4

Transition amplitudes

Ultimately our fundamental goal in particle physics is to understand the dynamics, i.e. to have a theory from which we can actually calculate transition amplitudes. Tests of the theory will involve, at the crudest level, measurements of differential cross-sections or decay rates but, at a more sophisticated and more probing level, measurements of all kinds of spin-dependent phenomena. On the one hand, given a dynamical theory it is probably simplest to calculate the helicity transition amplitudes and from them the formulae for the spin-dependent observables that can be tested against experimental data. On the other hand, in the absence of a theory it would seem best to try to obtain information on the behaviour of the transition amplitudes from a sufficiently large number of different independent measurements. In this way one would hope to be led to deduce the nature of the underlying dynamics.

In both these situations it is important to bear in mind that certain properties are intrinsic to transition amplitudes, i.e. they do not depend upon detailed dynamical theory but rather follow from very general conservation laws, principally from the conservation of angular momentum.

The study of reactions thus divides into two phases:

(1) the general properties of transition amplitudes and the connection between their behaviour and the underlying dynamics; and
(2) the relationship between transition amplitudes and observables.

In this chapter we concentrate upon the former. The latter will be discussed in Chapter 5.

4.1 Helicity amplitudes for elastic and pseudoelastic reactions

Many kinds of transition amplitude can be found in the literature, but it seems to us that helicity amplitudes are generally the simplest and most useful amplitudes, and we shall therefore concentrate almost exclusively on

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them. (However, in some circumstances other types of transition amplitude can be valuable, in particular transversity amplitudes, so we include a brief discussion of these in Appendix 4.)

We consider reactions of the type

$$A + B \to C + D$$

where C and D may be stable or unstable particles. The particles have arbitrary spins s_A, s_B, s_C, s_D.

In defining the scattering amplitudes we shall utilize the simple *helicity states* discussed in Section 1.2, which differ slightly from those of the original Jacob–Wick paper (Jacob and Wick, 1959). We do not adopt the convention that deals asymmetrically with the particles and distinguishes 'particle 2' in the reaction.[1] Nevertheless our *helicity amplitudes* will be almost identical to the Jacob–Wick amplitudes at $\phi = 0$, the difference being an irrelevant constant factor. Our amplitudes will have a simpler ϕ-dependence and this will lead to simpler properties of the final state density matrices.

As in Section 1.2 we define single-particle helicity states $|\mathbf{p};\lambda\rangle \equiv |p,\theta,\phi;\lambda\rangle$ normalized as follows:

$$\langle \mathbf{p}';\lambda'|\mathbf{p};\lambda\rangle = (2\pi)^3 \times 2E\delta^3(\mathbf{p}' - \mathbf{p}). \tag{4.1.1}$$

A two-particle state, or indeed an N-particle state, is defined as a direct product of one-particle states. Thus our two-particle CM helicity state with relative momentum $\mathbf{p}' = (p',\theta,\phi)$ is

$$|\mathbf{p}';\lambda_C\lambda_D\rangle = |p',\theta,\phi;\lambda_C\rangle \otimes |p',\pi-\theta,\phi+\pi;\lambda_D\rangle. \tag{4.1.2}$$

For consistency, the initial state with A along OZ and relative momentum $\mathbf{p} = (p,0,0)$ is then

$$|\mathbf{p};\lambda_A\lambda_B\rangle = |p,0,0;\lambda_A\rangle \otimes |p,\pi,\pi;\lambda_B\rangle. \tag{4.1.3}$$

The transition amplitudes are essentially the matrix elements of the $\hat{S}$-operator taken between initial and final CM helicity states. We shall write these as $H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta,\phi)$ and they will be normalized in such a way that for an *unpolarized initial state* the invariant differential cross-section is given by

$$\frac{d\sigma}{dt} = \frac{1}{(2s_A+1)(2s_B+1)} \sum_{\text{all }\lambda} |H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta)|^2 \tag{4.1.4}$$

[1] So long as one works *only* in the CM the Jacob–Wick convention is sensible, but the moment one wishes to transform to other systems, e.g. to the Lab, the asymmetric treatment of the particle becomes a nuisance. Indeed Wick himself discarded the convention in later papers. (Wick, 1962).

where t is the invariant square of the 4-momentum transfer,

$$t \equiv (p_C - p_A)^2. \tag{4.1.5}$$

For photons, the factor $2s+1$ is replaced by 2 in (4.1.4). Here, as throughout, $H(\theta)$ means $H(\theta, \phi = 0)$.

Our amplitudes are then related to those of Jacob–Wick as follows. For $\phi = 0$,

$$H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta) = \exp\left[i\pi(s_B - s_D)\right]\sqrt{\frac{\pi}{pp'}}f_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta) \tag{4.1.6}$$

in which the *constant* phase factor is basically irrelevant.

However, the ϕ-dependence of our amplitudes is simpler than in Jacob–Wick. We have

$$H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta, \phi) = \exp\left\{i\phi(\lambda_A - \lambda_B)\right\} H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta). \tag{4.1.7}$$

With our normalization and conventions the partial-wave expansion is

$$H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta, \phi) = e^{i\pi(s_B - s_D)}\sqrt{\frac{\pi}{pp'}}\frac{e^{i\phi\lambda}}{p} \times \sum_j \left(J + \tfrac{1}{2}\right)\langle\lambda_C\lambda_D|\hat{T}^j(E)|\lambda_A\lambda_B\rangle d^J_{\lambda\mu}(\theta) \tag{4.1.8}$$

where

$$\lambda = \lambda_A - \lambda_B \qquad \mu = \lambda_C - \lambda_D \qquad \hat{S} = 1 + i\hat{T} \tag{4.1.9}$$

and the *partial-wave amplitudes* are identical to those of Jacob–Wick.

4.2 Symmetry properties of helicity amplitudes

We now list the symmetry properties of the $H_{\{\lambda\}}$ when the reaction possesses certain invariant properties.

4.2.1 Parity

Let η_j be the intrinsic parity of particle j and suppose that invariance under space inversion holds. Then, using also rotational invariance, one finds

$$H_{-\lambda_C-\lambda_D;-\lambda_A-\lambda_B}(\theta, \phi) = \eta e^{-i\pi\mu} H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta, \pi - \phi) \tag{4.2.1}$$

where μ is defined in (4.1.9) and

$$\eta = \frac{\eta_C\eta_D}{\eta_A\eta_B}(-1)^{s_A+s_B-s_C-s_D}. \tag{4.2.2}$$

Taking $\phi = 0$ and using (4.1.7) yields a condition on the $\phi = 0$ amplitudes:

$$H_{-\lambda_C-\lambda_D;-\lambda_A-\lambda_B}(\theta) = \eta(-1)^{\lambda-\mu} H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta). \tag{4.2.3}$$

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4.2.2 Time reversal

If time-reversal invariance holds then the sets of helicity amplitudes $H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta)$ for the process $A + B \to C + D$ and $H'_{\lambda_A\lambda_B;\lambda_C\lambda_D}(\theta)$ for the process $C + D \to A + B$ are related by

$$H'_{\lambda_A\lambda_B;\lambda_C\lambda_D}(\theta) = (-1)^{2(s_B-s_D)}(-1)^{\lambda-\mu}H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta). \tag{4.2.4}$$

If the reaction is an *elastic*, one $A + B \to A + B$, then (4.2.4) constitutes a set of relations amongst the amplitudes for the reaction:

$$H_{\lambda_A\lambda_B;\lambda'_A\lambda'_B}(\theta) = (-1)^{\lambda-\mu}H_{\lambda'_A\lambda'_B;\lambda_A\lambda_B}(\theta). \tag{4.2.5}$$

4.2.3 Identical particles

If C and D are identical particles with $s_C = s_D = s'$ then correctly symmetrized final states

$$\begin{aligned}\frac{1}{\sqrt{2}}\Big(&|C;\theta,\phi;\lambda_C\rangle \otimes |D;\pi-\theta,\phi+\pi;\lambda_D\rangle \\ &+(-1)^{2s}|C;\pi-\theta,\phi+\pi;\lambda_D\rangle \otimes |D;\theta,\phi;\lambda_C\rangle\Big)\end{aligned} \tag{4.2.6}$$

must be used instead of (4.1.2); similarly for the initial states if $A = B$.

Let $\mathscr{P}_{12}$ be the operator that exchanges the space and spin quantum numbers of the first and second particles in the state. Under this exchange for particles C and D one finds, using the definition of the helicity amplitudes, that for $\phi = 0$

$$H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta) \to (-1)^{2s'}\exp[i\pi(\lambda_A-\lambda_B)]\,H_{\lambda_D\lambda_C;\lambda_A\lambda_B}(\pi-\theta) \tag{4.2.7}$$

and a similar result for $A \leftrightarrow B$.

The correctly symmetrized amplitudes for processes involving identical particles, either fermions or bosons, are then as follows (we label the helicities a, b, c, d for simplicity):

For $A + A \to C + D$

$$H^{\mathscr{S}}_{cd,aa'}(\theta) = \frac{1}{\sqrt{2}}\left[H_{cd,aa'}(\theta) + (-1)^{c-d}H_{cd,a'a}(\pi-\theta)\right]. \tag{4.2.8}$$

For $A + B \to C + C$

$$H^{\mathscr{S}}_{cc',ab}(\theta) = \frac{1}{\sqrt{2}}\left[H_{cc',ab}(\theta) + (-1)^{a-b}H_{c'c,ab}(\pi-\theta)\right] \tag{4.2.9}$$

and for $A + A \to C + C$

$$\begin{aligned}H^{\mathscr{S}}_{cc',aa'}(\theta) = \tfrac{1}{2}\Big[&H_{cc',aa'}(\theta) + (-1)^{c-c'+a-a'}H_{c'c,a'a}(\theta) \\ &+ (-1)^{a-a'}H_{c'c,aa'}(\pi-\theta) + (-1)^{c-c'}H_{cc',a'a}(\pi-\theta)\Big]\end{aligned} \tag{4.2.10}$$

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The correctly symmetrized amplitudes have the following properties:

For $A + A \to C + D$

$$H^{\mathscr{S}}_{cd,aa'}(\theta) = (-1)^{c-d} H^{\mathscr{S}}_{cd,a'a}(\pi - \theta) \tag{4.2.11a}$$

For $A + B \to C + C$

$$H^{\mathscr{S}}_{cc',ab}(\theta) = (-1)^{a-b} H^{\mathscr{S}}_{c'c,ab}(\pi - \theta) \tag{4.2.11b}$$

For $A + A \to C + C$, both the above apply and, in addition,

$$H^{\mathscr{S}}_{cc',aa'}(\theta) = (-1)^{c-c'+a-a'} H^{\mathscr{S}}_{c'c,a'a}(\theta). \tag{4.2.12}$$

Note that if the particles belong to a multiplet of some internal symmetry group, so that we are dealing with an internal state vector (or wave function) that has a definite symmetry under interchange of the internal quantum numbers of the particles, then this symmetry factor (± 1) must be inserted on the right-hand side of (4.2.11a,b). For example, for a state of definite isospin I a factor $(-1)^{I+1}$ should be inserted. The symmetry (4.2.11a,b) forces certain amplitudes to vanish at 90° in the CM as follows:

For $A + A \to C + D$

$$H^{\mathscr{S}}_{cd,aa'}(\pi/2) = 0 \qquad \text{if } a = a' \quad \text{and} \quad (-1)^{c-d} = -1. \tag{4.2.13}$$

For $A + B \to C + C$

$$H^{\mathscr{S}}_{cc',ab}(\pi/2) = 0 \qquad \text{if } c = c' \quad \text{and} \quad (-1)^{a-b} = -1 \tag{4.2.14}$$

and, as before, both apply to $A + A \to C + C$.

Again, if the state has a definite symmetry under interchange of internal quantum numbers then the symmetry factor must be included in (4.2.13) and (4.2.14). Thus, for definite isospin (4.2.13) becomes $(-1)^{c-d+I+1} = -1$, etc.

There exist powerful phenomenological consequences of the symmetry conditions. We give some classical examples.

(i) *Elastic proton–proton scattering*. In the conventional notation

$$\phi_1(\theta) \equiv H_{++;++}(\theta) \qquad \phi_2(\theta) \equiv H_{++;--}(\theta) \qquad \phi_3(\theta) \equiv H_{+-;+-}(\theta)$$
$$\phi_4(\theta) \equiv H_{+-;-+}(\theta) \qquad \phi_5(\theta) \equiv H_{++;+-}(\theta), \tag{4.2.15}$$

we find

$$\begin{aligned} \phi^{\mathscr{S}}_{1,2} &= \phi_{1,2}(\theta) + \phi_{1,2}(\pi - \theta) \qquad & \phi^{\mathscr{S}}_5 &= \phi_5(\theta) - \phi_5(\pi - \theta) \\ \phi^{\mathscr{S}}_3 &= \phi_3(\theta) - \phi_4(\pi - \theta) \qquad & \phi^{\mathscr{S}}_4 &= \phi_4(\theta) - \phi_3(\pi - \theta), \end{aligned} \tag{4.2.16}$$

an immediate consequence of which (see subsection 5.4.1(ii)) is that the polarizing power which is proportional to ϕ_5, vanishes at $\theta = 90°$.

Also note that we have

$$\phi^{\mathscr{S}}_3(\pi - \theta) = -\phi^{\mathscr{S}}_4(\theta). \tag{4.2.17}$$

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(*ii*) *Resonance decaying into two identical particles.* As explained in subsection 8.2.1 the decay amplitude for a resonance of spin J into two particles is obtained by just keeping the term with the relevant J in the partial-wave expansion (4.1.8). In addition the partial-wave amplitude is then independent of the helicity of the resonance. Aside from a normalization constant, one has for a spin-J resonance $E \to C + D$, with helicities e, c, d,

$$H_{cd,e}(\theta) = M_E(c,d) d^J_{e,c-d}(\theta) \tag{4.2.18}$$

where the $M_E(c,d)$ are dynamics-dependent parameters that depend only on the helicities of C and D.

For the correctly symmetrized amplitudes for

$$E \to C + C$$

one finds from (4.2.11a,b) and (4.2.18), upon using

$$d^J_{\lambda\mu}(\pi - \theta) = (-1)^{J+\lambda} d^J_{\lambda -\mu}(\theta), \tag{4.2.19}$$

that

$$M^{\mathscr{S}}_E(c, c') = (-1)^J M^{\mathscr{S}}_E(c', c) \tag{4.2.20}$$

from which we see that

$$M^{\mathscr{S}}_E(\lambda, \lambda) = 0 \qquad \text{if } J \text{ is odd.} \tag{4.2.21}$$

A classical example is the decay of a massive spin-1 particle into two photons. To conserve the z-component of angular momentum in the rest frame of the particle we must have $|J_z| \leq 1$, so the photons can only have the same helicity, as shown in Fig. 4.1.

Thus, by (4.2.21), a massive spin-1 particle cannot decay into two photons, a result originally due to Landau (1948) and Yang (1950). The result (4.2.21) is thus a generalization of the Landau–Yang theorem.

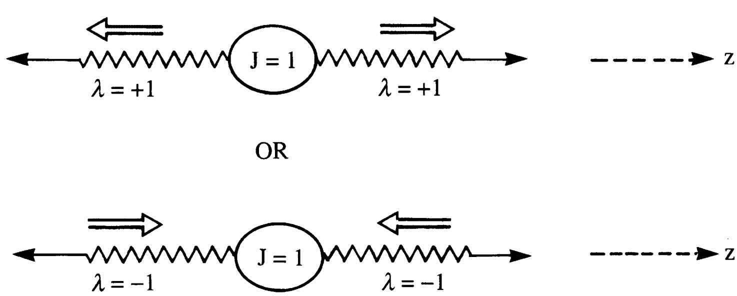

Fig. 4.1. Possible helicities for $J = 1$ decay into two photons.

4.2.4 Charge conjugation

For interactions that are invariant under charge conjugation $\mathscr{C}$, the most interesting cases, as regards helicity dependence, are reactions of the type

$$A + \overline{A} \to D + \overline{D}$$

or the decay of a resonance with definite charge parity of the type

$$E \to D + \overline{D}.$$

Since charge conjugation on a state of the type $|A\overline{A}\rangle$ is equivalent to exchanging the space and spin quantum numbers of the particles together with interchanging their order in the state, we have that

$$\mathscr{C}|A\overline{A}\dots\rangle = (-1)^{2s_A}\mathscr{P}_{12}|A\overline{A}\dots\rangle \tag{4.2.22}$$

and analogously to (4.2.12) we find for $A + \overline{A} \to D + \overline{D}$

$$H_{d\bar{d};a\bar{a}}(\theta) = (-1)^{\lambda-\mu} H_{\bar{d}d;\bar{a}a}(\theta) \tag{4.2.23}$$

where $\lambda = a - \bar{a}$ and $\mu = d - \bar{d}$.

In the case that A is its own antiparticle, i.e.

$$A + \overline{A} \to D + \overline{D}$$

with $\overline{A} = A$, one has also

$$H_{d\bar{d};a\bar{a}}(\theta) = (-1)^{a-\bar{a}} H_{\bar{d}d;a\bar{a}}(\pi - \theta). \tag{4.2.24}$$

For a resonance E of spin J that is an eigenstate of $\mathscr{C}$ with charge parity η_C one finds for $E \to D + \overline{D}$

$$M_E(d, \bar{d}) = \eta_C(-1)^J M_E(\bar{d}, d) \tag{4.2.25}$$

so that $M_E(d, \bar{d} = d) = 0$ if $\eta_C(-1)^J$ is odd.

4.3 Some analytic properties of the helicity amplitudes

An important consequence of the analytic structure of the $H_{\{\lambda\}}$ is that some amplitudes must vanish in the forward or backward direction. This is summarized by writing (Wang, 1966)

$$H_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta) = (\sin\theta/2)^{|\lambda-\mu|}(\cos\theta/2)^{|\lambda+\mu|}\tilde{H}_{\lambda_C\lambda_D;\lambda_A\lambda_B}(\theta) \tag{4.3.1}$$

where $\tilde{H}_{\{\lambda\}}$ is, in general, finite and non-zero at $\theta = 0$ and $\theta = \pi$.

In particular dynamical models the helicity amplitudes may vanish *more rapidly* as $\theta \to 0$ or π (Leader, 1968). Equation (4.3.1) gives the minimum requirement on this vanishing in the forward and backward direction.

If however there are *dynamical* singularities, e.g. at $t = 0$ in the Coulomb scattering of two charged particles, then $\tilde{H}$ may be singular at $\theta = 0$ or π. In that case the *relative* vanishing of different helicity amplitudes must be at least as fast as given by the $\sin\theta/2$, $\cos\theta/2$ factors in (4.3.1). For example, the electromagnetic (one-photon-exchange) contribution to proton–proton scattering at small t gives

$$\begin{aligned} \phi_1^{\text{em}} &= H_{1/2\ 1/2;1/2\ 1/2} \propto \frac{1}{t} \\ \phi_5^{\text{em}} &= H_{1/2\ 1/2;1/2-1/2} \propto \frac{1}{m\sqrt{-t}} \end{aligned} \tag{4.3.2}$$

the ratio being in accordance with (4.3.1).

There are other kinematic points at which analyticity imposes some particular behaviour, namely the thresholds $s = (m_A + m_B)^2$, $s = (m_C + m_D)^2$, the pseudothresholds $s = (m_A - m_B)^2$, $s = (m_C - m_D)^2$ and the origin $s = 0$. The detailed discussion of Cohen-Tannoudji *et al.* (1968a, b) showed that the behaviour of the helicity amplitudes in the neighbourhood of thresholds and pseudothresholds is complicated and involves constraint equations tying together the behaviour of several different amplitudes. (In Appendix 4 we shall see that, on the contrary, the behaviour of transversity amplitudes is simple at these points while it is complicated at $\theta = 0$ or π.)

At high energies, the behaviour of $H_{\{\lambda\}}$ at thresholds and pseudothresholds is unimportant. If however we construct models of the *t-channel* helicity amplitudes (see below) then care must be taken, because, for them, the singularities occur at points $t = (m_A \pm m_C)^2$ and $t = (m_B \pm m_D)^2$, some of which may be close to the physical scattering region. Care too must be taken to satisfy the constraints at $\theta = 0$ or π. Observed effects originating from the kinematic singularities must not be attributed to the dynamics, and models should be constructed so as to satisfy the constraints automatically.

4.4 Crossing for helicity amplitudes

The amplitudes for the three reactions

$$\begin{aligned} &A + B \to C + D && s\text{-channel} \\ &\overline{D} + B \to C + \overline{A} && t\text{-channel} \\ &\overline{C} + B \to \overline{A} + D && u\text{-channel} \end{aligned} \tag{4.4.1}$$

all depend upon the Mandelstam variables

$$\begin{aligned} s &= (p_A + p_C)^2 \\ t &= (p_A - p_C)^2 \\ u &= (p_A - p_D)^2 \end{aligned} \tag{4.4.2}$$

with

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2 \tag{4.4.3}$$

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and are described by just one set of analytic functions evaluated in different regions of the variables s, t, u. The reaction amplitudes for any one reaction channel are obtained by analytic continuation from the amplitudes of any other channel. The set of relations amongst the amplitudes constitute the 'crossing relations' (Trueman and Wick, 1964).

For the t- and u-channel reactions, the variables t and u respectively play the rôle of the square of the CM energy, just as s does for the s-channel reaction. Let $H_{\lambda_C\lambda_D;\lambda_A\lambda_B}$ denote the helicity amplitudes for the s-channel reaction and let us denote by $H^{(t)}_{\lambda_C\lambda_{\bar{A}};\lambda_{\bar{D}}\lambda_B}$, $H^{(u)}_{\lambda_{\bar{A}}\lambda_D;\lambda_{\bar{C}}\lambda_B}$ the helicity amplitudes for the t-channel and u-channel reactions, all with $\phi = 0$. Then the $t \to s$ crossing relation states that

$$\begin{aligned} H_{\lambda_C\lambda_D;\lambda_A\lambda_B} = & \, d^{s_C}_{\mu_C\lambda_C}(\psi_C) d^{s_D}_{\mu_{\bar{D}}\lambda_D}(\psi_D) \\ & \times d^{s_A}_{\mu_{\bar{A}}\lambda_A}(\psi_A) d^{s_B}_{\mu_B\lambda_B}(\psi_B) H^{(t)}_{\mu_C\mu_{\bar{A}};\mu_{\bar{D}}\mu_B} \end{aligned} \tag{4.4.4}$$

where the $t \to s$ crossing angles ψ_i are given by

$$\begin{aligned} \cos\psi_A &= -\frac{(s+m_A^2-m_B^2)(t+m_A^2-m_C^2)+2m_A^2\Delta}{\mathscr{S}_{AB}\mathscr{T}_{AC}} \\ \sin\psi_A &= \frac{m_A\mathscr{S}_{CD}}{\sqrt{s}\mathscr{T}_{AC}}\sin\theta \\ \cos\psi_B &= \frac{(s+m_B^2-m_A^2)(t+m_B^2-m_D^2)+2m_B^2\Delta}{\mathscr{S}_{AB}\mathscr{T}_{BD}} \\ \sin\psi_B &= \frac{m_B\mathscr{S}_{CD}}{\sqrt{s}\mathscr{T}_{BD}}\sin\theta \\ \cos\psi_C &= \frac{(s+m_C^2-m_D^2)(t+m_C^2-m_A^2)-2m_C^2\Delta}{\mathscr{S}_{CD}\mathscr{T}_{AC}} \\ \sin\psi_C &= \frac{m_C\mathscr{S}_{AB}}{\sqrt{s}\mathscr{T}_{AC}}\sin\theta \\ \cos\psi_D &= -\frac{(s+m_D^2-m_C^2)(t+m_D^2-m_B^2)+2m_D^2\Delta}{\mathscr{S}_{CD}\mathscr{T}_{BD}} \\ \sin\psi_D &= \frac{m_D\mathscr{S}_{AB}}{\sqrt{s}\mathscr{T}_{BD}}\sin\theta. \end{aligned} \tag{4.4.5}$$

Here

$$\begin{aligned} \mathscr{S}_{ij}^2 &\equiv [s-(m_i-m_j)^2][s-(m_i+m_j)^2] \\ \mathscr{T}_{ij}^2 &\equiv [t-(m_i-m_j)^2][t-(m_i+m_j)^2] \\ \Delta &\equiv m_B^2+m_C^2-m_A^2-m_D^2 \end{aligned} \tag{4.4.6}$$

and θ is the s-channel CM scattering angle.

For the crossing from $u \to s$ we have

$$H_{\lambda_C\lambda_D;\lambda_A\lambda_B} = d^{s_C}_{\mu_{\bar{C}}\lambda_C}(\chi_C)d^{s_D}_{\mu_D\lambda_D}(\chi_D) \times d^{s_A}_{\mu_{\bar{A}}\lambda_A}(\chi_A)d^{s_B}_{\mu_B\lambda_B}(\chi_B)H^{(u)}_{\mu_{\bar{A}}\mu_D;\mu_{\bar{C}}\mu_B} \tag{4.4.7}$$

where each χ_i is obtained from the ψ_i of eqn (4.4.5) by the substitutions

$$t \to u \qquad m_C \to m_D \qquad m_D \to m_C.$$

Note that for a massless particle the crossing rules simplify greatly. If under crossing the particle remains a particle its crossing matrix is simply $d^s_{\mu\lambda}(0) = \delta_{\mu\lambda}$. If an antiparticle crosses into a particle then the crossing matrix is $d^s_{\bar{\mu}\lambda}(\pi) = (-1)^{s+\bar{\mu}}\delta_{\bar{\mu},-\lambda}$.

4.5 Transition amplitudes in field theory

Consider now the calculation of the matrix elements of some operator in quantum field theory. All operators are expressed in terms of products of fields and the particle states are reduced to the vacuum state by the action of the field operators, as shown in eqn (2.4.10), for example. One sees that each particle or antiparticle in a matrix element will give rise to one or other wave-function factor. Thus a general transition amplitude involving particles A, B, ... and antiparticles $\overline{C}$, $\overline{D}$, ... will always be of the form

$$\langle B,\ldots,\overline{D},\ldots|\,S\,|A,\ldots,\overline{C},\ldots\rangle = \bar{u}_\alpha(B)\cdots\bar{v}_\beta(C)M_{\alpha\ldots\beta\ldots,\gamma\ldots\delta\ldots}u_\gamma(A)\cdots v_\delta(D). \tag{4.5.1}$$

4.6 Structure of matrix elements

The matrix M, which is a function only of the momenta of the particles, will be shown to have simple Lorentz transformation properties. It is therefore possible, in any given case, to write down its most general structure consistent with these properties (and with the requirements of invariance under the discrete transformation). The M's are referred to as M-functions in the literature (Stapp, 1962). We shall not give a general discussion of the theory of M-functions but will illustrate their use in some cases of particular importance.

4.6.1 Matrix elements of a vector current

As a prototype example we shall examine the matrix elements of a 4-vector current $j^\mu(x)$ taken between states of a spin-1/2 Dirac particle. This is germane to the study of the electromagnetic form factors of a nucleon. The method used works equally well for any 'current' that has a well-defined law of transformation under Lorentz transformations, e.g. a scalar, spinor, vector etc.

Under the Lorentz transformation $S \xrightarrow{l} S^l$ let

$$x^\mu \to x'^\mu = (l^{-1}x)^\mu \equiv \Lambda^\mu{}_\nu x^\nu \tag{4.6.1}$$

(with our conventions $\Lambda^\mu{}_\nu = \Lambda^\mu{}_\nu(l^{-1})$; see eqns (1.2.10), (1.2.14)) and, analogously to (2.4.1),

$$j^\mu(x) \to j'^\mu(x')$$

where

$$j'^\mu(x) = U(l)j^\mu(x)U(l^{-1}) = \Lambda^\mu{}_\nu j^\nu(lx). \tag{4.6.2}$$

Consider the 'vertex'

$$\Gamma^\mu(\mathbf{p}_2\lambda_2;\mathbf{p}_1\lambda_1) \equiv \langle \mathbf{p}_2;\lambda_2|j^\mu(0)|\mathbf{p}_1;\lambda_1\rangle \tag{4.6.3}$$

$$= \bar{u}_\alpha(\mathbf{p}_2,\lambda_2)M^\mu_{\alpha\beta}(\mathbf{p}_2,\mathbf{p}_1)u_\beta(\mathbf{p}_1,\lambda_1)$$

$$= \bar{u}(\mathbf{p}_2,\lambda_2)M^\mu(\mathbf{p}_2,\mathbf{p}_1)u(\mathbf{p}_1,\lambda_1) \tag{4.6.4}$$

where u, $\bar{u}$ are Dirac four-component spinors for particles of definite helicity. Our aim is to study the structure of the 4×4 matrices M^μ.

The transformation matrix $D_{nm}(l^{-1})$ that appears in (2.4.16) is customarily denoted by the 4×4 matrix S_{nm} for the case of Dirac particles. Thus, from (2.4.16) and (2.4.18) we have

$$\begin{aligned} u(l^{-1}\mathbf{p},\lambda')\mathscr{D}^{(1/2)}_{\lambda'\lambda}(r) &= Su(\mathbf{p},\lambda) \\ \mathscr{D}^{(1/2)}_{\lambda\lambda'}(r^{-1})\bar{u}(l^{-1}\mathbf{p},\lambda') &= \bar{u}(\mathbf{p},\lambda)S^{-1}. \end{aligned} \tag{4.6.5}$$

Now using (4.6.2) we insert

$$\Lambda^\mu{}_\nu j^\nu(0) = U(l)j^\mu(0)U(l^{-1})$$

into (4.6.3) and obtain, using (4.6.4),(2.1.1) and (2.1.9)

$$\begin{aligned} &\bar{u}(\mathbf{p}_2,\lambda_2)\Lambda^\mu{}_\nu M^\nu(\mathbf{p}_2,\mathbf{p}_1)u(\mathbf{p}_1,\lambda_1) \\ &\quad= \langle \mathbf{p}_2;\lambda_2|U(l)j^\mu(0)U(l^{-1})|\mathbf{p}_1;\lambda_1\rangle \\ &\quad= \mathscr{D}^{(1/2)}_{\lambda_2\lambda_2'}(r^{-1})\langle l^{-1}\mathbf{p}_2;\lambda_2'|j^\mu(0)|l^{-1}\mathbf{p}_1;\lambda_1'\rangle\mathscr{D}^{(1/2)}_{\lambda_1'\lambda_1}(r) \\ &\quad= \mathscr{D}^{(1/2)}_{\lambda_2\lambda_2'}(r^{-1})\bar{u}(l^{-1}\mathbf{p}_2,\lambda_2') \\ &\qquad\quad \times M^\mu(l^{-1}\mathbf{p}_2,l^{-1}\mathbf{p}_1)u(l^{-1}\mathbf{p}_1,\lambda_1')\mathscr{D}^{(1/2)}_{\lambda_1'\lambda_1}(r) \\ &\quad= \bar{u}(\mathbf{p}_2,\lambda_2)S^{-1}M^\mu(l^{-1}\mathbf{p}_2,l^{-1}\mathbf{p}_1)Su(\mathbf{p}_1,\lambda_1). \end{aligned} \tag{4.6.6}$$

Thus we end up with the requirement on M^μ

$$\Lambda^\mu{}_\nu M^\nu(\mathbf{p}_2,\mathbf{p}_1) = S^{-1}M^\mu(l^{-1}\mathbf{p}_2,l^{-1}\mathbf{p}_1)S. \tag{4.6.7}$$

The next step is to note that M, being a 4×4 matrix, can be written as a superposition of the complete set of 16 Dirac matrices, which comprises:

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the scalar I; the vector γ^μ; the tensor $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$; the axial vector $\gamma^\mu\gamma_5$; and the pseudoscalar γ_5. They have the transformation properties

$$\begin{aligned} &S^{-1}IS = I \qquad S^{-1}\gamma_5 S = |\Lambda|\gamma_5 \qquad S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu \\ &S^{-1}\gamma^\mu\gamma_5 S = |\Lambda|\Lambda^\mu{}_\nu\gamma^\nu\gamma_5 \qquad S^{-1}\sigma^{\mu\nu}S = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\sigma^{\alpha\beta} \end{aligned} \tag{4.6.8}$$

where $|\Lambda| = \det(\Lambda^\mu{}_\nu)$.

When l corresponds to the operation of space inversion, $j^\mu(x)$ transforms as a true vector under

$$x \to x' = l_{\mathscr{P}}^{-1}x = (t, -\mathbf{x}) = (g^{\mu\mu})x^\mu \quad \text{(no sum on } \mu)$$

$$\mathscr{P}^{-1}j^\mu(x)\mathscr{P} = (g^{\mu\mu})j^\mu(t - \mathbf{x}). \tag{4.6.9}$$

Using (2.3.7) and the fact that $S = \gamma^0$ for space inversion, one finds that

$$(g^{\mu\mu})M^\mu(\mathbf{p}_2, \mathbf{p}_1) = \gamma^0 M^\mu(-\mathbf{p}_2, -\mathbf{p}_1)\gamma^0 \tag{4.6.10}$$

must be satisfied.

It is simple to check that the following all satisfy (4.6.7) and (4.6.10); here we write $q^\mu \equiv p_2^\mu - p_1^\mu$:

$$\begin{gathered} Iq^\mu \qquad \gamma^\mu \qquad \sigma^{\mu\nu}q_\nu \\ I(p_1 + p_2)^\mu \qquad \sigma^{\mu\nu}(p_1 + p_2)_\nu \qquad \epsilon^\mu{}_{\nu\rho\sigma}p_1^\nu p_2^\rho\gamma_5\gamma^\sigma. \end{gathered}$$

However, since M^μ is sandwiched between Dirac spinors, use of the Dirac equation enables the latter three forms to be expressed in terms of the first three.

In addition the current is conserved, i.e. $\partial_\mu j^\mu(x) = 0$, so that, upon using the fact that translations are generated by the momentum operator $[\hat{P}_\alpha, f(x)] = -i\partial_\alpha f(x)$, we find

$$q_\mu\langle\mathbf{p}_2; \lambda_2|j^\mu(0)|\mathbf{p}_1; \lambda_1\rangle = 0, \tag{4.6.11}$$

which is incompatible with a term of the form Iq^μ. Thus we are left with γ^μ and $\sigma^{\mu\nu}q_\nu$.

Finally, under time reversal $x \to x' = l_{\mathscr{T}}^{-1}x = (-t, \mathbf{x})$

$$\mathscr{T}^{-1}j^\mu(x)\mathscr{T} = (g^{\mu\mu})j^\mu(-t, \mathbf{x}). \tag{4.6.12}$$

Using (2.3.17), and remembering that $\mathscr{T}$ is an anti-linear operator (see the discussion in subsection 2.3.2) we have

$$\begin{aligned} &\bar{u}(\mathbf{p}_2, \lambda_2)\; g^{\mu\mu}\; M^\mu(\mathbf{p}_2, \mathbf{p}_1)u(\mathbf{p}_1, \lambda_1) \\ &\quad = \langle\mathbf{p}_2; \lambda_2|\mathscr{T}^{-1}j^\mu(0)\mathscr{T}|\mathbf{p}_1; \lambda_1\rangle \\ &\quad = \langle\mathscr{T}(\mathbf{p}_2; \lambda_2)|j^\mu(0)|\mathscr{T}(\mathbf{p}_1; \lambda_1)\rangle^* \\ &\quad = e^{i\pi(\lambda_1-\lambda_2)}\langle p_2, \pi - \theta_2, \phi_2 + \pi; \lambda_2|j^\mu(0)|p_1, \pi - \theta_1, \phi_1 + \pi; \lambda_1\rangle^* \\ &\quad = \bar{u}(\mathbf{p}_2, \lambda_2)(\gamma^3\gamma^1)^\dagger M^{\mu*}(-\mathbf{p}_2, -\mathbf{p}_1)\gamma^3\gamma^1 u(\mathbf{p}_1, \lambda_1), \end{aligned} \tag{4.6.13}$$

where we have used (2.4.29) with $T = \gamma^3\gamma^1$. Thus we need

$$(\gamma^3\gamma^1)^\dagger M^{\mu *}(-\mathbf{p}_2, -\mathbf{p}_1)\gamma^3\gamma^1 = (g^{\mu\mu})M^\mu(\mathbf{p}_2, \mathbf{p}_1). \tag{4.6.14}$$

It is easy to check that

$$(\gamma^3\gamma^1)^\dagger \gamma^{\mu *}\gamma^3\gamma^1 = (g^{\mu\mu})\gamma^\mu \tag{4.6.15}$$

and

$$(\gamma^3\gamma^1)^\dagger \sigma^{\mu\nu *}(q^0, -\mathbf{q})_\nu \gamma^3\gamma^1 = -(g^{\mu\mu})\sigma^{\mu\nu} q_\nu \tag{4.6.16}$$

so that (4.6.14) is satisfied by the forms γ^μ or $i\sigma^{\mu\nu}q_\nu$, times any real scalar function. Conventionally one writes

$$\begin{aligned}&\langle \mathbf{p}_2; \lambda_2 | j^\mu(0) | \mathbf{p}_1; \lambda_1\rangle \\ &\quad = \bar{u}(\mathbf{p}_2, \lambda_2)\left[F_1(q^2)\gamma^\mu + \frac{\kappa}{2m}F_2(q^2)i\sigma^{\mu\nu}q_\nu\right]u(\mathbf{p}_1, \lambda_1)\end{aligned} \tag{4.6.17}$$

where κ is the anomalous magnetic moment of the fermion of mass m, and $F_{1,2}$ are the Dirac form factors.

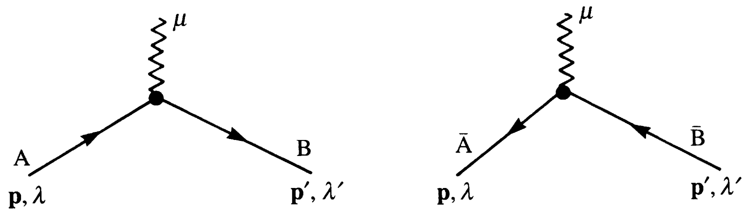
The approach used in this section can be applied to the analysis of the matrix elements of any operator that has a well-defined behaviour under Lorentz transformations. If parity and/or time-reversal invariance are broken one simply does not impose the restrictions (4.6.10) and/or (4.6.14).

The analysis that utilizes Lorentz invariance etc. to expose the essential structure of the matrix elements in (4.6.17) is akin to the familiar use of the Wigner–Eckhardt theorem to express a set of matrix elements in terms of just the reduced matrix elements. Thus these 16 matrix elements ($\mu = 0, 1, 2, 3; \lambda_1 = \pm 1/2, \lambda_2 = \pm 1/2$) are expressed in terms of just two independent functions $F_{1,2}$. The *dynamics*, therefore, is entirely contained in these functions.

4.6.2 Vector and axial-vector coupling

The two most fundamental theories at the present time are the electroweak theory of Glashow, Salam and Weinberg and quantum chromodynamics, and some aspects of these will be discussed in detail in Chapters 9 and 10. For a general introduction the reader is referred to Leader and Predazzi (1996). Here we note that these theories contain only vector and axial-vector couplings of the various gauge bosons to the spin-1/2 fermions. It is thus important to have a detailed understanding of the properties and the structure of these vertices.

Firstly we consider the relationship between the expressions for the Feynman diagram vertices shown below involving incoming and outgoing spin-1/2 fermions A, B or antifermions $\bar{A}$, $\bar{B}$.



Here the vertex is either γ^μ or $\gamma^\mu\gamma_5$. The transition amplitudes $A \to B$ or $\overline{A} \to \overline{B}$ will involve, see eqn (4.5.1),

$$\Gamma_{B\leftarrow A}(\mathbf{p}',\mathbf{p}) = \bar{u}_B(\mathbf{p}',\lambda')\,\{\gamma^\mu \text{ or } \gamma^\mu\gamma_5\}\,u_A(\mathbf{p},\lambda) \tag{4.6.18}$$

and

$$\Gamma_{\bar{B}\leftarrow\bar{A}}(\mathbf{p}',\mathbf{p}) = \bar{v}_A(\mathbf{p},\lambda)\,\{\gamma^\mu \text{ or } \gamma^\mu\gamma_5\}\,v_B(\mathbf{p}',\lambda'). \tag{4.6.19}$$

Using the charge conjugation result (2.4.35), we have

$$u(\mathbf{p},\lambda) = C\bar{v}(\mathbf{p},\lambda) \tag{4.6.20}$$

with

$$C = i\gamma^2\gamma^0. \tag{4.6.21}$$

Adding the fact that

$$C\gamma^\mu C^{-1} = -\gamma^{\mu T} \tag{4.6.22}$$

where $\gamma^{\mu T}$ is the transpose of γ^μ, one arrives at

$$\Gamma_{\bar{B}\leftarrow\bar{A}}(\mathbf{p}',\mathbf{p}) = \bar{u}_B(\mathbf{p}',\lambda')\,\{\gamma^\mu \text{ or } -\gamma^\mu\gamma_5\}\,u_A(\mathbf{p},\lambda). \tag{4.6.23}$$

Comparing with (4.6.18) we see that the amplitudes for $A \to B$ and $\overline{A} \to \overline{B}$ are equal for the vector coupling and opposite in sign for the axial-vector coupling. This will be helpful in comparing, for example,

$$\nu_e + n \to e^- + p$$

with

$$\bar{\nu}_e + p \to e^+ + n.$$

Next we consider the detailed helicity dependence of the vector and axial-vector vertices.

The four-component Dirac spinors which are constructed in accordance with eqns (2.4.14) and (2.4.15) and which respect eqns (4.6.20), (4.6.21) can be written

$$u(\mathbf{p},\lambda) = \frac{1}{\sqrt{E+m}}\begin{pmatrix} E+m \\ 2p\lambda \end{pmatrix}\chi_\lambda(\hat{\mathbf{p}}) \tag{4.6.24}$$

$$v(\mathbf{p},\lambda) = \frac{1}{\sqrt{E+m}}\begin{pmatrix} -2p\lambda \\ E+m \end{pmatrix}\chi_{-\lambda}(\hat{\mathbf{p}}) \tag{4.6.25}$$

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where $\hat{\mathbf{p}} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$, $\lambda = \pm 1/2$ and $\chi_\lambda(\hat{\mathbf{p}})$ is a two-component spinor. In (4.6.24) and (4.6.25) both $E + m$ and $2p\lambda$ are of course understood to be multiplied by $\chi(\hat{\mathbf{p}})$ to yield a four-component spinor. One has

$$\chi_\lambda(\hat{\mathbf{p}}) = e^{-i\phi\sigma_z/2} e^{-i\theta\sigma_y/2} \chi_\lambda. \tag{4.6.26}$$

where λ is + or − and

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.6.27}$$

Explicitly,

$$\chi_+(\hat{\mathbf{p}}) = \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{i\phi/2}\sin\theta/2 \end{pmatrix} \qquad \chi_-(\hat{\mathbf{p}}) = \begin{pmatrix} -e^{-i\phi/2}\sin\theta/2 \\ e^{i\phi/2}\cos\theta/2 \end{pmatrix}. \tag{4.6.28}$$

Let us for brevity put

$$\begin{aligned} V^\mu_{\lambda'\lambda} &\equiv N\bar{u}(\mathbf{p}', \lambda')\gamma^\mu u(\mathbf{p}, \lambda) \\ A^\mu_{\lambda'\lambda} &\equiv N\bar{u}(\mathbf{p}', \lambda')\gamma^\mu\gamma_5 u(\mathbf{p}, \lambda), \end{aligned} \tag{4.6.29}$$

with $N = [(E'+m')(E+m)]^{-1/2}$ included to make the result dimensionless, and let us define the angular function

$$h^\mu_{\lambda\lambda'} \equiv \chi^\dagger_{\lambda'}(\hat{\mathbf{p}}')\sigma^\mu\chi_\lambda(\hat{\mathbf{p}}) \tag{4.6.30}$$

where

$$\sigma^\mu \equiv (I, \boldsymbol{\sigma}). \tag{4.6.31}$$

One finds

$$V^0_{\lambda'\lambda} = N^2\left[(E' + m')(E + m) + 4pp'\lambda\lambda'\right] h^0_{\lambda'\lambda} \tag{4.6.32}$$
$$V^j_{\lambda'\lambda} = 2N^2\left[(E' + m')p\lambda + (E + m)p'\lambda'\right] h^j_{\lambda'\lambda} \tag{4.6.33}$$
$$A^0_{\lambda'\lambda} = 2N^2\left[(E' + m')p\lambda + (E + m)p'\lambda'\right] h^0_{\lambda'\lambda} \tag{4.6.34}$$
$$A^j_{\lambda'\lambda} = N^2\left[(E' + m')(E + m) + 4pp'\lambda\lambda'\right] h^j_{\lambda'\lambda}. \tag{4.6.35}$$

We see that only two different energy-dependent factors occur. So we may write

$$V^0_{\lambda'\lambda} = E_{\lambda'\lambda}h^0_{\lambda'\lambda}, \qquad A^0_{\lambda'\lambda} = F_{\lambda'\lambda}h^0_{\lambda'\lambda} \tag{4.6.36}$$
$$V^j_{\lambda'\lambda} = F_{\lambda'\lambda}h^j_{\lambda'\lambda}, \qquad A^j_{\lambda'\lambda} = E_{\lambda'\lambda}h^j_{\lambda'\lambda} \tag{4.6.37}$$

with

$$E_{\lambda'\lambda} \equiv \frac{1}{(E' + m')(E + m)}\left[(E' + m')(E + m) + 4pp'\lambda\lambda'\right] \tag{4.6.38}$$

and

$$F_{\lambda'\lambda} \equiv \frac{2}{(E'+m')(E+m)}\left[(E'+m')p\lambda + (E+m)p'\lambda'\right]. \tag{4.6.39}$$

In dealing with QCD and the parton model we shall be particularly interested in situations in which $E \gg m$ and $E' \gg m'$, corresponding to the partons being essentially massless. In this limit

$$E_{\lambda'\lambda} = 1 + 4\lambda\lambda' + O(m/E) \tag{4.6.40}$$
$$F_{\lambda'\lambda} = 2\,(\lambda + \lambda') + O(m/E). \tag{4.6.41}$$

We have then the remarkable result that for $\lambda' = -\lambda\,(=\pm 1/2)$

$$V^{\mu}_{-\lambda,\lambda} = A^{\mu}_{-\lambda,\lambda} = 0 + O(m/E). \tag{4.6.42}$$

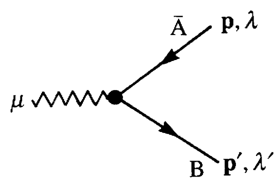
Thus the vector and axial-vector couplings approximately conserve helicity for a fast-moving particle. The helicity is exactly conserved for a massless fermion, e.g. for a neutrino. The impact of this in the parton model is dramatic, since in that model one is supposed to view the collision from an 'infinite momentum frame', i.e. from a frame in which all particles are moving at 'infinite' (i.e. very high) speeds.

For the helicity non-flip matrix elements one has the simple results

$$V^{\mu}_{\lambda,\lambda} = 2\left(h^{0}_{\lambda,\lambda}, 2\lambda h^{j}_{\lambda,\lambda}\right) + O(m/E) \tag{4.6.43}$$
$$A^{\mu}_{\lambda,\lambda} = 2\lambda V^{\mu}_{\lambda,\lambda} + O(m/E). \tag{4.6.44}$$

An analogous simplification arises if we consider the creation or annihilation of a fermion and antifermion via vector or axial-vector coupling in the limit $E \gg m$.

Consider the creation process



If we define

$$\overline{V}^{\mu}_{\lambda',\lambda} \equiv N\bar{u}_B(\mathbf{p}'\lambda')\gamma^{\mu}v_A(\mathbf{p},\lambda) \tag{4.6.45}$$
$$\overline{A}^{\mu}_{\lambda',\lambda} \equiv N\bar{u}_B(\mathbf{p}'\lambda')\gamma^{\mu}\gamma_5 v_A(\mathbf{p},\lambda) \tag{4.6.46}$$

then we find

$$\overline{V}^{0}_{\lambda'\lambda} = F_{\lambda',-\lambda}h^{0}_{\lambda',-\lambda} \qquad \overline{A}^{0}_{\lambda'\lambda} = E_{\lambda',-\lambda}h^{0}_{\lambda',-\lambda} \tag{4.6.47}$$
$$\overline{V}^{j}_{\lambda'\lambda} = E_{\lambda',-\lambda}h^{j}_{\lambda',-\lambda} \qquad \overline{A}^{j}_{\lambda'\lambda} = F_{\lambda',-\lambda}h^{j}_{\lambda',-\lambda}. \tag{4.6.48}$$

It follows from (4.6.40) and (4.6.41) that in the limit $E \gg m$

$$\overline{V}^{\mu}_{\lambda,\lambda} = \overline{A}^{\mu}_{\lambda,\lambda} = 0 + O(m/E). \tag{4.6.49}$$

Hence the amplitude for producing the fermion and the antifermion with equal helicity is of order m/E. For opposite helicities the result takes the simple form, analogous to (4.6.43) and (4.6.44),

$$\overline{V}^{\mu}_{\lambda,-\lambda} = 2\left(2\lambda h^{0}_{\lambda,\lambda}, h^{j}_{\lambda,\lambda}\right) + O(m/E) \tag{4.6.50}$$

$$\overline{A}^{\mu}_{\lambda,-\lambda} = 2\lambda \overline{V}^{\mu}_{\lambda,-\lambda} + O(m/E). \tag{4.6.51}$$

In a similar way, in the annihilation of a fermion–antifermion pair the matrix element of the form $\bar{v}(\gamma^{\mu}$ or $\gamma^{\mu}\gamma_5)u$ will vanish for $E \gg m$ unless the fermion and the antifermion have opposite helicities.

4.6.3 Chirality

Let us consider now the connection between these results and the concept of *chirality*. A Dirac spinor is said to be either right-handed (R) or left-handed (L) if it is an eigenvector of γ_5. By convention

$$\begin{aligned} \gamma_5 u_R &= u_R \qquad & \gamma_5 u_L &= -u_L \\ \gamma_5 v_R &= -v_R \qquad & \gamma_5 v_L &= v_L. \end{aligned} \tag{4.6.52}$$

An arbitrary spinor can always be split up into right-handed and left-handed pieces by noting that

$$\gamma_5(1 \pm \gamma_5) = \pm(1 \pm \gamma_5),$$

so that

$$u_{R,L} \equiv \tfrac{1}{2}(1 \pm \gamma_5)u \qquad v_{R,L} \equiv \tfrac{1}{2}(1 \mp \gamma_5)v \tag{4.6.53}$$

satisfy (4.6.52), and then

$$u = u_R + u_L \qquad v = v_R + v_L. \tag{4.6.54}$$

It is clear from (4.6.24), (4.6.25) that $u(\mathbf{p},\lambda)$, $v(\mathbf{p},\lambda)$ are not eigenvectors of

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

However, when $m = 0$ they do become chiral states and we have

$$\left.\begin{aligned} u_R(\mathbf{p}) &= u(\mathbf{p}, 1/2) \qquad & u_L(\mathbf{p}) &= u(\mathbf{p}, -1/2) \\ v_R(\mathbf{p}) &= v(\mathbf{p}, 1/2) \qquad & v_L(\mathbf{p}) &= v(\mathbf{p}, -1/2) \end{aligned}\right\} \quad (m = 0). \tag{4.6.55}$$

Clearly we should expect (4.6.55) to hold also for massive particles in the limit $m/E \to 0$. Upon splitting $u(\mathbf{p},\lambda)$, $v(\mathbf{p},\lambda)$ into their right- and

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left-handed pieces, as in (4.6.53), (4.6.54), we find

$$u(\mathbf{p},1/2) = \frac{E+m+p}{2[E(E+m)]^{1/2}} \times \left[u_R(\mathbf{p},1/2) + \frac{E+m-p}{E+m+p}u_L(\mathbf{p},1/2)\right] \tag{4.6.56}$$

and

$$u(\mathbf{p},-1/2) = \frac{E+m+p}{2[E(E+m)]^{1/2}} \times \left[u_L(\mathbf{p},-1/2) + \frac{E+m-p}{E+m+p}u_R(\mathbf{p},-1/2)\right] \tag{4.6.57}$$

Thus as $m/E \to 0$ we get

$$\begin{aligned} u(\mathbf{p},1/2) &= u_R(\mathbf{p})\,[1+O(m/E)] \\ u(\mathbf{p},-1/2) &= u_L(\mathbf{p})\,[1+O(m/E)] \end{aligned} \tag{4.6.58}$$

with analogous results for $v(\mathbf{p})$.

The result (4.6.42) can now be understood from a different point of view. Let us denote chirality eigenstates by $u_\eta(\mathbf{p})$, with $\eta = +1/-1$ corresponding to R/L, so that (4.6.52) reads

$$\begin{aligned} u_\eta(\mathbf{p}) &= \eta\gamma_5 u_\eta(\mathbf{p}) \\ v_\eta(\mathbf{p}) &= -\eta\gamma_5 v_\eta(\mathbf{p}), \end{aligned} \tag{4.6.59}$$

and let us consider the vector and axial-vector matrix elements for states of definite chirality. One has for example

$$\begin{aligned} \bar{u}_{\eta'}(\mathbf{p}')\gamma^\mu u_\eta(\mathbf{p}) &= \eta\bar{u}_{\eta'}(\mathbf{p}')\gamma^\mu\gamma_5 u_\eta(\mathbf{p}) \\ &= -\eta\bar{u}_{\eta'}(\mathbf{p}')\gamma_5\gamma^\mu u_\eta(\mathbf{p}). \end{aligned}$$

From (4.6.59) we have $\bar{u}_{\eta'}\gamma_5 = -\eta'\bar{u}_{\eta'}$, so the right-hand side is

$$\eta\eta'\bar{u}_{\eta'}(\mathbf{p}')\gamma^\mu u_\eta(\mathbf{p}),$$

which is our initial expression multiplied by $\eta\eta'$, so that we must have $\eta\eta' = +1$ for a non-zero matrix element, i.e. $\eta' = \eta$. The same result holds for $\gamma^\mu\gamma_5$.

Thus for *massless* fermions, γ^μ and $\gamma^\mu\gamma_5$ *exactly* conserve chirality. The conservation of the helicity in the limit $m/E \to 0$ follows because of the identification of helicity and chirality in this limit, as shown in (4.6.58).

For fermion-antifermion annihilation or creation, i.e. matrix elements of the type $\bar{v}(\gamma^\mu$ or $\gamma^\mu\gamma_5)u$ or $\bar{u}(\gamma^\mu$ or $\gamma^\mu\gamma_5)v$ one finds that in the massless case the fermion and the antifermion must have opposite chirality, which coincides with our results (4.6.49)–(4.6.51) that the amplitude for annihilation or creation with equal helicities is $O(m/E)$ compared with the opposite helicity case.

These results will play a seminal rôle in our study of the electroweak theory and QCD, where the couplings are just γ^μ and $\gamma^\mu\gamma_5$. (A more general version of these results is given in Section 10.4.) Note, for comparison, that the other couplings (I, γ_5, $\sigma^{\mu\nu}$) can be shown to flip helicity in the limit $m/E \to 0$, in contrast to (4.6.42).

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