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ON POINTS WITH POSITIVE DENSITY OF THE DIGIT SEQUENCE IN INFINITE ITERATED FUNCTION SYSTEMS

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Abstract

Let $\{f_n\}_{n\geq 1}$ be an infinite iterated function system on $[0, 1]$ and let Λ be its attractor. Then, for any $x \in \Lambda$, it corresponds to a sequence of integers $\{a_n(x)\}_{n\geq 1}$, called the digit sequence of x, in the sense that

$$x = \lim_{n\to\infty} f_{a_1(x)} \circ \cdots \circ f_{a_n(x)}(1).$$

In this note, we investigate the size of the points whose digit sequences are strictly increasing and of upper Banach density one, which improves the work of Tong and Wang and Zhang and Cao.

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1. Introduction

We follow the notation used in [6] by Jordan and Rams. Let $\{f_n\}_{n\geq 1}$ be a sequence of C^1 functions with $f_n : [0, 1] \to [0, 1]$ satisfying the following.

(i) *Contraction property:* there exists an integer m and a real number $\rho \in (0, 1)$ such that for any $(a_1, \ldots, a_m) \in \mathbb{N}^m$ and $x \in [0, 1]$,

$$0 < |(f_{a_1} \circ \cdots \circ f_{a_m})'(x)| \leq \rho < 1.$$

(ii) *Separation condition:* for any $i \neq j \in \mathbb{N}$, $f_i((0, 1)) \cap f_j((0, 1)) = \emptyset$.

(iii) *Regular property:* if there exists a sequence $\xi = \{\xi_n\}_{n\geq 1}$ such that, for any $\epsilon > 0$, there exist $c_1(\epsilon)$ and $c_2(\epsilon)$ with $0 < c_1(\epsilon) \leq 1 \leq c_2(\epsilon)$ such that, for any $n \in \mathbb{N}$ and $x \in [0, 1]$,

$$\frac{c_1(\epsilon)}{\xi_n^{1+\epsilon}} \leq |f_n'(x)| \leq \frac{c_2(\epsilon)}{\xi_n^{1-\epsilon}},$$

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then we call $([0,1],\{f_n\}_{n\geq 1})$, or simply $\{f_n\}_{n\geq 1}$, an ξ-regular infinite iterated function system (ξ-regular iIFS). When $\xi_n = n^d$, the ξ-regular iIFS $\{f_n\}_{n\geq 1}$ is referred to as a d-decaying system, as defined by Jordan and Rams [6]. It is called a *Gauss-like* system if the system also fulfils the following.

(iv) $\overline{\bigcup_{n=1}^{\infty} f_n([0,1])} = [0,1]$ and, when $i < j$, $f_i(x) > f_j(x)$.

There is a natural projection $\Pi : \mathbb{N}^{\mathbb{N}} \to [0,1]$ defined as

$$\Pi(\underline{a}) = \lim_{n\to\infty} f_{a_1} \circ \cdots \circ f_{a_n}(1)$$

for any $\underline{a} = \{a_n\}_{n\geq 1} \in \mathbb{N}^{\mathbb{N}}$. Let Λ be the attractor of the iIFS $\{f_n\}_{n\geq 1}$, that is to say,

$$\Lambda = \Pi(\mathbb{N}^{\mathbb{N}}).$$

For each $x \in \Lambda$, it corresponds to a sequence of integers $\{a_n\}_{n\geq 1}$ in the sense that

$$x = \lim_{n\to\infty} f_{a_1} \circ \cdots \circ f_{a_n}(1).$$

We call $\{a_n\}_{n\geq 1}$ the digit sequence of x. It should be pointed out that the digit sequence of one point may not be unique. By ignoring at most a countable number of points, there is a one-to-one correspondence between a real number in $[0,1]$ and a sequence of integers. The study of the question of whether and when a subset of integers contains arbitrarily long arithmetic progressions is of important theoretical value, because of its close connections with number theory, dynamical systems and ergodic theory. A famous result due to Szemerédi [8] states that an integer subset contains arbitrarily long arithmetic progressions if it is of positive upper Banach density. This is proved again by Furstenberg [4] using ergodic theory. For recent progress, see the work of Green and Tao [5] and references therein.

Inspired by Szemerédi's theorem and the correspondence between a real number and a sequence of integers, the size of the set

$$\mathbb{E}_A = \{x \in \Lambda : \{a_n(x)\}_{n\geq 1} \text{ is strictly increasing and contains arbitrarily long arithmetic progressions}\}$$

was studied by Tong and Wang [9] in the case of continued fractions. Here the maps $f_n : [0,1] \to [0,1]$ can be defined by $f_n(x) = 1/(x+n)$ for each $n \in \mathbb{N}$. Its Hausdorff dimension (denoted by $\dim_{\mathrm{H}}$) is $1/2$. A similar result was obtained by Zhang and Cao [10] in the case of Lüroth expansion, where the maps $f_n : [0,1] \to [0,1]$ are defined by $f_n(x) = x/(n(n+1)) + 1/(n+1)$ for each $n \in \mathbb{N}$. The continued fraction system and the Lüroth system are both special 2-decaying Gauss-like systems.

In this note, we consider the above set for a general d-decaying Gauss-like system. We also discuss what happens for Gauss-like systems that are not d-decaying iIFSs. Actually, we study the set of points whose digit sequences are strictly increasing and of upper Banach density one as a subset of integers, namely,

$$\mathbb{E}_S := \{x \in \Lambda : \{a_n(x)\}_{n\geq 1} \text{ is strictly increasing and } \{a_n(x)\}_{n\geq 1} \text{ is of upper Banach density one as a subset of integers}\},$$

where the *upper Banach density* is defined as follows.

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DEFINITION 1.1. Let S be a subset of integers. The upper Banach density of S is defined as

$$\bar{d}_B(S) := \limsup_{N\to\infty} \frac{1}{N} \sup_{M\in\mathbb{N}} \#\{n \in S : M \leq n < M + N\},$$

where # denotes the cardinality of a finite set.

Our main results are the following.

THEOREM 1.2. *Suppose $\{f_n\}_{n\geq 1}$ is a d-decaying Gauss-like system. Then*

$$\dim_{\mathrm{H}} \mathbb{E}_S = \dim_{\mathrm{H}} \mathbb{E}_A = \frac{1}{d}.$$

The convergence exponent

$$s_0 = \inf\Big\{s \geq 0 : \sum_{n\in\mathbb{N}} \frac{1}{\xi_n^s} < \infty\Big\}$$

plays an important role in depicting the Hausdorff dimension. In fact, $1/2$ and $1/d$ are just the convergence exponents of the continued fraction and the d-decaying systems.

We also mention that the convergence exponent has many applications in the multifractal analysis of Birkhoff average in iIFSs (see [3, 7]).

For a general Gauss-like iIFS without d-decaying assumption, the above result may not hold any more. Actually the following theorem applies.

THEOREM 1.3. *There exists a Gauss-like iIFS with the convergence exponent $s_0 > 0$ such that* $\dim_{\mathrm{H}} \mathbb{E}_S = 0$.

Noting that $\{a_n\}_{n\geq 1}$ is strictly increasing implies that $a_n(x) > \psi(n)$ for all $n \in \mathbb{N}$ if one takes $\psi(n) = n/2$. Then the above Theorem 1.3 is a direct consequence of a result of Cao, Wang and Wu, as follows.

THEOREM A [1]. *For any function $\psi : \mathbb{N} \to \mathbb{R}^+$ with $\psi(n) \to \infty$ as $n \to \infty$ and $s_0 \in [0, 1]$, there exists a Gauss-like system $\{f_n\}_{n\geq 1}$ with the convergence exponent s_0 such that*

$$\dim_{\mathrm{H}}\{x \in \Lambda : a_n(x) > \psi(n) \text{ for all } n \in \mathbb{N}\} = 0.$$

In fact, from Theorems 1.2 and 1.3, for the lower bound of the dimension of $\mathbb{E}_S$, there exist two Gauss-like systems that share the same convergence exponent s_0, but the dimension of $\mathbb{E}_S$ in different systems may have different values (furthermore, one is zero and the other is s_0), and so does $\mathbb{E}_A$.

For the upper bound of the dimension of $\mathbb{E}_S$ and $\mathbb{E}_A$, suppose that $\{f_n\}_{n\geq 1}$ is a Gauss-like system and that $\{\xi_n\}_{n\geq 1}$ is increasing with the convergence exponent s_0. Then

$$\dim_{\mathrm{H}} \mathbb{E}_S \leq \dim_{\mathrm{H}} \mathbb{E}_A \leq s_0, \tag{1.1}$$

which follows from the following Theorem B, since the fact that $\{a_n\}_{n\geq 1}$ is strictly increasing implies that $a_n(x) \to \infty$ as $n \to \infty$.

THEOREM B [1]. *Suppose that $\{f_n(x)\}_{n\geq 1}$ is an ξ-regular iIFS with the convergence exponent s_0. Define*

$$\mathbb{E} = \{x \in \Lambda : a_n(x) \to \infty \text{ as } n \to \infty\}.$$

Then $\dim_{\mathrm{H}} \mathbb{E} = s_0$.

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However, there do exist cases such that the inequality (1.1) is an equality, for example continued fractions [9], Lüroth expansions [10] and general d-decaying Gauss-like systems, by combining Theorem 1.2 above.

For more dimensional results concerning iIFSs, one is referred to [1, 6] and the references therein.

2. Preliminaries

We begin with some notation. For each $a_1, \dots, a_n \in \mathbb{N}$, we define

$$I_n(a_1, \dots, a_n) = \{x \in \Lambda : a_k(x) = a_k, 1 \le k \le n\},$$

that is, the collection of points whose digit sequences begin with $a_1, \dots, a_n$. We call $I_n(a_1, \dots, a_n)$ an nth order basic interval.

Since $f_k \in C^1$ for each $k \ge 1$, it follows that the length of an nth order basic interval verifies

$$\prod_{k=1}^{n} \frac{c_1(\epsilon)}{\xi_{a_k}^{1+\epsilon}} \le |I_n(a_1, \dots, a_n)| \le \prod_{k=1}^{n} \frac{c_2(\epsilon)}{\xi_{a_k}^{1-\epsilon}}. \tag{2.1}$$

Such an estimation is essential to all the arguments below.

For some set of fine structure in fractal geometry, the following lemma is an important tool to compute the Hausdorff dimension, which will be used to obtain a lower bound of Hausdorff dimension of the constructed subset.

Lemma 2.1 [2]. *Suppose that the fractal set $F \subset [0,1]$ has the general Cantor set construction, and that each $(k-1)$th level interval contains at least $m_k \ge 2$ kth level intervals $(k = 1, 2, \dots)$ that are separated by gaps of at least ε_k, where $0 < \varepsilon_{k+1} < \varepsilon_k$ for each k. Then*

$$\dim_{\mathrm{H}} F \ge \liminf_{k\to\infty} \frac{\log(m_1 m_2 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}.$$

To determine the lower bound of $\dim_{\mathrm{H}} \mathbb{E}_S$, we need a proper 'seed' set. Based on this set, we can construct a large enough subset contained in the set $\mathbb{E}_S$.

Lemma 2.2. *Suppose that $\{f_n\}_{n\ge1}$ is a d-decaying Gauss-like system. For any $a > 4$, define*

$$\widetilde{E} = \{x \in \Lambda : a^n < a_n(x) \le 2a^n \text{ for all } n \ge 1\}.$$

Then $\dim_{\mathrm{H}} \widetilde{E} = 1/d$.

Proof. Note that $a^n < a_n(x) \le 2a^n$ implies that $a_n(x) \to \infty$ as $n \to \infty$, $\dim_{\mathrm{H}} \widetilde{E} \le 1/d$, which follows from Theorem B.

It is obvious that the set

$$F = \{x \in \Lambda : a^n < a_n(x) \le 2a^n \text{ and } a_n(x) \text{ is even for all } n \ge 1\}$$

is a Cantor subset of $\widetilde{E}$.

Each $(k-1)$th level interval contains $([a^k] + 1)/2$ kth level intervals when $[a^k]$ is odd and $[a^k]/2$ kth level intervals when $[a^k]$ is even, where $[x]$ denotes the largest integer not greater than x.

The gap between kth level intervals is not less than $\min\{|I_k(a_1,\dots,a_k)| : a^i < a_i \le 2a^i, 1 \le i \le k\}$. By the inequality (2.1), for any $\epsilon > 0$, there exist $c_1(\epsilon)$ and $c_2(\epsilon)$ with $0 < c_1(\epsilon) \le 1 \le c_2(\epsilon)$ such that, for any $(a_1,\dots,a_k)$ with $a^i < a_i \le 2a^i (1 \le i \le k)$,

$$|I_k(a_1,\dots,a_k)| \ge \frac{(c_1(\epsilon))^k}{(a_1\cdots a_k)^{d(1+\epsilon)}} \ge \frac{(c_1(\epsilon))^k}{2^{kd(1+\epsilon)}a^{k(k+1)d(1+\epsilon)/2}}.$$

Take

$$m_1 = 2 \quad \text{and} \quad m_k = \frac{a^k}{3} \quad \text{for } k \ge 2$$

and

$$\varepsilon_k = \frac{(c_1(\epsilon))^k}{2^{kd(1+\epsilon)}a^{k(k+1)d(1+\epsilon)/2}} \quad \text{for } k \ge 1.$$

Then $0 < \varepsilon_{k+1} < \varepsilon_k, k \ge 1$. By Lemma 2.1,

$$\begin{aligned}\dim_{\mathrm{H}} F &\ge \liminf_{k\to\infty} \frac{\log(m_1\cdots m_{k-1})}{-\log(m_k\varepsilon_k)} \\ &= \liminf_{k\to\infty} \frac{\log\Big(2\dfrac{a^2}{3}\cdots\dfrac{a^{k-1}}{3}\Big)}{-\log\Big(\dfrac{a^k}{3}\dfrac{(c_1(\epsilon))^k}{2^{kd(1+\epsilon)}a^{k(k+1)d(1+\epsilon)/2}}\Big)}. \end{aligned} \tag{2.2}$$

Since

$$\log\Big(2\frac{a^2}{3}\cdots\frac{a^{k-1}}{3}\Big) = \frac{(k-2)(k+1)}{2}\log a - (k-2)\log 3 + \log 2$$

and the fact that the denominator in (2.2) equals

$$\frac{k(k+1)d(1+\epsilon)-2k}{2}\log a + kd(1+\epsilon)\log 2 - k\log c_1(\epsilon) + \log 3,$$

we can obtain $\dim_{\mathrm{H}} F \ge 1/(d(1+\epsilon))$. Letting $\epsilon \to 0$, we get that $\dim_{\mathrm{H}} F \ge 1/d$. Thus, $\dim_{\mathrm{H}} \widetilde{E} \ge \dim_{\mathrm{H}} F \ge 1/d$. This finishes the proof. □

To end this section, we state an auxiliary lemma for the proof, which establishes the relationship between the Hausdorff dimension of a set and that of its image under a Hölder map.

Lemma 2.3 [2]. *Let $F \subset \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$ satisfies a Hölder condition*

$$|f(x) - f(y)| \le c|x-y|^\alpha \quad (x, y \in F).$$

Then $\dim_{\mathrm{H}} f(F) \le (1/\alpha)\dim_{\mathrm{H}} F$.

3. Proof of the main theorem

In this section, we give the proof of Theorem 1.2. Recall that

$$\mathbb{E}_S = \{x \in \Lambda : \{a_n(x)\}_{n\ge1} \text{ is strictly increasing and } \bar{d}_B(\{a_n(x)\}_{n\ge1}) = 1\}$$

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and

$$\mathbb{E}_A = \{x \in \Lambda : \{a_n(x)\}_{n\geq 1} \text{ is strictly increasing and contains arbitrarily long arithmetic progressions}\}$$

are the sets in question.

The upper bound is easily available from Theorem B by noting that $\{a_n(x)\}_{n\geq 1}$ being strictly increasing implies that $a_n(x) \to \infty$ as $n \to \infty$. That is

$$\dim_{\mathrm{H}} \mathbb{E}_S \leq \dim_{\mathrm{H}} \mathbb{E}_A \leq \dim_{\mathrm{H}} \mathbb{E} = \frac{1}{d}.$$

To determine the lower bound of $\dim_{\mathrm{H}} \mathbb{E}_S$, we construct a subset of the target set $\mathbb{E}_S$ by inserting a group of words on $\mathbb{N}$ at the appropriate positions in the digit sequences of the points in $\widetilde{E}$. By the appropriate choice of these positions, we will find a Hölder function between this subset and $\widetilde{E}$.

For any rational number $\epsilon \in (0, 1)$, fix an integer $a > \max\{4, c_2(\epsilon)/c_1(\epsilon)\}$, and choose a sequence of integers $\{n_k\}_{k\geq 1}$ such that, for all $k \geq 1$,

$$n_1 \geq \frac{10}{\epsilon}, \quad n_{k+1} \geq \sum_{i=1}^{k} n_i; \quad \epsilon \cdot n_k \in \mathbb{N}. \tag{3.1}$$

For any $y \in \widetilde{E}$, we construct a point in $\mathbb{E}_S$ in the following way. For any $k \geq 1$, let $W_k = \{2a^{n_k} + 1, \ldots, 2a^{n_k} + \epsilon n_k\}$ be the word with length ϵn_k. Suppose that the digit sequence of y is

$$a_1(y), \ldots, a_{n_1}(y), a_{n_1+1}(y), \ldots, a_{n_2}(y), a_{n_2+1}(y), \ldots.$$

Then determine a new sequence as

$$a_1(y), \ldots, a_{n_1}(y), W_1, a_{n_1+1}(y), \ldots, a_{n_2}(y), W_2, a_{n_2+1}(y), \ldots,$$

which is obtained by inserting the words $\{W_k\}_{k\geq 1}$ into the digit sequence of y at the positions $\{n_k\}_{k\geq 1}$. Denote the corresponding point in Λ by $x = x(y)$ and denote the collection of those x by $\widetilde{E}_\epsilon$, that is,

$$\widetilde{E}_\epsilon = \{x \in \Lambda : x = x(y), y \in \widetilde{E}\}.$$

At first, we verify that the point $x = x(y)$ belongs to $\mathbb{E}_S$.

Lemma 3.1. *For any $\epsilon \in (0, 1)$,*

$$\widetilde{E}_\epsilon \subset \mathbb{E}_S.$$

Proof. Fix $\epsilon \in (0, 1)$. Let $x \in \widetilde{E}_\epsilon$ and let $y \in \widetilde{E}$ be the point corresponding to x.

In order to show the strictly increasing property of $\{a_n(x)\}_{n\geq 1}$, by the definition of $\widetilde{E}$, $\widetilde{E}_\epsilon$ and W_k, we only need to show that, for any $k \geq 1$,

$$a_{n_k}(y) < 2a^{n_k} + 1 \quad \text{and} \quad 2a^{n_k} + \epsilon n_k < a_{n_k+1}(y).$$

The first one follows from the fact that $a^k < a_k(y) \leq 2a^k$. Note that the conditions of $\{n_k\}_{k\geq 1}$ in (3.1) and $a > 4$, $\epsilon \in (0, 1)$ give $2a^{n_k} + \epsilon n_k < 3a^{n_k} < a^{n_k+1} < a_{n_k+1}(y)$.

For any $k \geq 1$, take $M_k = 2a^k + 1$. We verify that

$$\#\{i \in \{a_n(x)\}_{n\geq 1} : M_k \leq i < M_k + \epsilon n_k\} = \epsilon n_k,$$

by the definition of W_k and the construction of $\widetilde{E}_\epsilon$. Therefore

$$\frac{1}{\epsilon n_k} \sup_{M\in\mathbb{N}} \#\{i \in \{a_n(x)\}_{n\geq 1} : M \leq i < M + \epsilon n_k\} = 1,$$

which implies that the upper Banach density of the digit sequence $\{a_n(x)\}_{n\geq 1}$ of x is one.

So $x \in \mathbb{E}_S$. □

Now we estimate the Hausdorff dimension of $\widetilde{E}_\epsilon$, by Lemma 2.3. Clearly, the correspondence between $y \in \widetilde{E}$ and $x = x(y) \in \widetilde{E}_\epsilon$ is one-to-one. This enable us to define an onto map between $\widetilde{E}$ and $\widetilde{E}_\epsilon$ as

$$f : \widetilde{E}_\epsilon \to \widetilde{E}, \quad x = x(y) \to y.$$

Then we are led to estimate the Hölder exponent of f. Let $x_1, x_2 \in \widetilde{E}_\epsilon$ be close enough such that

$$a_i(x_1) = a_i(x_2) \quad 1 \leq i \leq n_1.$$

Also let y_1, y_2 be the corresponding points in $\widetilde{E}$, respectively. Assume that n is the minimal positive integer such that $a_{n+1}(y_1) \neq a_{n+1}(y_2)$. Without loss of generality, we can assume that $a_{n+1}(y_1) < a_{n+1}(y_2)$ (the other case can be treated similarly). So

$$a_i(y_1) = a_i(y_2) \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad a_{n+1}(y_1) < a_{n+1}(y_2).$$

Let $k \geq 1$ be the integer such that $n_k \leq n < n_{k+1}$. Then, by the definition of the map f, it follows that

$$a_i(x_1) = a_i(x_2) \quad 1 \leq i \leq n + \epsilon n_1 + \cdots + \epsilon n_k$$

and

$$a_{n+\epsilon n_1+\cdots+\epsilon n_k+1}(x_1) = a_{n+1}(y_1) < a_{n+1}(y_2) = a_{n+\epsilon n_1+\cdots+\epsilon n_k+1}(x_2).$$

Moreover, it should be noticed that if $n < n_{k+1} - 1$, then

$$a_{n+\epsilon n_1+\cdots+\epsilon n_k+2}(x_i) = a_{n+2}(y_i) < 2a^{n+2} \quad i = 1, 2.$$

If $n = n_{k+1} - 1$, then

$$a_{n+\epsilon n_1+\cdots+\epsilon n_k+2}(x_i) = 2a^{n_{k+1}} + 1 < 2a^{n_{k+1}+1} = 2a^{n+2} \quad i = 1, 2.$$

In other words, we always have that

$$a_{n+\epsilon n_1+\cdots+\epsilon n_k+1}(x_i) \leq a_{n+\epsilon n_1+\cdots+\epsilon n_k+2}(x_i) < 2a^{n+2} \quad i = 1, 2. \tag{3.2}$$

Since $x_1, x_2 \in I(a_1(x_1), \ldots, a_{n+\epsilon n_1+\cdots+\epsilon n_k}(x_1))$ and $a_{n+\epsilon n_1+\cdots+\epsilon n_k+1}(x_1) < a_{n+\epsilon n_1+\cdots+\epsilon n_k+1}$ (x_2), which implies that x_1 is on the right side of x_2 and that x_1, x_2 are separated by the interval

$$I(a_1(x_1), \ldots, a_{n+\epsilon n_1+\cdots+\epsilon n_k}(x_1), a_{n+\epsilon n_1+\cdots+\epsilon n_k+1}(x_1), a_{n+\epsilon n_1+\cdots+\epsilon n_k+2}(x_1) + 1),$$

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(2.1) and (3.2) give

$$|x_1 - x_2| \geq \frac{(c_1(\epsilon))^{n+\epsilon n_1+\cdots+\epsilon n_k+2}}{(a_1(x_1)\cdots a_{n+\epsilon n_1+\cdots+\epsilon n_k}(x_1)\cdot 2a^{n+2}\cdot 2a^{n+2})^{d(1+\epsilon)}}.$$

Since

$$a_1(x_1)\cdots a_{n+\epsilon n_1+\cdots+\epsilon n_k}(x_1) = \prod_{i=1}^{n} a_i(y)\cdot \prod_{i=1}^{\epsilon n_1}(2a^{n_1}+i)\cdots \prod_{i=1}^{\epsilon n_k}(2a^{n_k}+i)$$

and

$$\prod_{i=1}^{n} a_i(y)\cdot \prod_{i=1}^{\epsilon n_1}(2a^{n_1}+i)\cdots \prod_{i=1}^{\epsilon n_k}(2a^{n_k}+i) \leq \prod_{i=1}^{n}(2a^i)\cdot (2a^{n_1}+\epsilon n_1)^{\epsilon n_1}\cdots (2a^{n_k}+\epsilon n_k)^{\epsilon n_k},$$

it follows that

$$|x_1 - x_2| \geq \frac{(c_1(\epsilon))^{n+\epsilon n_1+\cdots+\epsilon n_k+2}}{(a^2\cdots a^{n+1}\cdot a^{\epsilon n_1(n_1+1)}\cdots a^{\epsilon n_k(n_k+1)}\cdot a^{2n+5})^{d(1+\epsilon)}},$$

by noting that $a > 4$ and $\epsilon \in (0,1)$. So

$$|x_1 - x_2| \geq \frac{(c_1(\epsilon))^{n+\epsilon n_1+\cdots+\epsilon n_k+2}}{a^{(\frac{n^2}{2}+\frac{7n}{2}+\epsilon(n_1^2+\cdots+n_k^2)+\epsilon(n_1+\cdots+n_k)+5)d(1+\epsilon)}}.$$

By the choice of n_k, that is, the formula (3.1),

$$|x_1 - x_2| \geq \frac{(c_1(\epsilon))^{n+\epsilon n_1+\cdots+\epsilon n_k+2}}{a^{((\frac{1}{2}+4\epsilon)n^2+(\frac{7}{2}+2\epsilon)n+5)d(1+\epsilon)}} \geq \frac{(c_1(\epsilon))^{4n}}{a^{(\frac{1}{2}+5\epsilon)n^2 d(1+\epsilon)}}.$$

Note that $y_1, y_2 \in I(a_1(y_1), \ldots, a_n(y_1))$. By (2.1) and $a > \max\{4, c_2(\epsilon)/c_1(\epsilon)\}$,

$$|y_1 - y_2| \leq \frac{(c_2(\epsilon))^n}{(a_1(y_1)\cdots a_n(y_1))^{d(1-\epsilon)}} \leq \frac{(c_2(\epsilon))^{4n}}{a^{\frac{n(n+1)}{2}d(1-\epsilon)}} \leq \frac{(c_1(\epsilon))^{4n}}{a^{\frac{n^2}{2}d(1-2\epsilon)}}.$$

It can be verified that

$$|y_1 - y_2| < |x_1 - x_2|^{\frac{1-2\epsilon}{1+21\epsilon}}.$$

Therefore, by Lemma 2.3, we arrive at

$$\dim_{\mathrm{H}} \mathbb{E}_S \geq \dim_{\mathrm{H}} \widetilde{E}_\epsilon \geq \frac{1-2\epsilon}{1+21\epsilon}\dim_{\mathrm{H}} \widetilde{E} = \frac{1-2\epsilon}{d(1+21\epsilon)}.$$

Letting $\epsilon \to 0$, we finally get that

$$\dim_{\mathrm{H}} \mathbb{E}_S \geq \frac{1}{d}.$$

This completes the proof of Theorem 1.2.

References

[1] C. Y. Cao, B. W. Wang and J. Wu, 'The growth speed of digits in infinite iterated function systems', *Stud. Math.* **217** (2013), 139–158.

[2] K. J. Falconer, *Fractal Geometry, Mathematical Foundations and Applications* (John Wiley & Sons, Chichester, 2003).

[3] A. H. Fan, L. M. Liao and J. H. Ma, 'The frequency of the digits in continued fractions', *Math. Proc. Cambridge Philos. Soc.* **148** (2010), 179–192.

https://doi.org/10.1017/S1446788716000288 Published online by Cambridge University Press

[4] H. Furstenberg, 'Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions', *J. Anal. Math.* **31** (1977), 204–256.
[5] B. Green and T. Tao, 'The primes contain arbitrarily long arithmetic progressions', *Ann. of Math. (2)* **167** (2008), 481–547.
[6] T. Jordan and M. Rams, 'Increasing digit subsystems of infinite iterated function systems', *Proc. Amer. Math. Soc.* **140** (2012), 1267–1279.
[7] L. M. Liao, J. H. Ma and B. W. Wang, 'Dimension of some non-normal continued fraction sets', *Math. Proc. Cambridge Philos. Soc.* **145** (2010), 215–225.
[8] E. Szemerédi, 'On sets of integers containing no k elements in arithmetic progression', *Acta Arith.* **27** (1975), 299–345.
[9] X. Tong and B. W. Wang, 'How many points contains arithmetic progressions in continued fraction expansion?', *Acta Arith.* **139** (2009), 369–376.
[10] Z. L. Zhang and C. Y. Cao, 'On points contain arithmetic progressions in their Lüroth expansion', *Acta Math. Sci.* **36B** (2016), 257–264.

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