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REPRESENTATIONS AND SUNSPOT STABILITY

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By endowing his agents with simple forecasting models, or *representations*, M. Woodford ("Learning to Believe in Sunspots," *Econometrica* 58, 277–307, 1990) found that finite state Markov sunspot equilibria may be stable under learning. We show that common factor representations generalize to all sunspot equilibria the representations used by Woodford. We find that if finite state Markov sunspots are stable under learning then *all* sunspots are stable under learning, provided common factor representations are used.

Keywords: Indeterminacy, Sunspot Equilibria, E-Stability

1. A BRIEF HISTORY OF STABLE SUNSPOTS

Sunspot equilibria provide avenues through which agents' expectations can drive fluctuations in real economic activity. Interest in these equilibria developed through the work of Shell (1977), Azariadis (1981), Cass and Shell (1983), and Guesnerie (1986), but remained couched primarily in the theoretical literature until Benhabib and Farmer (1994) and Farmer and Guo (1994) demonstrated the existence of sunspot equilibria in RBC-type models modified to incorporate externalities or monopolistic competition: see Farmer (1999) for a detailed development. These authors, and many other since, have used calibrated DSGE models to argue that fluctuations in agents' expectations explain at least part of the business cycle. These arguments have been extended to New Keynesian monetary models: Clarida et al. (2000) and Lubik and Schorfheide (2004) suggest that passive monetary policy in the seventies produced an economic environment conducive to sunspot equilibria and the associated high volatility.

The simple existence of sunspot equilibria in a model does not imply their relevance: it may not be possible for agents to coordinate their behavior appropriately. A benchmark coordination device in macroeconomics is stability under learning: for details, see Evans and Honkapohja (2001).

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In an OLG model, Woodford (1990) found that finite state Markov sunspots may be stable under learning, thus lending credence to the relevance of sunspots to applied models, and in part inspiring the work of Farmer and others. However, Evans and Honkapohja (2001) found that the sunspot equilibria studied by Farmer and Guo, which, unlike Woodford's finite state sunspots, are driven by martingale difference sequences with continuous support, are not stable under learning. This finding was further supported by Evans and McGough (2005a) and Duffy and Xiao (2007), who searched for stable sunspots in a host of RBC-type models and found none.

The stability of equilibria may depend on the values of the model's parameters; this has been known since Bray and Savin (1986). However, that Woodford's sunspots are stable and Farmer's sunspots are not cannot be explained so easily. In the linearized version of Woodford's model, Farmer's sunspots are *never* stable: see Evans and Honkapohja (2003). Perhaps then the explanation lies in the stochastic properties of the equilibria in question; after all, Woodford's sunspots are finite state and Farmer's sunspots have continuous support. However, as we will see below, the stochastic nature of an equilibrium has no impact on its stability.

The stability of sunspot equilibria turns on the way they are viewed by private agents. More specifically, a given equilibrium may often be associated with a particular recursive formulation: for example, Farmer considers equilibria in AR(1) form. We call these recursions *representations*. When stability under learning is investigated, a representation specifies a natural functional form for the forecasting model that agents estimate and use to form their expectations; and, in the case of indeterminacy, a given equilibrium may have several natural recursions associated with it.[1] It is known that the stability of a particular equilibrium may depend in part on the form of the forecasting model used by agents. Evans and Honkapohja (2003) provide an early example of this phenomenon: in a forward-looking linear model they show that, under a natural representation, finite state Markov sunspot equilibria are stable for a subset of the parameter region, but that if agents use an AR(1) representation to form forecasts then these equilibria are unstable. In Evans and McGough (2005b), we obtain another striking example: we investigate the relationship between representations and E-stability in models with lags and under the assumption that the extrinsic sunspot process has continuous support; we find that sunspot equilibria previously thought to be unstable under learning become stable if agents use a forecasting model consistent with what we call a *common-factor representation*.

The notion of a representation allows us to fully investigate why, in a linearized version of Woodford's model, Farmer's sunspots are never stable and Woodford's sunspots sometimes are. In this paper we show that the representation used in Woodford's analysis is, in fact, a special case of a common-factor representation we developed in Evans and McGough (2005b); indeed, common-factor representations generalize to all sunspot equilibria the learning mechanism used by Woodford for finite state Markov processes.[2] We conclude that whenever these finite state Markov sunspots are stable under learning, all sunspot equilibria will be stable, provided a common-factor representation is used.

2. EXPECTATIONAL STABILITY AND ADAPTIVE LEARNING

Although stability under adaptive learning has become the dominant equilibrium selection mechanism in macroeconomics, its language and techniques are still relatively new and warrant discussion. We couch this discussion within the context of the nonstochastic linearized version of Woodford's model, as it is sufficient both for the development of adaptive learning analysis and for our main results, which are presented in the next section. The model is given by

$$y_t = \beta E_t y_{t+1}, \tag{1}$$

where $y_t \in \mathbf{R}$. A rational expectations equilibrium (REE) is any bounded stochastic process y_t satisfying (1).[3] Sunspot equilibria exist in this model provided that $|\beta| > 1$: we consider this case in detail in Section 3 below. Here, for our introductory discussion on learning, we assume that $|\beta| < 1$, in which case the unique REE is given by $y_t = 0$.

Fully rational agents know that $y_{t+1} = 0$ and they form expectations accordingly. We back off the assumption of full rationality and instead model our agents as adaptive learners: we assume that agents form expectations using a forecasting model, or perceived law of motion (PLM), which has a functional form consistent with the equilibrium under examination; specifically,

$$\text{PLM}: y_t = a + \varepsilon_t, \tag{2}$$

where a is the perceived conditional mean and ε_t is a perceived error term.

As adaptive learners, agents use past data to estimate their forecasting models; but, as discussed below, the E-stability principle allows us to bypass this estimation procedure and instead focus on a stylized notion of learning. Given the perceived law of motion (2), agents form the forecast $E_t y_{t+1} = a$. This forecast may be imposed on the model (1), thus generating the time t value of y. This value identifies the relationship between y_t and the regressors in the agents' forecasting model; this relationship is known as the actual law of motion (ALM):[4]

$$\text{ALM}: y_t = \beta a. \tag{3}$$

The ALM defines a function, known as the T-map, that takes perceived coefficients to actual coefficients; in this case, $T(a) = \beta a$. A fixed point of the T-map indicates the alignment of perceived and actual coefficients and thus corresponds to a rational expectations equilibrium.

Under adaptive learning, agents use recursive least squares (RLS) or other updating algorithms to reestimate their forecasting model. The economy's rational expectations equilibrium is said to be *stable under learning* if these estimates converge to the associated fixed point of the T-map. Even within the context of a simple linear model such as (1), asymptotic analysis of the agents' estimators is nontrivial and relies on the theory of stochastic recursive algorithms: see Evans and Honkapohja (2001) for details. However, stylized learning via the notion of E-stability provides a simple, tractable alternative, which, according to the E-stability principle, provides conditions sufficient to guarantee stability under adaptive learning.

E-stability analysis proceeds as follows: using the T-map, we may write down the ordinary differential equation $\dot{a} = T(a) - a$. Notice that a rest point a^* of this ODE corresponds to an REE of (1). We say that the REE is *E-stable* if it corresponds to a Lyapunov stable fixed point of the ode. The *E-stability principle* states that E-stable REE are locally learnable under least squares or related learning algorithms.

The appeal of E-stability is in part due to the tractable nature of its computation: a sufficient condition for Lyapunov stability is that the eigenvalues of DT (the T-map's derivative evaluated at the fixed point) have real part less than unity.[5] This formulation also provides a converse to the E-stability principle: if DT has an eigenvalue with real part larger than one, then the agents' estimators do not converge to the T-map's fixed point. Thus we say that a rational expectations equilibrium is E-stable if all eigenvalues of DT have real part less than one and we say that it is E-unstable if at least one eigenvalue has real part greater than one. For the univariate model under consideration in this section, E-stability analysis is particularly simple to perform: $DT = \beta \in (-1, 1)$ so that the unique REE is always stable under learning.

3. REPRESENTATIONS AND SUNSPOT EQUILIBRIA

We return to the model (1), reproduced here for convenience,

$$y_t = \beta E_t y_{t+1},$$

and now examine the case $|\beta| > 1$. Let y_t be an REE, and set $\varepsilon_t = y_t - E_{t-1} y_t$. Then y_t satisfies the recursion

$$y_t = \beta^{-1} y_{t-1} + \varepsilon_t. \tag{4}$$

We call this recursion the *general form representation* of the equilibrium y_t. Because y_t is bounded, we know that either $\varepsilon_t = 0$ (so that $y_{t+k} = 0$ for $k \in \mathbf{Z}$) or $|\beta| > 1$; in the latter case, ε_t can be any martingale difference sequence (MDS) with uniformly bounded support. For the remainder of the paper we assume $|\beta| > 1$. Note that y_t is an REE of (3) if and only if there exists an MDS ε_t such that y_t satisfies (4). The MDS ε_t captures variation in y_t resulting from fluctuations in agents' expectations; it is often called a sunspot, and the associated REE y_t is often called a sunspot equilibrium. If ε_t has continuous support then so does y_t and (4) describes a standard AR(1) process, which is the type of sunspot solution studied in Farmer (1999).

Let y_t be an REE of (3), and consider its stability by providing agents with a PLM consistent with the representation (4),

$$y_t = \theta' X_t = \theta_1 + \theta_2 y_{t-1} + \theta_3 \varepsilon_t,$$

where $X_t = (1, y_{t-1}, \varepsilon_t)'$ is the vector of regressors. The actual law of motion is

$$y_t = T(\theta)' X_t = \beta(1+\theta_2)\theta_1 + \beta\theta_2^2 y_{t-1} + \beta\theta_2\theta_3\varepsilon_t,$$

giving the T-map

$$\begin{aligned} \theta_1 &\to \beta(1+\theta_2)\theta_1, \\ \theta_2 &\to \beta\theta_2^2, \\ \theta_3 &\to \beta\theta_2\theta_3. \end{aligned} \qquad \textbf{(5)}$$

Computing the Jacobian yields

$$DT = \begin{pmatrix} \beta(1+\theta_2) & \beta\theta_1 & 0 \\ 0 & 2\beta\theta_2 & 0 \\ 0 & \beta\theta_3 & \beta\theta_2 \end{pmatrix}. \qquad \textbf{(6)}$$

Evaluating DT at the fixed point $\theta^* = (0, \beta^{-1}, \theta_3)$ corresponding to the sunspot equilibrium identified above yields an eigenvalue equal to two; thus, by the converse of the E-stability principle, the REE is unstable.[6] We obtain the well-known result that y_t is not stable under learning, and because y_t was arbitrary, no sunspot equilibria of (3) are learnable—at least if agents use a PLM consistent with general form representations.

Equilibria generated by coordination on an arbitrary MDS were the type studied by Farmer and others in applied models; however, Woodford had a different type of sunspot in mind. Take as primitive a two-state Markov process $s_t \in \{\bar{s}_1, \bar{s}_2\} \equiv \{0, 1\}$, with transition matrix π: thus,

$$\pi_{ij} = \text{prob}\{s_t = \bar{s}_j | s_t = \bar{s}_i\}.$$

For any $\bar{y} \in \mathbf{R}^2$, we may construct the associated Markov process

$$y_t = \bar{y}_i \Leftrightarrow s_t = i - 1, \quad \text{for } i = 1, 2. \qquad \textbf{(7)}$$

Evans and Honkapohja (2003) showed that y_t is an REE of (3) if and only if the following two conditions hold:

$$\pi_{11} + \pi_{22} = 1 + \beta^{-1}, \qquad \textbf{(8)}$$

$$\bar{y}_2(1 - \pi_{11}) = -\bar{y}_1(1 - \pi_{22}). \qquad \textbf{(9)}$$

In case y_t is an REE, we call (7) its *natural representation*.[7] These y_t are the two-state Markov sunspots for the linear model (3) analogous to those studied in Woodford (1990).

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To analyze the stability under learning of these types of equilibria we follow Evans and Honkapohja (2003): we specify a PLM consistent with (7), that is, we assume that agents observe s_t, know that the transition matrix for s_t is π, and believe the equilibrium of the model is a two-state Markov process; but we assume they do not know the values of y_t in these two states. Instead, we provide agents with perceived values for the states, and thus we may identify their perceptions with points $\tilde{y} \in \mathbf{R}^2$. For given perceptions $\tilde{y}$, the PLM is formally given by[8]

$$y_t = \tilde{y}_i \Leftrightarrow s_t = i - 1, \quad \text{for} \quad i = 1, 2. \tag{10}$$

To obtain the ALM, notice that if agents observe $s_t = 0$ then they will forecast y_{t+1} as

$$\hat{E}_t(y_{t+1}|s_t = 0) = \pi_{11}\tilde{y}_1 + (1 - \pi_{11})\tilde{y}_2.$$

It follows that if $s_t = 0$ then

$$y_t = \beta\left(\pi_{11}\tilde{y}_1 + (1 - \pi_{11})\tilde{y}_2\right).$$

A symmetric computation holds for $s_t = 1$, so that if agents have perceptions given by $\tilde{y}$ then the economy follows a two-state Markov process with transition matrix π and states $\hat{y} \in \mathbf{R}^2$ given by

$$\hat{y} = \beta\pi\tilde{y} \equiv T_N(\tilde{y}). \tag{11}$$

We call $T_N : \mathbf{R}^2 \to \mathbf{R}^2$ the "natural T-map."

When perceptions and truth coincide, that is, when $T_N(\tilde{y}) = \tilde{y}$, an REE is identified. It follows that $\tilde{y}$ is an REE if and only if the matrix $\beta\pi$ has a unit eigenvalue and $\tilde{y}$ is an associated eigenvector: these conditions are guaranteed by the restrictions (8) and (9).

Now notice that $DT_N = \beta\pi$. The eigenvalues of DT_N evaluated at a fixed point are unity and β, which means that the associated REE are E-stable precisely when $\beta < -1$. Appealing to the E-stability principle, we obtain Evans and Honkapohja's conclusion that the two-state Markov sunspot equilibria y_t are stable under learning provided $\beta < -1$. This is the result for the linear model analogous to the celebrated stability result of Woodford (1990).

Above we noted that if y_t is an REE then there is an MDS ε_t such that y_t satisfies (4); thus finite state Markov sunspot equilibria must also have general form representations. Also, even in this case, if agents use a forecasting model consistent with (4) then the equilibrium is unstable. We conclude, as did Evans and Honkapohja, that stability of finite state Markov sunspots under learning is representation-dependent.

We now turn to the main question of this paper: "What's so special about finite state Markov sunspots?" The answer, of course, is "Nothing." We will now show that if $\beta < -1$ then all sunspots are stable under learning, provided agents have perceptions consistent with Woodford's natural PLM. To facilitate the argument, we first show how to write Woodford's finite state PLM in a way that naturally generalizes to all sunspot equilibria, regardless of support cardinality; and to do this, we begin by constructing the general form representation of a two-state Markov sunspot equilibrium $\bar{y} \in \mathbf{R}^2$ (the equilibrium is again associated with the fundamental process s_t with transition matrix π).

Define a stochastic process $\varepsilon_t(s_{t-1}, s_t)$ as follows:

s_{t-1}	s_t	$\varepsilon_t(s_{t-1}, s_t)$
0	0	$(1-\beta^{-1})\bar{y}_1$
0	1	$\bar{y}_2 - \beta^{-1}\bar{y}_1$
1	0	$\bar{y}_1 - \beta^{-1}\bar{y}_2$
1	1	$(1-\beta^{-1})\bar{y}_2$

Because the two-state process $(\bar{y}, \pi)$ satisfies the restrictions (8) and (9), it can be shown that ε_t is a martingale difference sequence, and further, by construction, the two-state process $(\bar{y}, \pi)$ solves

$$y_t = \beta^{-1} y_{t-1} + \varepsilon_t. \tag{12}$$

Equation (12) is the general form representation of the two-state sunspot equilibrium $\bar{y}$.

Using the lag operator, we may solve (12) for y_t to obtain

$$\begin{aligned} y_t &= \eta_t, \\ \eta_t &= (1-\beta^{-1}L)^{-1}\varepsilon_t. \end{aligned} \tag{13}$$

Equation (13) is the *common-factor representation* of the two-state sunspot equilibrium $\bar{y}$. Because y_t is a two-state Markov process, it follows that η_t is a two-state Markov process. We think of η_t as a serially correlated extrinsic noise process on which agents coordinate to form expectations, and we call it a *common-factor sunspot*.[9]

That an REE may have multiple representations raises an obvious question: is one representation more natural than another? This question cannot be answered from the perspective of rationality: the stochastic structure of the equilibrium is independent of the representation, so rational agents will arrive at the same forecasts regardless of the representation they are assumed to use. To compare different representations of a given equilibrium, then, we must consider the behavior of, and the information available to, an adaptive learner.

A representation provides the functional form of the forecasting model used by the adaptive agent. Within the context of (3), a general form representation requires that agents view a nonforecastable sunspot shock, and that they regress current y on lagged y and the sunspot: this has some appeal in that little structure is placed on the extrinsic sunspot shock, but it also is somewhat awkward in the sense that agents incorporate a lag into their regression model when no such lag is indicated by the economic model.[10]

Within the context of (3), a common-factor representation requires that agents view and regress y on an exogenous serially correlated shock. The simplicity of the model under examination makes this representation and associated forecasting model seem somewhat forced; however, the common factor representation has a natural, and quite general interpretation. Every linear model has a finite number of minimal state-variable (MSV) solutions, which are usually viewed as fundamental; for the model (3) there is a unique MSV solution, given by $y_t = 0$. A common-factor representation is obtained by simply appending a common-factor sunspot to an MSV solution. In particular, when using a common-factor representation, agents include in their regression model only those lagged endogenous variables indicated by the economic model.[11] For an extended discussion of common-factor representations and their relation to minimal state variable solutions, see Evans and McGough (2005b).

To study the stability under learning of y_t, we provide agents with a PLM consistent with (13):

$$y_t = a + b\eta_t. \tag{14}$$

Notice that because η_t is a two-state Markov process, the perceptions identified by this PLM are entirely analogous to the perceptions identified by Woodford's natural PLM (10): agents believe the economy follows a two-state Markov process. We compute

$$E_t y_{t+1} = a + bE_t\eta_{t+1},$$

and because

$$E_t\eta_{t+1} = \beta^{-1}\eta_t,$$

we find that the T-map associated with a common factor representation is

$$T_{CF}(a, b) = (\beta a, b).$$

It follows that the eigenvalues of DT_{CF} are 1 and β. We conclude that if agents use common-factor representations to form their forecasting models then y_t is stable under learning provided that $\beta < -1$, just as in Woodford.

Our stability argument was made in the context of an MDS constructed to replicate the two-state Markov sunspot $(\bar{y}, \pi)$; however, *nothing* in the argument relied on the two-state nature of the equilibrium: indeed, the common-factor representation was intentionally constructed to be independent of the specific properties of the MDS ε_t. To see this, let y_t be *any* REE of the model, and let ε_t be the associated martingale difference sequence. Let $\eta_t = (1 - \beta^{-1}L)^{-1}\varepsilon_t$. Then $y_t = \eta_t$. Again, provide agents with the PLM (14). Exactly the same T-map obtains. We conclude that if $\beta < -1$ then *all* sunspot equilibria are stable under learning, provided that agents use a common-factor representation as their forecasting model.

Although this result generalizes Woodford's stability result to all sunspots, we have yet to formally establish the connection between the stability of common-factor representations and the stability of natural representations. Recall our

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primitive assumption that agents believe the economy follows a two-state Markov process with transition matrix π, where π satisfies (8). Let Γ be the set of all two-state Markov processes with transition matrix π and notice that Γ may be identified with $\mathbf{R}^2$. We think of Γ as the set of all possible agent beliefs.

We may also think of Γ as the set of all PLMs consistent with natural representations: simply recall that s_t is the primitive sunspot with transition matrix π, let $y \in \Gamma$, and see equation (7). Although the identification of Γ with this set is, in a sense, trivial, distinguishing the sets will aid clarity; therefore, let Γ_N be the set of all PLMs consistent with natural representations [as captured by $\mathbf{R}^2$ together with equation (7)] and let $S_N : \Gamma \to \Gamma_N$ be the identity map on $\mathbf{R}^2$. The map S_N, then, takes an agent's beliefs to the associated natural PLM.

Finally, we must characterize the set of all PLMs consistent with common-factor representations. To this end, let y_t be a two-state Markov sunspot equilibrium with transition matrix π and states $\bar{y} \in \mathbf{R}^2$ satisfying (9), let ε_t be the MDS such that y_t solves (4), and let η_t be the common factor sunspot generated by ε_t. Recall that η_t is a two-state Markov process with $\eta_t = \bar{y}_i$ if $s_t = i - 1$, for $i = 1, 2$. Now let Γ_{CF} be the set of PLMs (a, b) consistent with the common-factor representation of y_t, and notice that a PLM is uniquely determined by its coefficients: $a + b\eta_t = c + d\eta_t \Leftrightarrow a = c$ and $b = d$. Recalling that Γ is just $\mathbf{R}^2$, we may define $S_{CF} : \Gamma \longrightarrow \Gamma_{CF}$ by

$$S_{CF}(y) = \left(\frac{y_1(\bar{y}_2 - \bar{y}_1) - \bar{y}_1(y_2 - y_1)}{\bar{y}_2 - \bar{y}_1}, \frac{y_2 - y_1}{\bar{y}_2 - \bar{y}_1} \right)'.$$

Straightforward computation shows that S_{CF} is a bijection. Finally, define $S : \Gamma_N \longrightarrow \Gamma_{CF}$ by $S = S_{CF} \circ S_N^{-1}$ and note that S is bijective by construction.

PROPOSITION 1. *The following diagram commutes:*

$$\begin{tikzcd}
& \Gamma_N \arrow[r, "T_N"] \arrow[d, "S"'] & \Gamma_N \arrow[d, "S"] \arrow[dr, "S_N^{-1}"] & \\
\Gamma \arrow[ur, "S_N"] \arrow[r, "S_{CF}"'] & \Gamma_{CF} \arrow[r, "T_{CF}"'] & \Gamma_{CF} \arrow[r, "S_{CF}^{-1}"'] & \Gamma
\end{tikzcd}$$

A commutative diagram is an efficient way to make statements about equivalence of functions. To understand the diagram's meaning, start with a point in any set (the "initial" set) and pick any other set (the "final" set) that can be reached from the initial set by following a path of arrows. Under the composition of the functions corresponding to the arrows in your path, the point you chose in the initial set is mapped to a point in the final set. Because the diagram commutes, the path of arrows you chose to reach the final set is irrelevant. An important example is

$$S_N^{-1} \circ T_N \circ S_N = S_{CF}^{-1} \circ T_{CF} \circ S_{CF}. \tag{15}$$

To prove this statement, note that S_N is the identity map; so it suffices to show that for any point $y \in \mathbf{R}^2$ it follows that

$$S_{CF} \circ T_N(y) = T_{CF} \circ S_{CF}(y).$$

Notice that

$$\hat{y} = T_N(y) = (\beta[\pi_{11}y_1 + (1-\pi_{11})y_2], \beta[(1-\pi_{22})y_1 + \pi_{22}y_2])'$$

so that, using (8), $\hat{y}_2 - \hat{y}_1 = y_2 - y_1$. Using this, direct computation yields

$$S_{CF} \circ T_N(y) =$$
$$\left(\frac{1}{\bar{y}_2 - \bar{y}_1}\left[\beta(\pi_{11}y_1 + (1-\pi_{11})y_2)(\bar{y}_2 - \bar{y}_1) - \bar{y}_1(y_2 - y_1)\right], \frac{y_2 - y_1}{\bar{y}_2 - \bar{y}_1}\right)'.$$

Using restrictions (8) and (9), we obtain that

$$\pi_{11}(\bar{y}_2 - \bar{y}_1) = \bar{y}_2 - \beta^{-1}\bar{y}_1.$$

Combining, we get

$$\begin{aligned}
&S_{CF} \circ T_N(y)\\
&= \left(\frac{1}{\bar{y}_2 - \bar{y}_1}\left[\beta\pi_{11}(\bar{y}_2 - \bar{y}_1)(y_1 - y_2) + \beta y_2(\bar{y}_2 - \bar{y}_1) - \bar{y}_1(y_2 - y_1)\right], \frac{y_2 - y_1}{\bar{y}_2 - \bar{y}_1}\right)'\\
&= \left(\frac{1}{\bar{y}_2 - \bar{y}_1}\left[(\beta\bar{y}_2 - \bar{y}_1)(y_1 - y_2) + \beta y_2(\bar{y}_2 - \bar{y}_1) - \bar{y}_1(y_2 - y_1)\right], \frac{y_2 - y_1}{\bar{y}_2 - \bar{y}_1}\right)'\\
&= \left(\frac{1}{\bar{y}_2 - \bar{y}_1}\left[\beta(y_2(\bar{y}_2 - \bar{y}_1) - \bar{y}_2(y_2 - y_1))\right], \frac{y_2 - y_1}{\bar{y}_2 - \bar{y}_1}\right)'\\
&= \left(\frac{1}{\bar{y}_2 - \bar{y}_1}\left[\beta(-y_2\bar{y}_1 + \bar{y}_2 y_1)\right], \frac{y_2 - y_1}{\bar{y}_2 - \bar{y}_1}\right)'\\
&= \left(\frac{1}{\bar{y}_2 - \bar{y}_1}\left[\beta(y_1(\bar{y}_2 - \bar{y}_1) - \bar{y}_1(y_2 - y_1))\right], \frac{y_2 - y_1}{\bar{y}_2 - \bar{y}_1}\right)'\\
&= T_{CF} \circ S_{CF}(y).
\end{aligned}$$

The commutativity of this diagram allows us to make several precise statements about the relationship between common factor representations and Woodford's natural representations.

COROLLARY 2. *PLMs consistent with common factor representations and PLMs consistent with natural representations identify the same set of agent beliefs.*

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COROLLARY 3. *Viewed as acting on agents' beliefs, the maps T_N and T_{CF} coincide.*

Corollary 2 acknowledges that S is bijective and Corollary 3 is an interpretation of equation (15). Taken together, these corollaries indicate the sense in which, when restricted to finite state Markov sunspot equilibria, common-factor representations may be identified with Woodford's natural representations; importantly, however, common-factor representations may be used to analyze the stability of any sunspot equilibrium. In this sense, we may view common-factor representations as a generalization of Woodford's natural representations to all sunspot equilibria.

Finally, by applying the chain rule and the inverse function theorem to equation (15), we find that the eigenvalues of DT_N and DT_{CF} coincide. Thus,

COROLLARY 4. *Common-factor representations and natural representations have the same stability properties.*

Furthermore, because the eigenvalues of the associated T-maps are independent of the cardinality of the sunspot's support, we may conclude that whenever finite state Markov sunspots are stable under learning, all sunspot equilibria are stable under learning, provided common-factor representations are used for the stability analysis.[12]

It may be useful to conclude this section with a summary of the key results on the stability of stationary sunspot equilibria for the model (3): (i) if $\beta > 1$ then sunspot equilibria are not stable under learning, regardless of the representation used by agents; (ii) if $\beta < -1$ then all sunspot equilibria are stable under learning, provided agents use common-factor (or natural) representations of the solutions, and (iii) although stability of sunspot equilibria depends on the parameter region and on the representation of the equilibrium, it does not depend on whether the sunspot solution has finite support.

4. CONCLUSIONS

The implications for learning stability, of distinguishing between REE and their representations, can be striking. For the forward-looking model, such as the linearization of the OG setup used by Woodford (1990), Evans and Honkapohja (2003) show that for a subset of the indeterminacy region, finite-state sunspot solutions can be stable under learning for a natural representation, whereas they are not stable when put into an AR(1) representation. One might think that this is a reflection of the type of sunspot equilibria being considered. In particular, one might hypothesize that in this model a sunspot solution with continuous support would never be stable under learning, because such a solution cannot be represented as a function of an exogenous finite-state Markov process. Our central finding is that this conjecture is incorrect. We show that for the subset of the indeterminacy region in which finite-state sunspot solutions are stable under learning using the natural representation, any sunspot equilibrium is also

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stable under learning, provided agents use what is known as the common-factor representation of the sunspot solution.

NOTES

1. A linear model is indeterminate if it admits multiple REE.

2. In this way, we provide a converse to the result of Evans and Honkapohja (2003): they find that finite state sunspot equilibria are unstable if agents use an AR(1) representation; we show that all sunspot equilibria are stable if agents use a common-factor representation, and provided the model's parameters are in the correct subset of the indeterminacy region.

3. We consider only doubly infinite processes.

4. When trying to learn the unique REE of the model (1) under the assumption that $|\beta| < 1$, agents regress on a constant; in general, the PLM, and hence the ALM, will condition on several explanatory variables.

5. In case the stability of sunspot equilibria in linear models is being analyzed, the appropriate E-stability condition is that the eigenvalues of DT be less than *or equal* to unity: for details see Evans and Honkapohja (2001, 2003).

6. Here, θ_3 can be any real number, reflecting the fact that if ε_t is an MDS then so too is $\theta_3\varepsilon_t$.

7. The term "natural representation" emphasizes the simple connection between the equilibrium dynamics of the REE and the forecasting model implied by the representation (7): agents believe that the economy transitions between unknown states as the sunspot s_t fluctuates, and their forecasting model naturally reflects these beliefs.

8. Incorporating an error into the PLM, as in (2), makes realistic the assumption that agents perceive the economy as a two-state process even though the observations do not support this perception. We omit the error term here without loss of generality to facilitate exposition. Also, stability under learning of a two-state Markov sunspot equilibrium does not require that agents know the transition matrix π: agents could use standard econometric techniques to estimate π and the stability conditions would be unaltered.

9. The name "common-factor representation" comes from their construction, which may be thought of as obtained by dividing out the common factor $(1-\beta^{-1}L)$. This construction is less trivial in higher dimensions and with the incorporation of lags into the reduced-form model. Also, in higher-dimensional models, the "factors" may be complex—this sometimes occurs, for example, in nonconvex business-cycle models; we have developed the construction of common-factor representations in the case of complex factors, but the paper detailing the analysis is still being written.

10. In fact, it is the inclusion of the unnecessary lagged endogenous variable in the representation that drives the instability result; in particular, reduced-form models with lags may have stable equilibria when agents use AR(1) representations.

11. One drawback of the common-factor representation is that the common-factor sunspot is required to satisfy a knife-edge "resonance frequency" condition—for the model (3), the common-factor sunspot's serial correlation must be β^{-1}—but this condition is an artifact of our focus on linear models: in the nonlinear case, open sets of resonance frequencies exist.

12. Although this conclusion applies only to the simple model considered in this paper, we conjecture that it holds more generally.

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