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HOW TO OBTAIN BATHTUB-SHAPED FAILURE RATE MODELS FROM NORMAL MIXTURES

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We obtain some techniques to study the shape of reliability functions (failure rate, mean residual life, etc.) by using the s-equilibrium distribution of a renewal process defined by Fagiuoli and Pellerey (Naval Res. Logist., 1993). We apply these techniques to study how to obtain distributions with bathtub shaped failure rate (BFR) from mixtures of two positive truncated normal distributions.

1. INTRODUCTION

In reliability theory and survival analysis, a positive random variable X usually represents the life length of a unit or a component in a system. Let us suppose that X is an absolutely continuous random variable with density function $f(t)$. In this context, the distribution function $F(t) = \Pr(X \le t)$ represents the probability of failure before time t and the reliability function $R(t) = \Pr(X \ge t)$ represents the probability of correct functioning at time t.

The most used functions to describe the aging of the units are the failure rate $r(t) = f(t)/R(t)$ and the mean residual life $\mu(t) = E(X - t | X \ge t)$. It is well known that both of them uniquely determine the distribution function (i.e., they have all of the information about the model). For example, the inversion formula for $r(t)$ is

$$R(t) = \exp\left(-\int_0^t r(x)\,dx\right). \tag{1}$$

These functions are also used to compare and classify different units. The definitions for the likelihood ratio ($\leq_{\mathrm{lr}}$), failure rate ($\leq_{\mathrm{fr}}$), and mean residual life ($\leq_{\mathrm{mrl}}$) orders and increasing (decreasing) failure rate (IFR (DFR)) and decreasing (increasing) mean residual life (DMRL (IMRL)) classes can be found in [20]. It is well known that

$$X \leq_{\mathrm{lr}} Y \Rightarrow X \leq_{\mathrm{fr}} Y \Rightarrow X \leq_{\mathrm{mrl}} Y$$

and

$$X \text{ IFR (DFR)} \Rightarrow X \text{ DMRL (IMRL)}$$

Throughout, increasing (decreasing) means nondecreasing (nonincreasing).

In this context, we use positive random variables and, hence, the Normal distribution $N(\mu,\sigma)$ is replaced by the positive truncated Normal distribution $N^+(\mu,\sigma)$, defined by the density

$$f(t) = c\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) \quad \text{for } t > 0,$$

where

$$c = c(\mu,\sigma) = \left(\int_0^\infty \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx\right)^{-1} = \frac{1}{\sqrt{2\pi\sigma^2}}\,\frac{1}{\Phi(\mu/\sigma)}$$

and $\Phi(x)$ is the standard Normal distribution. Note that if $X \equiv N^+(\mu,\sigma)$, then $E(X) > \mu$ and $\mathrm{Var}(X) < \sigma^2$. If $\mu + 3\sigma \gg 0$, then $c \cong 1/\sqrt{2\pi\sigma^2}$, $E(X) \cong \mu$, and $\mathrm{Var}(X) \cong \sigma^2$. The failure rate for the positive truncated Normal coincides with the failure rate for the Normal distribution (Mill's ratio) for $t > 0$ and, hence, it is increasing.

However, in practice, the failure rates estimated from datasets have a bathtub shape, denoted by BFR; that is, they first strictly decrease from 0 to t_1, then they are constant from t_1 to t_2 ($t_1 \leq t_2$), and, finally, they increase from t_2 to ∞. The failures in the first period are called early failures and they are due to manufacturing defects. The failures in the second period (useful life period) are called chance failures and the failures in the last period (wear out) are due to the aging process. The times t_1 and t_2 are called change points.

Analogously, a model has an upside-down failure rate (UFR) if $r(t)$ strictly increases from 0 to t_1, it is constant from t_1 to t_2 ($t_1 \leq t_2$) and it decreases from t_2 to ∞. Similar classes (BMRL and UMRL) can be defined for the mean residual life. We use the following notation. A model is IBFR when r increases in $(0, t_0)$ and it has a bathtub shape in (t_0,∞). Analogously, a model is BBFR when it is BFR in $(0, t_0)$ and BFR in (t_0,∞).

In the last 20 years, many theoretical models with BFRs have been proposed from different methods (see the reviews in [14,17]). In this article, we show that BFRs can be obtained from a mixture of two positive truncated normal models with IFRs. In our opinion, this is the correct way to obtain a BFR distribution because it explains why this shape arises naturally from a mixture of two different kind of units (units with and without manufacturing defects).

Few articles study how to obtain BFR distributions from mixtures. Krohn [13] proposed obtaining BFR models from a mixture of three Weibull distributions (a DFR Weibull for the early failures, an exponential for the useful life period, and an IFR Weibull for the wear period). Vaupel and Yashin [22] obtained plots of BFR models from mixtures of an exponential and a model with linear failure rate. Unfortunately, these results are not true when $t \to \infty$ since the failure rate of the mixture is equal to the failure rate of strong units when $t \to \infty$ (see [3]). Glaser [8], Gupta and Warren [9], and Wondmagegnehu [23] studied special cases of Weibull and gamma mixtures. Other interesting articles that study, in general, the shape of the failure rate obtained from a mixture are [1,3,4,10,15,21].

The article is organized as follows. In Section 2, we introduce the model. In Sections 3 and 4, we obtain some general techniques to determine the shape of $r(t)$ and $\mu(t)$ and then, in Section 5, we study the mixtures of positive truncated Normal distributions. Finally, in Sections 6 and 7, we give some examples and remarks on this model from a practical point of view.

2. THE MODEL

Let us suppose that an absolutely continuous positive random variable X_p is a mixture of two random variables X_1 and X_0. Hence, the density f_p is a mixture of two densities f_1 and f_0,

$$f_p(t) = pf_1(t) + (1-p)f_0(t) \quad \text{for } t \geq 0, \tag{2}$$

where $0 \leq p \leq 1$, the reliability (or survival function) is also a mixture,

$$R_p(t) = pR_1(t) + (1-p)R_0(t),$$

and the failure rate and the mean residual life are dynamic mixtures,

$$r_p(t) = p(t)r_1(t) + (1-p(t))r_0(t), \tag{3}$$

$$\mu_p(t) = p(t)\mu_1(t) + (1-p(t))\mu_0(t), \tag{4}$$

where

$$0 \leq p(t) = \frac{pR_1(t)}{pR_1(t) + (1-p)R_0(t)} \leq 1 \tag{5}$$

Hence,

$$\min\{r_1(t), r_0(t)\} \leq r_p(t) \leq \max\{r_1(t), r_0(t)\}. \tag{6}$$

Moreover, from [3], we have, under some mild conditions,

$$\lim_{t\to\infty} r_p(t) = \lim_{t\to\infty} \min(r_1(t), r_0(t)). \tag{7}$$

It is well known that if X_0 and X_1 are DFR, then X_p is also DFR. However, the result is not true for IFR distributions.

If X_1 represents correct manufactured units and X_0 represents units with manufacturing defects, then the following practical assumptions can be made:

(i) $p > p_0$ (e.g., $p > 0.5$);
(ii) $X_1 \geq_{\text{fr}} X_0$;
(iii) X_1 and X_0 are IFR;
(iv) $r_1(0) < r_0(0)$.

Do these conditions imply a BFR mixture? In general, the answer to this question is negative, but we are interested in studying it in mixtures of positive truncated normal models. Note that condition (ii) implies $R_1(t) \geq R_0(t)$ and $E(X_1) \geq E(X_0)$. Condition (iv) implies $f_0(0) > 0$. Note that in the IFR Weibull model, this condition does not hold, and, hence, we cannot obtain a BFR mixture.

3. GENERAL RESULTS

Glaser [8] used the function $\eta(t) = -f'(t)/f(t)$ to study the failure rate shape. Glaser's result is very useful since, in many models, it is easier to study $\eta(t)$ than $r(t)$. For example, in the normal model, $r(t)$ does not have an explicit expression but $\eta(t) = (t-\mu)/\sigma^2$, and hence, from Glaser's result, $r(t)$ is increasing. The function $\eta(t)$ can be also used to characterize the likelihood ratio order ($X \leq_{\text{lr}} Y \Leftrightarrow \eta_X(t) \geq \eta_Y(t)$) and the increasing and decreasing likelihood ratio classes (ILR, DLR) by the monotonicity $\eta(t)$. The ILR class is usually defined by the logconcavity of the density function (see [20, p. 405]) and it is equivalent to the class of PF_2 densities defined in [2]. From Glaser's result, we have that ILR (DLR) implies IFR (DFR). Analogously, we can define the BLR class when $\eta(t)$ has a bathtub shape, and the ULR, IBLR, DULR, BBLR, and UULR classes (we use the same notation for the classes defined by $\eta(t)$ as that used for the classes defined by $r(t)$ or $\mu(t)$).

Recently, Gupta and Warren [9] have extended Glaser's result, showing that $r'(t) = 0$ has at most one solution on the closed interval $[z_{k-1}, z_k]$, where $z_0 = 0 < z_1 < \cdots < z_n$ are the zeros of $\eta'(t)$. They also showed that $r'(t) = 0$ does not have any solution in (z_n, ∞). In Theorem 4.3 of [9] they extend Glaser's result, showing that IBLR (DULR) implies IFR, BFR, or IBFR (DFR, UFR, or DUFR). Similar results were obtained independently by Block, Savits, and Singh [5]. We have extended this result in the following lemma.

LEMMA 1: *If $r'(t)$ is continuous in $(0,\infty)$ and $r(t)$ strictly increases (decreases) in (a,b) and strictly decreases (increases) in (b,c), then $\eta(t)$ strictly increases (decreases) at b.*

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The proof is obtained from

$$r'(t) = r(t)(r(t) - \eta(t)) = r^2(t) + \frac{f'(t)}{R(t)} \tag{8}$$

and

$$\lim_{t\to\infty} \frac{1}{\mu(t)} = \lim_{t\to\infty} r(t) = \lim_{t\to\infty} \eta(t). \tag{9}$$

Thus, if X is BBLR, then it is IFR, BFR, IBFR, or BBFR (i.e., r has an easier shape than η with the same monotonicity at the end).

The equilibrium distribution of a renewal process associated with a positive random variable X with mean μ is determined by the density

$$f^*(t) = \frac{R(t)}{\mu} \quad \text{for } t > 0.$$

We denote by X^* a random variable having this density. It is well known (see, e.g., [6,11]) that $r^*(t) = 1/\mu(t)$. Moreover, we note that $\eta^*(t) = r(t)$.

If $E(X^s) < \infty$, Fagiuoli and Pellerey [6], defined the s-equilibrium distribution of a renewal process associated to X as the distribution of $X^{*\overset{s}{\cdots}*}$. If $X_{(s)}$ denotes a random variable having the s-equilibrium distribution, then $X_{(0)} =_{\text{st}} X$, $X_{(1)} =_{\text{st}} X^*$, $X_{(2)} =_{\text{st}} X^{**}$, and so forth, where $=_{\text{st}}$ denotes equality in law. They also defined the s-CLASS by

$$X \text{ s-CLASS} \Leftrightarrow X_{(s)} \text{ CLASS}.$$

For example, 1-IFR = DMRL. Analogously, the s-order is defined by

$$X \leq_{\text{s-order}} Y \Leftrightarrow X_{(s)} \leq_{\text{order}} Y_{(s)}$$

and the s-function by

$$s - \mathit{function}_X(t) = \mathit{function}_{X_{(s)}}(t).$$

For example, the 1-lr order is the hr order, the 2-lr order is the mrl order, and the 1-failure rate is $r_{(1)}(t) = 1/\mu(t)$. Analogously, we note that if $E(X^s) < \infty$ for $s = 2,3,\ldots,$ then

$$\eta_{(s)}(t) = r_{(s-1)}(t) = \frac{1}{\mu_{(s-2)}(t)} \tag{10}$$

and

$$X \leq_{\text{s-lr,st,mrl}} Y \Rightarrow X \leq_{(s+1)\text{-lr,st,mrl}} Y.$$

Hence, Glaser's and Gupta and Warren's results can be applied to $\eta_{(s)}(t)$ and $r_{(s)}(t) = \eta_{(s+1)}(t)$, obtaining respectively the two following theorems.

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THEOREM 2: *If* $E(X^{s+1}) < \infty$ *for* $s = 0,1,2,\ldots$, *then*

1. $\eta_{(s)}$ *increasing (decreasing)* $\Rightarrow \eta_{(s+1)}$ *increasing (decreasing);*
2. $\eta_{(s)}$ *bathtub (upside-down)* $\Rightarrow \eta_{(s+1)}$ *bathtub or increasing (upside-down or decreasing).*

THEOREM 3: *If* $E(X^{s+1}) < \infty$ *for* $s = 0,1,2,\ldots$, *then* $\eta'_{(s+1)}(t) = 0$ *has at most one solution on the closed interval* $[z_{k-1}, z_k]$, *where* $z_0 = 0 < z_1 < \cdots < z_n$ *are the zeros of* $\eta'_{(s)}(t)$ *and* $\eta'_{(s+1)}(t) = 0$ *does not have any solution in* (z_n,∞).

Remark 4: Note that the shape of $\eta_{(s+1)}$ ($r_{(s+1)}$ or $\mu_{(s+1)}$) is easier than $\eta_{(s)}$ ($r_{(s)}$ or $\mu_{(s)}$). For example, if $\eta_{(s)}$ is increasing (decreasing), then $\eta_{(s+k)}$ is increasing (decreasing) for $k = 1,2,\ldots$ In particular, if $s = 1$, from (10), the shape of $\mu(t)$ can be obtained from the shape of $r(t)$. For example, from Glaser's result, we obtain the well-known result *IFR* (*DFR*) $\Rightarrow$ *DMRL* (*IMRL*) and, if $r(t)$ is BFR (UFR), then $\mu(t)$ is DMRL or UMRL (IMRL or BMRL) and the change point for $\mu(t)$ is smaller than the change point for $r(t)$ (another well-known result). Theorem 3 in [16] and Theorem 2 in [7] are now immediate. Moreover, the formulas for $\eta(t)$ and $r(t)$ can be translated to $r(t)$ and $\mu(t)$. For example, (8) gives

$$\left(\frac{1}{\mu(t)}\right)' = \frac{1}{\mu(t)}\left(\frac{1}{\mu(t)} - r(t)\right). \tag{11}$$

Remark 5: If X has a decreasing density function $f(t)$, then we can define the preceding equilibrium distribution $X_{(-1)}$ by the reliability $R_{(-1)}(t) = f(t)/f(0)$. Obviously, $X^*_{(-1)} =_{\text{st}} X$. Hence, we can study the shape of $\eta(r,\mu)$ from the shape of $\eta_{(-1)}(t) = -f''(t)/f'(t)$. Analogously, if the ith derivative $f^{(i)}(t) \le 0$ for $i = 1,\ldots,s$, then we can define $X_{(-s)}$ and use $\eta_{(-s)}$ to study $\eta(r,\mu)$.

Remark 6: The results given by Rojo [18] for age-smooth distributions can be also translated to the equilibrium distributions since it is also age-smooth. For example,

$$\limsup_{t\to\infty} \frac{R_1(t)}{R_2(t)} < \infty \Rightarrow \lim_{t\to\infty} \mu_1(t) \le \lim_{t\to\infty} \mu_2(t)$$

((iii) in Theorem 2.1 in [18]) can be obtained (by using (ii) in Theorem 2.1 in [18]) from the mild condition

$$\limsup_{t\to\infty} \frac{\displaystyle\int_t^\infty R_1(u)\,du}{\displaystyle\int_t^\infty R_2(u)\,du} < \infty \Rightarrow \lim_{t\to\infty} \mu_1(t) \le \lim_{t\to\infty} \mu_2(t).$$

Lemma 2.1 in [18] can be also translated to η and r when f is decreasing and $X_{(-1)}$ is age-smooth of index $-\rho$.

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4. GENERAL RESULTS FOR MIXTURES

To apply Glaser's and Gupta and Warren's results to mixtures, we note that from (2), $\eta_p(t) = -f_p'(t)/f_p(t)$ is also a dynamic mixture:

$$\eta_p(t) = w(t)\eta_1(t) + (1 - w(t))\eta_0(t), \tag{12}$$

where

$$0 \le w(t) = \frac{pf_1(t)}{pf_1(t) + (1-p)f_0(t)} \le 1. \tag{13}$$

Moreover, we note that the equilibrium distribution of a mixture is another mixture (with different weights) of the equilibrium distributions of the components; that is,

$$f_p^*(t) = \frac{R_p(t)}{\mu_p} = p^* f_1^*(t) + (1-p^*)f_0^*(t),$$

where

$$0 \le p^* = \frac{p\mu_1}{p\mu_1 + (1-p)\mu_0} \le 1.$$

Analogously, the s-equilibrium distribution of a mixture is another mixture of the s-equilibrium distributions of the components. Hence, from the results given in the preceding section, the results obtained for the failure rate of mixtures can be translated to the mean residual life of mixtures by using (10). Analogously, the results obtained for $\eta(t)$ in mixtures can be translated to $r(t)$, $1/\mu(t)$, and, in general, $\eta_{(s)}(t)$. For example, the mixture of two IMRL distributions is also IMRL. Moreover, Theorems 2.1 and 2.2 obtained by Block and Joe [3] can be applied to $\mu(t)$, obtaining, under some conditions, the following theorem.

Theorem 7: *The asymptotic behavior of the mean residual life of the mixture is equal to that of the mean residual life of stronger components; that is, if $\mu_1(t) \ge \mu_0(t)$, then*

$$\lim_{t\to\infty} \mu_p(t)/\mu_1(t) = 1. \tag{14}$$

We use the following notation $g(t) \nearrow (\searrow)\, c$ as $t \to \infty$, when $\lim_{t\to\infty} g(t) = c$ and there exists t' such that $g(t)$ increases (decreases) for $t > t'$.

We have obtained the following results.

Proposition 8: *If $0 < p < 1$ and $r_1(t) \le r_0(t)$ for $t \ge t_1$, then the following hold:*

1. *$R_0(t)/R_1(t) \searrow K \ge 0$ as $t \to \infty$.*
2. *$p(t) \nearrow K' = p/[p + (1-p)K] \in (0,1]$ as $t \to \infty$.*
3. *If $\liminf_{t\to\infty} r_0(t)/r_1(t) > 1$, then $K = 0$ and $K' = 1$.*

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4. *If* $\lim_{t\to\infty} r_0(t)/r_1(t) = c$, *then* $\lim_{t\to\infty} f_0(t)/f_1(t) = K$ *and* $\lim_{t\to\infty} w(t) = K'$. *Moreover,* $K = 0$ *or* $c = 1$.
5. *If* $\lim_{t\to\infty} \eta_0(t)/\eta_1(t) = c'$ *and* $\lim_{t\to\infty} r_0(t)/r_1(t) = c$, *then* $\lim_{t\to\infty} f_0'(t)/f_1'(t) = K$. *Moreover,* $K = 0$ *or* $c' = 1$.
6. *If* $\mu_1(t) \geq \mu_0(t)$, $\eta_0(t)/\eta_1(t)$ *decreases to* c', *and* $f_0(t)/f_1(t)$ *decreases to* K, *then* $\eta_p(t)/\eta_1(t)$ *decreases to 1.*

PROOF: From (1), we have

$$R_0(t)/R_1(t) = \exp\left(\int_0^t (r_1(x) - r_0(x))\,dx\right).$$

If $r_1(t) \leq r_0(t)$ for $t \geq t_1$, then $R_0(t)/R_1(t)$ decreases to $K \geq 0$. Hence, we obtain item 2 from (5).

Moreover, if $\liminf_{t\to\infty} r_0(t)/r_1(t) > 1$, then there exist $d > 1$ and $t_2 > 0$ such that $r_0(t) > dr_1(t) > r_1(t)$ for $t > t_2$. Thus, we have

$$\frac{R_0(t)}{R_1(t)} \leq \exp\left(\int_0^{t_2} (r_1(x) - r_0(x))\,dx\right)\exp\left((1-d)\int_{t_2}^t r_1(x)\,dx\right).$$

From (1), it is easy to show that $\int_{t_2}^\infty r_1(x)\,dx = \infty$, and, hence, $K = 0$ and $K' = 1$.

Moreover, as

$$\frac{f_0(t)}{f_1(t)} = \frac{r_0(t)}{r_1(t)}\,\frac{R_0(t)}{R_1(t)}$$

if $\lim_{t\to\infty} r_0(t)/r_1(t) = c$, then

$$\lim_{t\to\infty} \frac{f_0(t)}{f_1(t)} = Kc$$

and, from L'Hôpital, we obtain

$$K = \lim_{t\to\infty} \frac{R_0(t)}{R_1(t)} = \lim_{t\to\infty} \frac{f_0(t)}{f_1(t)} = Kc$$

and $K = 0$ or $c = 1$. Moreover, $\lim_{t\to\infty} w(t) = K'$.

To obtain item 5, we note that

$$\lim_{t\to\infty} \frac{f_0'(t)}{f_1'(t)} = \lim_{t\to\infty} \frac{f_0(t)}{f_1(t)}\,\frac{\eta_0(t)}{\eta_1(t)} = Kc'$$

and

$$K = \lim_{t\to\infty} \frac{f_0(t)}{f_1(t)} = \lim_{t\to\infty} \frac{f_0'(t)}{f_1'(t)} = Kc'$$

hold. Moreover, item 6 is obtained from

$$\frac{\eta_p(t)}{\eta_1(t)} = 1 + (1 - w(t))\left(\frac{\eta_0(t)}{\eta_1(t)} - 1\right). \qquad \blacksquare$$

Remark 9: If $r_1(t) \le r_0(t)$ for all t, then $R_0(t)/R_1(t)$ decreases and $p(t)$ increases for all t. Moreover, note that

$$\frac{r_p(t)}{r_1(t)} = 1 + (1 - p(t))\left(\frac{r_0(t)}{r_1(t)} - 1\right) \qquad \textbf{(15)}$$

holds, and, hence, if r_0/r_1 decreases, then r_p/r_1 decreases to 1 for all $p > 0$. This result was given by Block and Joe [3]. Also note that if r_0/r_1 decreases, then f_0/f_1 decreases and w increases. Block and Joe's result can be also obtained from item 6 and (10). Analogously, from Section 3 results, we obtain that if $\mu_0(t) \le \mu_1(t)$ and $\mu_0(t)/\mu_1(t)$ increases, then $\mu_p(t)/\mu_1(t)$ increases to 1.

Moreover, we have the following result.

PROPOSITION 10: *If (2) holds, then*

$$r_p'(t) = p(t)r_1'(t) + (1 - p(t))r_0'(t) - p(t)(1 - p(t))(r_1(t) - r_0(t))^2. \qquad \textbf{(16)}$$

PROOF: From (3) and (5), we have

$$r_p'(t) = p'(t)(r_1(t) - r_0(t)) + p(t)r_1'(t) + (1 - p(t))r_0'(t)$$

and

$$p'(t) = \frac{p(1-p)R_0(t)R_1(t)}{(pR_1(t) + (1-p)R_0(t))^2}\,(r_0(t) - r_1(t))$$

and, hence, (16) holds. $\blacksquare$

Remark 11: We note that expression (3.2) in Gupta and Warren [9] is wrong. In particular, from (16), we obtain the following well-known result: If both X_1 and X_0 are DFR, then the mixture is DFR. We can obtain similar results for $\eta(t)$, $\mu(t)$, and, in general, $\eta_{(s)}(t)$.

5. MIXTURES OF POSITIVE TRUNCATED NORMAL DISTRIBUTIONS

First, we give some properties for positive truncated Normal distributions. Some of these properties are also true for (untruncated) Normal distributions.

PROPOSITION 12. *If $X \equiv N^+(\mu,\sigma)$, then the following hold:*

1. $\eta(t) = (t - \mu)/\sigma^2$.
2. $r'(t) = (r(t) - (t - \mu)/\sigma^2)r(t)$.
3. $r(t) > (t - \mu)/\sigma^2$.
4. $E(X) = \mu + \sigma^2 r(0)$.
5. *$r(t)$ increases to ∞ as $t \to \infty$.*

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6. $\mu(t)$ *decreases to* 0 *as* $t \to \infty$.
7. $\lim_{t\to\infty} r(t)/t = 1/\sigma^2$.
8. $r(t) - (t-\mu)/\sigma^2$ *decreases to* 0 *as* $t \to \infty$.
9. $\lim_{t\to\infty} t\mu(t) = \sigma^2$.
10. $\lim_{t\to\infty}(1/\mu(t)) - (t-\mu)/\sigma^2 = 0$.
11. $\lim_{t\to\infty} r'(t) = 1/\sigma^2$.

PROOF: From the definitions and (8), items 1 and 2 are immediate. Moreover, from Kotz and Shanbhag [12] (see also [19]), we have

$$\mu(t) = \mu - t + \sigma^2 r(t) > 0 \tag{17}$$

for $t \geq 0$, and, hence, items 3 and 4 hold.

From items 2 and 3, $r(t)$ increases and $\mu(t)$ decreases for all t. Moreover, from item 3 and (9), $\lim_{t\to\infty} r(t) = \infty$ and $\lim_{t\to\infty} \mu(t) = 0$. Hence,

$$\lim_{t\to\infty} \frac{r(t)}{t} = \lim_{t\to\infty} \frac{f(t)}{tR(t)} = \lim_{t\to\infty} \frac{f'(t)}{R(t) - tf(t)} = \lim_{t\to\infty} \frac{\eta(t)}{t - 1/r(t)} = \frac{1}{\sigma^2}.$$

Analogously, from (17), we obtain

$$r(t) - \frac{t-\mu}{\sigma^2} = \frac{\mu(t)}{\sigma^2},$$

which decreases to 0 as $t \to \infty$.

Applying L'Hôpital, we have

$$\lim_{t\to\infty} \frac{1}{t\mu(t)} = \lim_{t\to\infty} \frac{t^{-1}R(t)}{\displaystyle\int_t^\infty R(x)\,dx} = \lim_{t\to\infty} \frac{1}{t^2} + \frac{r(t)}{t} = \frac{1}{\sigma^2}$$

and, hence, item 9 holds.

Analogously, from (17), we obtain

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{\mu(t)} - \frac{t}{\sigma^2} &= \lim_{t\to\infty} \frac{\sigma^2 R(t) - t\displaystyle\int_t^\infty R(x)\,dx}{\sigma^2 \displaystyle\int_t^\infty R(x)\,dx} \\
&= \lim_{t\to\infty} \frac{\sigma^2 f(t) - tR(t) + \displaystyle\int_t^\infty R(x)\,dx}{\sigma^2 R(t)} \\
&= \lim_{t\to\infty} \frac{\sigma^2 r(t) + \mu(t) - t}{\sigma^2} \\
&= \lim_{t\to\infty} \frac{2\mu(t) - \mu}{\sigma^2} \\
&= -\frac{\mu}{\sigma^2}
\end{aligned}$$

and, hence, item 10 holds. Finally, from item 2 and (17), we have

$$r'(t) = \frac{r(t)\mu(t)}{\sigma^2}$$

and, hence,

$$\lim_{t\to\infty} r'(t) = \lim_{t\to\infty} \frac{r(t)}{t\sigma^2}\, t\mu(t) = \frac{1}{\sigma^2}$$

holds. ■

Remark 13: Note that the asymptotic behavior of a normal failure rate is equivalent to a linear failure rate. If $g(t) = r(t) - (t-\mu)/\sigma^2$ and $r^*(t)$ is the failure rate of a standard normal, then $g(\mu + k\sigma) = (r^*(t) - k)/\sigma$. Hence, for $k = 3$, we obtain $g(t) \le 0.283099/\sigma$ for $t \ge \mu + 3\sigma$.

We consider now a mixture of two positive truncated Normal distributions $X_i \equiv N^+(\mu_i, \sigma_i)$, $i = 0,1$. First, we note that the failure rate of the mixture of truncated Normal distributions is not equal to the failure rate of a mixture of (untruncated) Normal distributions in $t > 0$. We have obtained the following properties.

PROPOSITION 14: *If X_p is a mixture of $X_0 \equiv N^+(\mu_0, \sigma_0)$ and $X_1 \equiv N^+(\mu_1, \sigma_1)$, with $0 < p < 1$ and $X_1 \ge_{\text{fr}} X_0$, then the following hold:*

1. $\sigma_1^2 \ge \sigma_0^2$.
2. $\lim_{t\to\infty} r_0(t)/r_1(t) = \sigma_1^2/\sigma_0^2 \ge 1$.
3. *If $\sigma_1^2 > \sigma_0^2$, then $R_0(t)/R_1(t)$ decreases to 0 and $p(t)$ increases to 1 as $t \to \infty$.*
4. *If $\sigma_1^2 > \sigma_0^2$, then $f_0(t)/f_1(t) \searrow 0$ and $w(t) \nearrow 1$ as $t \to \infty$.*
5. *If $\sigma_1^2 = \sigma_0^2$, then $\mu_0 < \mu_1$, $f_0(t)/f_1(t)$ decreases to 0, $w(t)$ increases to 1, $R_0(t)/R_1(t)$ decreases to 0, and $p(t)$ increases to 1 as $t \to \infty$.*
6. *If $(t-\mu_1)/\sigma_1^2 < (t-\mu_0)/\sigma_0^2$, then $w(t)$ increases.*
7. *If $\mu_0 < \mu_1$ $(>)$, then $r_0(t)/r_1(t) \searrow \sigma_1^2/\sigma_0^2$ $(\nearrow)$ and $r_p(t)/r_1(t) \searrow 1$ as $t \to \infty$.*
8. $\eta_p'(t) = w(t)(1/\sigma_1^2) + (1-w(t))(1/\sigma_0^2) - w(t)(1-w(t))\big((t-\mu_1)/\sigma_1^2 - (t-\mu_0)/\sigma_0^2\big)^2$.
9. $\eta_p(t) \nearrow \infty$ *as* $t \to \infty$.

PROOF: Items 1–3 can be obtained from Propositions 8 and 12. Items 4–6 can be obtained from

$$\frac{f_0(t)}{f_1(t)} = \frac{c_0}{c_1} \exp\left(\frac{(t-\mu_1)^2}{2\sigma_1^2} - \frac{(t-\mu_0)^2}{2\sigma_0^2}\right).$$

To obtain item 7, we note that the asymptotic behavior of r_0/r_1 is equal to that of

$$\frac{t-\mu_0}{t-\mu_1}\,\frac{\sigma_1^2}{\sigma_0^2},$$

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which decreases (increases) to σ_1^2/σ_0^2 when $\mu_0 < \mu_1$ (>). The property for r_p/r_1 is obtained from (15). Moreover, from (12),

$$\eta_p(t) = w(t)\left(\frac{t-\mu_1}{\sigma_1^2} - \frac{t-\mu_0}{\sigma_0^2}\right) + \frac{t-\mu_0}{\sigma_0^2},$$

where

$$w(t) = \frac{1}{1+\alpha(t)},$$

$$\alpha(t) = \frac{1-p}{p}\frac{f_0(t)}{f_1(t)} = \frac{1-p}{p}\frac{c_0}{c_1}\exp\left(\frac{(t-\mu_1)^2}{2\sigma_1^2} - \frac{(t-\mu_0)^2}{2\sigma_0^2}\right) \geq 0$$

and

$$\alpha'(t) = \alpha(t)\left(\frac{t-\mu_1}{\sigma_1^2} - \frac{t-\mu_0}{\sigma_0^2}\right).$$

Thus, differentiating, we have

$$\eta_p'(t) = -\frac{\alpha(t)}{(1+\alpha(t))^2}\left(\frac{t-\mu_1}{\sigma_1^2} - \frac{t-\mu_0}{\sigma_0^2}\right)^2 + w(t)\frac{1}{\sigma_1^2} + (1-w(t))\frac{1}{\sigma_0^2}$$

and, hence, result 8 holds. Moreover, $\lim_{t\to\infty}\eta_p'(t) = 1/\sigma_1^2 > 0$ because

$$\lim_{t\to\infty}(1-w(t))\left(\frac{t-\mu_1}{\sigma_1^2} - \frac{t-\mu_0}{\sigma_0^2}\right)^2 = 0.$$

Thus, result 9 holds. ■

COROLLARY 15: *If X_p is a mixture of $X_0 \equiv N^+(\mu_0,\sigma_0)$ and $X_1 \equiv N^+(\mu_1,\sigma_1)$, with $0 < p < 1$, $\sigma_1 = \sigma_0$, and $\delta = \sigma_0^2/(\mu_0-\mu_1)^2$, then*

1. *If $\delta > \frac{1}{4}$, then X_p is IFR.*
2. *If $\delta \leq \frac{1}{4}$, $w(0) \geq \frac{1}{2}$, and $w(0)(1-w(0)) < \delta$, then X_p is IFR.*
3. *If $\delta \leq \frac{1}{4}$, $w(0) \geq \frac{1}{2}$, and $w(0)(1-w(0)) \geq \delta$, then X_p is IFR or BFR.*
4. *If $\delta \leq \frac{1}{4}$, $w(0) < \frac{1}{2}$, and $w(0)(1-w(0)) > \delta$, then X_p is IFR or BFR.*
5. *If $\delta \leq \frac{1}{4}$, $w(0) < \frac{1}{2}$, and $w(0)(1-w(0)) \leq \delta$, then X_p is IFR, BFR, or IBFR.*

Moreover, the change points of η_p are determined by

$$w(t)(1-w(t)) = \delta.$$

PROOF: If $\sigma_1 = \sigma_0$, then from the preceding proposition,

$$\sigma_0^2\eta_p'(t) = 1-(1-w(t))w(t)\frac{(\mu_0-\mu_1)^2}{\sigma_0^2}$$

and $\eta_p'(t) \geq 0$ if and only if $(1-w(t))w(t) \leq \delta$.

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As $x(1-x) \leq \frac{1}{4}$ for $0 < x < 1$, then $\eta_p'(t) > 0$ when $\delta > \frac{1}{4}$. Hence, from Glaser's result, the mixture is IFR. As $w(t)$ increases, the same result holds when $\delta \leq \frac{1}{4}$, $w(0) \geq \frac{1}{2}$, and

$$w(0)(1 - w(0)) < \delta.$$

If $\delta \leq \frac{1}{4}$, $w(0) \geq \frac{1}{2}$, and

$$w(0)(1 - w(0)) \geq \delta,$$

then there exist $z_1 > 0$ such that $w(z_1)(1 - w(z_1)) = \delta$, $\eta_p'(z_1) = 0$, $\eta_p'(t) < 0$ for $0 < t < z_1$ and $\eta_p'(t) > 0$ for $t > z_1$. Thus, from Glaser's result, X_p is BFR or IFR.

The same result holds when $\delta \leq \frac{1}{4}$, $w(0) < \frac{1}{2}$, and

$$w(0)(1 - w(0)) > \delta.$$

If $\delta \leq \frac{1}{4}$, $w(0) < \frac{1}{2}$, and

$$w(0)(1 - w(0)) \leq \delta,$$

then there exist $z_1 < z_2$ such that $w(z_i)(1 - w(z_i)) = \delta$, $i = 1{,}2$, $\eta_p'(t) > 0$ for $0 < t < z_1$, $\eta_p'(t) < 0$ for $z_1 < t < z_2$, and $\eta_p'(t) > 0$ for $t > z_1$. From Theorem 4.3 in [9], X_p is IFR, BFR, or IBFR. ■

We have obtained a general result when the variances are not equal and $\eta_0(t) \geq \eta_1(t)$.

Corollary 16: *If X_p is a mixture of $X_0 \equiv N^+(\mu_0, \sigma_0)$ and $X_1 \equiv N^+(\mu_1, \sigma_1)$, with $0 < p < 1$, $\sigma_1 > \sigma_0$, and $\eta_0(t) \geq \eta_1(t)$ for $t > 0$, then the following hold:*

1. *If $w(0) \geq x_1$ and $\gamma(0) \geq 0$, then X_p is IFR.*
2. *If $w(0) \geq x_1$ and $\gamma(0) < 0$, then X_p is IFR or BFR.*
3. *If $w(0) < x_1$ and $\gamma(t_1) \geq 0$, then X_p is IFR.*
4. *If $w(0) < x_1$ and $\gamma(t_1) < 0$, then X_p is IFR, BFR, or IBFR,*

where

$$0 < x_1 = \frac{a - \sqrt{a^2 - ab + b^2}}{a - b} < 1,$$

$$a = \frac{1}{\sigma_1^2}, \qquad b = \frac{1}{\sigma_0^2},$$

$$\gamma(t) = \frac{w(t)/\sigma_1^2 + (1 - w(t))/\sigma_0^2}{w(t)(1 - w(t))} - \left(\frac{t - \mu_1}{\sigma_1^2} - \frac{t - \mu_0}{\sigma_0^2}\right)^2,$$

and t_1 is uniquely determined by $w(t_1) = x_1$.

Proof: First, we note that $\eta_0(t) \geq \eta_1(t)$ ($X_1 \geq_{\mathrm{lr}} X_0$) implies $X_1 \geq_{\mathrm{fr}} X_0$. Hence, from the preceding proposition, $w(t)$ increases, $\lim_{t\to\infty} w(t) = 1$, and $\eta_p'(t) \geq 0$ ($\leq$) if and only if $\gamma(t) \geq 0$ ($\leq$). Thus,

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$$\gamma'(t) = \frac{\eta_0(t) - \eta_1(t)}{w(t)(1 - w(t))}\, p(w(t)), \tag{18}$$

where

$$p(x) = (b - a)x^2 + 2ax - b$$

and $0 < a = 1/\sigma_1^2 < b = 1/\sigma_0^2$. Thus, $p(0) = -b$ and $p(1) = a$, and $p(x) = 0$ has a unique solution x_1 in (0,1). Hence, as w increases, if $w(0) \geq x_1$, then $\gamma(t)$ increases for all t and we have properties 1 and 2. If $w(0) < x_1$, then $\gamma(t)$ has a minimum at $t_1 > 0$, where t_1 is uniquely determined by $w(t_1) = x_1$, and we obtain properties 3 and 4. ■

The following corollary completes the possible cases.

Corollary 17: *If X_p is a mixture of $X_0 \equiv N^+(\mu_0, \sigma_0)$ and $X_1 \equiv N^+(\mu_1, \sigma_1)$, with $0 < p < 1$, $\sigma_1 > \sigma_0$ and*

$$t_0 = \frac{\sigma_1^2 \mu_0 - \sigma_0^2 \mu_1}{\sigma_1^2 - \sigma_0^2} > 0,$$

then the following hold:

1. *If $w(t_0) \geq x_1$, then X_p is IFR.*
2. *If $w(t_0) < x_1$, $w(0) < x_1$, $\gamma(0) \geq 0$, and $\gamma(t_1) \geq 0$, then X_p is IFR.*
3. *If $w(t_0) < x_1$, $w(0) < x_1$, $\gamma(0) < 0$, and $\gamma(t_1) \geq 0$, then X_p is IFR or BFR.*
4. *If $w(t_0) < x_1$, $w(0) < x_1$, $\gamma(0) \geq 0$, and $\gamma(t_1) < 0$, then X_p is IFR, BFR, or IBFR.*
5. *If $w(t_0) < x_1$, $w(0) < x_1$, $\gamma(0) < 0$, and $\gamma(t_1) < 0$, then X_p is IFR, BFR, IBFR, or BBFR.*
6. *If $w(t_0) < x_1$, $w(0) \geq x_1$, $\gamma(0) > 0$, $\gamma(t_1) \geq 0$, and $\gamma(t_2) \geq 0$, then X_p is IFR.*
7. *If $w(t_0) < x_1$, $w(0) \geq x_1$, $\gamma(0) \leq 0$, and $\gamma(t_2) \geq 0$, then X_p is IFR or BFR.*
8. *If $w(t_0) < x_1$, $w(0) \geq x_1$, $\gamma(0) > 0$, and $\gamma(t_i) \geq 0$ for $i = 1$ or $i = 2$, then X_p is IFR, BFR, or IBFR.*
9. *If $w(t_0) < x_1$, $w(0) \geq x_1$, $\gamma(0) \leq 0$, and $\gamma(t_2) < 0$, then X_p is IFR, BFR, IBFR, or BBFR.*
10. *If $w(t_0) < x_1$, $w(0) \geq x_1$, $\gamma(0) > 0$, $\gamma(t_1) < 0$, and $\gamma(t_2) < 0$, then X_p is IFR, BFR, IBFR, BBFR, or IBBFR.*

where t_1 and t_2 are uniquely determined by $w(t) = x_1$ and $t_1 < t_2$.

Proof: First, we note that

$$w'(t) = w(t)(1 - w(t))(\eta_0(t) - \eta_1(t)),$$

and, hence, w decreases in $(0, t_0)$ and increases in (t_0, ∞). Moreover, $0 < p < 1$ and $\sigma_1 > \sigma_0$ imply that $\lim_{t\to\infty} w(t) = 1$. Hence, taking into account that $\gamma(t_0) \geq 0$, we have the following cases:

If $w(t_0) \geq x_1$, from (18), the sign of $\gamma'(t)$ is the same as that of $\eta_0(t) - \eta_1(t)$, and we obtain item 1.

If $w(t_0) < x_1$ and $w(0) < x_1$, then $w(t) = x_1$ has a unique positive solution $t_1 > t_0$. Hence, from (18), γ increases in $(0, t_0)$, decreases in (t_0, t_1), and increases in (t_1, ∞), and we obtain items 2–6.

If $w(t_0) < x_1$ and $w(0) \geq x_1$, then $w(t) = x_1$ has two positive solutions t_1 and t_2 such that $t_1 < t_0 < t_2$. Hence, from (18), γ decreases in $(0, t_1)$, increases in (t_1, t_0), decreases in (t_0, t_2), and increases in (t_2, ∞), and we obtain items 6–10. ∎

Remark 18: Note that

$$\frac{t-\mu_0}{\sigma_0^2} = \frac{t-\mu_0}{\sigma_0^2} \Leftrightarrow t = \frac{\sigma_1^2\,\mu_0 - \sigma_0^2\,\mu_1}{\sigma_1^2 - \sigma_0^2},$$

and, hence, the preceding corollary includes all possible cases. Moreover, $r_p'(0) > 0$ $(<)$ if and only if $r_p(0) > \eta_p(0)$ $(<)$. We also note that from the results given in Section 3, we have the shape and some information about the change points of $\mu(t)$.

Moreover, we have the following properties.

PROPOSITION 19: *If X_p is a mixture of $X_0 \equiv N^+(\mu_0, \sigma_0)$ and $X_1 \equiv N^+(\mu_1, \sigma_1)$ with $0 < p < 1$ and $X_1 \geq_{\text{fr}} X_0$, then the following hold:*

1. $r_p'(t) = p(t)(r_1(t) - (t-\mu_1)/\sigma_1^2)r_1(t) + (1-p(t))(r_0(t) - (t-\mu_0)/\sigma_0^2) \times r_0(t) - p(t)(1-p(t))(r_1(t) - r_0(t))^2$
2. $r_p'(t) = (p(t)r_1(t) + (1-p(t))r_0(t))^2 - (p(t)[(t-\mu_1)/\sigma_1^2]r_1(t) + (1-p(t))[(t-\mu_0)/\sigma_0^2]r_0(t))$.
3. *If* $0 \leq t \leq \min(\mu_1, \mu_0)$, *then* $r_p'(t) > 0$.
4. $r_p(t) \nearrow \infty$ *as* $t \to \infty$.

PROOF: The proof of item 1 is immediate from (16) and item 2 can be obtained from item 1 since

$$r_p'(t) = p^2(t)r_1^2(t) + (1-p(t))^2 r_0^2(t) + 2p(t)(1-p(t))r_1(t)r_0(t)$$
$$- p(t)\,\frac{t-\mu_1}{\sigma_1^2}\,r_1(t) - (1-p(t))\,\frac{t-\mu_0}{\sigma_0^2}\,r_0(t).$$

Hence, item 3 is obtained from item 2. Moreover, as $\lim_{t\to\infty} p(t) = 1$ and $\lim_{t\to\infty} R_p(t)/R_0(t) = \infty$, then

$$\lim_{t\to\infty}(1-p(t))(r_1(t) - r_0(t))^2 = \lim_{t\to\infty} p\,\frac{(r_1(t) - r_0(t))^2}{R_p(t)/R_0(t)},$$

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which is equal to 0 ($\sigma_1 = \sigma_0$) or ∞/∞ ($\sigma_1 > \sigma_0$). In the second case, we have

$$\lim_{t\to\infty} p\,\frac{(r_1(t)-r_0(t))^2}{R_p(t)/R_0(t)} = 2p\lim_{t\to\infty}\frac{(r_1(t)-r_0(t))(r_1'(t)-r_0'(t))}{(-f_p(t)R_0(t)+f_0(t)R_p(t))/R_0^2(t)}$$
$$= 2p\lim_{t\to\infty}\frac{R_0(t)}{R_p(t)}\,\frac{(r_1(t)-r_0(t))(r_1'(t)-r_0'(t))}{r_0(t)-r_p(t)} = 0,$$

since, from Proposition 12, $r_p \to r_1$ and $r_i' \to 1/\sigma_i^2$, as $t \to \infty$. Hence, from (16)

$$\lim_{t\to\infty} r_p'(t) = \frac{1}{\sigma_1^2} > 0$$

and property 4 holds. ■

Remark 20: From the preceding proposition, if $f_0(0) \gg f_1(0) \cong 0$ (i.e., $\mu_1 + 3\sigma_1 \gg 0$), then $r_p'(0) < 0$ if and only if

$$-\frac{\mu_0}{\sigma_0^2} > (1-p)f_0(0), \tag{19}$$

which implies $\mu_0 < 0$. This condition is equivalent to comparing $\eta_0(0)$ with $r_p(0) \cong (1-p)r_0(0)$. In particular, if $\eta_0(0) \cong r_0(0)$ (or if X_0 has a linear failure rate), then (19) holds. Also note that if $\mu_0 < 0$ and $p \to 1$, then (19) holds. Note that in the case $w(0) \cong 0$, w increases and $\eta_p'(0) \cong 1/\sigma_0^2 > 0$. Hence, if (19) holds, then X_p is BFR.

Remark 21: Corollaries 15–17 can also be applied to (untruncated) Normal distributions $X_i \equiv N(\mu_i, \sigma_i)$ (the failure rate shape is the same as that of the translated models $X_i \equiv N^+(\mu_i + c, \sigma_i)$, where $c > \max(\mu_0 + 3\sigma_0, \mu_1 + 3\sigma_1)$). In this case, we have that r_p' increases from $(-\infty, \min(\mu_0, \mu_1))$. Moreover, $\eta_0(t) \geq \eta_1(t)$ implies $\sigma_0 = \sigma_1$. Thus, if $\sigma_1 = \sigma_0$ ($\mu_0 < \mu_1$), then $w(-\infty) = 0$ and X_p is IFR or IBFR. If $\sigma_1 > \sigma_0$, then $w(-\infty) = 1$ and $\gamma(-\infty) = \infty$, and, hence, X_p is IFR, IBFR, or IBBFR.

Remark 22: Corollaries 15–17 can also be used to study the shape of mixtures of linear failure rates which have the same shape as the translated models $X_i \equiv N^+(\mu_i - c, \sigma_i)$, where $c > 0$ verifies $r_i(t+c) \cong \eta_i(t+c)$ for $t > 0$ and $i = 1,2$. Note that this model includes the exponential distribution. Thus, if X_0 and X_1 are two models with linear failure rates $r_i(t) = a_i t + b_i$, for $t > 0$, where $a_i, b_i \geq 0$ and $i = 1,2$, we have the following cases. If $a_0 = a_1$ and $b_0 > b_1$, then the shape of the failure rate of the mixture is obtained from Corollary 15 by using

$$\delta = \frac{a_0}{(b_0 - b_1)^2}$$

and

$$w(0) = \frac{pb_1}{pb_1 + (1-p)b_0}.$$

Analogously, if $a_0 > a_1$ and $b_0 > b_1$, then the shape of the failure rate of the mixture is obtained from Corollary 16 by using

$$x_1 = \frac{a_1 - \sqrt{a_1^2 - a_1 a_0 + a_0^2}}{a_1 - a_0} < 1,$$

$$w(t) = \frac{pr_1(t)\exp(-a_1 t^2/2 - b_1 t)}{pr_1(t)\exp(-a_1 t^2/2 - b_1 t) + (1-p)r_0(t)\exp(-a_0 t^2/2 - b_0 t)}$$

$$\gamma(t) = \frac{a_1 w(t) + a_0(1 - w(t))}{w(t)(1 - w(t))} - (r_1(t) - r_0(t))^2.$$

Finally, if $a_0 > a_1$ and $b_0 < b_1$, then the shape of the failure rate of the mixture is obtained from Corollary 17 and $t_0 = (b_1 - b_0)/(a_0 - a_1)$.

6. EXAMPLES

In this section, we give some examples showing that all of the different shapes given in Section 5 for the failure rate can be obtained from the mixture of positive truncated Normal distributions. We pay special attention to BFR models.

Example 23: If $X_1 \equiv N^+(3,3)$ and $X_0 \equiv N^+(6,3)$, then $r_p(t)$ increases in $(0,\infty)$ for all p, since $\sigma_0 = \sigma_1 = 3$ and

$$\delta = \sigma_0^2/(\mu_0 - \mu_1)^2 = 1 > \frac{1}{4}.$$

The failure rates for $p = 0$, 0.2, 0.4, 0.6, 0.8, 1 are given in Figure 1.

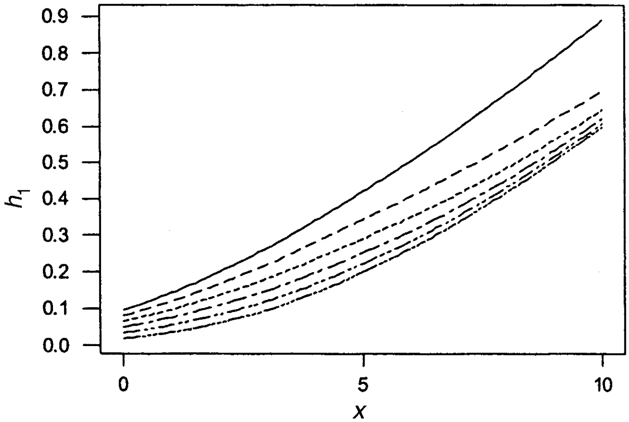

Figure 1. Failure rates for the mixture of $N^+(3,3)$ and $N^+(6,3)$ with $p = 0$, 0.2, 0.4, 0.6, 0.8, and 1.

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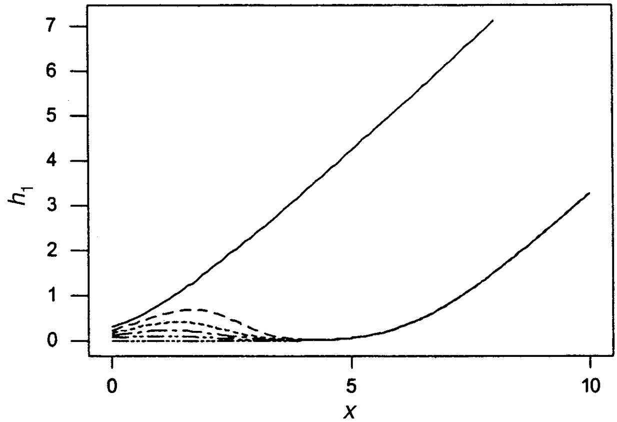

FIGURE 2. Failure rates for the mixture of $N^+(1,1)$ and $N^+(7,1)$ with $p = 0$, 0.2, 0.4, 0.6, 0.8, and 1.

Example 24: If $X_1 \equiv N^+(9,1)$ and $X_0 \equiv N^+(3,1)$, then

$$\delta = \frac{\sigma_0^2}{(\mu_0 - \mu_1)^2} = \frac{1}{36} < \frac{1}{4},$$

$$w(0) \cong 0,$$

$$w(0)(1 - w(0)) \cong 0,$$

and, hence, X_p is IFR or IBFR. The mixture failure rates are given in Figure 3. Note that we have a BFR from (c_p, ∞). Thus, a censure in $(0, c_p)$ gives practical BFR.

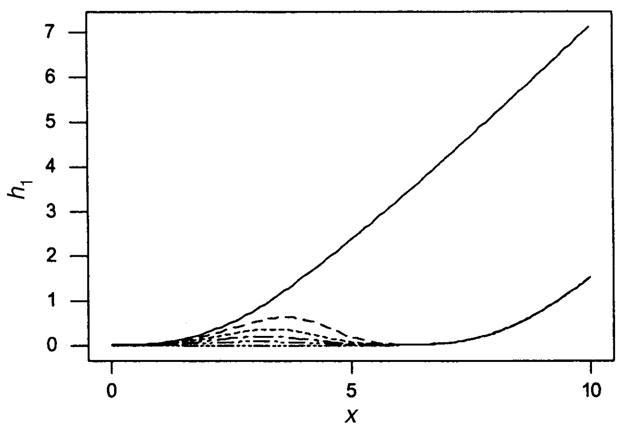

FIGURE 3. Failure rates for the mixture of $N^+(3,1)$ and $N^+(9,1)$ with $p = 0$, 0.2, 0.4, 0.6, 0.8, and 1.

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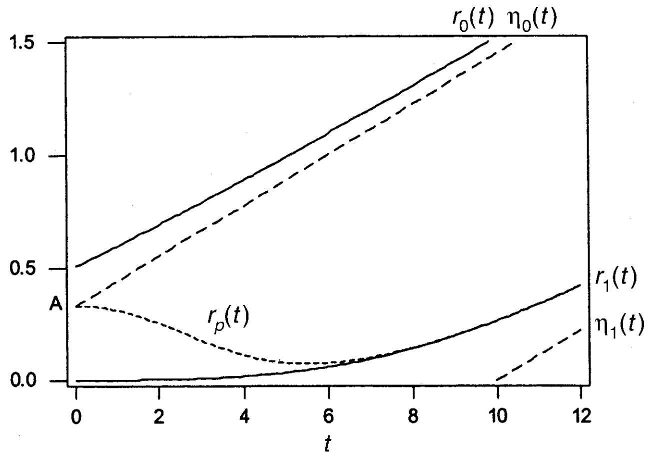

FIGURE 4. Failure rate for the mixture of $N^+(-3,3)$ and $N^+(10,3)$ with $p = 0.34432$.

This is equivalent to censuring approximately 60% of early failures represented by X_0. We obtain similar results if we consider more truncated normal distributions for early failures, but in this case, c_p is closer to zero (see Fig. 2).

Example 25: To obtain a BFR mixture, we need $\mu_0 < 0$ and (19). Thus, if we consider $N^+(-3,3)$ and $N^+(10,3)$, (19) implies $p > 0.34432$ and, in this case, we obtain a BFR mixture. If $p \leq 0.34432$, then X_p is IBFR. In Figure 4, we give r_0, r_1, η_0, η_1, and r_p for $p = 0.34432$. Note that (19) is equivalent to comparing $\eta_0(0) = \frac{1}{3}$ with $r_p(0) \cong (1-p)r_0(0) \cong (1-p)/2$ (A in Figure 4). We give the mixture failure rates for $p = 0.2, 0.4, 0.6$, and 0.8 in Figure 5. Observe that a truncated normal $N^+(-3,3)$ is very similar to a linear failure rate.

Example 26: In Figure 6, we show the failure rates obtained from a mixture of $N^+(5,1)$ and $N^+(-1,3)$ for $p = 0.2, 0.4, 0.6$, and 0.8. The mixture is BBFR for $p = 0.8$ and IBBFR for $p = 0.4$.

7. CONCLUSIONS

First, we note that to have a BFR mixture from positive truncated normal models, we need $\mu_0 < 0$. From a practical point of view, this is equivalent to assuming a censored burn-in period larger than μ_0 for the units with manufacturing defect which, in practice, is quite usual (we test at the factory until, at least, the observed mean time of failure for this kind of unit). This condition holds if we suppose a linear failure rate for X_0. Thus, under assumptions (i)–(iv) given in Section 2 and the conditions $\mu_0 < 0$, $r_1(0) \cong 0$ and (19), we obtain a BFR mixture from positive truncated normal models.

Sometimes, we have bathtub estimated failure rates from models which do not have BFR due to some practical considerations. For example, in practice, it is very

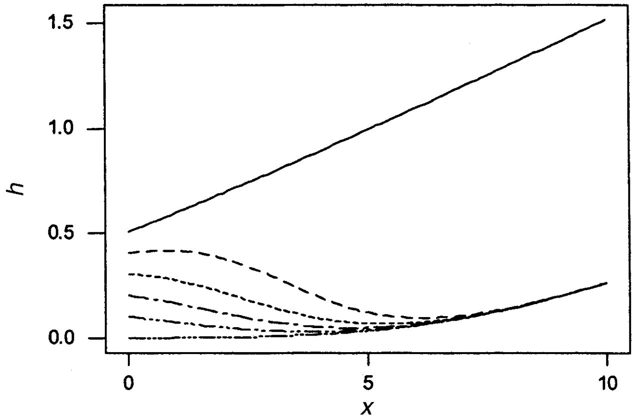

FIGURE 5. Failure rates for the mixture of $N^+(-3,3)$ and $N^+(10,3)$ with $p = 0$, 0.2, 0.4, 0.6, 0.8, and 1.

difficult to estimate $r(t)$ when $R(t)$ is small. It is also difficult to estimate $r(t)$ near $t = 0$ (especially when $\mu_0 \ll \mu_1$). Hence, models such as that of Figure 2, are, in practice, BFR.

The main conclusion is that BFR models appear naturally from mixtures of usual IFR models. We think that this is the correct way to obtain the BFR model, since it explains the reason for the shape of the failure rate from the use of two different populations. Moreover, we can use the extensive literature on this topic.

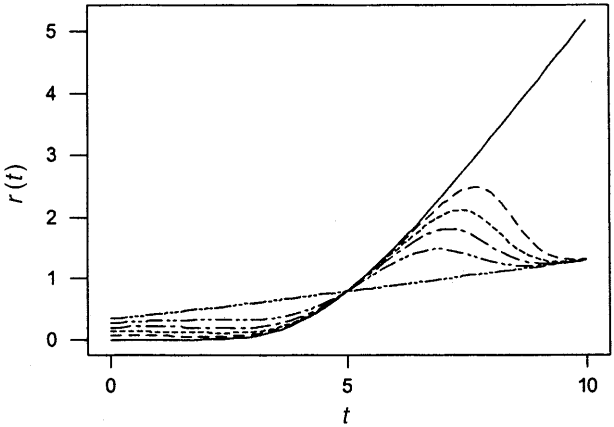

FIGURE 6. Failure rates for the mixture of $N^+(5,1)$ and $N^+(-1,3)$ with $p = 0, 0.2$, 0.4, 0.6, 0.8, and 1.

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