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JACKSON NETWORKS IN NONAUTONOMOUS RANDOM ENVIRONMENTS

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Abstract

We investigate queueing networks in a random environment. The impact of the evolving environment on the network is by changing service capacities (upgrading and/or degrading, breakdown, repair) when the environment changes its state. On the other side, customers departing from the network may enforce the environment to jump immediately. This means that the environment is nonautonomous and therefore results in a rather complex two-way interaction, especially if the environment is not itself Markov. To react to the changes of the capacities we implement randomised versions of the well-known deterministic rerouteing schemes 'skipping' (jump-over protocol) and 'reflection' (repeated service, random direction). Our main result is an explicit expression for the joint stationary distribution of the queue-lengths vector and the environment which is of product form.

Keywords: Randomised random walk; Jackson network; skipping; processes in random environment; reflection; product form steady state; breakdown of nodes, degrading service; speed-up of service

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1. Introduction

Queueing networks with product form steady state have found many fields of application, e.g. production systems, telecommunications, and computer system modelling. The success of this class of models stems from the simple structure of the steady state distribution which provides access to easy performance evaluation procedures. Starting from the work of Jackson [13], various generalisations have been developed. A branch which has recently found interest is queueing networks in a random environment with product form steady state, and we will contribute to this research area.

In this introduction we will give examples of predecessors, i.e. queueing systems in a random environment and provide a survey of the state of the art. Thereafter we discuss two essentials for our results and presentation: the construction of rerouteing schemes available in the literature to deal with blocking or breakdown of nodes, and the distinction between autonomous and nonautonomous environments. We end with a sketch of our achievements and of the structure of the paper.

Predecessors. For birth–death processes (and $M/M/1/\infty$ queues) in a random environment there is a long history of investigations; see, e.g. [4], [5], [10], [21], and [37]. Related research

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is on service systems under external influences which cause the service process to break down or decrease availability of servers; see, e.g. [39] and the recent survey [19]. The results in these papers most often lack the elegance of Jackson's product form steady state and the simplicity of the steady states of birth–death processes.

Exceptions are [20], [25], and [28], where the environment of a production model (queue) is an associated inventory, and [9] and [26], where the influence of the environment results in randomly occurring breakdowns of the server, and [15], where the environment of a sensor node encompasses the node's neighbours, their status, etc. Queues in a general environment were investigated in [16]. In these papers the two-dimensional steady state distribution of queueing-environment processes factorises into the product of the marginal steady state distributions. Essentially, in equilibrium and in the long run the states for the queue and the environment decouple.

In [36] environment states are called 'background states' which govern transition rates of the queue and on the other side are influenced by the state the queue. A decomposition was proved for the joint queueing-environment process and supplementary variables for nonexponential service times and holding times for the background states.

State of the art. Zhu [38] was the first to find product form steady state distributions for Jackson networks in a random Markovian environment. Economou [7], Balsamo and Marin [1], and Tsitsiashvili *et al.* [32] continued the investigations. The procedure in these papers for a network of single exponential servers is as follows (explained in terms of Zhu's notation). The key ingredients for node i in a Jackson network are an external Poisson-λ_i arrival stream, exponential-μ_i service times, and a Markovian routeing scheme, which produces a total arrival rate η_i. With $\eta_i/\mu_i =: \rho_i$ a local marginal stationary distribution is

$$\xi_i(n) = (1-\rho_i)(\rho_i)^n, \qquad n \in \mathbb{N}_0. \tag{1.1}$$

In [38] these parameters depend on the environment's state, say k: $\lambda_i(k)$, $\mu_i(k)$, $\eta_i(k)$, and, with the additional assumption that utilisations $\rho_i(k) := \eta_i(k)/\mu_i(k) =: \rho_i$ do not depend on k, the local steady state is (1.1) again. Zhu and his followers do not explain how independence of k for utilisations emerges.

A network of parallel nodes each with a local environment (= dedicated inventory) and a common global environment (= replenishment system) with product form steady state was described in [23]. The environment in this setting is not itself Markov.

In [11] a random walk and other systems of statistical mechanics were embedded into environments which were represented by Jackson networks. In reversed interpretation the Jackson networks are in an environment from statistical mechanics. The authors derived under certain reversibility and local balance conditions explicit expressions for the system's stationary distribution which resemble product form equilibria.

Rerouteing schemes. The quest for readjustment rules which generate approximately a behaviour similar to the independence of k for $\rho_i(k) := \eta_i(k)/\mu_i(k) =: \rho_i$ has a long history, especially in the control of communications networks. There the term 'rerouteing' describes the necessity to react, e.g. to buffer overflow, broken down nodes, and (partial) degrading of transmission lines. To be more concrete: rerouteing schemes are stylised policies in communications networks which mimic an exchange of routeing tables or of rules for dynamic traffic reallocation as a reaction to changes of service capacities (e.g. by degradation of servers) or of the network's load situation, due to varying environment conditions; see [12], [22], [30], and the references therein.

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In networks with blocking or with unreliable servers, it is often possible to obtain explicit product form stationary distributions by implementing a clever rerouteing regime for customers who find at a node, selected for his/her next entrance, the buffer full or the node broken down. Examples are described in [26] under the heading of 'skipping', 'repeated service-random destination', and 'stalling'. The first regime is called 'jump-over protocol' by other authors, see [33], the second regime is 'reflection'.

The bulk of applications exploiting these schemes can be found in [34] and in [3, Chapter 1,9]. These rerouteing schemes maintain $\rho_i(k) =: \rho_i$ for all k as Zhu requires it, but only for those nodes i which are not blocked, respectively not broken down.

Autonomous versus nonautonomous environment. An environment is *autonomous* if its describing process is itself Markov and its stationary distribution can be calculated without reference to the network process. So, the flow of influence in the coupled system is unidirectional. See [1], [7], [32], and [38] for research into autonomous environments.

An environment is *nonautonomous* if its describing process is not Markov, because its transition rates depend on the network's actual state, or because jumps of the network process may enforce the environment to change its status. So networks in nonautonomous environments can be characterised by bidirectional influence and interaction. Nonautonomous environments occur with various applications. For birth–death processes see [5] with autonomous environment and its follower [6], where the environment experiences 'feedback', i.e. its development depends on the state of the birth–death process. A nonstandard construction can be found in [24], where for a referenced node of a Jackson network the other nodes are considered as its environment, which generally is nonautonomous. The background states in [36] are nonautonomous. Typical scenarios for nonautonomous environments of queueing systems are described in various settings in [11], [15]–[17], [23], and [28]. Our research in this paper focuses on nonautonomous environments.

Structure of the paper and overview of our results. We start in Section 2 with a construction of routeing chains for the selection of individual customers' itineraries in the network and suitable modifications of these in terms of general random walks. These schemes generalise the aforementioned schemes from the literature: skipping (jump-over protocol) and reflection (repeated service-random destination), and provide a much more flexible readjustment of routeing.

In Section 3 we show how these rerouteing schemes can be used to readjust the routeing in a network, when service capacities at the nodes are changed. The aim is to maintain the utilisations ρ_i of the nodes. This is possible as long as the servers are not completely down. We indicate briefly how this aim emerges from optimisation issues: for typical cost functions the queue-length distributions are decisive, which leads to the desire to maintain an optimally designed (1.1), whenever such an optimal point is found.

Our main achievements are presented in Section 4. For Jackson networks in a nonautonomous random environment we derive an explicit expression of product form for the joint queueing-environment process. This generalises all the mentioned results for Jackson networks in an autonomous random environment, and incorporates furthermore the case where nodes may break down completely, which is not covered by the results of Zhu and his followers. We study in depth 'randomised skipping' and 'randomised reflection' to react to influences of the environment on the network's capacities, which may decrease gradually or totally at some nodes, but which may increase as well at other nodes. We indicate that more general rerouteing schemes can be used for readjustment of routeing to obtain similar results.

Notation and conventions. Throughout $\mathbb{R}_0^+ = [0, \infty)$, $\mathbb{R}^+ = (0, \infty)$, $\mathbb{N} = 1, 2, 3, \ldots$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and $A \subsetneq B$ means A is strict subset of B, and $A \subset B$ means $A \subsetneq B$ or $A = B$.

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The node set of our graphs (networks) is $\widetilde{J} := \{1, \ldots, J\}$, and the 'extended node set' is $\widetilde{J}_0 := \{0, 1, \ldots, J\}$, where '0' refers to the source and sink of the network. The vector $\boldsymbol{e}_j$ is the standard jth base vector in $\mathbb{N}_0^{\widetilde{J}}$ if $1 \le j \le J$, and $\boldsymbol{n} = (n_j : j \in \widetilde{J})$ is usually the joint queue-length vector of some queueing network. Set the indicator function $\mathbf{1}_A = 1$ if A is true, and the Kronecker delta $\delta_{xy} := \mathbf{1}_{\{x=y\}}$. For any finite index set $\widetilde{F} = \{0, 1, \ldots, F\}$ and any $\boldsymbol{\alpha} = (\alpha_j : j \in \widetilde{F})$, we define diagonal matrices $I_{\boldsymbol{\alpha}}$ and $I_{(1-\boldsymbol{\alpha})}$: $I_{\boldsymbol{\alpha}}(i, j) := \mathbf{1}_{\{i=j\}}\alpha_i$, respectively $I_{1-\boldsymbol{\alpha}}(i, j) := \mathbf{1}_{\{i=j\}}(1 - \alpha_i)$, $i, j \in \widetilde{F}$. For $f, g : \widetilde{F} \to \mathbb{R}$ with countable $\widetilde{F}$, we define pointwise multiplication $f \bullet g$ by $f \bullet g(i) = f(i)g(i)$, and diagonal matrix $I_{f \bullet g}$ by $I_{f \bullet g}(i, j) := \mathbf{1}_{\{i=j\}} f(i)g(i)$, $i, j \in \widetilde{F}$. For $\boldsymbol{x} = (x_j : j \in \widetilde{F})$, we define $\|\boldsymbol{x}\|_\infty := \sup_{j \in \widetilde{F}} |x_j|$.

2. Randomised random walks

The Markov chain which we consider will represent in network applications the chain that describes routeing of customers on the nodes of a network. We construct randomised versions of the described (deterministic) rerouteing schemes from the literature.

Let $X = (X_n : n \in \mathbb{N}_0)$ be a homogeneous irreducible Markov chain on a finite state space $\widetilde{F}$ with one-step transition probability matrix $r = (r(i, j) : i, j \in \widetilde{F})$ and (unique) steady state distribution $\eta = (\eta_i : i \in \widetilde{F})$. An intuitive description of the modification of X which we will construct is in terms of a random walk on the 'nodes' ($\equiv$ states) in $\widetilde{F}$ governed by r. The general principle is as follows. The transition matrix r will be used as a 'candidate generating matrix' for the next state of the random walker (RW). The candidate state, say j, will be accepted with state dependent probability $\alpha_j \in [0, 1]$. We develop different policies to continue when the proposed state is not accepted. We let $\boldsymbol{\alpha} := (\alpha_j : j \in \widetilde{F})$ denote in any case the vector of 'acceptance probabilities', and $B(\boldsymbol{\alpha}) = \{j \in \widetilde{F} : \alpha_j = 0\}$ is in any case the 'taboo set for the RW'.

Definition 2.1. The RW selects his/her itinerary under r and constraints $\boldsymbol{\alpha}$ by randomised skipping with acceptance probabilities $\boldsymbol{\alpha}$ as follows. If the RW is in state $i \in \widetilde{F}$ and selects (with probability $r(i, j)$) destination $j \in \widetilde{F}$, a Bernoulli experiment is performed with success (acceptance) probability α_j, independent of the past, given j. If the experiment is successful ($= 1$), the jump is accepted, immediately performed, and the RW settles down at j for at least one time slot. If the experiment is not successful ($= 0$), the jump is not accepted and the RW only performs an imaginary jump to j, spends no time there, and jumps on immediately, governed by the matrix r, i.e. with probability $r(j, l)$ he/she selects another possible successor state l; thereafter a Bernoulli experiment is performed with success probability α_l, independent of the past, given l, and so on.

Example 2.1. If for $\varnothing \ne B \subsetneq \widetilde{F}$ we set $\alpha_j = 0$ if $j \in B$, and $\alpha_j = 1$ if $j \in \widetilde{F} \setminus B$, we have 'skipping over taboo set B' as described in the literature in various applications: a jump to $j \in B$ is never accepted, while a proposed jump to $j \in \widetilde{F} \setminus B$ will be accepted with probability 1.

This scheme is also known as 'jump-over protocol for B', or 'skipping B'. In queueing networks this scheme to modify a Markov chain was found independently several times in order to resolve blocking; see, e.g. [33], [9], and [29, Chapter 3.6]. As a general methodology skipping was introduced by Schassberger [27]. For unreliable networks this scheme was introduced in [9] and [26].

Randomised skipping generates a Markov chain $X^{(\boldsymbol{\alpha})}$, with transition matrix $r^{(\boldsymbol{\alpha})}$. Details of the construction and proof of the following theorem can be found in [18].

Theorem 2.1. *The Markov chain modification $X^{(\boldsymbol{\alpha})}$ of X with transition matrix $r^{(\boldsymbol{\alpha})}$ under randomised skipping with taboo set $B(\boldsymbol{\alpha}) = \{j \in \widetilde{F} : \alpha_j = 0\} \subsetneq \widetilde{F}$ is irreducible on $\widetilde{F} \setminus B(\boldsymbol{\alpha})$, the states in $B(\boldsymbol{\alpha})$ are inessential, and it holds that*

$$r^{(\boldsymbol{\alpha})} = \sum_{k=0}^{\infty} (rI_{(1-\boldsymbol{\alpha})})^k rI_{\boldsymbol{\alpha}} = (I - rI_{(1-\boldsymbol{\alpha})})^{-1} rI_{\boldsymbol{\alpha}}. \tag{2.1}$$

We denote the steady state of $r^{(\boldsymbol{\alpha})}$ by $\eta^{(\boldsymbol{\alpha})} = (\eta^{(\boldsymbol{\alpha})}(j) \colon j \in \widetilde{F})$, which has support $\widetilde{F} \setminus B(\boldsymbol{\alpha})$. We will study the relation between the steady state η of r and $\eta^{(\boldsymbol{\alpha})}$.

Proposition 2.1. *Let x be a solution of $xr = x$. Then $y := xI_{\boldsymbol{\alpha}}$ solves the balance equation $yr^{(\boldsymbol{\alpha})} = y$ of the modified Markov chain under randomised skipping.*

Proof. From $xr = x$, we obtain

$$xr\underbrace{(I_{\boldsymbol{\alpha}} + I_{(1-\boldsymbol{\alpha})})}_{=I} = x \quad \Longleftrightarrow \quad xrI_{\boldsymbol{\alpha}} = x - xrI_{1-\boldsymbol{\alpha}},$$

which yields

$$xrI_{\boldsymbol{\alpha}} = x(I - rI_{(1-\boldsymbol{\alpha})}) \quad \Longleftrightarrow \quad x\underbrace{(I - rI_{(1-\boldsymbol{\alpha})})(I - rI_{(1-\boldsymbol{\alpha})})^{-1}}_{=I} rI_{\boldsymbol{\alpha}} = \underbrace{x(I - rI_{(1-\boldsymbol{\alpha})})}_{=y}$$

and with $r^{(\boldsymbol{\alpha})} = (I - rI_{(1-\boldsymbol{\alpha})})^{-1} rI_{\boldsymbol{\alpha}}$ and $y := x(I - rI_{(1-\boldsymbol{\alpha})})$, we obtain a required solution of $yr^{(\boldsymbol{\alpha})} = y$. This is $x(I - rI_{(1-\boldsymbol{\alpha})}) = x - xI_{(1-\boldsymbol{\alpha})} = xI_{\boldsymbol{\alpha}}$. □

Proposition 2.2. *Let y be a solution of $yr^{(\boldsymbol{\alpha})} = y$ then $x := y(I - rI_{(1-\boldsymbol{\alpha})})^{-1}$ is a solution of $xr = x$ and it holds that $y = (\alpha_j x_j \colon j \in \widetilde{F})$.*

Proof. We have

$$y\underbrace{(I - rI_{(1-\boldsymbol{\alpha})})^{-1} rI_{\boldsymbol{\alpha}}}_{=r^{(\boldsymbol{\alpha})}} = y\underbrace{(I - rI_{(1-\boldsymbol{\alpha})})^{-1}(I - rI_{(1-\boldsymbol{\alpha})})}_{=I}.$$

So $x = y(I - rI_{(1-\boldsymbol{\alpha})})^{-1}$ fulfills

$$xrI_{\boldsymbol{\alpha}} = x(I - rI_{(1-\boldsymbol{\alpha})}) \quad \Longleftrightarrow \quad xr\underbrace{(I_{\boldsymbol{\alpha}} + I_{(1-\boldsymbol{\alpha})})}_{=I} = x,$$

which is $xr = x$. The explicit expression follows from $x(I - rI_{(1-\boldsymbol{\alpha})}) = y$ as in Proposition 2.1. □

Theorem 2.2. *If η is the unique steady state of X then the unique steady state of $X^{(\boldsymbol{\alpha})}$ is, with normalisation constant $C^{(\boldsymbol{\alpha})} = (\eta I_{\boldsymbol{\alpha}} \boldsymbol{e}) = \langle \eta, \boldsymbol{\alpha} \rangle$ and support $\widetilde{F} \setminus B(\boldsymbol{\alpha})$,*

$$\eta^{(\boldsymbol{\alpha})} = (C^{(\boldsymbol{\alpha})})^{-1} (\eta_j \alpha_j \colon j \in \widetilde{F}).$$

Proof. The proof follows by applying Proposition 2.2 and the uniqueness of η as a stochastic solution of $xr = x$. □

Definition 2.2. The RW selects his/her itinerary under r and the constraints $\boldsymbol{\alpha}$ by randomised reflection with acceptance probabilities $\boldsymbol{\alpha}$ as follows. If the RW is in state $i \in \widetilde{F}$ and selects

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(with probability $r(i, j)$) destination $j \in \widetilde{F}$, a Bernoulli experiment is performed with success (acceptance) probability α_j, independent of the past, given j. If the experiment is successful (= 1), the jump is accepted, immediately performed, and the RW settles down at j for at least one time slot. If the experiment is not successful (= 0), the jump is not accepted, and the RW stays on at i for at least one further time slot. If this slot expires, with probability $r(i, l)$ the RW selects another successor state l; thereafter a Bernoulli experiment is performed with success probability α_l, and so on.

Example 2.2. If, for $\varnothing \neq B \subsetneq \widetilde{F}$ we set $\alpha_j = 0$ if $j \in B$, and $\alpha_j = 1$ if $j \in \widetilde{F} \setminus B$, we have reflection of the RW at taboo set B: a jump to $j \in B$ is never accepted, while a proposed jump to $j \in \widetilde{F} \setminus B$ will be accepted with probability 1.

This scheme is sometimes termed *repetitive service–random destination*. It is used to model communication protocols in systems with finite buffers or for ALOHA-type protocols; see [14, Section 5.11]. For networks with unreliable nodes this scheme was introduced in [9] and [26].

The principle in case of full buffer regulation is that whenever a packet is sent from node i to node j and the buffer for incoming packets at j is full, the packet is rejected (lost) and node i, which has saved a copy, tries to resend this packet (repetitive service), but not necessarily to j (random destination). This procedure is iterated until the packet is sent to a node with free buffer places.

Note, in our framework, (randomised) reflection of a customer's jump may be caused by rather different reasons, e.g. broken down nodes, partially degraded node capacity.

The transition matrix $r^{(\boldsymbol{\alpha})}$ of the Markov chain $X^{(\boldsymbol{\alpha})}$ under randomised reflection is, with taboo set $B(\boldsymbol{\alpha}) = \{j \in \widetilde{F} : \alpha_j = 0\} \subsetneq \widetilde{F}$,

$$r^{(\boldsymbol{\alpha})}(i, j) = \begin{cases} r(i, j)\alpha_j, & i, j \in \widetilde{F},\ i \neq j, \\ r(i, i) + \sum_{k \in \widetilde{F}} r(i, k)(1 - \alpha_j), & i \in \widetilde{F},\ i = j, \end{cases} \tag{2.2}$$

where $X^{(\boldsymbol{\alpha})}$, respectively $r^{(\boldsymbol{\alpha})}$, may be reducible even on $\widetilde{F} \setminus B(\boldsymbol{\alpha})$.

To apply the reflection principle a standard assumption is the reversibility of X for some probability measure $\eta = (\eta_i : i \in \widetilde{F})$. We set this assumption in force when investigating this protocol. By checking the local balance for $r^{(\boldsymbol{\alpha})}$, we obtain the following proposition.

Proposition 2.3. *If η is the steady state distribution of the reversible irreducible Markov chain X on finite state space $\widetilde{F}$ then under randomised reflection $X^{(\boldsymbol{\alpha})}$ is reversible with steady state $\eta^{(\boldsymbol{\alpha})}$ with support $\widetilde{F} \setminus B(\boldsymbol{\alpha})$ given by*

$$\eta^{(\boldsymbol{\alpha})} = (C^{(\boldsymbol{\alpha})})^{-1}(\eta_j \alpha_j : j \in \widetilde{F})$$

with $C^{(\boldsymbol{\alpha})} = (\eta I_{\boldsymbol{\alpha}} \boldsymbol{e}) = \langle \eta, \boldsymbol{\alpha} \rangle$.

3. Modification of Jackson networks with invariant utilisations

In this section we introduce standard Jackson networks and discuss optimal network design in connection with invariance of the utilisations under parameter changes, as required in the literature on networks in random environments. Thereafter, we will implement the rerouteing schemes (modification techniques) from Section 2 into a network where the service capacities are changed. We elaborate on the details of skipping and reflection and extract the general principles behind them without proof. For more details, see [18].

Standard Jackson networks. We consider a Jackson network [13] with node set $\widetilde{J} := \{1, \dots, J\}$. Customers arrive in independent external Poisson streams and at node j with intensity $\lambda_j \geq 0$, we set $\lambda = \lambda_1 + \cdots + \lambda_J > 0$. Customers are indistinguishable, follow the same rules, and request for exponentially(1)-distributed service at all nodes. All these requests constitute an independent family of variables which are independent of the arrival streams. Nodes are exponential single servers with state dependent service rates and infinite waiting room under a first-come–first-served regime. If at node i there are $n_i > 0$ customers, either in service or waiting, service is provided there with intensity $\mu_i(n_i) > 0$. Routeing is Markovian, a customer departing from node i immediately proceeds to node j with probability $r(i, j) \geq 0$, and departs from the network with probability $r(j, 0)$. Taking $r(0, j) = \lambda_j/\lambda$, $r(0, 0) = 0$, we assume that the extended routeing matrix $r = (r(i, j) : i, j \in \widetilde{J}_0)$ is irreducible. Then the traffic equation

$$\eta_j = \lambda_j + \sum_{i=1}^{J} \eta_i r(i, j), \qquad j \in \widetilde{J}, \tag{3.1}$$

has a unique solution which we denote by $\eta = (\eta_j : j \in \widetilde{J})$. We extend (3.1) to a steady state equation for a routeing Markov chain by

$$\eta_j = \sum_{i=0}^{J} \eta_i r(i, j), \qquad j = 0, 1, \dots, J,$$

which is solved by $\eta = (\eta_j : j = 0, 1, \dots, J)$, where $\eta_0 := \lambda$ and the other η_j are from (3.1). We use η in both meanings and emphasise the latter one by *extended traffic solution* η. Note that η is usually not a stochastic vector. Let $\boldsymbol{X} = (X(t) : t \geq 0)$ denote the vector process recording the queue lengths in the network. Then $X(t) = (X_1(t), \dots, X_J(t)) \in \mathbb{N}_0^{\widetilde{J}}$ reads: at time t there are $X_j(t)$ customers present at node j, either in service or waiting. The assumptions put on the system imply that $\boldsymbol{X}$ is a strong Markov process on state space $\mathbb{N}_0^{\widetilde{J}}$. For an ergodic network process $\boldsymbol{X}$ Jackson's theorem [13] states that the unique steady state and limiting distribution ξ on $\mathbb{N}_0^{\widetilde{J}}$ is with normalising constants $C(j)$ for the marginal (over nodes) distributions

$$\xi(\boldsymbol{n}) = \xi(n_1, \dots, n_J) = \prod_{j=1}^{J} \prod_{\ell=1}^{n_j} \frac{\eta_j}{\mu_j(\ell)} C(j)^{-1}, \qquad \boldsymbol{n} \in \mathbb{N}_0^{\widetilde{J}}. \tag{3.2}$$

Invariance of utilisations. We will investigate the problem of how to readjust routeing in Jackson networks when service capacities change. The aim is to maintain the utilisations $\rho_i = \eta_i/\mu_i$, respectively the ratios $\eta_j/\mu_j(n_j)$. This program is motivated by the following observation which with hindsight justifies Zhu's [38] and his followers' requirement to have $\rho_i = \eta_i(k)/\mu_i(k)$ for all environment states k.

Network design is an optimisation problem with the goal to clear the input traffic as efficiently as possible, in accordance with some cost criterion, and respecting the resources available. A standard problem is to distribute for prescribed capacities $\mu_i(\cdot)$ the offered load by optimal routeing. Typical cost functions rely on the (expected) queue lengths only, plus mean waiting times obtained by Little's formula from these. These are mainly determined by the utilisations $\rho_i = \eta_i/\mu_i$, respectively the ratios $\eta_j/\mu_j(n_j)$. This results (in our notation) in determining optimal routeing probabilities $r(i, j)$ and in the cost function occur the optimal ratios $\eta_i/\mu_i(n_i)$, respectively $\eta_i^{n_i}/(\prod_{\ell=1}^{n_i} \mu_i(\ell))$.

Such a design was performed in [2] and [31, Section 6.1] for a set of parallel stations, in [31, Chapter 7] for a general networks, and similarly by Whittle [35, Section 2], who searched for optimal 'nonadaptive routeing rules' (not depending on the actual queue lengths) based on cost functions which depend on mean queue lengths only.

We remark that this cost function may be inappropriate in some situations, especially when routeing decisions are costly. Nevertheless, we assume that (for given $\mu_j(n_j)$, $j \in \widetilde{J}$) according to some cost criterion, which relies on the mean local queue lengths, we have found optimal utilisations η_j/μ_j, respectively ratios $\eta_j/\mu_j(n_j)$, by an adequate routeing scheme r. Because of this property of the found utilisations it is advisable to maintain the ρ_i, and we will succeed in doing so.

The modification technique. If due to some external changes the service intensities $\mu_i(\cdot)$ at node i are changed by a factor $\gamma_i \in [0, \infty)$ for $i \in \widetilde{J}$, we have to react in different ways depending on the size of the γ_i.

Nodes may break down completely, i.e. $\gamma_\ell = 0$ for such node ℓ. Clearly, broken down nodes should not be visited any longer. Recall that in this case classical (deterministic) skipping and reflection can be successfully applied; see Examples 2.1 and 2.2.

On the other hand, nodes with degraded capacity should contribute to clearing the input traffic, possibly for a reduced portion of the offered load. From the side constraint to maintain at least approximately the ratios 'overall arrival rate/service rates', it is tempting to try rerouteing by randomised skipping or reflection with suitably selected 'acceptance probability vector' $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\gamma})$, where $\boldsymbol{\gamma} = (\gamma_i : i \in \widetilde{J})$. (If there is no ambiguity we will shortly write only $\boldsymbol{\alpha}$.) We will proceed as follows. If $\gamma_i \in [0, 1]$ for all $i \in \widetilde{J}$, i.e. nodes are degraded, the new routeing has two components:

- part of the total external arrival rate will be rejected, and
- the admitted load will be redistributed among the nodes which are not completely broken down in a way to meet exactly the old ratios.

We will show that randomised reflection and skipping with acceptance probability vector $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\gamma})$ work, where $\alpha_0 = 1$ and $\alpha_i := \gamma_i$, $i \in \widetilde{J}$ constitute the vector $\boldsymbol{\alpha}(\boldsymbol{\gamma}) = (\alpha_i, i \in \widetilde{J}_0)$.

If $\gamma_j \in [0, \infty)$ then we either speed up service at node j if $\gamma_j > 1$, or have a degraded server at node j if $\gamma_j < 1$. When at least one service rate increases, i.e. when $\|\boldsymbol{\gamma}\|_\infty > 1$, the mechanism to adapt the network's load and routeing is:

- we increase the total network input by a factor $\beta = \|\boldsymbol{\gamma}\|_\infty > 1$ to $\beta\lambda$, and
- redistribute the admitted load, choose, with $\alpha_0 = 1$, as acceptance probability vector $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\gamma})$ the relative service rate changes $\alpha_j := \gamma_j/\|\boldsymbol{\gamma}\|_\infty$, $j \in \widetilde{J}$.

We remark that if $\gamma_j > 1$, node j can process more load without being overloaded. However, this additional load departing from j can cause overload at other nodes. Therefore, some of the offered new total input of rate $\beta\lambda$ possibly will not be accepted after readjusting the routeing. Our randomised random walk algorithms from Section 2 will automatically compute the correct rejection rates for the external arrivals.

Definition 3.1. The modified routeing matrix is for randomised skipping, respectively reflection denoted by $r^{(\boldsymbol{\alpha})} = (r^{(\boldsymbol{\alpha})}(i, j) \colon i, j \in \widetilde{J}_0)$, and is given by (2.1), respectively (2.2). The set of 'blocked nodes' $B(\boldsymbol{\gamma}) \subset \widetilde{J}$ is defined by $j \in B(\boldsymbol{\gamma}) \colon \iff \gamma_j = 0$, and its complement is the set of the 'working nodes': $W(\boldsymbol{\gamma}) \subset \widetilde{J}$, is by $j \in W(\boldsymbol{\gamma}) \colon \iff \gamma_j > 0$.

When the service rates of a Jackson network are modified according to $\boldsymbol{\gamma}$ and routeing is adjusted according to $\boldsymbol{\alpha}(\boldsymbol{\gamma})$, we obtain a new Markovian network process on $\mathbb{N}_0^{\widetilde{J}}$ denoted by $\boldsymbol{X}^{(\gamma)} = (X^{(\boldsymbol{\gamma})}(t) \colon t \geq 0)$. Then $X_t^{(\boldsymbol{\gamma})} = (X_1^{(\boldsymbol{\gamma})}(t), \dots, X_J^{(\boldsymbol{\gamma})}(t)) \in \mathbb{N}_0^{\widetilde{J}}$ reads: at time t there are $X_j^{(\boldsymbol{\gamma})}(t)$ customers present at node j, either in service or waiting. The strict positive transition rates of the generator $Q^{\boldsymbol{X}^{(\gamma)}} =: Q^{(\boldsymbol{\gamma})} = (q^{(\boldsymbol{\gamma})}(\boldsymbol{n}, \boldsymbol{n}') \colon \boldsymbol{n}, \boldsymbol{n}' \in \mathbb{N}_0^{\widetilde{J}})$ are under both rerouteing regimes for $\boldsymbol{n} = (n_1, \dots, n_J) \in \mathbb{N}_0^{\widetilde{J}}$ given by

$$q^{(\boldsymbol{\gamma})}(\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}_i) = \beta\lambda r^{(\boldsymbol{\alpha})}(0, i), \qquad i \in \widetilde{J}_0,$$
$$q^{(\boldsymbol{\gamma})}(\boldsymbol{n}, \boldsymbol{n} - \boldsymbol{e}_j + \boldsymbol{e}_i) = \mathbf{1}_{\{n_j > 0\}} \gamma_j \mu_j(n_j) r^{(\boldsymbol{\alpha})}(j, i) \qquad i, j \in \widetilde{J}, i \neq j,$$
$$q^{(\boldsymbol{\gamma})}(\boldsymbol{n}, \boldsymbol{n} - \boldsymbol{e}_j) = \mathbf{1}_{\{n_j > 0\}} \gamma_j \mu_j(n_j) r^{(\boldsymbol{\alpha})}(j, 0), \qquad j \in \widetilde{J}.$$

Theorem 3.1. (i) Randomised skipping. *Let $\boldsymbol{X}$ be an ergodic Jackson network process with stationary distribution ξ from (3.2), where the service intensities $\mu_i(n_i)$ at node i are changed by a factor $\gamma_i \in [0, \infty)$ for $i \in \widetilde{J}$. Denote*

$$\beta := \begin{cases} 1 & \text{if } \|\gamma\|_\infty \leq 1, \\ \|\gamma\|_\infty & \text{if } \|\gamma\|_\infty > 1, \end{cases} \tag{3.3}$$

$$\alpha_0 = 1, \qquad \alpha_j = \begin{cases} \gamma_j & \text{if } \|\boldsymbol{\gamma}\|_\infty \leq 1, \\ \dfrac{\gamma_j}{\|\boldsymbol{\gamma}\|_\infty} & \text{if } \|\boldsymbol{\gamma}\|_\infty > 1, \end{cases} \quad \text{for all } j \in \widetilde{J}, \tag{3.4}$$

change routeing by randomised skipping with $\boldsymbol{\alpha} = (\alpha_i \colon i \in \widetilde{J}_0)$ according to Theorem 2.1, and change the total network input by factor β. Then ξ is a stationary distribution for $\boldsymbol{X}^{(\boldsymbol{\gamma})} = (X^{(\boldsymbol{\gamma})}(t) \colon t \geq 0)$ as well, and if $B(\boldsymbol{\gamma}) = \varnothing$ then $\boldsymbol{X}^{(\boldsymbol{\gamma})}$ is ergodic.

If $B(\boldsymbol{\gamma}) \neq \varnothing$ then $\boldsymbol{X}^{(\boldsymbol{\gamma})}$ is not irreducible on $\mathbb{N}_0^{\widetilde{J}}$ and its state space is divided into an infinite set of closed subspaces $\mathbb{N}_0^{W(\boldsymbol{\gamma})} \times \{(n_j \colon j \in B(\boldsymbol{\gamma}))\}$ for all $(n_j \colon j \in B(\boldsymbol{\gamma})) \in \mathbb{N}_0^{B(\boldsymbol{\gamma})}$. For any probability distribution φ on $\mathbb{N}_0^{B(\boldsymbol{\gamma})}$ there exists a stationary distribution $\xi_\varphi^{(\boldsymbol{\gamma})}$ for $\boldsymbol{X}^{(\boldsymbol{\gamma})}$, which is for $\boldsymbol{n} = (n_1, \dots, n_J) \in \mathbb{N}_0^{\widetilde{J}}$,

$$\xi_\varphi^{(\boldsymbol{\gamma})}(\boldsymbol{n}) = \xi_\varphi^{(\boldsymbol{\gamma})}(n_1, \dots, n_J) = \prod_{j \in W(\boldsymbol{\gamma})} \prod_{\ell=1}^{n_j} \frac{\eta_j}{\mu_j(\ell)} C(j)^{-1} \varphi(n_j \colon j \in B(\boldsymbol{\gamma})). \tag{3.5}$$

(ii) Randomised reflection. *If additionally to the assumptions of Theorem 3.1(i) the routeing chain r is reversible with respect to η then the rerouteing may be performed by randomised reflection according to Proposition 2.3 with $\boldsymbol{\alpha} = (\alpha_i \colon i \in \widetilde{J}_0)$. The results and equations of Theorem 3.1(i) carry over word by word.*

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Proof. The global balance equation $xQ^{(\boldsymbol{\gamma})} = 0$ for the joint queue-length process $X^{(\boldsymbol{\gamma})}$ of the modified system is in both settings for $\boldsymbol{n} = (n_1, \ldots, n_J) \in \mathbb{N}_0^{\widetilde{J}}$,

$$x(\boldsymbol{n})\Big(\sum_{j\in\widetilde{J}} \beta\lambda r^{(\boldsymbol{\alpha})}(0,j) + \sum_{j\in\widetilde{J}} \mathbf{1}_{\{n_j>0\}}\gamma_j\mu_j(n_j)(1-r^{(\boldsymbol{\alpha})}(j,j))\Big)$$
$$= \sum_{i\in\widetilde{J}} x(\boldsymbol{n}-\boldsymbol{e}_i)\mathbf{1}_{\{n_i>0\}}\beta\lambda r^{(\boldsymbol{\alpha})}(0,i) + \sum_{j\in\widetilde{J}} x(\boldsymbol{n}+\boldsymbol{e}_j)\gamma_j\mu_j(n_j+1)r^{(\boldsymbol{\alpha})}(j,0)$$
$$+ \sum_{j\in\widetilde{J}}\sum_{i\in\widetilde{J}\setminus\{j\}} x(\boldsymbol{n}-\boldsymbol{e}_i+\boldsymbol{e}_j)\mathbf{1}_{\{n_i>0\}}\gamma_j\mu_j(n_j+1)r^{(\boldsymbol{\alpha})}(j,i). \tag{3.6}$$

We consider the $B(\boldsymbol{\gamma}) \neq \varnothing$ case (the $B(\boldsymbol{\gamma}) = \varnothing$ case is proved similarly). Then for $i \in B(\boldsymbol{\gamma})$, we have $\gamma_i = \alpha_i = 0$ and $r^{(\boldsymbol{\alpha})}(j,i) = 0$ for all $j \in \widetilde{J}_0$, and (3.6) reduces to

$$x(\boldsymbol{n})\Big(\sum_{j\in W(\boldsymbol{\gamma})} \beta\lambda r^{(\boldsymbol{\alpha})}(0,j) + \sum_{j\in W(\boldsymbol{\gamma})} \mathbf{1}_{\{n_j>0\}}\gamma_j\mu_j(n_j)(1-r^{(\boldsymbol{\alpha})}(j,j))\Big)$$
$$= \sum_{i\in W(\boldsymbol{\gamma})} x(\boldsymbol{n}-\boldsymbol{e}_i)\mathbf{1}_{\{n_i>0\}}\beta\lambda r^{(\boldsymbol{\alpha})}(0,i) + \sum_{j\in W(\boldsymbol{\gamma})} x(\boldsymbol{n}+\boldsymbol{e}_j)\gamma_j\mu_j(n_j+1)r^{(\boldsymbol{\alpha})}(j,0)$$
$$+ \sum_{j\in W(\boldsymbol{\gamma})}\sum_{i\in W(\boldsymbol{\gamma})\setminus\{j\}} x(\boldsymbol{n}-\boldsymbol{e}_i+\boldsymbol{e}_j)\mathbf{1}_{\{n_i>0\}}\gamma_j\mu_j(n_j+1)r^{(\boldsymbol{\alpha})}(j,i).$$

Inserting $x(n_1,\ldots,n_J) = \prod_{j\in W(\boldsymbol{\gamma})}\prod_{k=1}^{n_j}(\eta_j/\mu_j(k))C(j)^{-1}\varphi(n_j\colon j\in B(\boldsymbol{\gamma}))$ for any probability density φ on $\mathbb{N}_0^{B(\boldsymbol{\gamma})}$, we see that $\prod_{j\in W(\boldsymbol{\gamma})} C(j)^{-1}\varphi(n_j\colon j\in B(\boldsymbol{\gamma}))$ cancels.

Multiplication with $(\beta\prod_{j\in W(\boldsymbol{\gamma})}\prod_{\ell=1}^{n_j}(\eta_j/\mu_j(\ell)))^{-1}$ yields

$$\Big(\sum_{j\in W(\boldsymbol{\gamma})} \lambda r^{(\boldsymbol{\alpha})}(0,j) + \sum_{j\in W(\boldsymbol{\gamma})} \mathbf{1}_{\{n_j>0\}}\frac{\gamma_j}{\beta}\mu_j(n_j)(1-r^{(\boldsymbol{\alpha})}(j,j))\Big)$$
$$= \sum_{i\in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i}\mathbf{1}_{\{n_i>0\}}\lambda r^{(\boldsymbol{\alpha})}(0,i) + \sum_{j\in W(\boldsymbol{\gamma})} \frac{\eta_j}{\mu_j(n_j+1)}\frac{\gamma_j}{\beta}\mu_j(n_j+1)r^{(\boldsymbol{\alpha})}(j,0)$$
$$+ \sum_{j\in W(\boldsymbol{\gamma})}\sum_{i\in W(\boldsymbol{\gamma})\setminus\{j\}} \frac{\mu_i(n_i)}{\eta_i}\mathbf{1}_{\{n_i>0\}}\frac{\eta_j}{\mu_j(n_j+1)}\frac{\gamma_j}{\beta}\mu_j(n_j+1)r^{(\boldsymbol{\alpha})}(j,i).$$

Using the fact that $\gamma_j/\beta = \gamma_j/\|\boldsymbol{\gamma}\|_\infty = \alpha_j$ for all $j \in \widetilde{J}$, we obtain

$$\Big(\sum_{j\in W(\boldsymbol{\gamma})} \lambda r^{(\boldsymbol{\alpha})}(0,j) + \sum_{j\in W(\boldsymbol{\gamma})} \mathbf{1}_{\{n_j>0\}}\alpha_j\mu_j(n_j)(1-r^{(\boldsymbol{\alpha})}(j,j))\Big)$$
$$= \sum_{i\in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i}\mathbf{1}_{\{n_i>0\}}\lambda r^{(\boldsymbol{\alpha})}(0,i) + \sum_{j\in W(\boldsymbol{\gamma})} \frac{\eta_j}{\mu_j(n_j+1)}\alpha_j\mu_j(n_j+1)r^{(\boldsymbol{\alpha})}(j,0)$$
$$+ \sum_{j\in W(\boldsymbol{\gamma})}\sum_{i\in W(\boldsymbol{\gamma})\setminus\{j\}} \frac{\mu_i(n_i)}{\eta_i}\mathbf{1}_{\{n_i>0\}}\frac{\eta_j}{\mu_j(n_j+1)}\alpha_j\mu_j(n_j+1)r^{(\boldsymbol{\alpha})}(j,i).$$

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Reordering and cancelling this yields

$$\left(\sum_{j \in W(\boldsymbol{\gamma})} \lambda r^{(\boldsymbol{\alpha})}(0, j) + \sum_{j \in W(\boldsymbol{\gamma})} \mathbf{1}_{\{n_j > 0\}} \alpha_j \mu_j(n_j) \right)$$
$$= \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} \mathbf{1}_{\{n_i > 0\}} \lambda r^{(\boldsymbol{\alpha})}(0, i) + \sum_{j \in W(\boldsymbol{\gamma})} \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} \mathbf{1}_{\{n_i > 0\}} \alpha_j \eta_j r^{(\boldsymbol{\alpha})}(j, i)$$
$$+ \sum_{j \in W(\boldsymbol{\gamma})} \eta_j \alpha_j r^{(\boldsymbol{\alpha})}(j, 0). \tag{3.7}$$

The first term on the left-hand side and the last term on the right-hand side equate because of

$$\sum_{j \in W(\boldsymbol{\gamma})} \lambda r^{(\boldsymbol{\alpha})}(0, j) = \lambda(1 - r^{(\boldsymbol{\alpha})}(0, 0)),$$

and with $(\eta_j \alpha_j : j \in \widetilde{J}_0)$, we have

$$\underbrace{\left(\sum_{j \in W(\boldsymbol{\gamma})} \eta_j \alpha_j r^{(\boldsymbol{\alpha})}(j, 0) + \eta_0 \alpha_0 r^{(\boldsymbol{\alpha})}(0, 0) \right)}_{= \eta_0 \alpha_0} - \eta_0 \alpha_0 r^{(\boldsymbol{\alpha})}(0, 0) = \lambda(1 - r^{(\boldsymbol{\alpha})}(0, 0)),$$

where we used the fact that $(\eta_j \alpha_j : j \in \widetilde{J}_0)$ is invariant for $r^{(\boldsymbol{\alpha})}$. So (3.7) reduces to

$$\sum_{j \in W(\boldsymbol{\gamma})} \mathbf{1}_{\{n_j > 0\}} \alpha_j \mu_j(n_j)$$
$$= \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} \mathbf{1}_{\{n_i > 0\}} \lambda r^{(\boldsymbol{\alpha})}(0, i) + \sum_{j \in W(\boldsymbol{\gamma})} \sum_{i \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} \mathbf{1}_{\{n_i > 0\}} \alpha_j \eta_j r^{(\boldsymbol{\alpha})}(j, i).$$

Take any $i \in W(\boldsymbol{\gamma})$ with $n_i > 0$ and considering the summands with this i, we have

$$\mathbf{1}_{\{n_i > 0\}} \alpha_i \mu_i(n_i) = \frac{\mu_i(n_i)}{\eta_i} \mathbf{1}_{\{n_i > 0\}} \lambda r^{(\boldsymbol{\alpha})}(0, i) + \sum_{j \in W(\boldsymbol{\gamma})} \frac{\mu_i(n_i)}{\eta_i} \mathbf{1}_{\{n_i > 0\}} \alpha_j \eta_j r^{(\boldsymbol{\alpha})}(j, i),$$

which is

$$\alpha_i \eta_i = \lambda r^{(\boldsymbol{\alpha})}(0, i) + \sum_{j \in W(\boldsymbol{\gamma})} \alpha_j \eta_j r^{(\boldsymbol{\alpha})}(j, i),$$

and recalling that $\eta_0 = \lambda$, $\alpha_0 = 1$, and $\alpha_j = 0$ for $j \in B(\boldsymbol{\gamma})$, we obtain

$$\alpha_i \eta_i = \sum_{j \in W(\boldsymbol{\gamma}) \cup \{0\}} \alpha_j \eta_j r^{(\boldsymbol{\alpha})}(j, i). \tag{3.8}$$

Any i will occur in such a procedure for some state vector with $n_i > 0$. Therefore, if (3.8) would hold, we would eventually arrive at $(\alpha_j \eta_j : j \in \widetilde{J}_0) r^{(\boldsymbol{\alpha})} = (\alpha_j \eta_j : j \in \widetilde{J}_0)$. Now (3.8) does hold for Theorem 3.1(i) by Proposition 2.1 and $\alpha_0 = 1$, and for Theorem 3.1(ii) by Proposition 2.3 and $\alpha_0 = 1$, which completes the proof. □

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Corollary 3.1. *In Theorem 3.1 take* $\boldsymbol{X}$ *ergodic with equilibrium* ξ *from (3.2).*

(i) *If after modification all nodes are still working, possibly with degraded capacity, i.e.* $B(\boldsymbol{\gamma}) = \varnothing$*, then in both cases of rerouteing,* $\boldsymbol{X}^{(\boldsymbol{\gamma})} = (X_t^{(\boldsymbol{\gamma})} \colon t \geq 0)$ *is ergodic with unique stationary and limiting distribution* ξ*.*

(ii) *If* $r^{(\boldsymbol{\alpha})}(0,0) > 0$ *then the effective arrival rate after modification is* $\beta\lambda(1 - r^{(\boldsymbol{\alpha})}(0,0))$*.*

Corollary 3.2. (Extension to general rerouteing.) *Let* $\boldsymbol{X}$ *be an ergodic Jackson network process with stationary distribution* ξ *from (3.2), where the service intensities* $\mu_i(n_i)$ *at node* i *are changed by a factor* $\gamma_i \in [0, \infty)$ *for* $i \in \widetilde{J}$*. Change routeing to follow some matrix* $r^{(\boldsymbol{\alpha})}$ *with invariant measure* $y = (\alpha_j \eta_j \colon j \in \widetilde{J}_0)$ *and increase the total network input by a factor* β*, where* $\alpha_0 = 1$*,* α_j*, and* β *are defined as in (3.4) and (3.3).*

Denote the resulting Markovian state process on $\mathbb{N}_0^{\widetilde{J}}$ *by* $\boldsymbol{X}^{(\boldsymbol{\gamma})} = (X^{(\boldsymbol{\gamma})}(t) \colon t \geq 0)$*.*

Then ξ *is a stationary distribution for* $\boldsymbol{X}^{(\boldsymbol{\gamma})}$ *as well, and if* $B(\boldsymbol{\gamma}) = \varnothing$ *then* $\boldsymbol{X}^{(\boldsymbol{\gamma})}$ *is ergodic.*

If $B(\boldsymbol{\gamma}) \neq \varnothing$ *then* $\boldsymbol{X}^{(\boldsymbol{\gamma})}$ *is not irreducible on* $\mathbb{N}_0^{\widetilde{J}}$ *and its state space is divided into an infinite set of closed subspaces* $\mathbb{N}_0^{W(\boldsymbol{\gamma})} \times \{(n_j \colon j \in B(\boldsymbol{\gamma}))\}$ *for all* $(n_j \colon j \in B(\boldsymbol{\gamma})) \in \mathbb{N}_0^{B(\boldsymbol{\gamma})}$*, and for any probability distribution* φ *on* $\mathbb{N}_0^{B(\boldsymbol{\gamma})}$ *there exists a stationary distribution* $\xi_\varphi^{(\boldsymbol{\gamma})}$ *for* $\boldsymbol{X}^{(\boldsymbol{\gamma})}$ *given in (3.5).*

4. Jackson networks in a random environment

In this section we apply the technique developed in Section 3 to control a Jackson network in a nonautonomous environment, i.e. with bidirectional interaction of a network and its environment. We implement randomised skipping and reflection as a rerouteing regime in response to the environment's changes, see Theorems 4.1 and 4.2, and thereafter extract the principles behind this in Corollary 4.2. This generalises the results of [38], [7], [1], and [32]. The dynamic of the interacting system is determined, on one hand, by the environment process $\boldsymbol{Y} = (Y(t) \colon t \geq 0)$, changes of which result in changes of the network's parameter, and, on the other hand, by the network process $\boldsymbol{X} = (X(t) \colon t \geq 0)$, where some jumps of $\boldsymbol{X}$ enforce the environment to immediately react to this jump. To be more precise, the environment space K is countable and whenever the environment at time t is in state $Y(t) = k$ it changes to $m \in K$ with rate $\nu(k, m)$, and we set $V = (\nu(k, m) \colon k, m \in K)$.

The network process $\boldsymbol{X}$ records the joint queue-length vector, and $X_j(t) = n_j$ is the queue length at node $j \in \widetilde{J}$. Whenever the environment's state is $k \in K$ and at node j a customer is served and leaves the network, this jump triggers with probability $R_j(k, m)$ the environment to jump from k to $m \in K$. We set $R_j = (R_j(k, m) \colon k, m \in K)$, $j \in \widetilde{J}$. Neither the generator matrix $V = (\nu(k, m) \colon k, m \in K)$ nor the stochastic matrices $R_j = (R_j(k, m) \colon k, m \in K)$, $j \in \widetilde{J}$ need to be irreducible or positive recurrent.

Associated with the environment state, $k \in K$ is a vector $\boldsymbol{\gamma}(k) \in [0, \infty)^{\widetilde{J}}$ which determines the factors by which the service capacities are changed, when the environment enters k (see $\boldsymbol{\gamma} \in [0, \infty)^{\widetilde{J}}$ in Section 3). This results in a service rate $\mu_j(n_j, k) = \gamma_j(k)\mu_j(n_j)$ if the queue length at j is n_j and the environment's state is k.

The network reacts to the impact of the environment in state k by modifying the routeing according to strategies described in Definitions 2.1 and 2.2, possibly by admitting more customers. The latter part of the strategy is set in force whenever in environment state k there exist some $\gamma_j(k) > 1$. In such state $k \in K$ the arrival rate to the network is increased by $\beta(\boldsymbol{\gamma}(k)) = \|\boldsymbol{\gamma}(k)\|_\infty$ from λ to $\lambda\beta(\boldsymbol{\gamma}(k))$.

4.1. Rerouteing by randomised skipping

In this section we modify routeing in reaction to the servers' changes of capacities by randomised skipping from Definition 2.1. We investigate this case in detail, other modifications will be described in less detail.

We need environment dependent rerouteing with acceptance probabilities $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\gamma}(k))$, modified rerouteing matrices $r^{(\boldsymbol{\alpha}(\boldsymbol{\gamma}(k)))}$, and overall load factors $\beta(\boldsymbol{\gamma}(k))$. To keep notation short we write (in the rest of the paper) $\boldsymbol{\alpha}(k) = (\alpha_j(k) : j \in \widetilde{J}_0)$ instead of $\boldsymbol{\alpha}(\boldsymbol{\gamma}(k))$, $r^{(\boldsymbol{\alpha}(k))}$ instead of $r^{(\boldsymbol{\alpha}(\boldsymbol{\gamma}(k)))}$, and $\beta(k)$ instead of $\beta(\boldsymbol{\gamma}(k))$. Randomised skipping from Definition 2.1 yields routeing regime $r^{(\boldsymbol{\alpha}(k))}$ from Theorem 2.1, and the total input rate is changed by a factor $\beta(k)$. We define $\boldsymbol{\alpha}$ and β similar to (3.3) and (3.4) for $k \in K$,

$$\beta(k) := \begin{cases} 1 & \text{if } \|\gamma(k)\|_\infty \le 1, \\ \|\gamma(k)\|_\infty & \text{if } \|\gamma(k)\|_\infty > 1, \end{cases} \tag{4.1}$$

$$\alpha_0(k) = 1, \qquad \alpha_j(k) = \begin{cases} \gamma_j(k) & \text{if } \|\boldsymbol{\gamma}(k)\|_\infty \le 1, \\ \dfrac{\gamma_j(k)}{\|\boldsymbol{\gamma}(k)\|_\infty} & \text{if } \|\boldsymbol{\gamma}(k)\|_\infty > 1, \end{cases} \qquad \text{for all } j \in \widetilde{J}. \tag{4.2}$$

We further define $B(\boldsymbol{\gamma}(k))$ and $W(\boldsymbol{\gamma}(k))$ similar to Definition 3.1 as a set of completely broken down nodes, respectively as a set of nodes which, although possibly being degraded or upgraded, can still serve customers under environment condition k.

With standard independence assumptions for interarrival and service times and of conditional independence of routeing and of jumps of the environment triggered by departing customers, the queue-lengths-environment process $\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y})$ on $E := \mathbb{N}_0^{\widetilde{J}} \times K$ is Markov. Its generator $Q^{\boldsymbol{Z}} = (q^{\boldsymbol{Z}}((\boldsymbol{n}, k), (\boldsymbol{n}', k')) : (\boldsymbol{n}, k), (\boldsymbol{n}', k') \in E)$ has strict positive transition rates for $(\boldsymbol{n}, k) = ((n_1, \dots, n_J), k) \in \mathbb{N}_0^{\widetilde{J}} \times K$, $j, i \in \widetilde{J}$ given as

$$\begin{aligned} q^{\boldsymbol{Z}}((\boldsymbol{n}, k), (\boldsymbol{n} + \boldsymbol{e}_i, k)) &= \beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0, i), \\ q^{\boldsymbol{Z}}((\boldsymbol{n}, k), (\boldsymbol{n} - \boldsymbol{e}_j + \boldsymbol{e}_i, k)) &= \mathbf{1}_{\{n_j > 0\}}\gamma_j(k)\mu_j(n_j) r^{(\boldsymbol{\alpha}(k))}(j, i), \qquad i \neq j, \\ q^{\boldsymbol{Z}}((\boldsymbol{n}, k), (\boldsymbol{n} - \boldsymbol{e}_j, m)) &= \mathbf{1}_{\{n_j > 0\}}\gamma_j(k)\mu_j(n_j) r^{(\boldsymbol{\alpha}(k))}(j, 0) R_j(k, m), \\ q^{\boldsymbol{Z}}((\boldsymbol{n}, k), (\boldsymbol{n}, m)) &= \nu(k, m), \qquad m \in K. \end{aligned} \tag{4.3}$$

Theorem 4.1. *Assume that the queue-lengths-environment process $\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y})$ is ergodic and that the pure Jackson network process $\boldsymbol{X}$ without environment is ergodic with stationary distribution ξ on $\mathbb{N}_0^{\widetilde{J}}$ from (3.2). Define the* reduced generator Q_{red} *as*

$$Q_{\text{red}} := \Big[V + \sum_{j \in \widetilde{J}} \eta_j I_{(\gamma_j \bullet r^{(\boldsymbol{\alpha}(\cdot))}(j,0))}(R_j - I) \Big], \tag{4.4}$$

where γ_j and $r^{(\boldsymbol{\alpha}(\cdot))}(j, 0)$ are for $j \in \widetilde{J}$ real-valued functions on K. Assume that the reduced generator equation $\theta Q_{\text{red}} = 0$ has a nonzero, nonnegative solution. Then Q_{red} is irreducible on K and the reduced generator equation $\theta Q_{\text{red}} = 0$ has a strictly positive stochastic solution which we denote by θ.

Furthermore, the queue-lengths-environment process $\boldsymbol{Z}$ has the unique steady state distribution $\pi = (\pi(\boldsymbol{n}, k) : \boldsymbol{n} \in \mathbb{N}_0^{\widetilde{J}}, k \in K)$ of product form given by

$$\pi(\boldsymbol{n}, k) = \xi(\boldsymbol{n})\theta(k), \qquad \boldsymbol{n} \in \mathbb{N}_0^{\widetilde{J}},\ k \in K. \tag{4.5}$$

Proof. The global balance equation of $\boldsymbol{Z}$ is

$$\pi(\boldsymbol{n},k)\Big(\sum_{i\in\widetilde{J}}\beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0,i)+\underbrace{\sum_{m\in K\setminus\{k\}}\nu(k,m)}_{-\nu(k,k)}$$
$$+\sum_{j\in\widetilde{J}}\mathbf{1}_{\{n_j>0\}}\gamma_j(k)\mu_j(n_j)(1-r^{(\boldsymbol{\alpha}(k))}(j,j))\Big)$$
$$=\sum_{m\in K\setminus\{k\}}\pi(\boldsymbol{n},m)\nu(m,k)+\sum_{i\in\widetilde{J}}\pi(\boldsymbol{n}-\boldsymbol{e}_i,k)\mathbf{1}_{\{n_i>0\}}\beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0,i)$$
$$+\sum_{i\in\widetilde{J}}\sum_{j\in\widetilde{J}\setminus\{i\}}\pi(\boldsymbol{n}-\boldsymbol{e}_i+\boldsymbol{e}_j,k)\mathbf{1}_{\{n_i>0\}}\gamma_j(k)\mu_j(n_j+1)r^{(\boldsymbol{\alpha}(k))}(j,i)$$
$$+\sum_{j\in\widetilde{J}}\sum_{m\in K}\pi(\boldsymbol{n}+\boldsymbol{e}_j,m)\gamma_j(m)\mu_j(n_j+1)r^{(\boldsymbol{\alpha}(m))}(j,0)R_j(m,k).$$

Inserting $\pi(\boldsymbol{n},k)=\xi(\boldsymbol{n})\theta(k)$, adding $\xi(\boldsymbol{n})\theta(k)\nu(k,k)$ on both sides, and rearranging terms and blowing up, we obtain

$$\theta(k)\Big[\xi(\boldsymbol{n})\Big(\sum_{i\in\widetilde{J}}\beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0,i)+\sum_{j\in\widetilde{J}}\mathbf{1}_{\{n_j>0\}}\gamma_j(k)\mu_j(n_j)(1-r^{(\boldsymbol{\alpha}(k))}(j,j))\Big)\Big]$$
$$=\theta(k)\Big[\sum_{i\in\widetilde{J}}\xi(\boldsymbol{n}-\boldsymbol{e}_i)\mathbf{1}_{\{n_i>0\}}\beta(k)\lambda r^{(\boldsymbol{\alpha}(k))}(0,i)$$
$$+\sum_{i\in\widetilde{J}}\sum_{j\in\widetilde{J}\setminus\{i\}}\xi(\boldsymbol{n}-\boldsymbol{e}_i+\boldsymbol{e}_j)\mathbf{1}_{\{n_i>0\}}\gamma_j(k)\mu_j(n_j+1)r^{(\boldsymbol{\alpha}(k))}(j,i)$$
$$+\sum_{j\in\widetilde{J}}\xi(\boldsymbol{n}+\boldsymbol{e}_j)\gamma_j(k)\mu_j(n_j+1)r^{(\boldsymbol{\alpha}(k))}(j,0)\Big]$$
$$+\sum_{m\in K}\xi(\boldsymbol{n})\theta(m)\nu(m,k)-\theta(k)\sum_{j\in\widetilde{J}}\xi(\boldsymbol{n}+\boldsymbol{e}_j)\gamma_j(k)\mu_j(n_j+1)r^{(\boldsymbol{\alpha}(k))}(j,0)$$
$$+\sum_{j\in\widetilde{J}}\sum_{m\in K}\xi(\boldsymbol{n}+\boldsymbol{e}_j)\theta(m)\gamma_j(m)\mu_j(n_j+1)r^{(\boldsymbol{\alpha}(m))}(j,0)R_j(m,k). \tag{4.6}$$

For each fixed environment state k the terms in squared brackets equate from Theorem 3.1, see (3.6), where for $B(\boldsymbol{\gamma}(k))$ we set in modified notation ($\varphi\to\varphi(k)$) from that theorem the specific probabilities

$$\varphi(k)(n_j\colon j\in B(\boldsymbol{\gamma}(k))):=\prod_{j\in B(\boldsymbol{\gamma}(k))}\prod_{\ell=1}^{n_j}\frac{\eta_j}{\mu_j(\ell)}C(j)^{-1},\qquad (n_j\colon j\in B(\boldsymbol{\gamma}(k)))\in\mathbb{N}_0^{B(\boldsymbol{\gamma}(k))}.$$

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Dividing by $\xi(\boldsymbol{n})$ and cancelling $\mu_j(n_j+1)$, we arrive at

$$0 = -\theta(k) \sum_{j\in\widetilde{J}} \eta_j \gamma_j(k) r^{(\boldsymbol{\alpha}(k))}(j,0) + \sum_{j\in\widetilde{J}} \sum_{m\in K} \theta(m)\eta_j\gamma_j(m) r^{(\boldsymbol{\alpha}(m))}(j,0) R_j(m,k) + \sum_{m\in K} \theta(m)\nu(m,k).$$

Rearranging terms, we have

$$\theta(k) \sum_{j\in\widetilde{J}} \eta_j \gamma_j(k) r^{(\boldsymbol{\alpha}(k))}(j,0) = \sum_{m\in K} \theta(m)\bigg(\nu(m,k) + \sum_{j\in\widetilde{J}} \eta_j\gamma_j(m) r^{(\boldsymbol{\alpha}(m))}(j,0) R_j(m,k)\bigg),$$

which finally leads for any prescribed $k \in K$ to

$$0 = \sum_{m\in K} \theta(m)\bigg(\nu(m,k) + \sum_{j\in\widetilde{J}} \eta_j\gamma_j(m) r^{(\boldsymbol{\alpha}(m))}(j,0)(R_j(m,k) - \delta_{mk})\bigg).$$

In matrix form this is (4.4). Then Q_{red} is obviously the generator of some Markov process.

It holds that Q_{red} is irreducible because otherwise $\boldsymbol{Z}$ would not be irreducible. If $\theta Q_{\text{red}} = 0$ has no stochastic solution, the global balance equation of $\boldsymbol{Z}$ would have a nontrivial nonnegative solution which cannot be normalised. This would contradict ergodicity. The same argument shows that the solution must be unique. □

The proof of Theorem 4.1 shows the following characterisation. A similar corollary for Theorem 4.2 and Corollary 4.2 below is valid but will not be stated separately.

Corollary 4.1. *Assume that the queue-lengths-environment process $\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y})$ is irreducible. Define Q_{red} as in (4.4). Then the following statements are equivalent:*

(i) *$\boldsymbol{Z}$ is ergodic with product form steady state distribution*

$$\pi(\boldsymbol{n},k) = \xi(\boldsymbol{n})\theta(k) = \prod_{j=1}^{J}\prod_{\ell=1}^{n_j} \frac{\eta_j}{\mu_j(\ell)} C(j)^{-1}\theta(k), \qquad \boldsymbol{n} \in \mathbb{N}_0^{\widetilde{J}},\ k \in K;$$

(ii) *for all $j \in \widetilde{J} : \sum_{n_j=0}^{\infty} \prod_{\ell=1}^{n_j} (\eta_j/\mu_j(\ell)) < \infty$ and $\theta Q_{\text{red}} = 0$ has a strictly positive stochastic solution.*

4.2. Rerouteing by randomised reflection

In this section we assume that r is reversible and the reaction to the servers' change of capacities is by randomised reflection according to Definition 2.2, which yields a routeing regime $r^{(\boldsymbol{\alpha}(k))}$ according to Proposition 2.3. We use $\boldsymbol{\alpha}(k)$ and $\beta(k)$ as defined in (4.2) and (4.1), and take $B(\boldsymbol{\gamma}(k))$ and $W(\boldsymbol{\gamma}(k))$ as in Definition 3.1.

Note that under randomised reflection, $r^{(\boldsymbol{\alpha}(k))}$ may be reducible on $W(\boldsymbol{\gamma}(k)) \cup \{0\}$. This does not destroy ergodicity of the Markovian system process $\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y})$ on $E := \mathbb{N}_0^{\widetilde{J}} \times K$. The generator $Q^{\boldsymbol{Z}}$ of $\boldsymbol{Z}$ is identical to that displayed in (4.3).

Theorem 4.2. *Assume that $\boldsymbol{Z}$ is ergodic and the pure Jackson network process $\boldsymbol{X}$ without environment is ergodic with stationary distribution ξ on $\mathbb{N}_0^{\widetilde{J}}$ from (3.2). Assume that the extended routeing matrix $r = (r(i,j) : i, j \in \widetilde{J}_0)$ is reversible for $\eta = (\eta_j : j \in \widetilde{J}_0)$. Assume*

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that $\theta Q_{\text{red}} = 0$ (with Q_{red} as in (4.4)) has a nonzero, nonnegative solution. Then Q_{red} is irreducible on K and $\theta Q_{\text{red}} = 0$ has a strictly positive stochastic solution which we denote by θ. Furthermore, the queue-lengths-environment process $\mathbf{Z}$ has the unique steady state distribution $\pi = (\pi(\boldsymbol{n}, k) : \boldsymbol{n} \in \mathbb{N}_0^{\widetilde{J}}, k \in K)$ of product form (4.5).

The proof of the theorem is along the same lines as the proof of Theorem 4.1. There, when manipulating (4.6), we had to refer to the properties of skipping in Theorem 3.1(i), which is now replaced by referring to the properties in Theorem 3.1(ii).

4.3. Rerouteing by general randomisation

The results of the previous sections lead us to extract general principles for randomised rerouteing. We use $\boldsymbol{\alpha}(k)$ and $\beta(k)$ as defined in (4.1) and (4.2), and take $B(\boldsymbol{\gamma}(k))$ and $W(\boldsymbol{\gamma}(k))$ as in Definition 3.1. For rerouteing regimes $r^{(\boldsymbol{\alpha}(k))}$, $k \in K$, for $\boldsymbol{\alpha}(k)$ with $\alpha_0(k) = 1$ and $\boldsymbol{\alpha}(k) \in [0, 1]^{\widetilde{J}_0}$, we require the properties described in Corollary 3.2.

Corollary 4.2. *The queue-lengths-environment process $\mathbf{Z} = (\boldsymbol{X}, \boldsymbol{Y})$ is Markov on $E := \mathbb{N}_0^{\widetilde{J}} \times K$, and its generator is identical to that displayed in (4.3). Assume that the rerouteing regimes $r^{(\boldsymbol{\alpha}(k))}$, $k \in K$, have invariant measures $y(k) = (\alpha_j(k)\eta_j : j \in \widetilde{J}_0)$. Assume that $\mathbf{Z}$ is ergodic and assume that the pure Jackson network process $\boldsymbol{X}$ without environment is ergodic with stationary and limiting distribution ξ on $\mathbb{N}_0^{\widetilde{J}}$ from (3.2). Assume that $\theta Q_{\text{red}} = 0$ (with Q_{red} as in (4.4)) has a nonzero, nonnegative solution. Then Q_{red} is irreducible on K and $\theta Q_{\text{red}} = 0$ has a strictly positive stochastic solution which we denote by θ. Furthermore, $\mathbf{Z}$ has the unique steady state distribution $\pi = (\pi(\boldsymbol{n}, k) : \boldsymbol{n} \in \mathbb{N}_0^{\widetilde{J}}, k \in K)$ of product form (4.5).*

We remark that Corollary 4.2 extends [38, Theorem 1], and [7, Corollary 5] to nonautonomous environments. Furthermore, in [7] equivalence of the existence of product form stationary distribution (4.5) and invariance of the ratios (overall arrival rate/service rates) is proved under the condition that for all environment states the solution of the traffic equations are strictly positive [8]. Our environment dependent traffic equations $\eta^{(\boldsymbol{\alpha}(k))} = \eta^{(\boldsymbol{\alpha}(k))} r^{(\boldsymbol{\alpha}(k))}$, $k \in K$, have in general no strict positive solutions because for $B(\boldsymbol{\gamma}(k)) \neq \varnothing$ and $j \in B(\boldsymbol{\gamma}(k))$, we have $\eta^{(\boldsymbol{\alpha}(k))}(j) = 0$. If there is some $k' \in K$ with $B(\boldsymbol{\gamma}(k')) = \varnothing$, we have $\eta^{(\boldsymbol{\alpha}(k'))}(j) > 0$. So the $\eta^{(\boldsymbol{\alpha}(k))}(j)/\mu_j(n_j, k)$ are not independent of k. Nevertheless, Theorems 4.1 and 4.2 as well as Corollary 4.2 prove product form steady state, which extends the previous results of Zhu and others even for the case of autonomous environments.

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