ torus link plus an

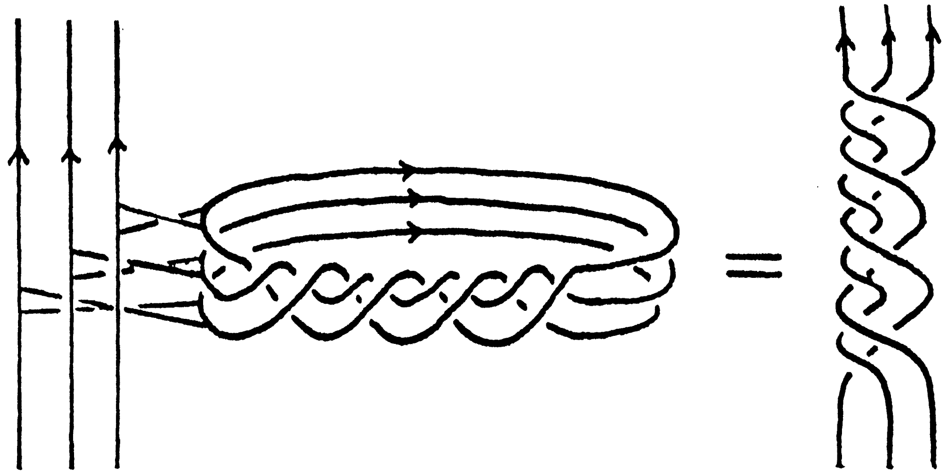

Figure 7

unknotted, unlinked component. By Lemma 2 (2) σ is unchanged, so $\sigma_{2n+1} = \sigma_{2n-1}(= \cdots = \sigma_1 = 1)$. Statement (1_n) of Theorem 2 follows immediately. The proof of (2_n) is similar; the only difference arises when we reverse the orientation of a component—in this case σ changes; i.e., $\sigma_{2n} = -\sigma_{2n-2}$. Hence $\sigma_{2n} = (-1)^{n-1}\sigma_2 = (-1)^n$ and statement (2_n) follows. ∎

5. **Another proof of Theorem 1.** The techniques used to prove Theorem 2 can be used to give another proof of Theorem 1.

If L is *proper* (i.e., each component has even linking number with the union of the remaining components), then the Arf or *Robertello* [19], Z_2-invariant $\mathrm{Arf}(L)$ is defined. For such a link Murakami [16] has shown that $V_L(i) = (-\sqrt{2})^{c_L-1}(-1)^{\mathrm{Arf}(L)}$, taking $i^{\frac{1}{2}} = e^{\pi i/4}$; $V_L(i) = 0$ if L is not proper (see [9] for another proof of Murakami's result). Since L is proper if and only if $\Delta^2_{2n+1}(L)$ or $\Delta^4_{2n}(L)$ is proper, Theorem 1 is equivalent to the following statement: *If L is any proper oriented link and n is any nonnegative integer, then*

$$(1_n) \quad \mathrm{Arf}\big(\Delta^2_{2n+1}(L)\big) - \mathrm{Arf}(L) = \left\lceil \frac{n}{2} \right\rceil \pmod 2$$

$$(2_n) \quad \mathrm{Arf}\big(\Delta^4_{2n}(L)\big) - \mathrm{Arf}(L) = n \pmod 2.$$

As in the proof of Theorem 2, $\Delta^2_{2n+1}(L)$ can be obtained by joining L and the $\big(2n+1, 2(2n+1)\big)$ torus link T with $2n+1$ bands (Figure 7), and by breaking all but one band, we obtain $L\#T$. Since banding together distinct components of any link does not affect the Arf invariant (see [9], for example) $\mathrm{Arf}(\Delta^2_{2n+1}(L)) = \mathrm{Arf}(L\#T) = \mathrm{Arf}(L) + \mathrm{Arf}(T)$. We will denote $\mathrm{Arf}(T)$ by a_{2n+1}. In order to compute a_{2n+1}, we proceed as in § 3: reverse the orientation of any strand of T; call the altered link T'. By the Jones reversing result $V_L(i) = (i)^{3(2n)}V_T(i) = (-1)^n V_T(i)$. In particular $a_{2n+1} - \mathrm{Arf}(T') = n \pmod 2$. Now consider any crossing of T' that involves the altered strand. If we change the crossing, the resulting link is no longer proper. If we smooth the crossing, we obtain a link T_0 which is the $\big(2n-1, 2(2n-1)\big)$ torus link plus an unknotted, unlinked component. Using the defining relation for the Jones polynomial ($t = i$) together with Murakami's result, we immediately find that $\mathrm{Arf}(T') = \mathrm{Arf}(T_0) = a_{2n-1}$. Hence $a_{2n+1} - a_{2n-1} = n \pmod 2$ which implies statement (1_n). The proof of (2_n) is similar; alternatively, one can now use Lemma 1. ∎

REMARK. The delta-moves in this paper can be regarded as a special case of $t_{2k,m}$-moves defined by R. Fox. Here $t_{2k,m}(L)$ is the link obtained by giving k full twists to any parallel collection of strands of L, provided the strands intersect a standard 2-disk transversely with oriented intersection number m. (See [18] for details.) The arguments above (§§3, 4) can be easily extended to prove results about these more general moves.

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