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ON SEMIDIRECTLY CLOSED NON-APERIODIC PSEUDOVARIETIES OF FINITE MONOIDS

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Abstract It is shown that, for every prime number p, the complete lattice of all semidirectly closed pseudovarieties of finite monoids whose intersection with the pseudovariety $\mathbf{G}$ of all finite groups is equal to the pseudovariety $\mathbf{G}_p$ of all finite p-groups has the cardinality of the continuum. Furthermore, it is shown, in addition, that the complete lattice of all semidirectly closed pseudovarieties of finite monoids whose intersection with the pseudovariety $\mathbf{G}$ of all finite groups is equal to the pseudovariety $\mathbf{G}_{\mathrm{sol}}$ of all finite solvable groups has also the cardinality of the continuum.

Keywords: finite monoids; pseudovarieties of finite monoids; pseudovarieties of finite groups; semidirect products of monoids; semidirectly closed pseudovarieties; finite inverse monoids; finite aperiodic monoids; finite $\mathcal{R}$-trivial monoids; finite simple groups; finite p-groups; finite solvable groups

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1. Introduction

The study of the operation of semidirect products of finite semigroups on the pseudovariety level has attracted considerable attention in recent years. See Chapter 10 in the book [1] by Almeida for an account of the hitherto development in this area. Consult also the paper [2] by Almeida and Weil for the current progress in this direction. The collection of all pseudovarieties of finite semigroups forms a monoid with respect to the mentioned operation of semidirect products. Particular attention has been paid to the identification of the idempotents of this monoid. These idempotents are exactly the semidirectly closed pseudovarieties of finite semigroups. Yet otherwise stated, a pseudovatiety of finite semigroups is said to be *semidirectly closed* if it is closed under the formation of semidirect products of its members. The study of semidirectly closed pseudovarieties of finite semigroups was initiated by Eilenberg in [4, Chapter V]. More recently, research in this area was pursued by Almeida in his book [1]. See Section 10.10 in [1] for an overview of the results obtained so far along this line.

The present paper can be viewed as a contribution to the solution of certain open problems posed in [1] regarding semidirectly closed pseudovarieties of finite semigroups. Although all problems posed in [1] in this connection are formulated as questions about pseudovarieties of finite semigroups, the particular problems we want to resolve in this paper can also be treated as questions about pseudovarieties of finite monoids. And it is this point of view, which is apparently more appropriate with regard to the answers obtained in this paper, that we will adopt here. Therefore, henceforth, in this paper, we will deal exclusively with pseudovarieties of finite monoids.

The family $\mathcal{SC}\ell$ of all semidirectly closed pseudovarieties of finite monoids forms a complete lattice under the ordering given by inclusion. For every semidirectly closed pseudovariety $\mathbf{H}$ of finite groups, we may consider the subfamily $\mathcal{SC}\ell_{\mathbf{H}}$ of all semidirectly closed pseudovarieties of finite monoids whose intersection with the pseudovariety $\mathbf{G}$ of all finite groups forms just the given pseudovariety $\mathbf{H}$. This subfamily $\mathcal{SC}\ell_{\mathbf{H}}$ constitutes a complete sublattice in the just mentioned complete lattice $\mathcal{SC}\ell$. Thus $\mathcal{SC}\ell_{\mathbf{H}}$ forms alone a complete lattice. The least element in this complete lattice is just the pseudovariety $\mathbf{H}$ itself and the greatest element is the pseudovariety $\overline{\mathbf{H}}$ of all finite monoids all of whose subgroups belong to $\mathbf{H}$.

Let us confine our attention, for the time being, to the case of non-trivial semidirectly closed pseudovarieties $\mathbf{H}$ of finite groups. Then the respective complete lattices $\mathcal{SC}\ell_{\mathbf{H}}$ consist of non-aperiodic semidirectly closed pseudovarieties of finite monoids. It is known that, for every such pseudovariety $\mathbf{H}$, the complete lattice $\mathcal{SC}\ell_{\mathbf{H}}$ is atomic and coatomic and it contains exactly one atom and one coatom. Let us denote provisionally this unique atom by $\mathbf{P}$, and let us also denote the unique coatom in question by $\mathbf{Q}$. Thus every pseudovariety in $\mathcal{SC}\ell_{\mathbf{H}}$, which is distinct from $\mathbf{H}$ and from $\overline{\mathbf{H}}$, must lie in the interval $[\mathbf{P}, \mathbf{Q}]$. It is also known that, if the pseudovariety $\mathbf{H}$ is distinct from the pseudovariety $\mathbf{G}$ of all finite groups, then the respective interval $[\mathbf{P}, \mathbf{Q}]$ is non-trivial, that is, the pseudovariety $\mathbf{P}$ is a proper subpseudovariety of $\mathbf{Q}$.

However, up until now, no additional information on the nature of the mentioned intervals $[\mathbf{P}, \mathbf{Q}]$ in the complete lattices $\mathcal{SC}\ell_{\mathbf{H}}$, for various non-trivial semidirectly closed pseudovarieties $\mathbf{H}$ of finite groups other than $\mathbf{G}$, has been available. It is the purpose of the present paper to fill in this gap to a certain extent. The principal results obtained in this paper state that, for every prime number p and for the semidirectly closed pseudovariety $\mathbf{G}_p$ of all finite p-groups, the interval $[\mathbf{P}, \mathbf{Q}]$ in the complete lattice $\mathcal{SC}\ell_{\mathbf{G}_p}$ has the cardinality of the continuum, and likewise, for the semidirectly closed pseudovariety $\mathbf{G}_{\mathrm{sol}}$ of all finite solvable groups, the interval $[\mathbf{P}, \mathbf{Q}]$ in the complete lattice $\mathcal{SC}\ell_{\mathbf{G}_{\mathrm{sol}}}$ also has the cardinality of the continuum.

So that's it as far as the main results gained in this paper are concerned. However, the interested reader may wish to learn more about the subject treated in this paper. For example, he or she may wonder what are the atom $\mathbf{P}$ and the coatom $\mathbf{Q}$ in the complete lattice $\mathcal{SC}\ell_{\mathbf{H}}$ like. Therefore, so as to provide the reader with the right perspective on the material presented in this paper, we have chosen to include a short exposition of the fundamentals of the subject in question in the remainder of this introduction.

Recall once more that, for every semidirectly closed pseudovariety $\mathbf{H}$ of finite groups, the family $\mathcal{SC}\ell_{\mathbf{H}}$ of all semidirectly closed pseudovarieties of finite monoids whose intersection with the pseudovariety $\mathbf{G}$ of all finite groups forms just the given pseudovariety $\mathbf{H}$ constitutes a complete sublattice in the complete lattice $\mathcal{SC}\ell$. In particular, if we take up in this way for **H** the pseudovariety of all trivial groups, we thus get the family of all semidirectly closed pseudovarieties of finite aperiodic monoids, which therefore creates a complete sublattice. The least element in this complete sublattice is the pseudovariety **I** of all trivial monoids and the greatest element is the pseudovariety **A** of all finite aperiodic monoids.

Every non-trivial pseudovariety of aperiodic monoids contains the two-element semilattice Y_2. It has been shown already by Stiffler in [8] that the semidirectly closed pseudovariety of finite monoids $[\![Y_2]\!]$ generated by this semilattice Y_2 is equal to the whole pseudovariety **R** of all $\mathcal{R}$-trivial monoids. Thus, of course, every non-trivial semidirectly closed pseudovariety of aperiodic monoids must contain this pseudovariety **R** of all $\mathcal{R}$-trivial monoids. On the other hand, let R_2 be the two-element right zero semigroup and let R_2^1 be the monoid obtained from the semigroup R_2 by adjoining an identity to it. From the well-known Krohn-Rhodes decomposition theorem, whose exposition can be found in [4, Chapter II], or, specifically for monoids, in [6, Chapter 4] by Lallement, for instance, it follows that the semidirectly closed pseudovariety of finite monoids $[\![R_2^1]\!]$ generated by the latter monoid R_2^1 is already equal to the entire pseudovariety **A** of all aperiodic monoids. Moreover, the mentioned monoid R_2^1 is prime, which means that if R_2^1 divides the semidirect products of any two finite monoids, then R_2^1 must divide at least one of these two factors. Hence it ensues that every semidirectly closed pseudovariety of finite monoids, which does not contain all aperiodic monoids, must be included in the exclusion $\ll R_2^1 \gg$ of the monoid R_2^1, that is, in the class $\ll R_2^1 \gg$ of all finite monoids, which do not have R_2^1 as a divisor, and that this class $\ll R_2^1 \gg$ is itself a semidirectly closed pseudovariety of finite monoids. Furthermore, it follows from the work of Stiffler [8] and it has been explicitly stated in [1, § 10.10] that the latter class $\ll R_2^1 \gg$ is equal to the pseudovariety **ER** consisting of those finite monoids whose idempotent generated submonoids are $\mathcal{R}$-trivial monoids. Consequently, every semidirectly closed pseudovariety of aperiodic monoids except the pseudovariety **A** alone is contained in the pseudovariety $\mathbf{A} \cap \mathbf{ER}$. In this way, we arrive at the chain

$$\mathbf{I} \subseteq \mathbf{R} \subseteq \mathbf{A} \cap \mathbf{ER} \subseteq \mathbf{A}$$

of semidirectly closed pseudovarieties of aperiodic monoids. In this chain, the inclusion $\mathbf{R} \subseteq \mathbf{A} \cap \mathbf{ER}$ is obviously proper, and every semidirectly closed pseudovariety of aperiodic monoids distinct from **I** and **A** must occur in the interval $[\mathbf{R}, \mathbf{A} \cap \mathbf{ER}]$. Let further B_2 be the five-element aperiodic Brandt semigroup and let B_2^1 be the monoid obtained from the semigroup B_2 by adjoining an identity to it. Then it turns out that every pseudovariety of aperiodic monoids included in the pseudovariety **ER**, but not included in the pseudovariety **R**, contains the monoid B_2^1. Hence it ensues that every semidirectly closed pseudovariety of aperiodic monoids distinct from **I** and **R** must contain the semidirectly closed pseudovariety $[\![B_2^1]\!]$ generated by the monoid B_2^1. Now one of the open questions posed in [1, § 10.10] can be stated as follows. Is it true that the pseudovariety $\mathbf{A} \cap \mathbf{ER}$ covers the pseudovariety **R** in the lattice of all semidirectly closed pseudovarieties of aperiodic monoids? That is, does the pseudovariety $\mathbf{A} \cap \mathbf{ER}$ coincide with the pseudovariety $[\![B_2^1]\!]$? However, this question has been in the end answered in the negative by Teixeira in [9]. In fact, she has shown in her paper that the interval $[\,[\![B_2^1]\!], \mathbf{A} \cap \mathbf{ER}]$ in the lattice of all semidirectly closed pseudovarieties of aperiodic monoids has actually the cardinality of the continuum.

The mentioned question for semidirectly closed pseudovarieties of aperiodic monoids being thus settled, we next turn to semidirectly closed non-aperiodic pseudovarieties of finite monoids, that is, to semidirectly closed pseudovarieties of finite monoids containing some non-trivial groups. The collection of all groups from such a semidirectly closed pseudovariety then forms itself a non-trivial semidirectly closed pseudovariety of finite groups. Thus let us now deal with an arbitrary non-trivial semidirectly closed pseudovariety $\mathbf{H}$ of finite groups. As it has been mentioned above already, the family $\mathcal{SC}\ell_{\mathbf{H}}$ of all semidirectly closed pseudovarieties of finite monoids whose intersection with the pseudovariety $\mathbf{G}$ of all finite groups forms exactly the given semidirectly closed pseudovariety $\mathbf{H}$ constitutes a complete sublattice in the complete lattice $\mathcal{SC}\ell$. As also mentioned above, the least element in this complete sublattice is just the pseudovariety $\mathbf{H}$ itself and the greatest element is the pseudovariety $\overline{\mathbf{H}}$ of all finite monoids all of whose subgroups belong to $\mathbf{H}$.

Every pseudovariety of finite monoids whose intersection with the pseudovariety $\mathbf{G}$ of all finite groups forms just the given semidirectly closed pseudovariety $\mathbf{H}$ but which is distinct from the pseudovariety $\mathbf{H}$ itself must contain the two-element semilattice Y_2. Therefore every semidirectly closed pseudovariety of finite monoids lying in $\mathcal{SC}\ell_{\mathbf{H}}$, which is distinct from the given pseudovariety $\mathbf{H}$, must contain aside from the pseudovariety $\mathbf{H}$ also the semidirectly closed pseudovariety $[\![Y_2]\!]$ generated by the mentioned semilattice Y_2, that is, the pseudovariety $\mathbf{R}$ of all $\mathcal{R}$-trivial monoids. However, as observed in [1, § 10.10], from the results of Stiffler established in [8] it follows that the inclusion of semidirect products $\mathbf{H} * \mathbf{R} \subseteq \mathbf{R} * \mathbf{H}$ of the just mentioned pseudovarieties holds. Hence it ensues straightforwardly that already the pseudovariety $\mathbf{R} * \mathbf{H}$ is semidirectly closed. Note also that, since the pseudovariety $\mathbf{H}$ is non-trivial, one can show that the pseudovariety $\mathbf{R} * \mathbf{H}$ contains the monoid B_2^1, and so it contains the entire semidirectly closed pseudovariety $[\![B_2^1]\!]$ generated by the monoid B_2^1. Every semidirectly closed pseudovariety of finite monoids lying in $\mathcal{SC}\ell_{\mathbf{H}}$, which is distinct from the pseudovariety $\mathbf{H}$, must contain the pseudovariety $\mathbf{R} * \mathbf{H}$. Furthermore, from the Krohn-Rhodes decomposition theorem it follows that, for our non-trivial semidirectly closed pseudovariety $\mathbf{H}$ of finite groups, the least semidirectly closed pseudovariety of finite monoids containing the pseudovariety $\mathbf{H}$ and the above-mentioned monoid R_2^1 is already equal to the entire pseudovariety $\overline{\mathbf{H}}$. Therefore every semidirectly closed pseudovariety of finite monoids lying in $\mathcal{SC}\ell_{\mathbf{H}}$, which is distinct from the pseudovariety $\overline{\mathbf{H}}$, must be contained in the exclusion $\ll R_2^1 \gg$ of the monoid R_2^1, that is, in the pseudovariety $\mathbf{ER}$. Consequently, every such semidirectly closed pseudovariety must be contained, in fact, in the pseudovariety $\overline{\mathbf{H}} \cap \mathbf{ER}$. In this way, we arrive at the chain

$$\mathbf{H} \subseteq \mathbf{R} * \mathbf{H} \subseteq \overline{\mathbf{H}} \cap \mathbf{ER} \subseteq \overline{\mathbf{H}}$$

of semidirectly closed pseudovarieties of finite monoids whose intersection with the pseudovariety $\mathbf{G}$ of all finite groups is equal to the pseudovariety $\mathbf{H}$. At this point, every semidirectly closed pseudovariety of finite monoids lying in $\mathcal{SC}\ell_{\mathbf{H}}$ and distinct from the pseudovarieties $\mathbf{H}$ and $\overline{\mathbf{H}}$ must occur in the interval $[\mathbf{R} * \mathbf{H}, \overline{\mathbf{H}} \cap \mathbf{ER}]$. Now another open question posed in [1, § 10.10] can be stated as follows. Is it true that the interval $[\mathbf{R} * \mathbf{H}, \overline{\mathbf{H}} \cap \mathbf{ER}]$ in the complete lattice $\mathcal{SC}\ell$ is always trivial? That is, is it true that, for every non-trivial semidirectly closed pseudovariety $\mathbf{H}$ of finite groups, the pseudovariety $\overline{\mathbf{H}} \cap \mathbf{ER}$ coincides with the pseudovariety $\mathbf{R} * \mathbf{H}$? From the results attained by Stiffler in [8] it ensues that, for $\mathbf{H}$ equal to the entire pseudovariety $\mathbf{G}$ of all finite groups,

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this question is answered positively, that is, the remarkable equality of pseudovarieties $\mathbf{ER} = \mathbf{R} * \mathbf{G}$ holds. However, against the expectations expressed in [1], it has been subsequently proved by Higgins and Margolis in [5] that the whole pseudovariety $\mathbf{G}$ is the sole exception for which the equality of the above-mentioned pseudovarieties holds. Or, to put it in different terms, Higgins and Margolis have proved in their paper that, for every non-trivial proper subpseudovariety $\mathbf{H}$ of finite groups, the pseudovariety $\mathbf{R} * \mathbf{H}$ is a proper subpseudovariety of the pseudovariety $\overline{\mathbf{H}} \cap \mathbf{ER}$.

Nevertheless, as also pointed out above, the situation just outlined leaves still some questions unanswered. For example, it is still unknown what is the actual cardinality of the interval $[\mathbf{R} * \mathbf{H}, \overline{\mathbf{H}} \cap \mathbf{ER}]$ in the complete lattice $\mathcal{SC}\ell$, for various non-trivial proper semidirectly closed subpseudovarieties $\mathbf{H}$ of finite groups. The present paper aspires to answering this question for some particular such pseudovarieties $\mathbf{H}$. Minimal non-trivial semidirectly closed pseudovarieties of finite groups are exactly the pseudovarieties $\mathbf{G}_p$ of all p-groups, for arbitrary prime numbers p. We intend to elaborate on the methods employed by Higgins and Margolis in [5] and to show hereby that, for every prime number p, the above interval where the pseudovariety $\mathbf{H}$ is equal to the pseudovariety $\mathbf{G}_p$ of all p-groups, that is, the interval $[\mathbf{R} * \mathbf{G}_p, \overline{\mathbf{G}_p} \cap \mathbf{ER}]$ has the cardinality of the continuum. As a larger semidirectly closed pseudovariety of finite groups we mention the pseudovariety $\mathbf{G}_{\mathrm{sol}}$ of all solvable groups. We intend to apply some rudimentary knowledge of finite simple group theory and number theory in order to show that the above interval where the pseudovariety $\mathbf{H}$ is equal to the pseudovariety $\mathbf{G}_{\mathrm{sol}}$ of all solvable groups, that is, the interval $[\mathbf{R} * \mathbf{G}_{\mathrm{sol}}, \overline{\mathbf{G}_{\mathrm{sol}}} \cap \mathbf{ER}]$ also has the cardinality of the continuum. As a small bonus, we conclude the present paper by showing that the uncountably many semidirectly closed pseudovarieties of finite monoids constructed so far are all not local in the sense of Tilson [10].

2. Preliminaries

As mentioned already in the introduction, we shall deal in this paper exclusively with pseudovarieties of finite monoids. Therefore we will allow only such semidirect products $S * T$ of two finite monoids S and T where the underlying left action of T on S is both left and right unitary. The semidirect product $\mathbf{U} * \mathbf{V}$ of two pseudovarieties $\mathbf{U}$ and $\mathbf{V}$ of finite monoids is then defined as the pseudovariety of finite monoids generated by the class of all such semidirect products $S * T$ where $S \in \mathbf{U}$ and $T \in \mathbf{V}$. It can be easily shown that then the pseudovariety $\mathbf{U} * \mathbf{V}$ consists, in fact, of all divisors of the just mentioned semidirect products $S * T$.

Rarely shall we meet in this paper also the notion of the Mal'cev product of two pseudovarieties of finite monoids. In fact, we shall make do here merely with the case when the latter of the two pseudovarieties in question is a pseudovariety of finite groups. Recall that a monoid S is said to be a co-extension of a group G by a monoid T, if there exists a surjective homomorphism $\vartheta : S \to G$ such that the submonoid $\vartheta^{-1}(1)$ of S is isomorphic to T. Now consider any pseudovariety $\mathbf{V}$ of finite monoids and any pseudovariety $\mathbf{H}$ of finite groups. Then the Mal'cev product $\mathbf{V} ⓜ \mathbf{H}$ of these two pseudovarieties is the pseudovariety of finite monoids generated by the class of all finite monoids S, which are co-extensions of a group G from $\mathbf{H}$ by a monoid T from $\mathbf{V}$. In fact, the Mal'cev product $\mathbf{V} ⓜ \mathbf{H}$ then consists exactly of all homomorphic images of the monoids S just specified.

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There is yet an alternative way how the Mal'cev product $\mathbf{V} ⓜ \mathbf{H}$ can be determined. Namely, this Mal'cev product consists precisely of all finite monoids S for which there exists a group G in $\mathbf{H}$ and a relational morphism $\varphi : S \multimap G$ such that the submonoid $\varphi^{-1}(1)$ of S belongs to $\mathbf{V}$.

Occasionally in this paper, we shall have to deal also with pseudovarieties of finite categories. Roughly speaking, by a category we mean a directed graph endowed with an associative partial binary operation of composition of consecutive edges admitting local identities. In more detail, a graph Γ consists of a set $V(\Gamma)$ of vertices and a set $E(\Gamma)$ of edges together with two mappings $\alpha, \omega : E(\Gamma) \to V(\Gamma)$ assigning to every edge $e \in E(\Gamma)$ its beginning $\alpha(e)$ and its end $\omega(e)$. For any vertices $u, v \in V(\Gamma)$, we write $\Gamma(u, v)$ for the set of all edges $e \in E(\Gamma)$ such that $\alpha(e) = u$ and $\omega(e) = v$. By a category we mean a graph C endowed with an associative partial binary operation of multiplication of edges which, for arbitrary vertices $u, v, w \in V(C)$, assigns to any edges $e \in C(u, v)$ and $f \in C(v, w)$ an edge $ef \in C(u, w)$ and which has, for every vertex $v \in V(C)$, an identity edge $1_v \in C(v, v)$ acting as an identity element with respect to the mentioned operation of multiplication. In this context, we also call the sets of edges of the form $C(u, v)$ the hom-sets of C. We also write shortly $C(v)$ instead of $C(v, v)$. For every vertex $v \in V(C)$, the set $C(v)$ together with the multiplication from C restricted to $C(v)$ forms a monoid. This monoid is called the local monoid of C at v.

If Γ and Δ are graphs, then a graph mapping $\varphi : \Gamma \to \Delta$ consists of two mappings $V(\Gamma) \to V(\Delta)$ and $E(\Gamma) \to E(\Delta)$, also denoted by φ, such that, for every edge $e \in E(\Gamma)$, one has $\alpha(\varphi(e)) = \varphi(\alpha(e))$ and $\omega(\varphi(e)) = \varphi(\omega(e))$. If C and D are categories, then a homomorphism $\psi : C \to D$ of these categories is a graph mapping such that, for any vertices $u, v, w \in V(C)$ and any edges $e \in C(u, v)$ and $f \in C(v, w)$, the equality $\psi(ef) = \psi(e)\psi(f)$ holds and, in addition, for any vertex $v \in V(C)$, also the equality $\psi(1_v) = 1_{\psi(v)}$ holds. Furthermore, we say that the mentioned homomorphism $\psi : C \to D$ is faithful if it is injective on hom-sets of C, that is, if for arbitrary vertices $u, v \in V(C)$, the restriction of ψ to $C(u, v)$ is injective. We say that the homomorphism $\psi : C \to D$ is a quotient homomorphism if it is bijective on the vertices of C, that is, if it determines a bijection of $V(C)$ onto $V(D)$, and if it is surjective on hom-sets, that is, if for arbitrary vertices $u, v \in V(C)$, the restriction of ψ to $C(u, v)$ is a surjection of $C(u, v)$ onto $D(\psi(u), \psi(v))$. In the end, we say that a category C divides a category D if there exists a category B, a faithful homomorphism $\varphi : B \to D$ and a quotient homomorphism $\psi : B \to C$. Then we write $C \prec D$. Note that then the underlying category B is isomorphic to a certain subcategory of the direct product of categories $C \times D$. Thus, if C and D are finite categories, then the mentioned category B must also be finite.

A class $\mathbf{W}$ of finite categories is said to be a pseudovariety of finite categories if it is closed under taking finitary direct products of finite categories and under division of finite categories. Every monoid S can be viewed as a category having a single vertex in such a way that the monoid S itself then becomes identified with the local monoid of the category under consideration at its unique vertex. Thus, as a particular case of the notion of divisibility of categories introduced at the end of the previous paragraph, we obtain what it means that a category C divides a monoid S. If this is the case, then we write $C \prec S$. Furthermore, in this manner, various classes of monoids can be viewed as classes of categories. For instance, every pseudovariety $\mathbf{V}$ of finite monoids can be viewed as a class of finite categories. In such a situation, it is possible to consider the pseudovariety

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$\mathbf{W}$ of finite categories generated by this pseudovariety $\mathbf{V}$ of finite monoids. Then $\mathbf{W}$ consists of all finite categories C such that $C \prec S$ for some finite monoid S from $\mathbf{V}$. This pseudovariety $\mathbf{W}$ is commonly denoted by $g\mathbf{V}$. Furthermore, it is customary to denote by $\ell\mathbf{V}$ the pseudovariety of all finite categories all of whose local monoids lie in $\mathbf{V}$. Then, of course, the inclusion $g\mathbf{V} \subseteq \ell\mathbf{V}$ holds. Following the terminology introduced by Tilson in [**10**, § 13], we say that our pseudovariety $\mathbf{V}$ of finite monoids is local if the equality $g\mathbf{V} = \ell\mathbf{V}$ holds.

We conclude our considerations in this section with the following quite familiar fact which, however, has not been proved thoroughly anywhere. If $\mathbf{V}$ is any local pseudovariety of finite monoids and if $\mathbf{H}$ is any pseudovariety of finite groups, then the equality of pseudovarieties $\mathbf{V} * \mathbf{H} = \mathbf{V} ⓜ \mathbf{H}$ holds. The inclusion $\mathbf{V} * \mathbf{H} \subseteq \mathbf{V} ⓜ \mathbf{H}$ can be verified straightforwardly. Namely, every semidirect product $T * G$ of a monoid $T \in \mathbf{V}$ and a group $G \in \mathbf{H}$ can be readily seen to be a co-extension of the group G by the monoid T. The verification of the reverse inclusion $\mathbf{V} ⓜ \mathbf{H} \subseteq \mathbf{V} * \mathbf{H}$ requires some apparatus, which can be taken over from Tilson's paper [**10**]. Thus let S be a co-extension of a group $G \in \mathbf{H}$ by a monoid $T \in \mathbf{V}$ and let $\vartheta : S \to G$ be a surjective homomorphism such that the submonoid $\vartheta^{-1}(1)$ of S is isomorphic to T. Then, as in [**10**, § 4], one can construct the derived category D_ϑ of the homomorphism ϑ. The local monoids of this derived category D_ϑ can be easily seen to be homomorphic images of the monoid T, and so they belong to $\mathbf{V}$. Therefore the derived category D_ϑ itself belongs to the pseudovariety $\ell\mathbf{V}$. However, since the pseudovariety $\mathbf{V}$ is local, this category D_ϑ belongs, in fact, to the pseudovariety $g\mathbf{V}$. Hence this category D_φ divides some monoid Y from $\mathbf{V}$. Now one can apply the derived category theorem provided in [**10**, § 5]. According to this theorem, the monoid S divides the wreath product $Y \circ G$ of the monoid Y and the group G. But this wreath product $Y \circ G$ evidently belongs to the pseudovariety $\mathbf{V} * \mathbf{H}$. Thus also the monoid S belongs to $\mathbf{V} * \mathbf{H}$.

3. The construct of Higgins and Margolis

The purpose of this section is to lay the groundwork to the deliberations evolved in the subsequent sections. This foundation of our future reasonings consists in a construction provided by Higgins and Margolis in [**5**]. They have conceived this structure in [**5**] for finite semigroups, but it can be straightforwardly carried over to finite monoids. We proceed to give the details of this construction hereinafter.

Take the set $X_n = \{1, 2, \ldots, n\}$ and consider arbitrary partial one-to-one mappings $\beta_1, \beta_2, \ldots, \beta_k$ of the set X_n into itself. Let U be the submonoid of the symmetric inverse monoid on the set X_n generated by the one-to-one mappings $\beta_1, \beta_2, \ldots, \beta_k$. Let $X'_n = \{1', 2', \ldots, n'\}$ be a disjoint copy of the set X_n. We build a submonoid $M(U)$ of the symmetric inverse monoid on the set $X_n \cup X'_n$ as follows. First, for every subset $Z = \{i_1, i_1, \ldots, i_r\}$ of the set X_n, denote by Z' the set $\{i'_1, i'_2, \ldots, i'_r\}$. Further on, for every $j \in \{1, 2, \ldots, k\}$, let D_j be the domain of the mapping β_j, let R_j be the range of the mapping β_j, and let β'_j be the partial one-to-one mapping of the set X_n into the set X'_n such that the domain of β'_j is the set D_j, the range of β'_j is the set R'_j, and the mapping β'_j itself is given by the formula $\beta'_j(h) = \beta_j(h)'$, for all $h \in D_j$. In addition, let α' be the mapping whose domain is the set X_n, whose range is the set X'_n, and which is itself given by the formula $\alpha'(h) = h'$, for all $h \in X_n$. Finally, let B_{2n} be the aperiodic Brandt

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semigroup consisting of all partial one-to-one mappings of the set $X_n \cup X'_n$ into itself of rank no more than 1, and let B^1_{2n} be the monoid obtained from the semigroup B_{2n} by adjoining the identity on the set $X_n \cup X'_n$ to it. Having all of these ingredients at hand, we let $M(U)$ be the submonoid of the symmetric inverse monoid on the set $X_n \cup X'_n$ generated by the set of one-to-one mappings $\{\alpha', \beta'_1, \beta'_2, \ldots, \beta'_k\} \cup B^1_{2n}$. Then $M(U)$ is, in fact, the union of the monoid B^1_{2n} and the set $\{\alpha', \beta'_1, \beta'_2, \ldots, \beta'_k\}$, and this latter set generates only a zero semigroup, since the domains and the ranges of all mappings in this set are subsets of the sets X_n and X'_n, respectively, and the sets X_n and X'_n are disjoint. Hence it readily follows that $M(U)$ is an aperiodic monoid with commuting idempotents. Notice yet in passing that the notation $M(U)$ might seem slightly misleading, since the monoid $M(U)$ depends not only on U but, more precisely, on the set $\{\beta_1, \beta_2, \ldots, \beta_k\}$ of generators of U. But we will encounter no true difficulty hereinafter resulting from this minor ambiguity.

The following statement comes from [**5**, Theorem 3.2] and it represents the principal result of [**5**, § 3].

Theorem 3.1. *If the monoid $M(U)$ divides some finite inverse monoid I, then the monoid U divides this inverse monoid I as well.*

In particular, we will apply this theorem in the situation when U will be a finite group G. Of course, for this purpose, this group G must be represented as a permutation group in accordance with Cayley's theorem. We will use the usual right regular representation of the group G by its right translations in a standard manner. That is, for every element $g \in G$, we consider the function $\rho_g : G \to G$, which maps every element $f \in G$ to the element fg and which is called the right translation of G by g. Then the mapping assigning to every element $g \in G$ the corresponding right translation ρ_g is an embedding of the group G into the symmetric group on the set G. This embedding is called the right regular representation of the group G. Let further G' be a disjoint copy of the set G. In such a situation, by $M(G)$ we will mean the submonoid $M(U)$ of the symmetric inverse monoid on the set $G \cup G'$ where U will be the subgroup of the symmetric group on the set G formed by all right translations ρ_g, for arbitrary elements $g \in G$.

In the introduction already, we have denoted by $\mathbf{R}$ the pseudovariety of all $\mathcal{R}$-trivial monoids. Furthermore, we denote by $\mathbf{Sl}$ the pseudovariety of all semilattice monoids. Now we are in a position to state and prove the following assertion.

Theorem 3.2. *Let $\mathbf{H}$ be any non-trivial pseudovariety of finite groups. Then, for every finite group G, we have $M(G) \in \mathbf{R} * \mathbf{H}$ if and only if $G \in \mathbf{H}$.*

Proof. Assume first that $G \in \mathbf{H}$. If this group G is trivial, then $M(G)$ is isomorphic to the monoid B^1_2, and as the pseudovariety $\mathbf{H}$ is non-trivial, this monoid B^1_2 belongs to $\mathbf{R} * \mathbf{H}$. Thus we may further assume that the group G is non-trivial. Once again, since the pseudovariety $\mathbf{H}$ is non-trivial, there exists a prime number p such that $\mathbf{H}$ contains the finite cyclic group of order p. Let us represent this finite cyclic group as the additive group $\mathbb{Z}_p = \{[0], [1], \ldots, [p-1]\}$ of all residue classes modulo p. Then, along with the mentioned group G, the pseudovariety $\mathbf{H}$ contains also the group $G \times \mathbb{Z}_p$. Consider next the relational morphism $\varphi : M(G) \twoheadrightarrow G \times \mathbb{Z}_p$ defined as follows. Recall that $M(G)$ is the union of the monoid B^1_{2n} and the set $\{\alpha', \beta'_1, \beta'_2, \ldots, \beta'_k\}$, where $n = |G|$, $k = |G|$, and $\{\beta_1, \beta_2, \ldots, \beta_k\}$

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is the set of all right translations of the group G. Since the group G is non-trivial, the mentioned union is, in fact, disjoint. Note that, for arbitrary elements $f, h \in G$, there exists a unique element $g \in G$ such that $fg = h$. The monoid B_{2n}^1 consists of the empty mapping $\emptyset$, the one-to-one mappings of rank 1, which are of the forms $\{(f,h)\}$, $\{(f,h')\}$, $\{(f',h)\}$, $\{(f',h')\}$, for any elements $f, h \in G$, and the identity on the set $G \cup G'$. Then we put $\varphi(\emptyset) = G \times \mathbb{Z}_p$, further $\varphi(f,h) = (g,[0])$, $\varphi(f,h') = (g,[1])$, $\varphi(f',h) = (g,[p-1])$, $\varphi(f',h') = (g,[0])$, where $g \in G$ is the unique element for which $fg = h$, and, at last, we let φ map the identity on the set $G \cup G'$ to the identity $(1,[0])$ of the group $G \times \mathbb{Z}_p$. Furthermore, we let $\varphi(\alpha') = (1,[1])$. As far as the mappings $\beta'_1, \beta'_2, \ldots, \beta'_k$ are concerned, we first recall once more that the initial mappings $\beta_1, \beta_2, \ldots, \beta_k$ are exactly the right translations ρ_g of G by g, for arbitrary elements $g \in G$. Therefore the set of mappings $\{\beta'_1, \beta'_2, \ldots, \beta'_k\}$ can be represented as the set $\{\rho'_g : g \in G\}$. Then, for every element $g \in G$, we put $\varphi(\rho'_g) = (g,[1])$. It can be checked straightforwardly that then φ indeed is a relational morphism of the monoid $M(G)$ onto the group $G \times \mathbb{Z}_p$. The preimage $\varphi^{-1}(1,[0])$ of the identity $(1,[0])$ of $G \times \mathbb{Z}_p$ consists of the idempotents $\emptyset$, $\{(f,f)\}$, $\{(f',f')\}$, for all elements $f \in G$, and the identity on the set $G \cup G'$. These idempotents form a semilattice submonoid in $M(G)$. Therefore the monoid $M(G)$ itself belongs to the Mal'cev product $\mathbf{Sl} ⓜ \mathbf{H}$ of the pseudovarieties $\mathbf{Sl}$ and $\mathbf{H}$. But $\mathbf{Sl}$ is a local pseudovariety according to Simon's theorem (see [4, Chapter VIII, Theorem 7.1] and [10, Example 15.6]). Therefore, in compliance with the conclusions made in the last paragraph of the previous section, we have the equality of pseudovarieties $\mathbf{Sl} * \mathbf{H} = \mathbf{Sl} ⓜ \mathbf{H}$. Consequently, the monoid $M(G)$ belongs to the semidirect product $\mathbf{Sl} * \mathbf{H}$ of the aforementioned pseudovarieties. But $\mathbf{Sl}$ is a subpseudovariety of $\mathbf{R}$, and so $\mathbf{Sl} * \mathbf{H}$ is a subpseudovariety of $\mathbf{R} * \mathbf{H}$. Thus the monoid $M(G)$ belongs to the semidirect product $\mathbf{R} * \mathbf{H}$, as desired.

Conversely, assume that G is a finite group such that the monoid $M(G)$ belongs to $\mathbf{R} * \mathbf{H}$. We wish to show that then the group G itself belongs to the pseudovariety $\mathbf{H}$. If the group G is trivial, then, of course, G belongs to $\mathbf{H}$. Thus we may further assume that the group G is non-trivial. The arguments in this part of the proof then generally follow those included in the proof of Theorem 5.3 in [5]. For the reader's convenience, however, we repeat here the details. Since the monoid $M(G)$ belongs to $\mathbf{R} * \mathbf{H}$, it divides some semidirect product of the form $T * K$ where T is an $\mathcal{R}$-trivial monoid and K is a group from $\mathbf{H}$. Thus there exists a submonoid V of $T * K$ and a surjective homomorphism $\eta : V \to M(G)$. Let further $\pi : T * K \to K$ be the semidirect product projection. Put $\sigma = \pi \circ \eta^{-1}$. Then $\sigma : M(G) \longrightarrow\!\!\!\!\!\!\!\!\ominus\ \ K$ is a relational morphism and $\sigma^{-1}(1) = \eta(\pi^{-1}(1) \cap V)$ is a homomorphic image of the submonoid $\pi^{-1}(1) \cap V$ of $\pi^{-1}(1)$. Here $\pi^{-1}(1)$ is a submonoid of $T * K$, which is isomorphic to the monoid T, and hence it is $\mathcal{R}$-trivial. Therefore also the submonoid $\sigma^{-1}(1)$ of the monoid $M(G)$ is $\mathcal{R}$-trivial. Recall now once again that the monoid $M(G)$ is the disjoint union of the monoid B_{2n}^1 and the set $\{\alpha', \beta'_1, \beta'_2, \ldots, \beta'_k\}$, where $n = |G|$, $k = |G|$, and $\{\beta_1, \beta_2, \ldots, \beta_k\}$ is the set of all right translations of the group G. Recall also from the previous paragraph that the pseudovariety $\mathbf{H}$ contains the cyclic group $\mathbb{Z}_p$ of order p for some prime number p. Consider now the relation $\tau \subseteq M(G) \times \mathbb{Z}_p$ such that $\tau(\alpha') = \tau(\beta'_1) = \tau(\beta'_2) = \ldots = \tau(\beta'_k) = [1]$, τ maps the identity on the set $G \cup G'$ to the element $[0]$, and τ relates all non-identity elements of the monoid B_{2n}^1 to the entire set $\mathbb{Z}_p$. Then $\tau : M(G) \longrightarrow\!\!\!\!\!\!\!\!\ominus\ \ \mathbb{Z}_p$ is evidently a relational morphism and the submonoid $\tau^{-1}([0])$ of $M(G)$ coincides with the monoid B_{2n}^1. Consider next the product relational morphism $\sigma \times \tau : M(G) \longrightarrow\!\!\!\!\!\!\!\!\ominus\ \ K \times \mathbb{Z}_p$ given by the prescription

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$(\sigma \times \tau)(s) = \sigma(s) \times \tau(s)$, for all elements $s \in M(G)$. Point yet that the identity of the group $K \times \mathbb{Z}_p$ is of the form $(1, [0])$. Then obviously $(\sigma \times \tau)^{-1}(1, [0]) = \sigma^{-1}(1) \cap \tau^{-1}([0])$. Thus, according to the above notes about the submonoids $\sigma^{-1}(1)$ and $\tau^{-1}([0])$, the submonoid $(\sigma \times \tau)^{-1}(1, [0])$ is $\mathcal{R}$-trivial and it is contained in the monoid B^1_{2n}. This submonoid $(\sigma \times \tau)^{-1}(1, [0])$ therefore consists of some regular elements of the monoid $M(G)$. It can be shown that, under these circumstances, this submonoid $(\sigma \times \tau)^{-1}(1, [0])$ is itself a regular monoid. Indeed, if m is the exponent of the finite group K, if s is any element in $(\sigma \times \tau)^{-1}(1, [0])$ and if t is any element in $M(G)$ such that $sts = s$, then $s(ts)^{pm} = s$ and the element $t(st)^{pm-1}$ can be seen to belong again to $(\sigma \times \tau)^{-1}(1, [0])$. Thus the initial element s is regular already in the submonoid $(\sigma \times \tau)^{-1}(1, [0])$. Consequently, this submonoid is regular and $\mathcal{R}$-trivial, whence it ensues that it consists merely of the idempotents of the monoid B^1_{2n}. But these idempotents form a semilattice monoid. This shows that the monoid $M(G)$ belongs to the Mal'cev product $\mathbf{Sl} \ⓜ \mathbf{H}$ of the foregoing pseudovarieties $\mathbf{Sl}$ and $\mathbf{H}$. Once more we evoke here the equality of pseudovarieties $\mathbf{Sl} * \mathbf{H} = \mathbf{Sl} \ⓜ \mathbf{H}$. Thus we see that the monoid $M(G)$ belongs to the semidirect product $\mathbf{Sl} * \mathbf{H}$ of the mentioned pseudovarieties. But this semidirect product consists of all divisors of inverse monoids of the form $S * F$ where S is a semilattice monoid and F is a group from $\mathbf{H}$. It can be readily shown that subgroups of such an inverse monoid $S * F$ are all isomorphic to subgroups of the group F, and therefore they belong to $\mathbf{H}$. Subgroups of arbitrary divisors of such an inverse monoid $S * F$ then also belong to $\mathbf{H}$. Consequently, the class of all groups contained in $\mathbf{Sl} * \mathbf{H}$ forms exactly the pseudovariety $\mathbf{H}$. Finally, the monoid $M(G)$ belongs to $\mathbf{Sl} * \mathbf{H}$, and hence it divides an inverse monoid of the form $S * F$ as above. But then, by Theorem 3.1, the group G itself divides this inverse monoid $S * F$. Thus the group G belongs to $\mathbf{Sl} * \mathbf{H}$, and by virtue of what has just been inferred, it must belong to $\mathbf{H}$, as desired. □

We conclude this section by indicating how can Theorem 3.2 be applied in order to show that, for certain non-trivial semidirectly closed pseudovarieties $\mathbf{H}$ of finite groups, there exist uncountably many semidirectly closed pseudovarieties of finite monoids contained in the interval $[\mathbf{R} * \mathbf{H}, \overline{\mathbf{H}} \cap \mathbf{ER}]$ that has been exhibited in the introduction. Assume that $\mathbf{K}$ is a semidirectly closed pseudovariety of finite groups containing the pseudovariety $\mathbf{H}$. Then $\mathbf{R} * \mathbf{K}$ is a semidirectly closed pseudovariety of finite monoids, and, consequently, $\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K})$ is also a semidirectly closed pseudovariety of finite monoids. Further on, on the one hand, one then has $\mathbf{R} * \mathbf{H} \subseteq \mathbf{R} * \mathbf{K}$, whence it ensues that $\mathbf{R} * \mathbf{H} \subseteq \overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K})$, and, on the other hand, one further has $\mathbf{R} * \mathbf{K} \subseteq \mathbf{R} * \mathbf{G} = \mathbf{ER}$, which entails that $\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K}) \subseteq \overline{\mathbf{H}} \cap \mathbf{ER}$. Thus the semidirectly closed pseudovariety $\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K})$ occurs in the interval $[\mathbf{R} * \mathbf{H}, \overline{\mathbf{H}} \cap \mathbf{ER}]$, which has been mentioned above. Recall that, for every finite group G, the monoid $M(G)$ is aperiodic, and hence it belongs to the pseudovariety $\overline{\mathbf{H}}$. Furthermore, this monoid $M(G)$ has the property that its idempotents form a semilattice monoid, and hence it belongs to the pseudovariety $\mathbf{ER}$. Therefore, for every finite group G, the monoid $M(G)$ belongs to the pseudovariety $\overline{\mathbf{H}} \cap \mathbf{ER}$. Let us now discuss for which finite groups G does the monoid $M(G)$ belong to the subpseudovarieties $\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K})$ of the pseudovariety $\overline{\mathbf{H}} \cap \mathbf{ER}$, for various semidirectly closed pseudovarieties $\mathbf{K}$ of finite groups that contain the pseudovariety $\mathbf{H}$. Since the monoid $M(G)$ belongs to the pseudovariety $\overline{\mathbf{H}}$, we have $M(G) \in \overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K})$ if and only if $M(G) \in \mathbf{R} * \mathbf{K}$. Now, since the pseudovariety $\mathbf{K}$ is also non-trivial, by Theorem 3.2, we have $M(G) \in \mathbf{R} * \mathbf{K}$

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if and only if $G \in \mathbf{K}$. Thus, altogether, we get that $M(G) \in \overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K})$ if and only if $G \in \mathbf{K}$. Therefore, for distinct semidirectly closed pseudovarieties $\mathbf{K}$ of finite groups containing the pseudovariety $\mathbf{H}$, the semidirectly closed pseudovarieties $\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K})$ of finite monoids lying in the interval $[\mathbf{R} * \mathbf{H}, \overline{\mathbf{H}} \cap \mathbf{ER}]$ will be likewise distinct. Consequently, in order to provide uncountably many semidirectly closed pseudovarieties of finite monoids lying in the just mentioned interval, it suffices to exhibit uncountably many semidirectly closed pseudovarieties $\mathbf{K}$ of finite groups containing the pseudovariety $\mathbf{H}$.

4. Uncountably many semidirectly closed non-aperiodic pseudovarieties

This is the central section of the present paper. We apply here the tools prepared in the preceding section in order to exhibit uncountably many semidirectly closed non-aperiodic pseudovarieties of finite monoids lying in the intervals that have been specified in the last paragraph of the introduction.

For every prime number p, we denote by $\mathbf{Ab}_p$ the pseudovariety of all elementary abelian p-groups. Furthermore, we have already denoted by $\mathbf{G}_p$ the pseudovariety of all finite p-groups. Note that this pseudovariety $\mathbf{G}_p$ is just the semidirectly closed pseudovariety of finite groups generated by the pseudovariety $\mathbf{Ab}_p$. Let further p be a fixed prime number. Then, for the corresponding pseudovariety $\mathbf{G}_p$, we have the subsequent finding.

Theorem 4.1. *The interval* $[\mathbf{R} * \mathbf{G}_p, \overline{\mathbf{G}_p} \cap \mathbf{ER}]$ *in the complete lattice of all semidirectly closed pseudovarieties of finite monoids has the cardinality of the continuum.*

Proof. According to the considerations performed at the close of the previous section, all that remains to be done is to show a continuum of semidirectly closed pseudovarieties of finite groups containing the pseudovariety $\mathbf{G}_p$. But this is easy to do. For every non-empty set Q of prime numbers, consider the semidirectly closed pseudovariety of finite groups $\mathbf{G}_Q$ generated by the collection of all pseudovarieties $\mathbf{Ab}_q$ with $q \in Q$. Then $\mathbf{G}_Q$ contains finite cyclic groups of orders q for all prime numbers $q \in Q$, but it contains no finite cyclic group of order q for any prime number $q \notin Q$. Indeed, the orders of finite groups which may appear in $\mathbf{G}_Q$ can only be divisible by a couple of prime numbers from the set Q. Thus, for distinct non-empty sets Q of prime numbers, the semidirectly closed pseudovarieties $\mathbf{G}_Q$ are distinct. In particular, for distinct sets Q of prime numbers containing the number p, we obtain distinct semidirectly closed pseudovarieties $\mathbf{G}_Q$ of finite groups containing the pseudovariety $\mathbf{G}_p$. This collection of semidirectly closed pseudovarieties has the cardinality of the continuum. □

We next continue considering the pseudovariety $\mathbf{G}_{\mathrm{sol}}$ of all finite solvable groups. This pseudovariety can be obtained as the semidirectly closed pseudovariety of finite groups generated by the collection of the pseudovarieties $\mathbf{Ab}_p$ for all prime numbers p. We are about to prove the following notable statement.

Theorem 4.2. *The interval* $[\mathbf{R} * \mathbf{G}_{\mathrm{sol}}, \overline{\mathbf{G}_{\mathrm{sol}}} \cap \mathbf{ER}]$ *in the complete lattice of all semidirectly closed pseudovarieties of finite monoids has the cardinality of the continuum.*

Proof. Once again, according to the considerations carried out at the close of the previous section, all that remains to be done is to show a continuum of semidirectly

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closed pseudovarieties of finite groups containing the pseudovariety $\mathbf{G}_{\mathrm{sol}}$. Observe that the statement that there is a continuum of semidirectly closed pseudovarieties of finite groups containing the pseudovariety $\mathbf{G}_{\mathrm{sol}}$ subsumes the statement appearing in the proof of Theorem 4.1 saying that there is a continuum of semidirectly closed pseudovarieties of finite groups containing the pseudovariety $\mathbf{G}_p$, for any given prime number p. In order to achieve our objective, we exhibit first a suitable family of finite simple groups lying beyond the pseudovariety $\mathbf{G}_{\mathrm{sol}}$.

Let us first recall some standard concepts from finite group theory. For arbitrary natural numbers n, k and for any prime number p, one denotes by $\mathrm{GL}(n, p^k)$ the general linear group consisting of all invertible $n \times n$ matrices with entries in the finite field of order p^k. This set of matrices forms a group under matrix multiplication. The center $\zeta\mathrm{GL}(n, p^k)$ of this group consists of all non-zero scalar matrices. The quotient group $\mathrm{GL}(n, p^k)/\zeta\mathrm{GL}(n, p^k)$ is called the projective general linear group and it is denoted by $\mathrm{PGL}(n, p^k)$. Further on, one denotes by $\mathrm{SL}(n, p^k)$ the special linear group consisting of all $n \times n$ matrices with entries in the finite field of order p^k whose determinant is equal to 1. The center $\zeta\mathrm{SL}(n, p^k)$ of this group is equal to the intersection $\mathrm{SL}(n, p^k) \cap \zeta\mathrm{GL}(n, p^k)$. The quotient group $\mathrm{SL}(n, p^k)/\zeta\mathrm{SL}(n, p^k)$ is called the projective special linear group and it is denoted by $\mathrm{PSL}(n, p^k)$. Now it is a standard fact from finite group theory that, assuming $n \geqslant 2$, this group $\mathrm{PSL}(n, p^k)$ is simple, except when $n = 2$, $k = 1$ and $p = 2$ or $p = 3$. See [**3**, § 6] for a proof of this fact, for instance.

We continue by selecting from the latter family of finite simple groups an infinite subset, which will be suitable for our purposes. We will consider further only the case of 2×2 matrices, that is, we will confine ourselves to the instance $n = 2$ in the previous deliberations. Moreover, we will consider further only finite fields of prime orders, that is, we will limit ourselves to the instance $k = 1$ in the above definitions. Thus we will think merely of projective special linear groups of the form $\mathrm{PSL}(2, p)$ where $p \geqslant 5$ is a prime number. As it is shown in [**3**, § 6], the order of such a group $\mathrm{PSL}(2, p)$ is equal to the number $\frac{1}{2}(p^3 - p)$. At last, we will still narrow the scope of possible values of the prime number p to a certain subsequence of prime numbers, which will be constructed as follows.

We will need Dirichlet's theorem on arithmetic progressions from number theory. According to this theorem, whenever a, b are two relatively prime natural numbers, there exist infinitely many prime numbers in the sequence of natural numbers $\{am + b\}_{m=1}^{\infty}$. See the paper [**7**] by Selberg for a relatively elementary proof of Dirichlet's theorem. Note also that this theorem can be easily seen to be equivalent to the seemingly weaker assertion that, for any two relatively prime natural numbers a, b, there exists at least one prime number in the sequence of natural numbers $\{am + b\}_{m=1}^{\infty}$.

Now the subsequence of prime numbers $\{q_i\}_{i=1}^{\infty}$ as advised in the last paragraph but one will be constructed by induction in the following manner. Firstly, let $q_1 = 5$. Secondly, assume that for some natural number j, the prime numbers $q_1, q_2, \ldots, q_j$ such that $q_1 < q_2 < \ldots < q_j$ have already been constructed. Then all of these prime numbers are odd, and hence the product $q_1 q_2 \ldots q_j$ is an odd number, and so it is relatively prime to the number 2. Therefore, by Dirichlet's theorem, there exists a natural number m such that $q_1 q_2 \ldots q_j m + 2$ is a prime number. Then put $q_{j+1} = q_1 q_2 \ldots q_j m + 2$. Then, of course, $q_j < q_{j+1}$. In this way, proceeding by induction, the entire subsequence of prime numbers $\{q_i\}_{i=1}^{\infty}$ is determined.

Now consider the projective special linear groups $\mathrm{PSL}(2, q_i)$, for all natural numbers i. For every subset A of the set $\{1, 2, \ldots\}$ of all natural numbers, consider the semidirectly closed pseudovariety of finite groups $\mathbf{G}_A$ generated by the family of all cyclic groups of prime orders and by the family of projective special linear groups of the form $\mathrm{PSL}(2, q_j)$, for all $j \in A$. Then, of course, the pseudovariety $\mathbf{G}_A$ contains the pseudovariety $\mathbf{G}_{\mathrm{sol}}$ of all finite solvable groups. We are going to show thereinafter that, for distinct subsets A of the set $\{1, 2, \ldots\}$ of all natural numbers, the corresponding pseudovarieties $\mathbf{G}_A$ are also distinct.

For this purpose, let us recall yet another familiar fact about finite simple groups. It turns out that every finite simple group H is prime, which means that, whenever H divides a semidirect product $K * L$ of two groups K and L, then either H divides K or H divides L. See [**6**, Chapter 4, § 1] by Lallement for a proof of this assertion. Hence it also readily follows that, for any class $\mathbf{C}$ of finite groups, the following statement holds true. If a finite simple group H belongs to the semidirectly closed pseudovariety of finite groups generated by the mentioned class $\mathbf{C}$, then the group H itself divides some group from this class $\mathbf{C}$.

Finally, we are in a position to complete our actual proof. We want to show that, for every subset A of the set $\{1, 2, \ldots\}$ of all natural numbers, the pseudovariety $\mathbf{G}_A$ of finite groups introduced in the last paragraph but one contains the finite simple groups $\mathrm{PSL}(2, q_j)$, for all $j \in A$, but it does not contain the finite simple groups $\mathrm{PSL}(2, q_{\bar{\jmath}})$, for any natural numbers $\bar{\jmath} \notin A$. The former of these two statements is clear. We proceed to verify the latter of these two statements. Thus let $\bar{\jmath} \notin A$ be any natural number and assume, by contradiction, that the finite simple group $\mathrm{PSL}(2, q_{\bar{\jmath}})$ belongs to the pseudovariety $\mathbf{G}_A$. We have seen above that the order of the group $\mathrm{PSL}(2, q_{\bar{\jmath}})$ is equal to the number $\frac{1}{2}(q_{\bar{\jmath}}^3 - q_{\bar{\jmath}})$. Consequently, the group $\mathrm{PSL}(2, q_{\bar{\jmath}})$ cannot divide any cyclic group of prime order, since its order is a composite number. The group $\mathrm{PSL}(2, q_{\bar{\jmath}})$ also cannot divide any finite simple group of the form $\mathrm{PSL}(2, q_i)$ with $i < \bar{\jmath}$, since the order of such a group is less than the order of the given group $\mathrm{PSL}(2, q_{\bar{\jmath}})$. Therefore, according to what has been said in the previous paragraph, our group $\mathrm{PSL}(2, q_{\bar{\jmath}})$ must divide some finite simple group of the form $\mathrm{PSL}(2, q_j)$ with $j \in A$ satisfying $j > \bar{\jmath}$. As we have also seen above, the order of this group $\mathrm{PSL}(2, q_j)$ is equal to the number $\frac{1}{2}(q_j^3 - q_j)$, that is, it is equal to the number $\frac{1}{2} q_j (q_j - 1)(q_j + 1)$. Thus the prime number $q_{\bar{\jmath}}$ must divide the number $q_j(q_j - 1)(q_j + 1)$. But $q_{\bar{\jmath}}$ cannot divide the number q_j, since q_j and $q_{\bar{\jmath}}$ are distinct prime numbers. Furthermore, by the construction of the sequence of prime numbers $\{q_i\}_{i=1}^{\infty}$, the number $q_j - 1$ is of the form $q_1 q_2 \ldots q_{j-1} m + 1$ for some natural number m, and the number $q_{\bar{\jmath}}$ occurs among the numbers $q_1, q_2, \ldots, q_{j-1}$. This shows that $q_{\bar{\jmath}}$ cannot divide the number $q_j - 1$. In a similar manner, the number $q_j + 1$ is of the form $q_1 q_2 \ldots q_{j-1} m + 3$ for the same natural number m. This yields that $q_{\bar{\jmath}}$ also cannot divide the number $q_j + 1$, since it occurs among the numbers $q_1, q_2, \ldots, q_{j-1}$ and it is greater than 3. All in all, this entails that the prime number $q_{\bar{\jmath}}$ cannot divide the number $q_j(q_j - 1)(q_j + 1)$, which is the desired contradiction. Therefore we may conclude that the pseudovariety $\mathbf{G}_A$ contains exactly those finite simple groups of the form $\mathrm{PSL}(2, q_j)$, for which $j \in A$. This ensures that, for distinct subsets A of the set $\{1, 2, \ldots\}$ of all natural numbers, the corresponding semidirectly closed pseudovarieties $\mathbf{G}_A$ are also distinct. In this way, we gain a collection of semidirectly closed pseudovarieties of finite groups containing the pseudovariety $\mathbf{G}_{\mathrm{sol}}$, which has the cardinality of the continuum. □

5. The non-locality

In this concluding section of the present paper we show that, for every proper non-trivial semidirectly closed pseudovariety $\mathbf{H}$ of finite groups and for every proper semidirectly closed pseudovariety $\mathbf{K}$ of finite groups containing the pseudovariety $\mathbf{H}$, the semidirectly closed pseudovariety of finite monoids $\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K})$ lying in the interval $[\mathbf{R} * \mathbf{H}, \overline{\mathbf{H}} \cap \mathbf{ER}]$, which has been discussed in the closing paragraph of Section 3, is not local. Thus, in particular, the uncountably many semidirectly closed pseudovarieties of finite monoids provided in Theorems 4.1 and 4.2 are all not local. This piece of information will prove useful in a sequel to the present paper.

As in Section 3, take again the set $X_n = \{1, 2, \ldots, n\}$ and consider arbitrary partial one-to-one mappings $\beta_1, \beta_2, \ldots, \beta_k$ of the set X_n into itself. Let U be the submonoid of the symmetric inverse monoid on the set X_n generated by the one-to-one mappings $\beta_1, \beta_2, \ldots, \beta_k$. Let also $X'_n = \{1', 2', \ldots, n'\}$ be a disjoint copy of the set X_n. In Section 3, we have built a submonoid $M(U)$ of the symmetric inverse monoid on the set $X_n \cup X'_n$. Now we will build in a similar fashion a category $C(U)$ having for its vertices the sets X_n and X'_n and having for its edges certain partial one-to-one mappings determined as follows. At first, as in Section 3, for every subset $Z = \{i_1, i_2, \ldots, i_r\}$ of the set X_n, denote by Z' the set $\{i'_1, i'_2, \ldots, i'_r\}$. Further on, for every $j \in \{1, 2, \ldots, k\}$, let D_j be the domain of the mapping β_j and let R_j be the range of the mapping β_j. Then, for every $j \in \{1, 2, \ldots, k\}$, we let β'_j be the partial one-to-one mapping of the set X_n into the set X'_n such that the domain of β'_j is the set D_j, the range of β'_j is the set R'_j, and the mapping β'_j itself is given by the formula $\beta'_j(h) = \beta_j(h)'$, for all $h \in D_j$. In addition, we let α' be the mapping whose domain is the set X_n, whose range is the set X'_n, and which is itself given by the formula $\alpha'(h) = h'$, for all $h \in X_n$. Furthermore, let B_n be the aperiodic Brandt semigroup consisting of all partial one-to-one mappings of the set X_n into itself of rank no more than 1, let B^1_n be the monoid obtained from the semigroup B_n by adjoining the identity on the set X_n to it, let B'_n be the aperiodic Brandt semigroup consisting of all partial one-to-one mappings of the set X'_n into itself of rank no more than 1, and let B'^1_n be the monoid obtained from the semigroup B'_n by adjoining the identity on the set X'_n to it. Having all that at hand, we determine the hom-sets of the category $C(U)$ in the following way. We let the hom-set $C(U)(X_n)$ be equal to the monoid B^1_n, we let the hom-set $C(U)(X'_n)$ be equal to the monoid B'^1_n, we let the hom-set $C(U)(X_n, X'_n)$ consist of the partial one-to-one mappings $\alpha', \beta'_1, \beta'_2, \ldots, \beta'_k$ together with all partial one-to-one mappings of the set X_n into the set X'_n of rank no more than 1, and we let the hom-set $C(U)(X'_n, X_n)$ consist of all partial one-to-one mappings of the set X'_n into the set X_n of rank no more than 1. Then it can be readily seen that $C(U)$ thus indeed becomes a category with respect to the usual composition of partial one-to-one mappings.

We next clarify what is the relationship between our present category $C(U)$ and the monoid $M(U)$ constructed in Section 3. On the one hand, the category $C(U)$ divides the monoid $M(U)$. In order to see this, consider, for each of the hom-sets $C(U)(X_n)$, $C(U)(X'_n)$, $C(U)(X_n, X'_n)$, and $C(U)(X'_n, X_n)$ of the category $C(U)$, the obvious identity mapping of this hom-set into the set $M(U)$. These identity mappings constitute together, in fact, a faithful homomorphism of the category $C(U)$ into the monoid $M(U)$. Thus, indeed, our category $C(U)$ divides the monoid $M(U)$.

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On the other hand, the monoid $M(U)$ can be produced from the category $C(U)$ in the following way. In order to present here how this can be done, we need the concept of the consolidation of a category introduced by Tilson in [**10**, § 3]. According to his definition, the consolidation of a category D is the semigroup S whose set of elements consists of the set $E(D)$ of all edges of D together with a new element 0. This new element 0 is a multiplicative zero in S, and the product of any two edges from $E(D)$ is as in D when it is defined and it is equal to 0 otherwise. Having this concept at hand, we claim that our initial monoid $M(U)$ can be seen to divide the monoid T^1 obtained by adjoining an identity to the consolidation T of our current category $C(U)$. Indeed, it is possible to draw out the monoid $M(U)$ from the monoid T^1 by deleting first the identities on the sets X_n and X'_n from T^1, and then by identifying the empty mappings from all of the hom-sets $C(U)(X_n)$, $C(U)(X'_n)$, $C(U)(X_n, X'_n)$, and $C(U)(X'_n, X_n)$ of the category $C(U)$ with the zero 0 of the consolidation T of $C(U)$.

Once again, as in Section 3, we will apply the above machinery in the situation when U will be a finite group G. For this purpose, this group G must be represented as a permutation group. Just as before, in the paragraph following Theorem 3.1 in Section 3, we will use in this connection the usual right regular representation of the group G by its right translations. Having this in view, by $C(G)$ we will mean the category $C(U)$ where U will be the subgroup of the symmetric group on the set G formed by all right translations of the group G by its arbitrary elements. Now we are ready to state and prove the following assertion.

Theorem 5.1. *Let $\mathbf{H}$ be any non-trivial pseudovariety of finite groups. Then, for every finite group G, we have $C(G) \in g(\mathbf{R} * \mathbf{H})$ if and only if $G \in \mathbf{H}$.*

Proof. In view of Theorem 3.2, we need to show that, for every finite group G, we have $M(G) \in \mathbf{R} * \mathbf{H}$ if and only if $C(G) \in g(\mathbf{R} * \mathbf{H})$. We have seen above that the category $C(G)$ divides the monoid $M(G)$. Thus if $M(G) \in \mathbf{R} * \mathbf{H}$, then certainly $C(G) \in g(\mathbf{R} * \mathbf{H})$. Consequently, we are left with the need to prove the converse statement.

Thus assume that $C(G) \in g(\mathbf{R} * \mathbf{H})$. Since the pseudovariety $\mathbf{H}$ is non-trivial, the pseudovariety $\mathbf{R} * \mathbf{H}$ contains the monoid B_2^1 obtained by adjoining an identity to the five-element aperiodic Brandt semigropup B_2. Now we are in a position to apply Proposition 13.4 from [**10**]. By this proposition, the monoid T^1 obtained by adjoining an identity to the consolidation T of the catgegory $C(G)$ belongs to the pseudovariety $\mathbf{R} * \mathbf{H}$. Or else, since $C(G) \in g(\mathbf{R} * \mathbf{H})$, $C(G)$ divides some monoid F from $\mathbf{R} * \mathbf{H}$. Then it is quite easy to see that the monoid T^1 obtained from the consolidation T of $C(G)$ divides the direct product $F \times B_2^1$. See also [**10**, Proposition 3.3]. Therefore this monoid T^1 indeed belongs to the pseudovariety $\mathbf{R} * \mathbf{H}$. We have seen above that the monoid $M(G)$ divides the mentioned monoid T^1. Hence it ensues that $M(G) \in \mathbf{R} * \mathbf{H}$, as desired. □

Finally let us return to what has been promised at the beginning of this section. We want to show that, for arbitrary proper non-trivial pseudovarieties $\mathbf{H}$ and $\mathbf{K}$ of finite groups such that $\mathbf{H} \subseteq \mathbf{K}$, the pseudovariety of finite monoids $\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K})$ is not local. For this purpose, let us take any finite group G such that $G \notin \mathbf{K}$. Consider the corresponding category $C(G)$. Then, by its construction, the local monoids of the category $C(G)$ are just the monoids B_n^1 and $B_n'^1$, where $n = |G|$. We have already mentioned above

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that the monoid B_2^1 belongs to the pseudovariety $\mathbf{R} * \mathbf{K}$. Since the monoids B_n^1 and $B_n'^1$ can be easily seen to divide a suitable finite direct power of B_2^1, also these monoids belong to the pseudovariety $\mathbf{R} * \mathbf{K}$. Moreover, the monoids B_n^1 and $B_n'^1$ are aperiodic, and hence they belong also to the pseudovariety $\overline{\mathbf{H}}$. Altogether these monoids B_n^1 and $B_n'^1$ belong to the pseudovariety $\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K})$. Therefore the category $C(G)$ belongs to the pseudovariety $\ell(\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K}))$. But, since $G \notin \mathbf{K}$, from Theorem 5.1 it follows that this category $C(G)$ does not belong to the pseudovariety $g(\mathbf{R} * \mathbf{K})$. Consequently, this category does not belong to the pseudovariety $g(\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K}))$. Thus $g(\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K}))$ is a proper subpseudovariety of $\ell(\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K}))$. This finding verifies that the pseudovariety $\overline{\mathbf{H}} \cap (\mathbf{R} * \mathbf{K})$ is not local.

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