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SOME QUARTIC DIOPHANTINE EQUATIONS IN THE GAUSSIAN INTEGERS

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Abstract

In this paper we examine solutions in the Gaussian integers to the Diophantine equation $ax^4 + by^4 = cz^2$ for different choices of a, b and c. Elliptic curve methods are used to show that these equations have a finite number of solutions or have no solution.

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1. Introduction and historical background

Through consideration of the question as to whether or not a right triangle with rational sides can have area the square of an integer, Fermat was led to the quartic equation $x^4 - y^4 = z^2$. Lagrange showed that this is equivalent to solving equations of the form $ax^4 + by^4 = cz^2$ [2]. Fermat considered the related equation $x^4 + y^4 = z^2$ and showed, by infinite descent, that this equation has no nontrivial rational solutions. Hilbert extended this result to Gaussian integers.

Pocklington proved by descent the impossibility of

$$x^4 - py^4 = z^2, \quad x^4 - p^2y^4 = z^2, \quad x^4 - y^4 = pz^2, \quad x^4 + 2y^4 = z^2,$$

where p is a prime of the form $8k + 3$. The local and global solvability of the Diophantine equations $ax^4 + by^4 + cz^2 = 0$ in the integers was studied in [1, Ch. 6]. Some of these fourth-degree Diophantine equations were studied in Chapter 4 of Mordell's book [3], as equations with only trivial solution in integers. In [7], Szabó studied the Diophantine equation $ax^4 + by^4 = cz^2$ for special integer values of a, b and c. Using elliptic curve techniques, Najman [4] proved that $x^4 + y^4 = iz^2$ has a finite number of solutions in the Gaussian integers and $x^4 - y^4 = iz^2$ has no solution in $\mathbb{Z}[i]$. Also, he gave a new proof of Hilbert's result. Using elliptic curves, Najman proved that the Diophantine equation $x^4 \pm y^4 = z^2$ has only trivial solutions in the Gaussian integers. Similarly, in this note we examine some Diophantine equations of degree four in $\mathbb{Z}[i]$, by using elliptic curve techniques.

Note 1.1. Note that the obvious mapping $z \mapsto iz$ shows that the nonsolvability of $az^4 + by^4 = cz^2$ over $\mathbb{Z}[i]$ implies the nonsolvability of $az^4 + by^4 = -cz^2$ and so only the former equation will be studied.

2. Elliptic curves

In this section we prove some results about the rank of elliptic curves over $\mathbb{Q}(i)$ for later use.

Let $E(\mathbb{Q})$ be an elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation of the form

$$E(\mathbb{Q}) : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Q}.$$

By the Mordell–Weil theorem, the set of rational points on $E(\mathbb{Q})$ is a finitely generated abelian group, that is,

$$E(\mathbb{Q}) \simeq E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r,$$

where $E(\mathbb{Q})_{\text{tors}}$ is a finite group called the torsion group and r is a nonnegative integer called the Mordell–Weil rank of $E(\mathbb{Q})$.

In order to determine the torsion subgroup of $E(\mathbb{Q}(i))$, we use the extended Lutz–Nagell theorem [6], which is a generalisation of the Lutz–Nagell theorem from $E(\mathbb{Q})$ to $E(\mathbb{Q}(i))$.

THEOREM 2.1 (Extended Lutz–Nagell theorem). *Let $E : y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}[i]$. If a point $(x, y) \in E(\mathbb{Q}(i))$ has finite order, then:*

(1) *both x and $y \in \mathbb{Z}[i]$; and*

(2) *either $y = 0$ or $y^2 \mid 4A^3 + 27B^2$.*

REMARK 2.2. It is well known (see, for example, [5]) that if an elliptic curve E is defined over $\mathbb{Q}$, then the rank of E over $\mathbb{Q}(i)$ is given by

$$\text{rank}(E(\mathbb{Q}(i))) = \text{rank}(E(\mathbb{Q})) + \text{rank}(E_{-1}(\mathbb{Q})),$$

where E_{-1} is the (-1)-*twist* of E over $\mathbb{Q}$. We also use this fact in the following proofs.

2-descent method. In this section we describe the method which we use for determining the rank of an elliptic curve. Let $E(\mathbb{Q})$ denote the group of rational points on the elliptic curve $E : y^2 = x^3 + ax^2 + bx$. Let Q^* denote the multiplicative group of nonzero rational numbers and Q^{*^2} the subgroup of squares of elements of Q^*. Define the group 2-descent homomorphism α from $E(\mathbb{Q})$ to Q^*/Q^{*^2} as follows:

$$\alpha(P) = \begin{cases} 1 \ (\text{mod} Q^{*^2}) & \text{if } P = O = \infty, \\ b \ (\text{mod} Q^{*^2}) & \text{if } P = (0, 0), \\ x \ (\text{mod} Q^{*^2}) & \text{if } P = (x, y) \text{ with } x \neq 0. \end{cases}$$

Similarly, take the isogenous curve $\widehat{E} : y^2 = x^3 - 2ax^2 + (a^2 - 4b)x$ with group of rational points $\widehat{E}(\mathbb{Q})$. The group 2-descent homomorphism $\widehat{\alpha}$ from $\widehat{E}(\mathbb{Q})$ to Q^*/Q^{*^2} is given by

$$\widehat{\alpha}(\widehat{P}) = \begin{cases} 1 \pmod{Q^{*^2}} & \text{if } \widehat{P} = O = \infty, \\ a^2 - 4b \pmod{Q^{*^2}} & \text{if } \widehat{P} = (0,0), \\ x \pmod{Q^{*^2}} & \text{if } \widehat{P} = (x,y) \text{ with } x \neq 0. \end{cases}$$

Proposition 2.3. *Using the above notation, the rank r of $E(\mathbb{Q})$ is determined by*

$$2^{r-2} = |\mathrm{Im}(\alpha)|\,|\mathrm{Im}(\widehat{\alpha})|.$$

Theorem 2.4 [1, Ch. 8]. *The group $\alpha(E(\mathbb{Q}))$ is equal to the classes modulo squares of 1, b and the positive and negative divisors b_1 of b such that the quartic equation*

$$N^2 = b_1M^4 + aM^2e^2 + \frac{b}{b_1}e^4$$

has a solution with M, N and e pairwise coprime integers such that $Me \neq 0$. If (M, N, e) is such a solution, then $P = (b_1M^2/e^2, b_1MN/e^3)$ is in $E(\mathbb{Q})$ and $\alpha(P) = b_1$.

Remark 2.5. A similar theorem is true for $\widehat{\alpha}$.

Now we are ready to prove some results about the rank of the elliptic curves, which we will use in the main results. In the following, we use the notation E_q for the elliptic curve $Y^2 = X^3 - qX$ and F_q for $Y^2 = X^3 + qx$.

Theorem 2.6.

(1) *For a prime integer $p \equiv 3 \pmod 8$, the rank of the elliptic curve $E_{p^3} : Y^2 = X^3 - p^3X$ is zero over $\mathbb{Q}(i)$ and its torsion group is isomorphic to $\{\infty, (0,0)\}$.*

(2) *For a prime integer $p \equiv 3 \pmod{16}$ and $F_{p^3} : Y^2 = X^3 + p^3X$, we have $F_{p^3}(\mathbb{Q}(i)) = \{\infty, (0,0)\}$.*

Theorem 2.7.

(1) *For a prime integer $p \equiv 7$ or $11 \pmod{16}$, the rank of the elliptic curve $F_p : Y^2 = X^3 + pX$ is zero in $\mathbb{Q}(i)$ and its torsion points are $\{\infty, (0,0)\}$.*

(2) *For a prime integer $p \equiv 3 \pmod 8$ and $E_p : Y^2 = X^3 - pX$, we have $E_p(\mathbb{Q}(i)) = \{\infty, (0,0)\}$.*

Remark 2.8. Obviously, the (-1)-*twist* of each of these curves is isomorphic to itself. By Remark 2.2, it is sufficient to show that each of these curves has zero rank in $\mathbb{Q}$.

Proof of Theorem 2.6(1). The quartic equation of the homogeneous space of E_{p^3} is

$$N^2 = b_1M^4 - \frac{p^3}{b_1}e^4,$$

where $b_1 \in \{\pm 1, \pm p, \pm p^2, \pm p^3\}$. By the definition of α, we have $1, -p \in \mathrm{Im}(\alpha)$. Considering b_1 mod squares, it is sufficient to consider $b_1 = -1$ and p. For $b_1 = -1$, we have $-M^4 + p^3e^4 = N^2$. Therefore,

$$-M^4 \equiv N^2 \pmod p \Longrightarrow -1 \equiv \left(\frac{N}{M^2}\right)^2 \pmod p \Longleftrightarrow p \equiv 1 \pmod 4$$

which is false. Also, $p \notin \mathrm{Im}(\alpha)$ since $\mathrm{Im}(\alpha)$ is a multiplicative group. So, $\mathrm{Im}(\alpha) = \{1, -p\}$.

Now consider the isogenous curve $\widehat{E_{p^3}} : \widehat{Y}^2 = \widehat{X}^3 + 4p^3\widehat{X}$. The biquadratic equation of the homogeneous space of this curve is

$$\widehat{N}^2 = b_1\widehat{M}^4 + \frac{4p^3}{b_1}\widehat{e}^4,$$

where $b_1 \in \{\pm 1, \pm 2, \pm 4, \pm p, \pm p^2, \pm p^3, \pm 2p, \pm 4p, \pm 2p^2, \pm 4p^2, \pm 2p^3, \pm 4p^3\}$. We have $1, p \in \mathrm{Im}(\widehat{\alpha})$. For negative b_1, the quartic equation has no solution. Considering b_1 mod squares, we have to examine the equation for $b_1 = 2$ and $2p$. In the former case, we have

$$2\widehat{M}^4 + 2p^3\widehat{e}^4 = \widehat{N}^2 \Rightarrow 2\widehat{M}^4 = \widehat{N}^2 \pmod p,$$

but then 2 is a square (mod p) so $p \equiv \pm 1 \pmod 8$, which is false. Since $\mathrm{Im}(\widehat{\alpha})$ is a multiplicative group, $2p \notin \mathrm{Im}(\widehat{\alpha})$. Therefore, $\mathrm{Im}(\widehat{\alpha}) = \{1, p\}$.

By Proposition 2.3, $\mathrm{rank}E_{p^3}(\mathbb{Q}) = 0$. Using the extended Lutz–Nagell theorem, $\Delta_{E_{p^3}} = -4p^9$ and so if (X, Y) is a torsion point,

$$Y^2 = 0 \quad \text{or} \quad ap^k,$$

where $a = \pm 1, \pm 2i, \pm 4$ and $k = 0, 2, 4, 8$. If $Y^2 = 4p^6$, then $4p^6 = X^3 - p^3X \Rightarrow 4p^6 = p^{3t}X'^3 - p^{t+3}X'$, where $p \nmid X'$ and $t \geq 1$. Suppose $t = 1$. Dividing both sides of the equation by p^3, we conclude that $p \mid 2$, which is a contradiction. Note that $t \geq 2$ yields $p \mid X'$, which is again a contradiction. Similarly, we can show that $Y^2 \neq \pm p^2, \pm p^4, \pm p^6, \pm 2i, \pm 2ip^2, \pm 2ip^4, \pm 2ip^6, \pm 4p^2, \pm 4p^4$. For $Y^2 = 4$, suppose that q is a prime divisor of x in $\mathbb{Z}[i]$. Then $q \mid 4$ and hence $q = \omega = 1 + i$. Comparing the powers of ω on both sides, we deduce that $Y^2 \neq 4$. In a similar way, we have $Y^2 \neq \pm 1, \pm 2i$. Only for $Y = 0$ do we have $X = 0$, which means that $E_{p^3}(\mathbb{Q}(i))_{Tor} = \{\infty, (0, 0)\}$. □

Proof of Theorem 2.6(2). The quartic equation of the homogeneous space of $F_{p^3} : Y^2 = X^3 + p^3X$ is

$$N^2 = b_1M^4 + \frac{p^3}{b_1}e^4,$$

where $b_1 \in \{\pm 1, \pm p, \pm p^2, \pm p^3\}$. By definition of α, we have $1, p \in \mathrm{Im}(\alpha)$. For negative b_1, the equation has no solution. Considering b_1 mod squares, we have $\mathrm{Im}(\alpha) = \{1, p\}$. The isogenous curve of F_{p^3} is $\widehat{F_{p^3}} : \widehat{Y}^2 = \widehat{X}^3 - 4p^3\widehat{X}$. The biquadratic equation of the homogeneous space of this curve is

$$\widehat{N}^2 = b_1\widehat{M}^4 - \frac{4p^3}{b_1}\widehat{e}^4,$$

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where $b_1 \in \{\pm 1, \pm 2, \pm 4, \pm p, \pm p^2, \pm p^3, \pm 2p, \pm 4p, \pm 2p^2, \pm 4p^2, \pm 2p^3, \pm 4p^3\}$. Since $1, -p \in \mathrm{Im}(\widehat{\alpha})$, it is sufficient to study this equation for $b_1 \in \{-1, \pm 2, p, \pm 2p\}$. Similarly to the first part of the theorem, we have $-1, 2, p, -2p \notin \mathrm{Im}(\widehat{\alpha})$. For $b_1 = 2p$, we have $2p\widehat{M}^4 - 2p^2\,\widehat{e}^4 = \widehat{N}^2$. Let $(\widehat{M}, \widehat{e}, \widehat{N})$ be a solution of this equation such that $\widehat{N} = p^\alpha \widehat{N}_0$, where $p \nmid \widehat{N}_0$ and $\alpha \gneqq 0$. Dividing both sides of the equation by p, we have $-2\widehat{M}^4 + 2p\,\widehat{e}^4 = p^{2\alpha-1}\widehat{N}_0^2$. So, $p \mid \widehat{M}$, which is impossible, since $(\widehat{M}, p) = 1$. Also, $-2 \notin \mathrm{Im}(\widehat{\alpha})$, because $-p \in \mathrm{Im}(\widehat{\alpha})$ and $\mathrm{Im}(\widehat{\alpha})$ is a multiplicative group. Now, Proposition 2.3 implies that $\mathrm{rank} F_{p^3}(\mathbb{Q}) = 0$. Similarly to the first part, $F_{p^3}(\mathbb{Q}(i))_{Tor} = \{\infty, (0,0)\}$. □

Proof of Theorem 2.7. It is sufficient to show that $\mathrm{rank}(F_p(\mathbb{Q})) = \mathrm{rank}(E_p(\mathbb{Q})) = 0$. The former is given in [6, Corollary 6.2.1, page 351]. The biquadratic equation of the homogeneous space of E_p is $N^2 = b_1 M^4 - pe^4/b_1$, where $b_1 \in \{\pm 1, \pm p\}$ and $\{1, -p\} \subset \mathrm{Im}(\alpha)$. If $b_1 = -1$,

$$-M^4 + pe^4 = N^2 \Longrightarrow -M \equiv N^2 \pmod{p} \Longrightarrow -1 \equiv \left(\frac{N}{M^2}\right)^2 \pmod{p}.$$

This implies that $p \equiv 1 \pmod 4$, which is not true. Also, $b_1 = p \notin \mathrm{Im}(\alpha)$ and thus $\mathrm{Im}(\alpha) = \{1, -p\}$. Consider the isogenous curve to E_{-p}, $\widehat{E_p} : \widehat{Y}^2 = \widehat{X}^3 + 4p\widehat{X}$, with the quartic equation $\widehat{N}^2 = b_1\widehat{M}^4 + (4p/b_1)\widehat{e}^4$ for its homogeneous space, where $b_1 \in \{\pm 1, \pm 2, \pm 4, \pm p, \pm 2p, \pm 4p\}$. Clearly, it has no solution for negative b_1 and $\{1, p\} \subset \mathrm{Im}(\widehat{\alpha})$. Let $b_1 = 2$; then

$$2\widehat{M}^4 + 2p\,\widehat{e}^4 = \widehat{N}^2.$$

This means that 2 is a square (mod p) or, equivalently, $p \equiv \pm 1 \pmod 8$, which is not true. So, 2 and consequently $2p$ are not in $\mathrm{Im}(\widehat{\alpha})$. Therefore, $\mathrm{Im}(\widehat{\alpha}) = \{1, p\}$ and $\mathrm{rank}(E_p(\mathbb{Q})) = 0$. Similarly to the proof of Theorem 2.6(1), the extended Lutz–Nagell theorem yields $\Delta_{E_p} = 4p^3$ and

$$Y^2 \in \{0, \pm 1, \pm 4, \pm p^2, \pm 2i, \pm 2ip^2, \pm 4p^2\}.$$

If $Y = 0$, we have $X = 0$. Comparing the powers of p and ω, we see that the other cases produce no solution in the Gaussian integers. This means that $E_p(\mathbb{Q}(i))_{Tor} = \{\infty, (0,0)\}$ and similarly for E_p. □

3. On the Diophantine equation $y^4 + dx^4 = cz^2$

In this section we study the equation $y^4 + dx^4 = cz^2$, where d is a power of an odd prime integer and c is a power of 2, ω and i. Not only do we prove insolubility of the equations in Gaussian integers, but we also prove it in $\mathbb{Q}(i)$.

Remark 3.1. For what follows, note that $\omega = 1 + i$ is a prime in the Gaussian integers.

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3.1. On the Diophantine equation $y^4 \pm p^3x^4 = z^2$. In this section p is a prime integer with $p \equiv 3 \pmod{16}$ or $p \equiv 3 \pmod 8$. We note that p is also prime in $\mathbb{Z}[i]$. A nontrivial solution of the Diophantine equations

$$y^4 + 4p^3x^4 = z^2, \quad -4y^4 + 4p^3x^4 = z^2, \quad y^4 - 4p^3x^4 = z^2$$

leads to a nontrivial solution of the Diophantine equations

$$y^4 - p^3x^4 = z^2, \quad y^4 + p^3x^4 = z^2,$$

respectively, since the first two equations are $y^4 - p^3(\omega x)^4 = z^2, (\omega y)^4 - p^3(\omega x)^4 = z^2$ and the third is $y^4 + p^3(\omega x)^4 = z^2$. Thus, it is enough to show that the last two equations have only trivial solutions in $\mathbb{Z}[i]$.

Theorem 3.2.

(1) *Let $p \equiv 3 \pmod 8$. The Diophantine equations $y^4 - p^3x^4 = \pm z^2$ and $y^4 + p^3x^4 = \pm iz^2$ have only trivial solutions in $\mathbb{Z}[i]$.*

(2) *For $p \equiv 3 \pmod{16}$, the Diophantine equations $y^4 + p^3x^4 = \pm z^2$ and $y^4 - p^3x^4 = \pm iz^2$ have only trivial solutions in $\mathbb{Z}[i]$.*

Proof. First suppose $p \equiv 3 \pmod 8$. Suppose that (x, y, z) is a nontrivial solution of the equation $y^4 \pm p^3x^4 = \pm z^2$. Dividing the equation by x^4 and considering the change of variables $s = y/x$ and $t = z/x^2$, we have $s^4 \pm p^3 = t^2$ for $s, t \in \mathbb{Q}(i)$. Let

$$X = s^2$$
$$X^2 \pm p^3 = t^2.$$

Multiplying these equations and letting $Y = st$, we have the elliptic curves $Y^2 = X^3 \pm p^3X$. By Theorem 2.6, the rank of these curves is zero over $\mathbb{Q}(i)$ and the only torsion point $(0, 0)$ on both of them leads to trivial solutions for the original equations.

Now suppose $p \equiv 3 \pmod{16}$. As in the first part of the proof, suppose that (x, y, z) is a nontrivial solution of the equations $y^4 \pm p^3x^4 = \pm iz^2$, so that

$$x^4 \pm p^3y^4 = iz^2 \Rightarrow \left(\frac{x}{y}\right)^4 \pm p^3 = i\left(\frac{z}{y^2}\right)^2 \Rightarrow s^4 \pm p^3 = it^2,$$

where $s = x/y$ and $t = z/y^2$. Let $r = s^2$; then $r^2 \pm p^3 = it^2$. Multiplying these equations together, we have $r^3 \pm p^3r = i(ts)^2$. Now, $X^3 \mp p^3X = Y^2$, using $X = ir$ and $Y = st$. On both of these curves, the only torsion point is $(0, 0)$ and this leads to trivial solutions for $y^4 \pm p^3x^4 = iz^2$. □

Corollary 3.3.

(1) *For $p \equiv 3 \pmod 8$, the Diophantine equations $y^4 - p^3x^4 = \pm 2^m z^2$ and $y^4 + p^3x^4 = \pm 2^m iz^2$ have only trivial solutions in $\mathbb{Q}(i)$ for any natural number m.*

(2) *For $p \equiv 3 \pmod{16}$, the Diophantine equations $y^4 + p^3x^4 = \pm 2^m z^2$ and $y^4 - p^3x^4 = \pm 2^m iz^2$ have only trivial solutions in $\mathbb{Q}(i)$ for any natural number m.*

(3) *For* $n \in \mathbb{N} \cup \{0\}$ *and* $p \equiv 3 \pmod 8$, *the Diophantine equations* $y^4 - p^3x^4 = 2^nz^4$ *and* $y^4 + p^3x^4 = 2^niz^4$ *have only trivial solutions in* $\mathbb{Q}(i)$.

(4) *For* $n \in \mathbb{N} \cup \{0\}$ *and* $p \equiv 3 \pmod{16}$, *the Diophantine equations* $y^4 + p^3x^4 = 2^nz^4$ *and* $y^4 - p^3x^4 = 2^niz^4$ *have only trivial solutions in* $\mathbb{Q}(i)$.

PROOF. In the equations $y^4 \pm p^3x^4 = \pm 2^mz^2$, let $m = 2k$ or $2k+1$. The equations become $y^4 \pm p^3x^4 = (2^kz)^2$ and $y^4 \pm p^3x^4 = i(i\omega 2^kz)^2$, respectively, with only trivial solutions.

Similarly, $y^4 \pm p^3x^4 = \pm 2^miz^2$ becomes $y^4 \pm p^3x^4 = (\omega 2^kz)^2$ if $m = 2k+1$ and $y^4 \pm p^3x^4 = i(2^kz)^2$ if $m = 2k$. Both have no nontrivial solutions by the theorem. □

3.2. On the Diophantine equation $y^4 \pm px^4 = z^2$. In this section p is a prime integer with $p \equiv 7$ or $11 \pmod{16}$. Note that p remains prime in $\mathbb{Z}[i]$. A nontrivial solution of the Diophantine equations

$$y^4 \pm 4px^4 = z^2, \quad -4y^4 + 4px^4 = z^2, \quad y^4 - 4px^4 = z^2$$

leads to a nontrivial solution of the Diophantine equations

$$y^4 - px^4 = z^2, \quad y^4 + px^4 = z^2,$$

respectively, since the first two equations are $y^4 - p(\omega x)^4 = z^2, (\omega y)^4 - p(\omega x)^4 = z^2$ and the third one is $y^4 - p(\omega x)^4 = z^2$. Thus, it is enough to show that the last two equations have only trivial solutions in $\mathbb{Z}(i)$.

THEOREM 3.4.

(1) *For* $p \equiv 7$ *or* $11 \pmod{16}$, *the Diophantine equations* $y^4 + px^4 = \pm z^2$ *and* $y^4 - px^4 = \pm iz^2$ *have only trivial solutions in* $\mathbb{Z}[i]$.

(2) *For* $p \equiv 3 \pmod 8$, *the Diophantine equations* $y^4 - px^4 = \pm z^2$ *and* $y^4 + px^4 = \pm iz^2$ *have only trivial solutions in* $\mathbb{Z}[i]$.

PROOF. First suppose $p \equiv 7$ or $11 \pmod{16}$. Suppose that (x, y, z) is a nontrivial solution of these equations. Dividing the equations by x^4 and considering the change of variables $s = y/x$ and $t = z/x^2$, we have $s^4 \pm p = t^2$ for $s, t \in \mathbb{Q}(i)$. Let

$$X = s^2,$$
$$X^2 \pm p = t^2.$$

Multiplying these equations together and letting $Y = st$, we obtain the elliptic curves $Y^2 = X^3 \pm pX$. By Theorem 2.7, the rank of these curves is zero over $\mathbb{Q}(i)$ and the only torsion point $(0, 0)$ leads to trivial solutions for the original equations.

Now suppose $p \equiv 3 \pmod 8$. As in the first part of the theorem, suppose that (x, y, z) is a nontrivial solution of these equations, so that

$$y^4 \pm px^4 = iz^2 \Rightarrow \left(\frac{y}{x}\right)^4 \pm p = i\left(\frac{z}{x^2}\right)^2 \Rightarrow s^4 \pm p = it^2,$$

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where $s = y/x$ and $t = z/x^2$. Let $r = s^2$; then $r^2 \pm p = it^2$. Multiplying these equations together, we have $r^3 \pm pr = i(ts)^2$. Now, $X^3 \mp pX = Y^2$ with $X = ir$ and $Y = st$. On both of these curves, the only torsion point is $(0, 0)$, which leads to trivial solutions for $y^4 \pm px^4 = iz^2$. □

As a result of this theorem, as in Corollary 3.3, we have the following result.

(1) For $p \equiv 7$ or $11 \pmod{16}$, the Diophantine equations $y^4 + px^4 = \pm 2^m z^2$, $y^4 + px^4 = \pm 2^n z^4$, $y^4 - px^4 = \pm 2^m iz^2$ and $y^4 - px^4 = 2^n iz^4$ have only trivial solutions in $\mathbb{Z}[i]$ for $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$.

(2) For $p \equiv 3 \pmod 8$, the Diophantine equations $y^4 - px^4 = \pm 2^m z^2$, $y^4 - px^4 = \pm 2^n z^4$, $y^4 + px^4 = \pm 2^m iz^2$ and $y^4 + px^4 = \pm 2^n iz^4$ have only trivial solutions in $\mathbb{Z}[i]$ for $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$.

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