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THE p-ADIC GROSS–ZAGIER FORMULA ON SHIMURA CURVES, II: NONSPLIT PRIMES – CORRIGENDUM

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The approximation argument used to prove the decay of intersection multiplicities is flawed. In this correction, we give an alternative argument in a similar spirit, based on an explicit form of the approximation that we deduce from [Dis22]. This argument requires some bounds on the ramification so that the main theorem is weakened.

Referring to the paragraph *The nonsplit case* in §1.1,[1] our general approximation result involved *local* arithmetic intersections, and so it does not imply the vanishing of *global* intersections with flat divisors in a proper local integral model. Instead, we revisit an idea of Perrin-Riou and apply an operator "$U_p - 1$". Since this acts as a difference operator on the Fourier coefficients of our generating series, we obtain the vanishing (up to multiples of p^s) once we prove, by inspection, that the relevant sequences of approximating vertical components are constant in the index s.

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[1] All references not accompanied by an external citation point to the original paper [Dis23], *except* references using a single number (e.g. "Proposition 3"), which are internal to the present correction.

Corrected statement

We denote by $S_{p,\mathrm{ns}}$ the set of places of F above p that are nonsplit in E. In Theorem B, the assumption that χ_p *is sufficiently ramified* should be replaced by the following assumption:

$$\textit{for each } v \in S_{p,\mathrm{ns}},\ v \textit{ is inert and } \chi_v \textit{ is unramified.} \tag{1}$$

Remark 1. It should be possible to prove the theorem also (at least) in the case where at some nonsplit places $v|p$, the representation π_v is unramified and χ_v is arbitrary. While in principle not more difficult than the case treated here, this case would require introducing a larger number of changes in the setup, making for a cumbersome text. We thus prefer to defer it to a future work under a different global approach.

The mistake

It occurs in Proposition 4.3.3, whose proof (with notation as in *loc. cit.*) correctly shows that

$$(\overline{z}\cdot D) = c[\kappa(y):\kappa]q_{F,v}^{s} + \rho(V\cdot D')_y. \tag{2}$$

The term $(V\cdot D')_y$ is a local intersection multiplicity at y, and it is not necessarily equal to the global intersection $(V\cdot D)$ on $\mathscr{X}$. Therefore, the corresponding terms in the formula displayed in the proof of Corollary 4.3.5 do not necessarily vanish, as the definition of flat extensions invoked in that proof only applies to global intersection pairings.

Correction

We explain the strategy to prove the statement under the hypothesis (1).

Setup. We discard Assumption 3.4.1 on χ_p; as in the corrected statement, we assume instead that v is inert and χ_v is unramified for all $v \in S_{p,\mathrm{ns}}$. We suppose that (ϕ, U) satisfy the assumptions of [Dis17, §6.1] as well as Assumption 3.4.2, and the following extra assumption. Let $T_{\iota_{\mathfrak{p}}}(\sigma^\vee)$ be a spherical $\sigma^\vee$-idempotent as in [Dis17, Proposition 2.4.4], which we may take to be of degree zero; by [Ram], we may and do assume that $T_{\iota_{\mathfrak{p}}}(\sigma^\vee)$ is supported at *split* places of F where all the data is unramified.

Assumption 2. We have

$$\phi = T_{\iota_{\mathfrak{p}}}(\sigma^\vee)\phi^\flat \tag{3}$$

for some $\phi^\flat$ satisfying the assumptions of [Dis17, §6.1].

This assumption will have the same effect as Assumption 3.4.1; namely, it ensures that the geometric kernel can be written in terms of height pairings of degree-zero divisors.

Denote by $\mathrm{T}(\sigma^\vee)$ the Hecke correspondence on X_U attached to $T_{\iota_{\mathfrak{p}}}(\sigma^\vee)$ via [Dis17, Lemma 5.2.2]; it has degree zero. Then by the definitions and [Dis17, Lemma 5.2.2],

$$^{\mathfrak{q}}\widetilde{Z}(\phi^\infty,\chi)_U = \langle {}^{\mathfrak{q}}\widetilde{Z}_*(\phi^{\flat,\infty})1, \mathrm{T}(\sigma^\vee)t_\chi\rangle,$$

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and in $\overline{\mathbf{S}}'$, we have the decomposition

$$^{\mathbf{q}}\widetilde{Z}(\phi^\infty,\chi) = \sum_v \widetilde{Z}(\phi^\infty,\chi)(v),$$

where

$$\widetilde{Z}(\phi^\infty,\chi)(v) = \sum_{w|v} \langle {}^{\mathbf{q}}\widetilde{Z}_*(\phi^{\flat,\infty})1, \mathrm{T}(\sigma^\vee)^{\mathrm{t}} t_\chi\rangle_{\ell,w}. \tag{4}$$

For $w \nmid p$, we may move the correspondence $\mathrm{T}(\sigma^\vee)^{\mathrm{t}}$ back to the left entry by interpreting the resulting pairing similarly to [YZZ12]. Namely, the local height pairing of two degree-zero divisors D_1, D_2 on X is, up to a factor $\ell(\varpi_w)$, the intersection multiplicity of flat extensions of D_1, D_2 to an integral model (see [Dis17, Proposition 4.2.2]). In turn, this arithmetic intersection pairing extends to divisors of arbitrary degree with disjoint supports by considering $\widehat{\xi}$-*admissible* (rather than flat) extensions as in [YZZ12, §7.1]. As a result, the pairing

$$\langle {}^{\mathbf{q}}\widetilde{Z}_*(\phi^\infty)1, t_\chi\rangle_{\ell,w}$$

is well-defined and it equals the w-term in (4). The fact that t_χ may not have degree zero introduces a term given by pairing with the Hodge class $\widehat{\xi}$, which however vanishes under our assumptions as in [YZZ12, Proposition 7.3.3]. Thus, the expression of [Dis17, (8.2.1)] for $\widetilde{Z}(\phi^\infty,\chi)(v)$ is still valid, and Theorem 3.6.1 continues to hold under our assumptions.

Theorem B is therefore still reduced to Proposition 3.6.2. For each nonsplit $v|p$, fix $m = m_v \geq r$, which is a multiple of the order of ϖ_v in the set (7) below. Define an operator $\mathscr{R}_v := \mathrm{U}_{v,*}^{m_v} - 1$. We will prove the following.

Proposition 3. *Let $v|p$. Under our running assumptions, the element*

$$\mathscr{R}_v \widetilde{Z}(\phi^\infty,\chi)(v) \in \overline{\mathbf{S}}'$$

is v-critical in the sense of (3.1.7).

Since $\ell_{\varphi^p,\alpha} \circ \mathscr{R}_v = (\alpha_v^m - 1)\ell_{\varphi^p,\alpha}$, and $\alpha_v^m - 1 \neq 0$ by our assumptions, the proposition still implies Proposition 3.6.2

Decay of intersection multiplicities

We prove Proposition 3. We fix an inert place v of F, and denote by w its extension to E.

Given our assumption that χ_v is unramified, we consider the action of $\mathscr{O}_{E,w}^\times$ on CM points; for the set $\Xi(\varpi_v^r)_a$ of Lemma 4.1.3, we have

$$[\Xi(\varpi_v^r)_a U_{F,v}^\circ]_U = \mathrm{rec}_{E_w}(\mathscr{O}_{E,w}^\times / \mathscr{O}_{F,v}^\times(1+\varpi_v^{r+s}\mathscr{O}_{E,w}))[x(\overline{b}_a)]_U,$$

where the Galois action is faithful, and $\overline{b}_a$ is any element of

$$q_v^{-1}(1 - a(1+\varpi_v^r\mathscr{O}_{F,v}))/(1+\varpi_v^{r+s}\mathscr{O}_{E,w}) \tag{5}$$

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In fact, let $\sqrt{\ }$ be the principal square root defined in a neighbourhood of $1 \in \mathscr{O}_{F,v}$. Then, if $v(a) \geq 1$ (or $v(a) \geq 2$ if $v|2$), we may and do fix $\overline{b}_a$ to be the class of

$$b_a := [\sqrt{1-a}].$$

Correspondingly, we define

$$H_{00}$$

to be the finite abelian extension of E with norm group $U_F^{\circ} U_T^v \mathscr{O}_{E,v}^{\times}$. It is contained in the extension H_0 defined before Proposition 4.1.4, and it is unramified at w. The study of intersection multiplicities of §4.3 then needs to take place in $\mathscr{X}$, the base change to $H_{00,\overline{w}}$ of the integral model $\mathscr{X}^{\natural}/\mathscr{O}_{F_v}$ of X_U defined by Carayol (we are renewing the notation: the model $\mathscr{X}$ considered in §4 is no longer in use). Note that under our assumption, $H_{00,\overline{w}/F_v}$ is unramified, so that $\mathscr{X}$ is still regular.

Consider Proposition 4.3.3. As noted above, its statement needs to be corrected by replacing (4.3.2) by

$$(\overline{z} \cdot D) = c[\kappa(y) : \kappa] q_{F,v}^s + \rho(V \cdot D')_{\mathscr{X}_y}, \tag{6}$$

where $V = V(z)$, $\rho = \rho(z)$. (The proof goes through verbatim in our renewed setup.)

The following is the new ingredient needed.

Proposition 4. *The sequence*

$$(V_s, \rho_s) := \big((V,\rho)([x(\overline{b}_{a\varpi_v^{ms}})]_U)\big)_{s \in \mathbf{N}}$$

is eventually constant.

Proof. We first consider V_s. By construction, it is the irreducible component of $\mathscr{X}_\kappa$ maximising the intersection multiplicity with the closure of the image $z_s \in X_{H_{00,\overline{w}}}$ of $[x(\overline{b}_{a\varpi_v^{ms}})]_U$. Here, κ is the residue field of $H_{00,\overline{w}}$. However, the irreducible components of $\mathscr{X}_\kappa$ are already defined over the residue field $\kappa^{\natural}$ of F_v. Therefore, V_s is the base-change of the component $V_s^{\natural} \subset \mathscr{X}_{\kappa^{\natural}}^{\natural}$ maximising the intersection multiplicity with the closure of the image $z_s^{\natural} \in X_{F_v}$ of z_s.

We explicitly compute $V_s^{\natural}$ in terms of the (equivalent) notions of geometric and algebraic *basins* of irreducible components introduced in [Dis22]. In fact, $V_s^{\natural}$ is, essentially by definition, the component through y to whose (geometric) basin the point $z_s^{\natural}$ belongs. First, recall from [Car86, Dis22] that

- the supersingular points in $\mathscr{X}_\kappa$ are parametrised by
$$B(v)^{\times} \backslash \mathbf{B}^{v\infty,\times} \times F_v^{\times} / (U^v \times q(U_v)); \tag{7}$$
- the irreducible components of $\mathscr{X}_{\kappa^{\natural}}$ are parametrised by $(\mathscr{O}_{F_v}/\varpi_v^r \mathscr{O}_{F_v})$-lines $L \subset (\varpi_v^{-r} \mathscr{O}_{F_v}/\mathscr{O}_{F_v})^2$;
- to a CM-by-E point $z \in X_{F_v}$ with sufficiently large conductor is attached an F_v-isomorphism $\tau \colon E_w \to F_v^2$, normalised so that
$$\mathscr{O}_{F,v}^2 \subset \tau(\mathscr{O}_{E,w}) \not\supset \varpi_v^{-1} \mathscr{O}_{F,v}^2, \tag{8}$$
and a corresponding line $L(\tau) = [\tau(\mathscr{O}_{E_w})] \subset (\varpi_v^{-r} \mathscr{O}_{F_v}/\mathscr{O}_{F_v})^2$.

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Then, in order to show the eventual constancy of $V_s^\natural$, we need to show that, for y_s the reduction of z_s and τ_s the invariant attached to $z_s^\natural$, we have $y_{s+1} = y_s$ and $L(\tau_{s+1}) = L(\tau_s)$ (for any sufficiently large s).

We have $[x(\overline{b}_{a\varpi_v^{m(s+1)}})]_U = [x(\overline{b}_{a\varpi_v^{ms}})h)]_U$ where, setting $b_s := b_{a\varpi_v^{ms}}$,

$$h = (1+\mathrm{j}b_s)^{-1}(1+\mathrm{j}b_{s+1}) = \frac{1}{a\varpi_v^{ms}}\begin{pmatrix}(1-b_s)(1+b_s) & [b_{s+1}(1-b_s)-b_s(1-b_{s+1})]\mathrm{T} \\ & (1+b_s)(1-b_{s+1})\end{pmatrix}$$

in $B_v^\times = \mathrm{GL}_2(F_v)$. The group $B_v^\times$ acts on the supersingular points via the map to the group (7) induced by the reduced norm q. By construction, $q(h) = \varpi_v^m$ has trivial image there; thus, $y_{s+1} = y_s$.

We have $b_s = 1 - 2^{-1}a\varpi_v^{ms} + O(\varpi_v^{ms})$, so that

$$h \equiv \begin{pmatrix}1 & \mathrm{T}/2 \\ & 0\end{pmatrix} \pmod{\varpi_v^m}.$$

By the construction in [Dis22, (1.2.1)], the group $B_v^\times$ acts on the invariant $F_v^\times\tau$ via left multiplication by h^{t}. Recalling the normalisation (8), we then have

$$L(\tau_{s+1}) = L\left(\begin{pmatrix} c & \\ c\mathrm{T}/2 & 0\end{pmatrix}\tau_s\right),$$

where $c \in F_v^\times$ is such that the matrix is integral and not divisible by ϖ_v. But this line is just the one spanned by $\binom{1}{0}$ (note that τ_s, as a surjective map to F_v^2, cannot be annihilated by a nonzero matrix over F_v). Thus, V_s is constant for $s \geq 2$.

We now show the eventual constancy of ρ_s. In fact, for large enough s, we have $\rho_s = (\overline{z}_s\cdot\Delta)/(V_s\cdot\Delta)$ for any divisor Δ whose support does not contain V_s. We take $\Delta = \mathrm{q}^*\Delta_0$, where $\mathscr{X}_0$ is as in the beginning of §4.3, q: $\mathscr{X} \to \mathscr{X}_{0,\mathscr{O}_{H_{00},\overline{w}}}$ is the projection, and Δ_0 is the Zariski closure of the canonical lift of $y := y_s = y_{s+1}$. By the projection formula, and with the notation of the proof of Lemma 4.3.2, the intersection $(\overline{z}_s\cdot\Delta)$ is a constant multiple of

$$(\overline{\mathrm{q}(z_s)}\cdot\Delta_0)_{\mathscr{X}_{0,\mathscr{O}^{\mathrm{un}}_{E,\overline{w}}},y} = \dim_k \mathscr{O}^{\mathrm{un}}_{E,\overline{w}}[[u]]/(\nu_s,u). \tag{9}$$

Here, by [Gro86], the local defining equation of the canonical lift Δ_0 is $u = 0$, and for the quasicanonical lift $\overline{\mathrm{q}(z_s)}$, it is $\nu_s(u) = 0$ for an Eisenstein polynomial ν_s. Thus, (9) equals 1 independently of s. This completes the proof. □

The following replaces Corollary 4.3.5.

Corollary 5. *If $D \in \mathrm{Div}^0(X_{H_0})_L$ is any degree-zero divisor, then for all sufficiently large s and all a,*

$$m_{\overline{w}}(\widetilde{Z}_{a\varpi^{m(s+1)}}(\phi^\infty)[1]_U, D) - m_{\overline{w}}(\widetilde{Z}_{a\varpi^{ms}}(\phi^\infty)[1]_U, D) = O(q_{F,v}^{ms}) \tag{10}$$

in L, where the implied constant can be fixed independently of a and s.

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Proof. Let $\widehat{D}$ be a flat extension of D to a divisor on $\mathscr{X}$ (with coefficients in L), and abbreviate $Z_{a,s} := \widetilde{Z}_{a\varpi^{ms}}(\phi^\infty)[1]_U$. Then by the corrected Proposition 4.3.3,

$$m_{\overline{w}}(Z_{a,s}, D) = (\overline{Z}_{a,s} \cdot \widehat{D}) = A_s q_{F,v}^{ms} + \sum_i \lambda_{i,s}(V_{i,s} \cdot D'_s) \qquad (11)$$

for some vertical components $V_{i,s} \subset \mathscr{X}$ and some A_s, $\lambda_{i,s} \in L$; here, we have written $\widehat{D} = c_s \mathscr{X}_\kappa + D'_s$, where D'_s is a divisor whose support does not contain V_s. By Proposition 4 (transported by Hecke correspondences away from v), all terms indexed by s are, in fact, eventually independent of s; thus, the second term of (11) gives vanishing contribution to (10). As remarked in Corollary 4.3.5, the constant $A = A_s$ is independent of a as well. □

Then the argument of the proof of Proposition 3.6.3 at the very end of the paper goes through to prove Proposition 3, with the following modifications: we apply the operator $(\mathscr{R}_v^{\mathrm{seq}}\star)_s := \star_{m(s+1)} - \star_{ms}$ to (4.4.2) and (4.4.4) (each viewed as a sequence $\star$ in s), and we use Corollary 5 instead of Corollary 4.3.5.

Erratum to [Dis17]

In Lemma 8.2.1 and in the Proof of Proposition 8.2.2, one should read '$F^\times \mathbf{A}^{S_1\infty,\times}$' in place of $\mathbf{A}^{S_1\infty,\times}$.

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