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Exact bounds for lift-to-drag ratios of profiles in the Helmholtz–Kirchhoff flow

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In this work we investigate limiting values of the lift and drag coefficients of profiles in the Helmholtz–Kirchhoff (infinite cavity) flow. The coefficients are based on the wetted arc length of profile surfaces. The problem is to find global minimum and maximum values of the drag coefficient C_D under a given lift coefficient C_L. We reduce the problem to a constrained problem of calculus of variations and solve it analytically. In so doing we do not only determine extremals but also strictly prove that these extremals realize global extrema. The proofs are based on non-trivial application of Jensen's inequality. The solution of the problem allows us to construct the domain of possible variations of coefficients C_L and C_D and define maximum and minimum values of the lift-to-drag ratios C_L/C_D for a given C_L.

Key words: Extremal problem, Ideal fluid, Potential flows, Helmholtz–Kirchhoff model, Cavity flows, Lift-to-drag ratio

1 Introduction

In the theory of aero and hydrofoils there are known two classical models for studying flows past a profile. For the first model the flow is continuous (Figure 1a), and for the second one the flow is separated with the formation of an infinite wake (Figure 1b). If we assume that the flow is steady, irrotational and incompressible, then for the first model the drag force $D = 0$ (d'Alembert's paradox) and the lift force L are defined by the well-known Kutta–Joukowskii theorem:

$$L = -\rho v_0 \Gamma, \quad \Gamma = \int_0^l (\mathbf{v} \cdot \boldsymbol{\tau})\, \mathrm{d}s. \tag{1.1}$$

Here ρ is the density of the fluid, v_0 is the velocity at infinity, Γ is the circulation around the profile, l is the perimeter of the profile surface, s is the arc abscissa of the profile contour, reckoned from the trailing edge point A, $(\mathbf{v} \cdot \boldsymbol{\tau})$ is the dot product of the velocity vector $\mathbf{v}$ at the point on the profile surface and the tangential unit vector $\boldsymbol{\tau}$, directed towards increase of s.

For the continuous model, point B with the arc abscissa $s = l$ coincides with point A for which $s = 0$. If l_1 is the arc abscissa of the stagnation point O and $v = |\mathbf{v}|$, then

$$(\mathbf{v} \cdot \boldsymbol{\tau}) = -v(s) \ \text{ for } \ 0 \leqslant s \leqslant l_1, \ \ (\mathbf{v} \cdot \boldsymbol{\tau}) = v(s) \ \text{ for } \ l_1 \leqslant s \leqslant l. \tag{1.2}$$

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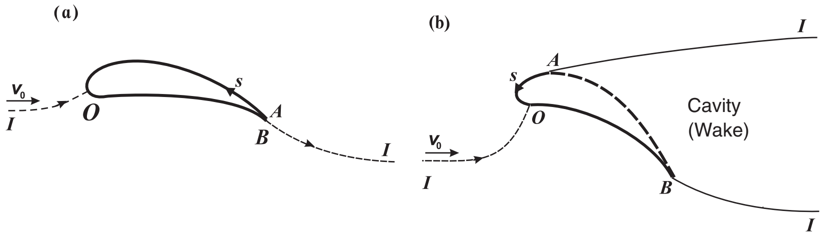

FIGURE 1. (a) Continuous flow over an aerofoil. (b) Helmholtz–Kirchhoff flow with an infinite cavity past a profile.

As one can see from (1.1) and (1.2), to compute the lift force for the continuous model one needs only to know the velocity distribution $v(s)$ along the profile surface and the arc abscissa l_1 of the critical point O. Moreover, if $v(s)$ is known, the contour of the profile can be restored by means of solving the so-called inverse boundary-value problem of aerodynamics [1]. The Kutta–Joukowskii theorem played an outstanding role in the theory of aerofoils and was used many times for aerodynamic shape optimization (see, for example, [1,2]).

Consider now the second classical model with an infinite wake. Initially, Kirchhoff [6] introduced this model in order to overcome d'Alembert's paradox, i.e. to have a non-zero drag force. In doing so, Kirchhoff used essentially the free streamline theory of Helmholtz [5]. Nowadays the Helmholtz–Kirchhoff model is treated as a limiting case of cavity flows, when the pressure in the cavity tends to the incident pressure, and the size of the cavity becomes infinitely large. In this paper we adhere to this treatment and call the model as the *Helmholtz–Kirchhoff* or *infinite cavity flow*. According to this model the flow detaches from the profile surface at points A and B, and an infinite cavity (wake) with a constant pressure, equal to the incident pressure, forms behind the profile. The velocity on the free streamlines AI and BI is constant and equals the incident velocity v_0. As previously, the stagnation point is denoted by O and the arc abscissa s is reckoned from point A (Figure 1b).

For the Helmholtz–Kirchhoff flow, formulae analogous to (1.1) have been recently obtained by Maklakov [12,13]. As in (1.1) the formulae express the lift force L and the drag force D in terms of the velocity distribution $v(s)$:

$$L = \rho v_0 \int_0^l (\mathbf{v} \cdot \boldsymbol{\tau}) \log \frac{v_0}{v} \, \mathrm{d}s, \quad D = \frac{\rho v_0}{4\pi} \left(\int_0^l \frac{v}{\sqrt{\varphi}} \log \frac{v_0}{v} \, \mathrm{d}s \right)^2, \tag{1.3}$$

where l is the length of the wetted arc AOB of the profile, $\varphi = \varphi(s)$ is the distribution of potential along AOB:

$$\varphi = \int_s^{l_1} v(s)\,\mathrm{d}s \ \text{ for } \ 0 \leqslant s \leqslant l_1, \quad \varphi = \int_{l_1}^{s} v(s)\,\mathrm{d}s, \ \text{ for } \ l_1 \leqslant s \leqslant l,$$

l_1 is the arc abscissa of the critical point O.

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In the theory of cavity flows the first Brillouin condition [3] plays an important role: the pressure in the cavity is minimal. Then the velocity on the free streamlines AI and BI is maximal, and therefore

$$v(s) \leqslant v_0, \quad 0 \leqslant s \leqslant l. \tag{1.4}$$

This implies that in formulae (1.3) the factor $\log \frac{v_0}{v} \geqslant 0$.

Owing to the simplicity of formulae (1.3) one can formulate different optimization problems in which one needs to determine a velocity distribution that satisfies the Brillouin condition (1.4) and has some optimal property. One of such problems has been solved by Maklakov in [12, 13]. Namely, it has been found that the velocity distribution under the Brillouin condition (1.4) provides a global maximum of the lift force. It has been established that for the profile of maximum lift the length $l_1 = 0$ (points A and O coincide) and the optimal velocity distribution $v(s) = \mathbf{e}^{-1} v_0 = \text{const}$, where $\mathbf{e}$ is the base of natural logarithms. It follows from (1.3) that $L_{\max} = \rho v_0^2 l\, \mathbf{e}^{-1}$ and this is the global maximum of the lift force. But such a formulation does not take into account at all the cavitation drag, which is defined by the second equation in (1.3). If $v(s) = \mathbf{e}^{-1} v_0$, then according to (1.3) the drag force $D = \rho v_0^2 l/(\pi\, \mathbf{e})$, and the lift-to-drag ratio of the profile of maximum lift is $\varkappa = L/D = \pi$. To obtain profiles with a greater lift-to-drag ratio, it seems to be natural to introduce the drag D in the optimization process.

At the end of the paper by Maklakov [12], as a variant of a further perspective direction of investigations, it has formulated the problem of finding a minimum of the drag force D under the constraints that the wetted arc length l and the lift force L are given. In this paper we present an exact analytical solution to this problem. Besides, for the sake of completeness, we find a maximum of the drag force D under the given values of l and L.

It is convenient to formulate the problem in terms of the lift and drag coefficients

$$C_L = \frac{2L}{\rho v_0^2 l} \quad \text{and} \quad C_D = \frac{2D}{\rho v_0^2 l}, \tag{1.5}$$

based on the wetted arc length l. Then the basic problem to be solved in the paper is as follows.

Basic problem *Let the lift coefficient C_L be given. Find a global minimum (maximum) of the drag coefficient C_D under the Brillouin condition* (1.4).

Solving the basic problem allows us to determine the functions $C_{D\min}(C_L)$ and $C_{D\max}(C_L)$, which define the global extrema of the drag coefficient. Thereby we define the domain of possible variations of coefficients C_L and C_D and determine upper and lower bounds for the lift-to-drag ratios:

$$\varkappa_{\max}(C_L) = C_L/C_{D\min}(C_L), \quad \varkappa_{\min}(C_L) = C_L/C_{D\max}(C_L).$$

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2 Auxiliary problem

Let $l_2 = l - l_1$ be the length of the arc OB. We introduce two dimensionless functions $u_1(\sigma)$ and $u_2(\sigma)$, $0 \leqslant \sigma \leqslant 1$, such that

$$\frac{v}{v_0} = \begin{cases} u_1(\frac{l_1 - s}{l_1}) \text{ on } OA \\ u_2(\frac{s - l_1}{l_2}) \text{ on } OB \end{cases}. \tag{2.1}$$

Since the velocity $v \geqslant 0$, the functions $u_1(\sigma)$ and $u_2(\sigma)$ are non-negative. Under the Brillouin condition (1.4) these satisfy the inequalities

$$u_1(\sigma) \leqslant 1, \quad u_2(\sigma) \leqslant 1. \tag{2.2}$$

By means of (1.3) and (1.5) we express the lift and drag coefficients in terms of $u_1(\sigma)$ and $u_2(\sigma)$:

$$C_L = 2\left\{(1-\varepsilon)\,\mathbf{I}[u_2] - \varepsilon\,\mathbf{I}[u_1]\right\}, \quad C_D = \frac{1}{2\pi}\left\{\sqrt{1-\varepsilon}\,\mathbf{J}[u_2] + \sqrt{\varepsilon}\,\mathbf{J}[u_1]\right\}^2, \tag{2.3}$$

where $\varepsilon = l_1/l$, $\mathbf{I}[u]$ and $\mathbf{J}[u]$ are non-linear functionals of $u(\sigma)$, $0 \leqslant \sigma \leqslant 1$:

$$\mathbf{I}[u] = -\int_0^1 u(\sigma)\log u(\sigma)\,\mathrm{d}\sigma, \quad \mathbf{J}[u] = -\int_0^1 \frac{u(\sigma)\log u(\sigma)\,\mathrm{d}\sigma}{\sqrt{\int_0^\sigma u(\sigma_1)\,\mathrm{d}\sigma_1}}. \tag{2.4}$$

As one can see from (2.4), under the Brillouin condition (2.2) the values of the functionals $\mathbf{I}[u]$ and $\mathbf{J}[u]$ at $u = u_1(\sigma)$ and $u = u_2(\sigma)$ are non-negative.

Let us rewrite $\mathbf{I}[u]$ and $\mathbf{J}[u]$ in terms of classical functionals of calculus of variations. To do so we transform function $u(\sigma)$ to $\lambda(\sigma)$:

$$\lambda(\sigma) = \sqrt{2\int_0^\sigma u(\sigma_1)\,\mathrm{d}\sigma_1}.$$

Then

$$\mathbf{I}[\lambda] = -\int_0^1 \lambda\lambda' \log(\lambda\lambda')\,\mathrm{d}\sigma, \quad \mathbf{J}[\lambda] = -\sqrt{2}\int_0^1 \lambda' \log(\lambda\lambda')\,\mathrm{d}\sigma. \tag{2.5}$$

It is clear that $\lambda(\sigma) \geqslant 0$. Besides, $u(\sigma) = \lambda(\sigma)\lambda'(\sigma)$, hence $\lambda'(\sigma) \geqslant 0$. In terms of $\lambda(\sigma)$ the Brillouin condition $u(\sigma) \leqslant 1$ is expressed as $\lambda(\sigma)\lambda'(\sigma) \leqslant 1$.

The solution of the basic problem, formulated in the Introduction, is based on solving the following auxiliary problem.

Auxiliary problem *Find the function $\lambda(\sigma)$, $\sigma \in [0, 1]$:*

$$\lambda(0) = 0, \quad \lambda'(\sigma) \geqslant 0, \tag{2.6}$$

which delivers a global minimum (maximum) to the functional $\mathbf{J}[\lambda]$ under the constraint $\mathbf{I}[\lambda] = q$ (q is given), and the complementary condition

$$\lambda(\sigma)\lambda'(\sigma) \leqslant 1. \tag{2.7}$$

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Let us find the exact upper and lower bounds of the functional $\mathbf{I}[\lambda]$. It follows from (2.6) and (2.7) that $\mathbf{I}[\lambda] \geqslant 0$. We note that

$$\mathbf{I}[\lambda] = -\int_0^1 G(\lambda\lambda')\,\mathrm{d}\sigma,$$

where

$$G(u) = u\log u. \tag{2.8}$$

The only minimum of the function $G(u)$ achieves at the point $u = 1/\mathbf{e}$ and $G(1/\mathbf{e}) = -1/\mathbf{e}$. Hence, the exact upper bound for the functional $\mathbf{I}[\lambda]$ is

$$\mathbf{I}[\lambda] \leqslant 1/\mathbf{e} = q_{\max}. \tag{2.9}$$

Equality in (2.9) holds if and only if $\lambda(\sigma)\lambda'(\sigma) = 1/\mathbf{e}$. This means that the global maximum of $\mathbf{I}[\lambda]$ is achieved by the function $\lambda(\sigma) = \sqrt{2\sigma/\mathbf{e}}$, which, as one can easily check, satisfies the constraints (2.6) and (2.7). The value of the functional $\mathbf{J}[\lambda]$ for this function is

$$\mathbf{J}[\lambda] = \mathbf{J}[\sqrt{2\sigma/\mathbf{e}}] = 2/\sqrt{\mathbf{e}} = \mathbf{J}_r. \tag{2.10}$$

The exact bounds for the functional $\mathbf{I}[\lambda]$ allow us to make more precise the formulation of the auxiliary problem, namely in the constraint $\mathbf{I}[\lambda] = q$ the value of q satisfies the inequalities $0 < q \leqslant q_{\max}$.

We denote by $\mathbf{J}_{\min}(q)$ and $\mathbf{J}_{\max}(q)$, correspondingly, the global minimum and maximum of the functional $\mathbf{J}[\lambda]$ for a given value of $\mathbf{I}[\lambda] = q$. The full solution to the auxiliary problem is given by the following theorem.

Theorem 1 (1) *The function* $\mathbf{J}_{\min}(q)$ *is defined by the parametric equations*

$$\begin{cases} q = q(b) = \dfrac{1}{2}\left[b^2 - k^2 + k(b-a) - k^2\log\dfrac{b}{a}\right], \\[2ex] \mathbf{J}_{\min} = \sqrt{2}\left(k\log\dfrac{b}{a} + b + k\right), \end{cases} \tag{2.11}$$

where $b \in (\sqrt{2/\mathbf{e}}, \sqrt{2})$,

$$k = K(b) = -\frac{(\mathbf{e}-1)b(b^2\mathbf{e}-2)}{2+(\mathbf{e}-2)b^2\mathbf{e}}, \qquad a = \frac{b^2\mathbf{e}-2}{(\mathbf{e}-1)b}. \tag{2.12}$$

The global minimum is achieved by the function

$$\lambda(\sigma) = \begin{cases} \sqrt{2\sigma} & \text{for } 0 \leqslant \sigma < \gamma, \\ -k + \sqrt{2c(\sigma-\gamma)+(a+k)^2} & \text{for } \gamma \leqslant \sigma \leqslant 1, \end{cases} \qquad \gamma = a^2/2, \tag{2.13}$$

where $c = \frac{2-b^2}{2+(\mathbf{e}-2)b^2\mathbf{e}}$.

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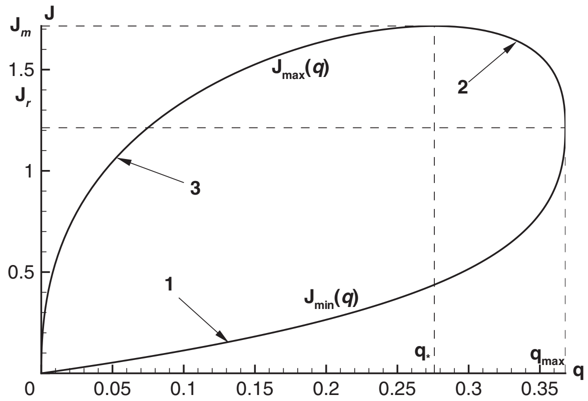

FIGURE 2. Dependencies of $\mathbf{J}_{\min}$ and $\mathbf{J}_{\max}$ on q.

(2) *The function* $\mathbf{J}_{\max}(q)$ *is defined by the parametric equations*

$$\begin{cases} q = q(b) = \dfrac{1}{2}\left[b^2 + k\,b - k^2 \log \dfrac{b+k}{k}\right], \\ \mathbf{J}_{\max} = \sqrt{2}\left(k \log \dfrac{b+k}{k} + b\right), \end{cases} \quad \textit{where } k = K_1(b) = -\frac{b(b^2\,\mathbf{e}-2)}{2(b^2\,\mathbf{e}-1)}, \tag{2.14}$$

$b \in (0, \sqrt{2/\mathbf{e}})$.

The global maximum is achieved by the functions

$$\lambda(\sigma) = -k - \sqrt{2c\sigma + k^2} \ \textit{for} \ q \in (0, q_*), \ \ b \in (0, \sqrt{1/\mathbf{e}}), \tag{2.15}$$

$$\lambda(\sigma) = \sigma/\sqrt{\mathbf{e}} \ \textit{for} \ q = q_*, \tag{2.16}$$

$$\lambda(\sigma) = -k + \sqrt{2c\sigma + k^2} \ \textit{for} \ q \in (q_*, q_{\max}], \ \ b \in (\sqrt{1/\mathbf{e}}, \sqrt{2/\mathbf{e}}), \tag{2.17}$$

where $q_* = \frac{3}{4\,\mathbf{e}}$, $c = \frac{b^2}{2(b^2\,\mathbf{e}-1)}$.

In the statements of the theorem the parameter $b = \lambda(1)$.

A proof of this theorem will be presented in Section 4. Right now we illustrate the theorem graphically, and thereafter demonstrate how the theorem can be used for solving the basic problem formulated in the Introduction.

In Figure 2, where the functions $\mathbf{J}_{\min}(q)$ and $\mathbf{J}_{\max}(q)$ are shown, one can distinguish three zones. In each of these zones the global extrema are achieved by the function $\lambda(\sigma)$ of different types. The first zone is the graph of $\mathbf{J}_{\min}(q)$ and the functions of the form (2.13) correspond to this zone. The graph of $\mathbf{J}_{\max}(q)$ is divided into two zones: second and third zones. The functions of the form (2.15) correspond to the second zone, and those of the form (2.17) correspond to the third one. Accordingly, in each of the zones the ranges of variations of the parameter $b = \lambda(1)$ are different. In zone 1 the parameter $b \in (\sqrt{2/\mathbf{e}}, \sqrt{2})$, in zone 2 the parameter $b \in (\sqrt{1/\mathbf{e}}, \sqrt{2/\mathbf{e}})$ and in zone 3 the parameter $b \in (0, \sqrt{1/\mathbf{e}})$.

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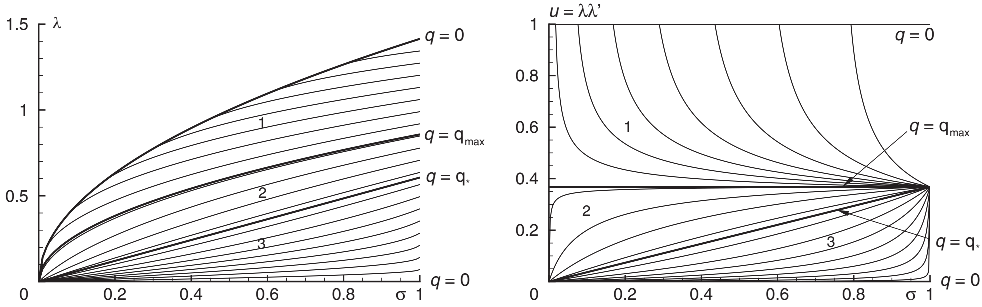

FIGURE 3. Functions $\lambda(\sigma)$ and $u = \lambda(\sigma)\lambda'(\sigma)$.

In Figure 3, on the left we demonstrate the functions $\lambda(\sigma)$, and on the right we show the functions $u(\sigma) = \lambda(\sigma)\lambda'(\sigma)$. We have constructed 19 curves varying uniformly the parameter $b = \lambda(1)$ with step $\sqrt{2}/20$. In addition, the bold lines in Figure 3 indicate the graphs for characteristic points of the functions $\mathbf{I}_{\min}(q)$ and $\mathbf{I}_{\max}(q)$: for the rightmost point of Figure 2 ($q = q_{\max}$), for the upward-most point of the figure ($q = q_*$) and the curves that correspond to the origin in Figure 2, when $q = 0$, $\mathbf{I}_{\min}(q) = \mathbf{I}_{\max}(q) = 0$.

Let $\mathbf{J}_m = 2\sqrt{2/\mathbf{e}}$. The value $\mathbf{J}_m$ is the maximum possible value of the functional $\mathbf{J}[\lambda]$ which is achieved only if $\mathbf{I}[\lambda] = q_*$. The value $\mathbf{J}_r$, defined by (2.10), is the value of $\mathbf{J}[\lambda]$ for the maximum possible value of $\mathbf{I}[\lambda] = q_{\max}$.

3 Solution of the basic problem

To solve the basic problem, formulated in the Introduction of the paper, we firstly prove the following.

Lemma 1 *The function* $\mathbf{J}_{\min}(q)$ *increases, whereas the function* $\mathbf{J}_{\max}(q)$ *increases on the segment* $(0, q_*)$ *and decreases on the segment* $(q_*, q_{\max})$.

Proof Making use of (2.11) and (2.14) leads to the equation

$$\frac{d\,\mathbf{J}_{\min,\max}(q)}{dq} = \frac{d\,\mathbf{J}_{\min,\max}(q)}{db} \bigg/ \frac{dq}{db} = -\frac{\sqrt{2}}{k}.$$

For the function $\mathbf{J}_{\min}(q)$, parameter k is always negative (see Theorem 1). This demonstrates the validity of the first part of the lemma. To prove the second part, it is enough to note that the parameter $k(q)$ is negative for $q \leqslant q_*$ and positive for $q \geqslant q_*$ (see again Theorem 1). Lemma 1 is proved. □

Consider now the first equation in (2.3). Taking into account that $\varepsilon = l_1/l \in [0, 1]$, we conclude that

$$C_L = 2\left\{(1-\varepsilon)\,\mathbf{I}[u_2] - \varepsilon\,\mathbf{I}[u_1]\right\} \leqslant 2\,\mathbf{I}[u_2]. \tag{3.1}$$

Since the maximum value of the functional $\mathbf{I}[u] = q_{\max} = 1/\mathbf{e}$, the lift coefficient $C_L \in (0, 2/\mathbf{e}]$.

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Theorem 2 *At a given value of the lift coefficient* $C_L \in (0, 2/\mathbf{e}]$, *the global minimum of the drag coefficient is* $C_{D\,\min} = \frac{1}{2\pi}\,\mathbf{J}^2_{\min}(C_L/2)$, *and the global maximum of the lift-to-drag ratio* $\varkappa_{\max} = 2\pi C_L/\,\mathbf{J}^2_{\min}(C_L/2)$.

Proof Let us set $q_1 = \mathbf{I}[u_1]$ and $q_2 = \mathbf{I}[u_2]$. It is clear that

$$\varepsilon \in [0,1), \quad 0 \leqslant q_1, q_2 \leqslant q_{\max}, \quad (1-\varepsilon)q_2 - \varepsilon q_1 = C_L/2 \in (0, q_{\max}], \tag{3.2}$$

$$C_{D\,\min} = \min_{\varepsilon, q_1, q_2} \frac{1}{2\pi} \left\{ \sqrt{1-\varepsilon}\,\mathbf{J}_{\min}(q_2) + \sqrt{\varepsilon}\,\mathbf{J}_{\min}(q_1) \right\}^2.$$

It is to be noted that $\varepsilon \neq 1$, since for $\varepsilon = 1$ the constraints (3.2) become contradictory. Expressing q_2 in terms of q_1 by means of the last relation in (3.2), we get

$$C_{D\,\min} = \min_{\varepsilon, q_1} \frac{1}{2\pi} \left\{ \sqrt{1-\varepsilon}\,\mathbf{J}_{\min}\left(\frac{C_L/2}{1-\varepsilon} + \frac{\varepsilon q_1}{1-\varepsilon} \right) + \sqrt{\varepsilon}\,\mathbf{J}_{\min}(q_1) \right\}^2.$$

By Lemma 1 the function to be minimized strictly increases with respect to q_1. Hence, the minimum is achieved at the point $q_1 = 0$, i.e.

$$C_{D\,\min} = \min_{\varepsilon} \frac{1}{2\pi} \left\{ \sqrt{1-\varepsilon}\,\mathbf{J}_{\min}\left(\frac{C_L/2}{1-\varepsilon} \right) \right\}^2. \tag{3.3}$$

To prove the theorem it needs to establish that the minimum is reached at the point $\varepsilon = 0$. This will be so if the function to be minimized in (3.3) strictly increases with respect to ε. Let

$$q = \frac{C_L/2}{1-\varepsilon}$$

and consider the function $\mathbf{J}^2_{\min}(q)/q$. It is easy to see that this function strictly increases. Indeed, calculating its derivative, we infer that the strict increase of $\mathbf{J}^2_{\min}(q)/q$ is equivalent to the inequality

$$q\,\mathbf{J}'_{\min}(q) - \frac{\mathbf{J}_{\min}(q)}{2} > 0.$$

By means of (2.11), after a little algebra, we deduce that

$$q\,\mathbf{J}'_{\min}(q) - \frac{\mathbf{J}_{\min}(q)}{2} = \frac{\sqrt{2}(2-b^2)}{b(\mathbf{e}-1)(b^2\,\mathbf{e}-2)}.$$

The last expression is positive because $b \in (\sqrt{2/\,\mathbf{e}}, \sqrt{2})$ (see Theorem 1). The strict increase of $\mathbf{J}^2_{\min}(q)/q$ leads to the strict increase of the function to be minimized in (3.3). Theorem 2 is proved. □

Theorem 2 solves the basic problem on the minimization of C_D and hence on maximization of the lift-to-drag ratio. The problem of finding the maximum of C_D (correspondingly, the minimum of the lift-to-drag ratio) turns out to be more complex. The difficulties are connected with a non-monotone behaviour of the function $\mathbf{J}_{\max}(q)$.

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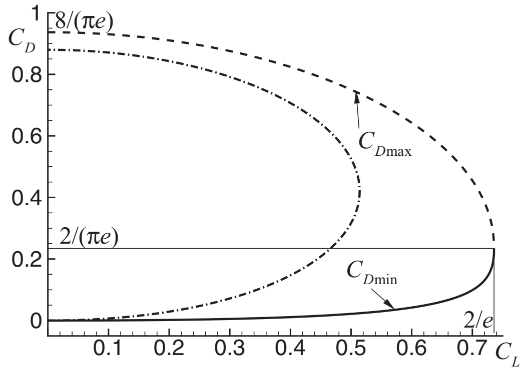

FIGURE 4. Dependencies $C_{D\,\min}$ and $C_{D\,\max}$ on C_L. The dash-and-dot line is the dependence C_D on C_L for a flat plate.

It is evident that

$$C_{D\,\max} = \max_{\varepsilon,q_1,q_2} \frac{1}{2\pi} \left\{ \sqrt{1-\varepsilon}\, \mathbf{J}_{\max}(q_2) + \sqrt{\varepsilon}\, \mathbf{J}_{\max}(q_1) \right\}^2, \tag{3.4}$$

where ε, q_1 and q_2 satisfy the constraints (3.2).

We have succeeded in obtaining analytical solutions to problems (3.2) and (3.4) only for the limiting values of $C_L = 0$ and $C_L = 2q_{\max} = 2/\mathbf{e}$. Indeed, the maximum of the function $\mathbf{J}_{\max}(q)$ is achieved at the point $q = q_* = 3/(4\,\mathbf{e})$, and this maximum equals $\mathbf{J}_m = 2\sqrt{2/\mathbf{e}}$. Therefore, for any functions $u_1(\sigma)$ and $u_2(\sigma)$

$$C_D \leqslant \max_{\varepsilon} \frac{1}{2\pi} \mathbf{J}_m^2 \left(\sqrt{1-\varepsilon} + \sqrt{\varepsilon} \right)^2 = \frac{1}{2\pi} \mathbf{J}_m^2 \left(\sqrt{1-\varepsilon} + \sqrt{\varepsilon} \right)^2_{|\varepsilon=\frac{1}{2}} = \frac{8}{\pi\,\mathbf{e}}.$$

The equality is only possible if $q_1 = q_2 = q_*$, $\varepsilon = 1/2$. But in this case, according to the last relation in (3.2), the lift coefficient $C_L = 0$. Therefore, at $C_L = 0$ the maximum drag coefficient is $C_{D\,\max} = 8/\pi\,\mathbf{e}$.

Let $C_L = 2q_{\max}$, then it follows from (3.1) that $\varepsilon = 0$, $q_2 = q_{\max}$ and q_1 can take any value. But $\mathbf{J}_{\min}(q_{\max}) = \mathbf{J}_{\max}(q_{\max}) = \mathbf{J}_r$, where the value of $\mathbf{J}_r$ is defined in (2.10). Hence, according to (3.4), at $C_L = 2q_{\max}$ we have

$$C_{D\,\min} = C_{D\,\max} = \frac{1}{2\pi} \mathbf{J}_r^2 = \frac{2}{\pi\,\mathbf{e}}.$$

So at $C_L = 2q_{\max}$ we have $\varkappa_{\min} = \varkappa_{\max} = \pi$.

For $0 < C_L < 2/\,\mathbf{e}$, the solution to problems (3.2) and (3.4) has been found numerically by means of the standard function **Maximize** of the package Mathematica 8.0.

In Figure 4 we demonstrate the dependencies of the minimal drag coefficient $C_{D\,\min}$ and the maximal drag coefficient $C_{D\,\max}$ on the lift coefficient C_L, and in Table 1 we show the maximal and minimal lift-to-drag ratios $\varkappa_{\max}$ and $\varkappa_{\min}$ for different C_L.

In Figure 5 we plot the values of ε, q_1 and q_2 versus C_L, which have been obtained in solving numerically problems (3.2) and (3.4). One can see from the figure that for any

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Table 1. *The values of $\varkappa_{\max}$ and $\varkappa_{\min}$ for different C_L*

C_L	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	$\frac{2}{e}$
$\varkappa_{\max}$	∞	224.88	99.1015	57.0649	35.9197	23.0608	14.1997	7.0821	π
$\varkappa_{\min}$	0	0.107495	0.219695	0.342541	0.48536	0.666406	0.933793	1.53824	π

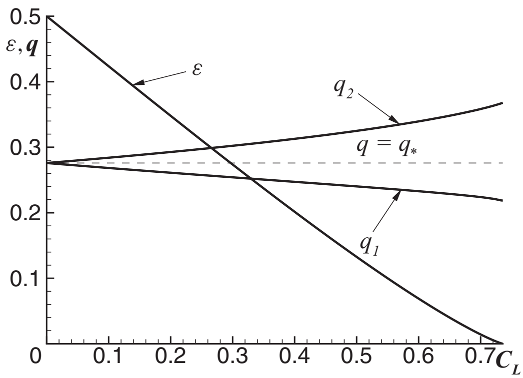

FIGURE 5. Dependencies ε, q_1 and q_2 on C_L.

C_L the values of q_1 and q_2 satisfy inequalities $q_1 < q_*$ and $q_2 > q_*$, and the value ε is an almost linear function that varies from $\varepsilon = 1/2$ at $C_L = 0$ to $\varepsilon = 0$ at $C_L = 2q_{\max}$.

We should note that at $C_L \to 0$ the maximal lift-to-drag ratio $\varkappa_{\max} \to +\infty$. Indeed, for flat plates the lift-to-drag ratio $\varkappa = \cot \alpha_a$, where α_a is the angle of attack. According to Rayleigh's well-known formula [3] the coefficient of the force, normal to the plate, is

$$C_N = \frac{2\pi \sin \alpha_a}{4 + \pi \sin \alpha_a}.$$

If $\alpha_a \to 0$, then $C_L = C_N \cos \alpha_a \to 0$, and $\varkappa = \cot \alpha_a \to +\infty$. Since the relation $C_{D\min}(C_L)/C_L$ defines the maximum possible lift-to-drag ratio, it is clear that $\varkappa_{\max} \to +\infty$ as $C_L \to 0$. In Figure 4 the dash-and-dot line demonstrates the dependence C_D on C_L for the flat plate. As one can see from the figure, the line lies entirely between the curves $C_{D\min}(C_L)$ and $C_{D\max}(C_L)$. It is worthy of note that for any profile in the Helmholtz–Kirchhoff flow, the point (C_L, C_D) always lies between the curves $C_{D\min}(C_L)$ and $C_{D\max}(C_L)$.

4 Proof of Theorem 1

4.1 Preliminary reasoning

Without the complementary condition (2.7) the auxiliary problem is a constrained problem of calculus of variations with a free right endpoint. Let us try to find extrema by the Lagrange multiplier rule without regard for the non-standard condition (2.7). To do so

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we construct the augmented cost functional

$$\mathbf{P}[\lambda] = -\int_0^1 \lambda'(\lambda + k)\log(\lambda\lambda')\,\mathrm{d}\sigma = -\int_0^1 E(\lambda, \lambda')\,\mathrm{d}\sigma, \tag{4.1}$$

where k is a real constant. We write the Euler equation [7]

$$E_\lambda - \frac{\mathrm{d}}{\mathrm{d}\sigma}E_{\lambda'} = 0,$$

which for the functional $\mathbf{P}[\lambda]$ takes the form

$$\frac{\lambda''(\lambda + k)}{\lambda'} + \lambda' = 0.$$

Integrating this equation yields

$$\lambda'(\sigma)[\lambda(\sigma) + k] = c, \tag{4.2}$$

where c is a constant. Because the right endpoint of the desired function $\lambda(\sigma)$ is free, there holds the relation $[E_{\lambda'}]_{\sigma=1} = 0$ (see [7]), which can be reduced to

$$\lambda(1)\lambda'(1) = 1/\mathbf{e}. \tag{4.3}$$

Equation (4.2) subject to the condition $\lambda(0) = 0$ can be easily integrated and has two solutions which are the functions of the form (2.15) for $k > 0$ and those of the form (2.17) for $k < 0$.

As follows from Theorem 1 and the results in Section 3, the functions of forms (2.15) and (2.17) do not define the minimum of the lift-to-drag ratio $\varkappa$, but its maximum, which we determine in the paper only for the sake of completeness. Thus, application of the classical approach to solving the basic problem, formulated in the Introduction, does not lead to finding extrema that are of most practical interest.

To prove Theorem 1 we use the technique developed earlier in [8]–[11], [14] for investigation of extremal problems of the jet and cavity theory. This technique is based on Jensen's inequality [4, theorem 204].

Let $f(x)$ and $g(x)$ be real functions defined in the interval $[x_1, x_2]$, and

$$f(x) \geqslant 0, \qquad \int_{x_1}^{x_2} f(x)\,\mathrm{d}x > 0.$$

If $G(u)$ is a strictly convex function, then there holds Jensen's inequality

$$\int_{x_1}^{x_2} f(x)G[g(x)]\,\mathrm{d}x \geqslant \left[\int_{x_1}^{x_2} f(x)\,\mathrm{d}x\right] G\left[\frac{\int_{x_1}^{x_2} f(x)g(x)\,\mathrm{d}x}{\int_{x_1}^{x_2} f(x)\,\mathrm{d}x}\right], \tag{4.4}$$

and the equality in (4.4) being possible if and only if $g(x) \equiv \text{const}$.

As the first example of the application of Jensen's inequality (4.4), let us obtain the exact upper bound of the functional $\mathbf{J}[\lambda]$. We denote $\lambda(1) = b$. By virtue of (2.7) it is

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evident that $b \leqslant \sqrt{2}$. We have

$$\mathbf{J}[\lambda] = -\sqrt{2}\left(\int_0^1 \lambda' \log \lambda \, \mathrm{d}\sigma + \int_0^1 \lambda' \log \lambda' \, \mathrm{d}\sigma\right) = \sqrt{2}\left[b - G(b) - \int_0^1 G(\lambda') \, \mathrm{d}\sigma\right],$$

where $G(u)$ is defined by (2.8).

We estimate the integral $\int_0^1 G(\lambda') \, \mathrm{d}\sigma$ from below by means of (4.4):

$$\int_0^1 G(\lambda') \, \mathrm{d}\sigma \geqslant G\left(\int_0^1 \lambda' \, \mathrm{d}\sigma\right) = G(b),$$

where the equality is possible if and only if $\lambda'(\sigma) = \text{const}$, i.e. for $\lambda(\sigma) = b\sigma$. From this, it follows that

$$\mathbf{J}[\lambda] \leqslant \sqrt{2}\,[b - 2G(b)] = G_1(b).$$

On the interval $b \in [0, \sqrt{2}]$, function $G_1(b)$ achieves its maximum at $b = 1/\sqrt{\mathbf{e}}$. Therefore, the functional $\mathbf{J}[\lambda]$ achieves the global maximum at $\lambda(\sigma) = \sigma/\sqrt{\mathbf{e}}$ and this maximum is $2\sqrt{2/\mathbf{e}}$. For the function $\lambda(\sigma) = \sigma/\sqrt{\mathbf{e}}$ we have $\lambda\lambda' = \sigma/\mathbf{e}$ and the constraints (2.6) and (2.7) are evidently valid. Thus,

$$\mathbf{J}[\lambda] \leqslant 2\sqrt{2/\mathbf{e}} = \mathbf{J}_m, \tag{4.5}$$

where the equality is only possible for $\lambda(\sigma) = \sigma/\sqrt{\mathbf{e}}$. For this function $\lambda(\sigma)$ we have $\mathbf{I}[\sigma/\sqrt{\mathbf{e}}] = \frac{3}{4\mathbf{e}} = q_*$.

We now formulate the following important lemma, whose proof is evident.

Lemma 2 *Let a function $\lambda(\sigma)$ deliver a global maximum (minimum) to the functional $\mathbf{P}[\lambda]$ under the constraints* (2.6) *and* (2.7). *Compute $q = \mathbf{I}[\lambda]$. Then for $k < 0$ the function $\lambda(\sigma)$ is the solution to the auxiliary problem on the minimum (maximum) of $\mathbf{J}[\lambda]$, and for $k > 0$ the function $\lambda(\sigma)$ is the solution to the auxiliary problem on the maximum (minimum) of $\mathbf{J}[\lambda]$.*

It follows from this lemma that determining global extrema of the functional $\mathbf{P}[\lambda]$ for different values of k leads to different solutions to the auxiliary problem. Value k plays the role of the parameter whose variation gives solutions of the constrained problem for different values of q.

4.2 Proof of the first part of Theorem 1

The proof is based on finding a global maximum of the functional $\mathbf{P}[\lambda]$ for $k < 0$. It is to be noted that for $k < 0$ the global maximum of the functional $\mathbf{P}[\lambda]$ is worthy of finding in the range $-\sqrt{2} < k < 0$. Indeed, assume $k \leqslant -\sqrt{2}$. By virtue of the constraint (2.7), function $\lambda(\sigma) \leqslant \sqrt{2\sigma}$. If $\lambda(\sigma) \equiv \sqrt{2\sigma}$, then $\lambda(\sigma)\lambda'(\sigma) \equiv 1$, $\mathbf{P}[\sqrt{2\sigma}] = 0$, and if $\lambda(\sigma) \not\equiv \sqrt{2\sigma}$, $\lambda(\sigma) \leqslant \sqrt{2\sigma}$, then $\mathbf{P}[\lambda] < 0$. Thus, the global maximum of the functional $\mathbf{P}[\lambda]$ for $k \leqslant -\sqrt{2}$ is zero. We drop the trivial case $\lambda(\sigma) \equiv \sqrt{2\sigma}$ from further consideration and suppose that k changes in the range $-\sqrt{2} < k < 0$.

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The proof consists of several steps.

Step 1. Let $k \in (-\sqrt{2}, 0)$ be a fixed value and choose $\gamma \in (k^2/2, 1)$. By means of Jensen's inequality we estimate the functional $\mathbf{P}[\lambda]$ from above by a function $H(a,b)$, where $a = \lambda(\gamma)$, $b = \lambda(1)$ (Lemma 3). It is of importance that for some class of functions $\lambda(\sigma)$ the non-strict inequality in the estimate turns into equality.

Step 2. We prove that the function $H(a,b)$ has a unique global maximum which can be found from the ordinary necessary conditions of extrema (Lemma 4).

Step 3. Let (a_*, b_*) be the point of global maximum of the function $H(a,b)$. Then for any function $\lambda(\sigma)$ that satisfies the constraints (2.6) and (2.7) there hold the inequalities

$$\mathbf{P}[\lambda] \leqslant H(a,b) \leqslant H(a_*, b_*), \tag{4.6}$$

where a_* and b_* depend on choosing γ. We demonstrate that parameter γ can be chosen in such a manner that there exists a unique function $\lambda(\sigma)$ so that $P[\lambda] = H(a_*, b_*)$, for all other functions $P[\lambda] < H(a_*, b_*)$ (Lemma 5). Hence, we find a global maximum of the functional $P[\lambda]$ under the constraints (2.6) and (2.7).

Step 4. Application of Lemma 2 finalises the proof.

Steps 1–3 allow us to reduce the problem of finding the global maximum of the non-linear functional $\mathbf{P}[\lambda]$ to the maximization of the function $H(a,b)$ of only two variables.

Now we shall realize step 1.

Lemma 3 *Let $k \in (-\sqrt{2}, 0)$. Choose $\gamma \in (k^2/2, 1)$ and consider any function $\lambda(\sigma)$ that satisfies the constraints* (2.6) *and* (2.7)*. Then for the functional* $\mathbf{P}[\lambda]$ *there holds the inequality*

$$\mathbf{P}[\lambda] \leqslant H[a,b] = \frac{1}{2}\left[(a+k)^2 \log f_1(a,b) - (b+k)^2 \log f_2(a,b) + k(b-a) + k^2 \log \frac{b}{a}\right], \tag{4.7}$$

where $a = \max[\lambda(\gamma), -k]$, $b = \max[\lambda(1), -k]$,

$$f_1(a,b) = \frac{a(b-a)(a+b+2k)(k+\sqrt{2\gamma})^2}{2(a+k)^3(1-\gamma)}, \tag{4.8}$$

$$f_2(a,b) = \frac{b(b-a)(a+b+2k)}{2(b+k)(1-\gamma)}. \tag{4.9}$$

For $\lambda(\gamma) > -k$ the equality in (4.7) *is possible if and only if $a = \sqrt{2\gamma}$ and on the interval $\sigma \in [\gamma, 1]$ the function $\lambda(\sigma)$ satisfies the differential equation* (4.2)*, i.e. for the functions of the form* (2.13)*, where*

$$c = \frac{(b-a)(a+b+2k)}{2(1-\gamma)}. \tag{4.10}$$

Proof Assume that $\lambda(1) \leqslant -k$. By virtue of (2.6), function $\lambda(\sigma) \leqslant -k$ everywhere, and $a = b = -k$. Because the constraint (2.7) is fulfilled, we have $\log(\lambda\lambda') \leqslant 0$ and the functional $\mathbf{P}[\lambda] \leqslant 0$. But at $a = b = -k$, the right-hand side of inequality (4.7) vanishes,

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and therefore for $\lambda(1) \leqslant -k$ inequality (4.7) is valid. Because of this, in further reasoning we suppose that $\lambda(1) = b > -k$.

Now assume that $\lambda(\gamma) < -k$. Then the function $\lambda(\sigma)$ can be reconstructed in such a manner that the value of the functional will not decrease, and the value of $b = \lambda(1)$ will not change, but $\lambda(\gamma) = -k$. Indeed, since $\lambda(1) = b > -k$, there exists a point $\sigma_1 \in (\gamma, 1)$ such that $\lambda(\sigma_1) = -k$, $\lambda(\sigma) < -k$ for $0 \leqslant \sigma < \sigma_1$. On the segment $(0, \sigma_1)$ the integrand in formula (4.1) is non-negative, that is the segment gives a non-positive contribution to the functional $\mathbf{P}[\lambda]$. Let us set that $\lambda(\sigma) = \sqrt{2\sigma}$ for $0 \leqslant \sigma \leqslant k^2/2$ and $\lambda(\sigma) = -k$ for $k^2/2 < \sigma \leqslant \sigma_1$. Then the contribution of the segment $(0, \sigma_1)$ is zero, hence the functional will not decrease, but now $\lambda(\gamma) = -k$.

So the class of admissible functions $\lambda(\sigma)$ can be contracted, namely it is possible to consider only the functions for which $\lambda(\sigma) = \sqrt{2\sigma}$ for $0 \leqslant \sigma \leqslant k^2/2$. In this case

$$\mathbf{P}[\lambda] = -\int_{k^2/2}^{1} \lambda'(\lambda + k)\log(\lambda\lambda')\,\mathrm{d}\sigma, \quad \lambda(k^2/2) = -k, \quad \lambda(\gamma) = a \geqslant -k. \tag{4.11}$$

Functional (4.11) can be represented in the following form:

$$\mathbf{P}[\lambda] = -\int_{k^2/2}^{\gamma} \lambda'(\lambda + k)\log(\lambda\lambda')\,\mathrm{d}\sigma - \int_{\gamma}^{1} \lambda'(\lambda + k)\log[\lambda'(\lambda + k)]\,\mathrm{d}\sigma + \int_{\gamma}^{1} \lambda'(\lambda + k)\log\frac{\lambda + k}{\lambda}\,\mathrm{d}\sigma. \tag{4.12}$$

Such a representation is correct since $\lambda(\sigma) + k \geqslant 0$ for $\sigma \in [k^2/2, 1]$.

The last integral in (4.12) can be calculated and expressed in terms of $a = \lambda(\gamma)$ and $b = \lambda(1)$:

$$\int_{\gamma}^{1} \lambda'(\lambda + k)\log\frac{\lambda + k}{\lambda}\,\mathrm{d}\sigma = \int_{a}^{b} (p + k)\log\frac{p + k}{p}\,\mathrm{d}p = M(b) - M(a),$$

where

$$M(p) = \frac{1}{2}\left[(p + k)^2\log\frac{p + k}{p} + kp + k^2\log p\right].$$

Because of this,

$$\mathbf{P}[\lambda] = -\int_{k^2/2}^{\gamma} f(\sigma)G(\lambda\lambda')\,\mathrm{d}\sigma - \int_{\gamma}^{1} G[\lambda'(\lambda + k)]\,\mathrm{d}\sigma + M(b) - M(a), \tag{4.13}$$

where $f(\sigma) = 1 + k/\lambda(\sigma)$ and $G(u)$ is defined by (2.8).

Let us set

$$\alpha = \int_{k^2/2}^{\gamma} f(\sigma)\,\mathrm{d}\sigma = \int_{k^2/2}^{\gamma} (1 + k/\lambda)\,\mathrm{d}\sigma.$$

Assuming that $\alpha > 0$, we estimate each of the integrals in (4.13) by means of Jensen's inequality (4.4) to obtain

$$\mathbf{P}[\lambda] \leqslant -\alpha\, G\left[\frac{1}{\alpha}\int_{k^2/2}^{\gamma} (1 + k/\lambda)\,\lambda\lambda'\,\mathrm{d}\sigma\right] - (1-\gamma)\, G\left[\frac{\int_{\gamma}^{1} \lambda'(\lambda + k)\,\mathrm{d}\sigma}{1 - \gamma}\right] + M(b) - M(a). \tag{4.14}$$

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Integrals on the right-hand side of (4.14) are calculated as

$$\int_{k^2/2}^{\gamma} (1+k/\lambda)\,\lambda\lambda'\,\mathrm{d}\sigma = \int_{-k}^{a} (p+k)\,\mathrm{d}p = \frac{(a+k)^2}{2},$$

$$\int_{\gamma}^{1} \lambda'(\lambda+k)\,\mathrm{d}\sigma = \int_{a}^{b} (p+k)\,\mathrm{d}p = \frac{1}{2}[(b+k)^2-(a+k)^2] = \frac{1}{2}(b-a)(b+a+2k).$$

This leads to the inequality

$$\mathbf{P}[\lambda] \leqslant \frac{(a+k)^2}{2}\log\frac{2\alpha}{(a+k)^2} - \frac{1}{2}\left[(b+k)^2-(a+k)^2\right]\log\frac{(b-a)(a+b+2k)}{2(1-\gamma)} + M(b) - M(a). \tag{4.15}$$

Thus, the right-hand side of (4.15) is a function of three variables: a, b, α (the parameters k and γ are fixed) and, as one can easily see, the right-hand side is a non-decreasing function of parameter α. We estimate the value of α from above. Since $k<0$ and $\lambda \leqslant \sqrt{2\sigma}$, we have

$$\alpha = \int_{k^2/2}^{\gamma}\left(1+\frac{k}{\lambda}\right)\mathrm{d}\sigma \leqslant \int_{k^2/2}^{\gamma}\left(1+\frac{k}{\sqrt{2\sigma}}\right)\mathrm{d}\sigma = \frac{(k+\sqrt{2\gamma})^2}{2}.$$

Inserting this estimate into (4.15), we obtain inequality (4.7) of the lemma deduced under the assumption that $\alpha>0$. But this inequality is also correct for $\alpha=0$. Indeed, if $\alpha=0$, then $\lambda(\sigma)=-k$ for $k^2/2\leqslant\sigma\leqslant\gamma$, $a=-k$, and the first summand in representation (4.12) vanishes. In relation (4.7) this summand is estimated by the inequality

$$-\int_{k^2/2}^{\gamma} \lambda'(\lambda+k)\log(\lambda\lambda')\,\mathrm{d}\sigma \leqslant \frac{(a+k)^2}{2}\log\frac{(k+\sqrt{2\gamma})^2}{(a+k)^2},$$

which turns into the equality $0=0$ at $a=-k$.

Now we should consider the question of equality in (4.7) for $a>-k$. If $a>-k$, then the term containing α on the right-hand side of inequality (4.15) is a strictly increasing function of parameter α. Because of this, equality in (4.7) is only possible if

$$\alpha = (k+\sqrt{2\gamma})^2/2. \tag{4.16}$$

Besides, from the condition of equality in Jensen's inequality (4.4) we must have

$$\lambda(\sigma)\lambda'(\sigma) = \text{const} \;\; \text{for} \;\; \sigma\in[k^2/2,\gamma].$$

Moreover, from the same condition, function $\lambda(\sigma)$ must satisfy differential equation (4.2) on the segment $\sigma\in[\gamma,1]$. Because $\lambda(k^2/2)=-k$, conditions $\lambda(\sigma)\lambda'(\sigma)=\text{const}$ for $\sigma\in[k^2/2,\gamma]$ and (4.16) can be simultaneously fulfilled if $\lambda(\sigma)=\sqrt{2\sigma}$ for $\sigma\in[k^2/2,\gamma]$, which is equivalent to the condition $a=\sqrt{2\gamma}$. Solving differential equation (4.2) subject to the conditions $\lambda(\gamma)=a$, $\lambda(1)=b$, we conclude that the function $\lambda(\sigma)$ is of the form (2.13). Lemma 3 is proved. □

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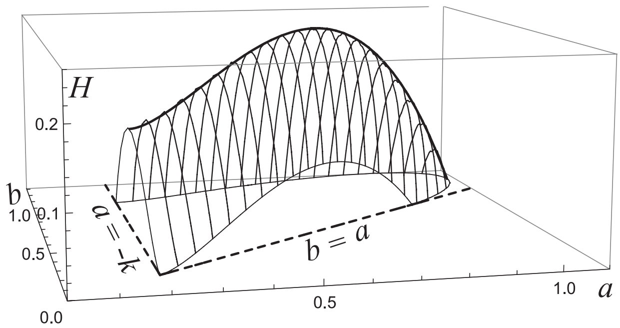

FIGURE 6. Graph of the function $H(a,b)$ at $k = -0.2$, $\gamma = 0.4$.

The graph of the function $H(a,b)$ at $k = -0.2$, $\gamma = 0.4$ is shown in Figure 6. As one can see from the figure, function $H(a,b)$ for these values of k and γ has a maximum at an inner point (a,b) of the angle S:

$$S = \{(a,b) \in S : \ a \geqslant -k, \ b \geqslant a\}. \tag{4.17}$$

To realize step 2 we need to prove this fact strictly for any $k \in (-\sqrt{2}, 0)$ and $\gamma \in (k^2/2, 1)$.

Lemma 4 *Consider in the plane ab an angle S. In the domain S the function $H(a,b)$ achieves a unique global maximum at an inner point (a,b) of the angle S, the coordinates a and b of this point being determined from the system of equations*

$$\begin{cases} f_1(a,b) = 1, \\ f_2(a,b) = \mathbf{e}^{-1}, \end{cases} \tag{4.18}$$

where $f_1(a,b)$ and $f_2(a,b)$ are defined by formulae (4.8) *and* (4.9) *respectively.*

Proof It is easy to see that the function $H(a,b)$ is continuous in the angle S up to the boundary. In the plane ab consider the family of triangles

$$T(\beta) = \{(a,b) \in T(\beta) : \ a \geqslant -k, \ b \geqslant a, \ \ b \leqslant \beta, \ \ \beta > -k\}.$$

Each triangle $T(\beta)$ is the intersection of the angle S and the half-plane $b \leqslant \beta$.

The function $H(a,b)$ is continuous in the triangle $T(\beta)$ up to the boundary, and therefore $H(a,b)$ achieves in this triangle maximum and minimum values. Let us demonstrate that for β sufficiently large, the maximum value cannot be achieved on the boundary of the triangle $T(\beta)$. To do so, we calculate the partial derivatives of the function $H(a,b)$:

$$\frac{\partial H(a,b)}{\partial a} = (a+k)\log f_1(a,b), \quad \frac{\partial H(a,b)}{\partial b} = -(b+k)[1 + \log f_2(a,b)] \tag{4.19}$$

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and establish that

$$\frac{\partial \log f_1(a,b)}{\partial a} = \frac{1}{a} - \frac{1}{b-a} + \frac{1}{a+b+2k} - \frac{3}{a+k}$$
$$= -\frac{2a(b+k)^2 - k(b-a)(a+b+2k)}{a(b-a)(a+k)(a+b+2k)} < 0, \quad (4.20)$$

$$\lim_{a\to -k} \log f_1(a,b) = +\infty, \quad \lim_{a\to b} \log f_1(a,b) = -\infty. \quad (4.21)$$

The first equality in (4.19) and relations (4.20) and (4.21) mean that for every fixed $b > -k$ the function $H(a,b)$ achieves a unique maximum with respect to a on the interval $a \in [-k, b]$, the point of maximum being located inside the interval. Hence, the maximum value of $H(a,b)$ cannot be achieved on the boundaries $a = -k$ and $b = a$ of the triangle $T(\beta)$.

Now we calculate

$$\frac{\partial \log f_2(a,b)}{\partial b} = \frac{1}{b} + \frac{1}{a+b+2k} + \frac{a+k}{(b-a)(b+k)} > 0, \quad (4.22)$$

$$\lim_{b\to a} \log f_2(a,b) = -\infty, \quad \lim_{b\to +\infty} \log f_2(a,b) = +\infty. \quad (4.23)$$

From the second equality in (4.19) and the relations (4.22) and (4.23), we infer that for every fixed $a \geqslant -k$ on the interval $b \in (a, +\infty)$ the function $H(a,b)$ first increases with respect to b, achieves a maximum and then decreases.

Let us prove that for β large enough, the maximum value of the function $H(a,b)$ cannot be achieved on the upper boundary $b = \beta$ of the triangle $T(\beta)$. To this end, from the equation $f_1(a_1, \beta) = 1$ we find the point a_1 at which the function $H(a, \beta)$ achieves its maximum with respect to a. If we prove that for this point there holds the inequality $\log f_2(a_1, \beta) + 1 > 0$, then $H(a_1, \beta)$ is not the maximum of the function $H(a,b)$ in the triangle $T(\beta)$ because

$$\lim_{b\to a_1} [-\log f_2(a_1, b) - 1] = +\infty, \quad -\log f_2(a_1, \beta) - 1 < 0,$$

therefore the derivative $\partial H(a,b)/\partial b$ changes its sign with respect to b on the interval $b \in (a_1, \beta)$, and hence the maximum of the function $H(a_1, b)$ with respect to b is located inside this interval.

To the contrary, assume that $\log f_2(a_1, \beta) + 1 \leqslant 0$. Then the relations

$$\begin{cases} \dfrac{\beta[(\beta+k)^2 - (a_1+k)^2]}{2(\beta+k)} \leqslant k_1, \\ \dfrac{a_1[(\beta+k)^2 - (a_1+k)^2]}{2(a_1+k)^3} = k_2 \end{cases} \quad (4.24)$$

hold simultaneously, where $k_1 = \mathbf{e}^{-1}(1-\gamma) > 0$, $k_2 = (1-\gamma)/(k+\sqrt{2\gamma})^2 > 0$.

From the first relation in (4.24) we find that

$$(a_1+k)^2 \geqslant (\beta+k)^2 - 2k_1(\beta+k)/\beta. \quad (4.25)$$

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We divide the first relation in (4.24) by the second and multiply the obtained inequality by $a_1(a_1+k)^{-1}/2$ to get

$$\frac{\beta(a_1+k)^2}{2(\beta+k)} \leqslant \frac{k_1}{2k_2}\,\frac{a_1}{a_1+k}. \tag{4.26}$$

Summing (4.26) and the first relation in (4.24) gives

$$\frac{\beta(\beta+k)}{2} \leqslant k_1 + \frac{k_1}{2k_2}\,\frac{a_1}{a_1+k}.$$

Because $a_1 \in (-k, \beta)$ and the right-hand side of (4.25) is strictly positive for β large enough, we can estimate the value of a_1+k from below to obtain

$$\frac{\beta(\beta+k)}{2} \leqslant k_1 + \frac{k_1}{2k_2}\,\frac{\beta}{\sqrt{(\beta+k)^2 - 2k_1(\beta+k)/\beta}}.$$

For β large enough, the last inequality cannot be fulfilled because as $\beta \to +\infty$ the left-hand side of the inequality tends to $+\infty$, but its right-hand side tends to a finite value of $k_1 + k_1/(2k_2)$. So for β large enough, system (4.24) becomes contradictory. Therefore, in the triangle $T(\beta)$ for β sufficiently large, the point of maximum of the function $H(a,b)$ is located inside the triangle, and at this point the necessary conditions of extremum (4.18) are fulfilled.

Now we prove that the system (4.18) has only one solution. The first equation of the system is quadratic with respect b. Solving the equation, we get

$$b = b(a) = (a+k)\sqrt{\frac{2(1-\gamma)}{(k+\sqrt{2\gamma})^2}\,\frac{a+k}{a} + 1} - k.$$

Further, we proceed in the same manner as in deducing relation (4.26), but now in (4.24) the inequality is changed by equality, β is changed by b and a_1 is changed by a. Doing so leads to the system that is equivalent to (4.18):

$$\begin{cases} b = b(a), \\ \dfrac{b(b+k)}{2} = k_1 + \dfrac{k_1}{2k_2}\,\dfrac{a}{a+k}. \end{cases} \tag{4.27}$$

The left-hand side of the second equation in (4.27) is a function that depends only on b and strictly increases with respect to b, whereas the right-hand side depends only on a and strictly decreases with respect to a. The function $b = b(a)$ strictly increases. Substituting $b(a)$ for b on the left-hand side of the second equation in (4.27), we obtain an equation in which the left-hand side strictly increases, and the right-hand side strictly decreases. Such an equation cannot have more than one solution.

Since for β large enough the solution (a,b) of the system (4.18) is independent of β, we conclude that at the point (a,b) the function $H(a,b)$ achieves its global maximum in the angle S. Lemma 4 is proved. □

To realize step 3 we prove the following.

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Lemma 5 *Let* $k \in (-\sqrt{2}, 0)$. *The global maximum of the functional* $\mathbf{P}[\lambda]$ *is achieved by the function* (2.13), *where the parameters a, c and* $b \in (\sqrt{2/\mathbf{e}}, \sqrt{2})$ *are uniquely determined by the system of equations*

$$\begin{cases} k = K(b) = -\dfrac{(\mathbf{e}-1)b(b^2\mathbf{e}-2)}{2+(\mathbf{e}-2)b^2\mathbf{e}}, \\ a = \dfrac{b^2\mathbf{e}-2}{(\mathbf{e}-1)b}, \\ c = \dfrac{2-b^2}{2+(\mathbf{e}-2)b^2\mathbf{e}}. \end{cases} \tag{4.28}$$

System (4.28) *is equivalent to two conditions*

$$\lambda(\gamma)\lambda'(\gamma+0) = 1, \quad \lambda(1)\lambda'(1) = 1/\mathbf{e}, \tag{4.29}$$

written in terms of the function $\lambda(\sigma)$ *defined by* (2.13).

The global maximum of $\mathbf{P}[\lambda]$ *is*

$$\mathbf{P}_{\max} = \frac{1}{2}\left[(b+k)^2 + k(b-a) + k^2 \log\frac{b}{a}\right]. \tag{4.30}$$

Proof Choose $\gamma \in (k^2/2, 1)$ and find the roots a_* and b_* of the system (4.18). Then for any function $\lambda(\sigma)$ which satisfies the constraints (2.6) and (2.7) there holds the estimate (4.6). Since by Lemma 4 we have $a_* > -k$, from Lemma 3 we deduce that the equality $\mathbf{P}[\lambda] = H(a_*, b_*)$ is only possible for the function $\lambda(\sigma)$ defined by (2.13) and (4.10).

It follows from (2.13) that

$$\gamma = a_*^2/2. \tag{4.31}$$

Hence, if initially γ is chosen so that after solving (4.18) equation (4.31) is satisfied, then with $\lambda(\sigma)$ defined by (2.13) and (4.10) all non-strict inequalities in (4.6) turn into equality and $P[\lambda] = H(a_*, b_*)$, for all other functions $P[\lambda] < H(a_*, b_*)$.

Inserting $\gamma = a^2/2$ into (4.18), we get the following system of equations for finding the parameters a_* and b_*:

$$\begin{cases} \dfrac{a(b-a)(a+b+2k)}{(a+k)(2-a^2)} = 1 \\ \dfrac{b(b-a)(a+b+2k)}{(b+k)(2-a^2)} = \dfrac{1}{\mathbf{e}} \end{cases}. \tag{4.32}$$

For shortness of writing, we omit ‘stars’ in the notations of a_* and b_* in (4.32) and further reasoning. It is easy to see that for $k < 0$, system (4.32) is equivalent to two first equations in (4.28). The third equation in (4.28) will be obtained if we insert k and a from (4.28) into (4.10).

From the conditions $k < 0$, $b > a$ it follows that $b \in (\sqrt{2/\mathbf{e}}, \sqrt{2})$. Besides, the function $K(b)$ decreases monotonically for $b > \sqrt{2/\mathbf{e}}$ and $K(\sqrt{2/\mathbf{e}}) = 0$, $K(\sqrt{2}) = -\sqrt{2}$, hence system (4.28) always has a unique solution for $k \in (-\sqrt{2}, 0)$, and the root $b \in (\sqrt{2/\mathbf{e}}, \sqrt{2})$.

The equivalence of (4.29) to (4.28) can be checked by direct calculations. Besides, after a little algebra it is possible to demonstrate that the function $\lambda(\sigma)$, defined by (4.28) and (2.13), satisfies the restrictions (2.6) and (2.7).

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It is to be noted that the second condition in (4.29) is the condition (4.3) of the 'free' right endpoint for extremals of the functional $\mathbf{P}[\lambda]$. The first condition in (4.29) means that the functions $u(\sigma) = \lambda(\sigma)\lambda'(\sigma)$, which define velocity distributions on the profile surface, are continuous.

Formula (4.30) follows from the relations (4.18) and the estimate (4.7). Lemma 5 is proved. □

To realize step 4, i.e. to finalize the proof of the first part of Theorem 1, we fix $k \in (-\sqrt{2}, 0)$ and find the parameters a, b and c from the system (4.28). By means of (2.13) we determine the corresponding function $\lambda(\sigma)$ and compute $q = \mathbf{I}[\lambda]$. Since $k < 0$, by Lemma 2 we infer that $\mathbf{J}_{\min}(q) = \mathbf{J}[\lambda]$. Then

$$q = \mathbf{P}_{\max} - k\,\mathbf{J}_{\min}/\sqrt{2}, \tag{4.33}$$

where $\mathbf{P}_{\max}$ is determined by (4.30). Now we calculate

$$\frac{\mathbf{J}_{\min}}{\sqrt{2}} = -\int_\gamma^1 \lambda' \log(\lambda\lambda')\,\mathrm{d}\sigma = -\int_\gamma^1 \lambda' \log\left[\frac{\lambda}{\lambda+k}\lambda'(\lambda+k)\right]\mathrm{d}\sigma.$$

Taking into account that the function $\lambda(\sigma)$ satisfies differential equation (4.2) and $\lambda(\gamma) = a$, $\lambda(1) = b$, we obtain

$$\frac{\mathbf{J}_{\min}}{\sqrt{2}} = \int_a^b \log\frac{p+k}{p\,c}\,\mathrm{d}p = k\log\frac{b+k}{a+k} + b\log\frac{b+k}{b\,c} - a\log\frac{a+k}{a\,c}.$$

Besides, for the function $\lambda(\sigma)$ relations (4.29) are valid. From this it follows that

$$\frac{a+k}{a\,c} = 1, \quad \frac{b+k}{b\,c} = \mathbf{e}, \quad \frac{b+k}{a+k} = \mathbf{e}\,\frac{b}{a},$$

and hence,

$$\frac{\mathbf{J}_{\min}}{\sqrt{2}} = k\log\frac{b}{a} + b + k.$$

Inserting this expression into (4.33) and taking into account (4.30), we come to formulae (2.11).

The function $q(b)$ in (2.11) decreases monotonically with respect to the parameter b on the interval $b \in (\sqrt{2/\mathbf{e}}, \sqrt{2})$, and

$$\lim_{b\to\sqrt{2/\mathbf{e}}} q(b) = q_{\max} = 1/\mathbf{e}, \quad \lim_{b\to\sqrt{2}} q(b) = 0.$$

Because of this, the parametric dependence (2.11) covers the whole range through which the parameter q is varied. The first part of Theorem 1 is proved.

4.3 Proof of the second part of Theorem 1

The proof is based on finding a global maximum of the functional $\mathbf{P}[\lambda]$ for $k > 0$ and a global minimum of this functional for $k < 0$. After determining these extrema we apply Lemma 2.

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We should note that the global minimum of the functional $\mathbf{P}[\lambda]$ is worthy of finding only for $k < 0$. Indeed, if $k \geqslant 0$, then the integrand in (4.1) is non-positive, whereas the functional $\mathbf{P}[\lambda]$ itself is non-negative. Therefore, for $\lambda(\sigma) \equiv \sqrt{2\sigma}$ the functional takes its global minimum value of $\mathbf{P}[\sqrt{2\sigma}] = 0$. As previously, we do not consider the trivial case $\lambda(\sigma) \equiv \sqrt{2\sigma}$.

Lemma 6 *Consider the functions $\lambda(\sigma)$ which satisfy the constraints* (2.6) *and* (2.7).

(1) *For $k > 0$ the functional $\mathbf{P}[\lambda]$ takes its global maximum with the function $\lambda(\sigma)$ of the form* (2.15). *The parameter $c > 0$ is uniquely determined either from the condition $\lambda(1)\lambda'(1) = 1/\mathbf{e}$ of free right endpoint or from the equivalent relation*

$$c = \frac{b^2}{2(b^2\,\mathbf{e} - 1)}, \tag{4.34}$$

where $b = \lambda(1)$ is the root of the equation

$$k = K_1(b) = -\frac{b(b^2\,\mathbf{e} - 2)}{2(b^2\,\mathbf{e} - 1)}, \tag{4.35}$$

and $b \in (\sqrt{1/\mathbf{e}}, \sqrt{2/\mathbf{e}})$.

(2) *For $k < 0$ the functional $\mathbf{P}[\lambda]$ takes its global minimum with the function $\lambda(\sigma)$ of the form* (2.17). *The parameter $c < 0$ is uniquely determined either from the condition $\lambda(1)\lambda'(1) = 1/\mathbf{e}$ or from the equivalent relation* (4.34)*, where $b = \lambda(1)$ is the root of equation* (4.35) *which belongs to the interval $b \in (0, \sqrt{1/\mathbf{e}})$.*

(3) *For both cases the extrema are defined by the formula*

$$\mathbf{P}_{\binom{\max}{\min}} = \frac{1}{2}\left[b^2 + 3kb + k^2 \log\frac{b+k}{k}\right]. \tag{4.36}$$

Proof Consider the first case $k > 0$. As previously, we denote $\lambda(1) = b$. Since $\lambda(\sigma) + k \geqslant 0$, the functional $\mathbf{P}[\lambda]$ can be represented in the following manner:

$$\begin{aligned}\mathbf{P}[\lambda] &= -\int_0^1 \lambda'(\lambda+k)\log[\lambda'(\lambda+k)]\,\mathrm{d}\sigma + \int_0^1 \lambda'(\lambda+k)\log\frac{\lambda+k}{\lambda}\,\mathrm{d}\sigma\\ &= -\int_0^1 G[\lambda'(\lambda+k)]\,\mathrm{d}\sigma + \int_0^b (p+k)\log\frac{p+k}{p}\,\mathrm{d}p,\end{aligned}$$

where $G(u)$ is defined by (2.8).

We estimate the first integral by Jensen's inequality and calculate analytically the second one. As a result, we obtain

$$\mathbf{P}[\lambda] \leqslant H_1(b) = \frac{1}{2}\left[kb + k^2\log\frac{b+k}{k} + b(b+2k)\log f(b)\right], \quad f(b) = \frac{2(b+k)}{b^2(b+2k)}. \tag{4.37}$$

Note that since the value of k is fixed, the function $H_1(b)$ depends only on b. We find the value of b, at which $H_1(b)$ achieves its maximum with respect to b. By differentiation,

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we have

$$\frac{\mathrm{d}H_1(b)}{\mathrm{d}b} = (b+k)[\log f(b) - 1], \tag{4.38}$$

where $b+k>0$ on the interval $b \in (0,+\infty)$. Besides,

$$\lim_{b\to+0} \log f(b) = +\infty, \quad \lim_{b\to+\infty} \log f(b) = -\infty, \quad \frac{\mathrm{d}\log f(b)}{\mathrm{d}b} = -\frac{2b^2+5\,bk+4k^2}{b(b+k)(b+2k)} < 0. \tag{4.39}$$

Hence, on the interval $b \in (0,+\infty)$ the function $H_1(b)$ firstly increases with increase of b, reaches its unique maximum at point b, which is a root of equation $f(b) = \mathbf{e}$, and then decreases. Equation $f(b) = \mathbf{e}$ is equivalent to (4.35), and, since $k \geqslant 0$, the root belongs to the interval $b \in (\sqrt{1/\mathbf{e}}, \sqrt{2/\mathbf{e}})$.

Let b be the root of equation (4.35). Then for any function $\lambda(\sigma)$, which satisfies the constraints (2.6), there holds the inequality $\mathbf{P}[\lambda] \leqslant H_1(b)$. The equality in this inequality is only possible if on the whole interval $\sigma \in [0,1]$ the function $\lambda(\sigma)$ satisfies the differential equation (4.2). Solving this equation under the conditions $\lambda(0)=0$, $\lambda(1)=b$, we find that the equality is only possible if $\lambda(\sigma)$ is defined by (2.15), where $c = b(b+2k)/2$. Inserting $k = K_1(b)$ in the last relation for c, we get equation (4.34).

It is easy to check that for the function $\lambda(\sigma)$, the constraints (2.6) and (2.7) are fulfilled. Moreover, condition (4.3) is also fulfilled. The first statement of Lemma 6 is proved.

We now prove the second part of the lemma. Let $k<0$ and assume that on the interval $[0,1]$ there exist points, where $\lambda(\sigma) > -k$. Then the function $\lambda(\sigma)$ can be reconstructed in such a manner that the value of the functional will not increase, but $\lambda(\sigma) \leqslant -k$ everywhere. Indeed, if the above assumption is true, then by monotonicity of $\lambda(\sigma)$ there exists a point $\sigma_1 \in (0,1)$ such that $\lambda(\sigma_1) = -k$, $\lambda(\sigma) \geqslant -k$ for $\sigma \in [\sigma_1, 1]$. On the segment $[\sigma_1,1]$ the integrand in (4.1) is non-positive, i.e. this segment gives a non-negative contribution to the functional $\mathbf{P}[\lambda]$. If we set $\lambda(\sigma) = -k$ for $\sigma \in [\sigma_1,1]$, then the contribution of the segment $[\sigma_1,1]$ will be zero and the functional will not increase, but $\lambda(\sigma) \leqslant -k$ everywhere.

So we can consider only the functions for which $\lambda(\sigma) \leqslant -k$. In this case the functional $\mathbf{P}[\lambda]$ can be represented as

$$\begin{aligned}\mathbf{P}[\lambda] &= -\int_0^1 \lambda'(\lambda+k)\log[-\lambda'(\lambda+k)]\,\mathrm{d}\sigma + \int_0^1 \lambda'(\lambda+k)\log\left(-\frac{\lambda+k}{\lambda}\right)\mathrm{d}\sigma \\ &= \int_0^1 G[-\lambda'(\lambda+k)]\,\mathrm{d}\sigma + \int_0^b (p+k)\log\left(-\frac{p+k}{p}\right)\mathrm{d}p,\end{aligned}$$

where, as previously, $b = \lambda(1)$. We estimate the first integral from below by means of Jensen's inequality (4.4) and calculate the second one analytically taking into account that $p+k \leqslant 0$. As a result, we have

$$\mathbf{P}[\lambda] \geqslant H_1(b). \tag{4.40}$$

Here the function $H_1(b)$ is again defined by formula (4.37). It is worthy of note that in inequality (4.37) the functional $\mathbf{P}[\lambda]$ is estimated from above, whereas in (4.40) the same functional is estimated from below by the same function $H_1(b)$, but in (4.37) the value of $k>0$, and now $k<0$. The first and third formulae in (4.39) retain their validity, and the

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second one should be changed by

$$\lim_{b \to -k} \log f(b) = -\infty.$$

Since $k + b < 0$, on the interval $b \in (0, -k)$, the function $H_1(b)$ first decreases with increase of b, reaches its unique minimum at point b, which is a root of equation $f(b) = \mathbf{e}$, and then increases. Equation $f(b) = \mathbf{e}$ is equivalent to (4.35), and, since $k < 0$, the root belongs to the interval $b \in (0, \sqrt{1/\mathbf{e}})$.

Repeating the final reasoning that we have used in proving the first statement of the lemma, we come to the second statement. Formula (4.36) follows from the equation $f(b) = \mathbf{e}$ and estimates (4.37) and (4.40). Thus, Lemma 6 is entirely proved. □

The second part of Theorem 1 follows from Lemmas 2 and 6. The only difficulty appears in the consideration of the special case $q = q_*$. But this case has been already investigated in deducing formula (4.5). Theorem 1 is proved.

5 Concluding remarks

In this work we are mainly concerned with the mathematical aspects of the formulated basic problem and do not find the profile shapes that realize the maximal lift-to-drag ratio. As follows from the proof of Theorem 2 for such shapes, the parameter $\varepsilon = 0$, i.e. the points A and O coincide and the segment OA entirely disappears. Moreover, according to the second statement of Theorem 1 on the initial part of the segment OB the velocity $v = v_0$. This means that on this part of OB the pressure is equal to that in the cavity, and thus this initial part is unloaded. Preliminary computations show that the flow over profiles with such a velocity distribution will be unrealizable physically (two-sheeted). This brings up the natural question: How can the point (C_L, C_D) be close to the boundary point $(C_L, C_{D\,\min})$ in Figure 4 to have a physically realizable flow? The solution of this question will be a subject of further investigations.

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