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Yet more on a stochastic economic model: Part 3C: stochastic bridging for share yields and dividends and interest rates

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Abstract
This is the third and last subpart of a long paper in which we consider stochastic interpolation for the Wilkie asset model, considering both Brownian bridges and Ornstein–Uhlenbeck (OU) bridges. In Part 3A, we developed certain properties for both these types of stochastic bridge, and in Part 3B we investigated retail prices and wages. In this paper, we investigate the remainder of many of our data series, relating to shares and interest rates. We conclude that, regardless of the form of the annual model, the monthly data within each year can be modelled by Brownian bridges, usually on the logarithm of the principal variable. But in no case is a simple Brownian bridge enough, and all series have their own peculiarities. Overall, however, our modelling produces simulations that are realistic in comparison with the known data. Many of our findings would apply to any similar model used for simulation over time. Our results have considerable importance for financial economics. We reconcile the conflict between the long-term mean-reverting modelling of Schiller and the short-term random walk modelling of Fama. This conclusion therefore has very wide significance.

Keywords
Wilkie model; Stochastic interpolation; Bridging; Shares; Interest rates

1. Introduction

1.1. In Parts 1 and 2 of this series of papers (Wilkie *et al*., 2011; Wilkie & Şahin, 2016), we updated the Wilkie model (see Wilkie 1986 and Wilkie 1995) to 2009, and described several aspects of using such an annual model. In the three subparts of this long paper, we consider ways to make the model applicable to shorter time steps, by stochastic interpolation between the points of a stochastically generated annual model.

1.2. In Part 3A, we described the basic ways of doing stochastic interpolation, by the familiar Brownian bridges (BBs), and rather less familiar Ornstein–Uhlenbeck (OU) bridges and went into the properties of these forms of bridging. In Part 3B, we considered models for retail prices and for wages within the Wilkie model and how stochastic interpolation could be applied to these series. In this paper, we do the same for the remaining series of the Wilkie model.

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1.3. In section 2 we consider dividend yields, in section 3 the share dividend index and the derived share price index. In section 4 we look at long-term interest rates, and in section 5 short-term ones, based on bank Base Rates. In section 6 we turn to the real yields on index-linked stocks, and in section 7 we draw some conclusions.

1.4. We find, as with retail prices and wages, that what seem to be the obvious methods are not always satisfactory. We also investigate the source data for each series, in so far as monthly data is available, and we find that the actual monthly data generally does not behave in the same way as the annual data. We produce plausible models in each case for monthly modelling.

1.5. None of the elements of the Wilkie model are simple random walks, for which a BB would be the obvious choice. Only one, that for the real yield on index-linked stocks, R, is a simple AR(1) model, for which an OU bridge might be appropriate; we come to that later. But all the other models are interconnected, and most have complications which make the choice of an appropriate bridge not so obvious.

1.6. We note, as we did in Part 3B, that in order to check the behaviour of any model it is essential to simulate projections of the future experience in both a deterministic way (with zero innovations, giving the expected values) and stochastically. The annual model can be projected in either way, and the monthly model within each of these in either way, giving us four possibilities. All should be investigated.

1.7. We conclude in section 7 with a summary of our findings, some comments on the real world, and some very important implications for financial economics. Our modelling reconciles the conflict between the long-term mean-reverting modelling of Schiller and others and the short-term random walk modelling of Fama and others. This conclusion has taken a great deal of analysis to justify, but it has wide significance.

2. Dividend Yield

2.1. The dividend yield model

2.1.1. We start the investment variables with the dividend yield, $Y(t)$, for which the formulae are as follows:

$$YN(t) = YA.YN(t-1) + YE(t)$$

$$YL(t) = YW.I(t) + YMUL + YN(t)$$

$$Y(t) = \exp(YL(t))$$

with parameters suggested in Part 1 (Wilkie *et al*., 2011):

$$YW = 1.55, YMU = 0.0375, YA = 0.63, YSD = 0.155$$

2.1.2. This is basically an AR(1) model for $YN(t)$, with the addition of a relatively small influence from $I(t)$. As we did with $WL(t)$ in Part 3B, we can split $YL(t)$ into two parts:

$$YL1(t) = YW.I(t)$$

$$YL2(t) = YMUL + YN(t)$$

We can investigate, and we can model, these parts separately, but since we assume that $I(t)$ and hence $YL1(t)$ are already given, this means investigating both YL and $YL2$. The natural bridging model for

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either of them is an OU one, to correspond with the annual AR(1) model for $YL2$, but we wish to look at them also as BBs, since, as we showed in section 2, there may not be much difference between these.

2.2. The dividend yield data

2.2.1. We have data available at apparent monthly intervals from 1923 to 2014, but in fact the data for 1923 is interpolated between December 1922 and December 1923, so we shall exclude this year and start in June 1924. There is also a small lacuna in 1929, where we have only quarterly figures, and have filled in the gaps with interpolated values. Since this was a rather eventful year, this is a pity, but overall it makes no noticeable difference. We show these monthly values in Figure 1. The graph is very well-behaved compared with those for some of the data series used in Part 3B.

2.2.2. We analyse YL and $YL2$, both as BBs and as OU bridges. The calculation of $YL2$ does not have the same difficulties as that of $WL2$, because it is not cumulative. The initial estimates of σ_m for these four combinations are shown in Table 1. They are not very different. Those for $YL2$ are a little larger than those for YL, showing that, although the rate of inflation is important in the annual model, dividend yields do not respond monthly to the values; indeed they could hardly do so in detail, since the rate of inflation for the year ending in a particular month is not known till the middle of the following month, and our dividend yields are those observed on the last day of the month stated; nevertheless, the market might pay attention to the forecast forthcoming rate, and for 11 out

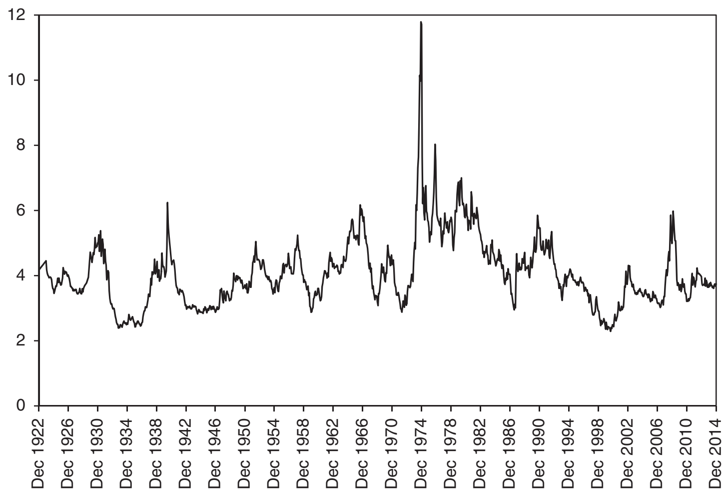

Figure 1. Dividend yield %, $Y(t)$, monthly from December 1922 to December 2014.

Table 1. Estimates of σ_m for $YL(t)$ and $YL2(t)$, 1923–2014.

	Brownian bridge	OU bridge
$YL(t)$	0.04981	0.04897
$YL2(t)$	0.05111	0.05024

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Table 2. Statistics for forwards deviations, Df_j, of YL, from June 1924 to June 2014; unadjusted and standardised.

Month	Unadjusted				Standardised			
	Mean of Df_j	T-ratio	Skewness	Kurtosis	Mean of Df_j	T-ratio	Skewness	Kurtosis
July	0.00207	0.46	0.27	3.57	−0.03095	−0.29	−0.22	2.69
August	−0.00595	−1.19	0.45	4.06	−0.23335	−2.26	0.27	2.39
September	0.00987	1.86	1.00	4.41	0.12922	1.28	0.24	2.51
October	−0.00607	−1.04	1.66	11.21	−0.21825	−2.12	0.44	2.91
November	0.00189	0.40	0.39	5.29	0.05453	0.60	0.20	2.44
December	−0.00978	−2.63	−0.70	5.41	−0.18428	−2.21	0.04	2.51
January	−0.01384	−2.20	−3.24	23.58	−0.24514	−2.36	0.03	2.79
February	0.00499	0.97	−0.97	5.59	0.20170	2.01	−0.19	2.33
March	0.00485	1.08	0.50	6.14	0.19833	2.11	−0.11	2.53
April	−0.01361	−3.22	0.03	3.47	−0.29664	−3.24	0.19	3.02
May	0.01538	3.37	0.36	3.06	0.36605	3.51	0.19	2.34
June	0.01021	2.13	0.52	7.87	0.25877	2.77	0.04	3.01

of the 12 months the change in retail prices is already known. However, there seems to be little advantage in considering $YL2$ further.

2.2.3. There is also rather little difference between the values of σ_m for the BB and OU models for YL. But there is a noticeable difference between the annual standard deviations, since for the BB, $\sigma_y = \sqrt{12}.\sigma_m = 0.17253$ and for the OU $\sigma_y = \sigma_m\ \sqrt{\{(1-\alpha_y^2)/(1-\alpha_m^2)\}} = 0.01397$, with $\alpha_y = 0.63$, and $\alpha_m = \alpha_y^{1/12} = 0.962229$. This, however, is quite consistent; a certain value for σ_m has different effects on the annual values σ_y with the two models. We note also that the annual standard deviation from the annual model, calculated as $\sqrt{(YW^2.QSD^2 + YSD^2)} = 0.16694$, is not very different from the value estimated from the BB.

2.2.4. We continue analysing the YL series as if it were a BB. In Table 2 we show statistics for the forwards deviations, first unadjusted, and then standardised by dividing each value of Df_j by the estimated value of σ_m for the same year calculated from the values of Df_j in that year. We consider first the unadjusted values. We see that the T-ratios for four months are significantly large, three negative, December, January and April, and one positive, May. But we see also very high kurtosis for almost every month (a normal distribution has kurtosis of 3.0) and rather high skewness for several months. We can identify specific months with large values of Df_j: January 1975, −0.41478; February 1975, −0.20019; October 1988, +0.29222; and June 1940, +0.20896 (note that a negative sign implies that share prices rose, a positive sign that they fell). Each of these values is a large multiple of the overall standard deviation.

2.2.5. When we standardise the values, the kurtosis is now almost always small, so we have possibly compressed the values too much, but more of the T-ratios appear significant. Standardising by the expected values of σ_m according to whatever regression formula we might choose might be better, but we have not got there yet.

2.2.6. We calculate the same table of unadjusted statistics assuming that the model is in fact an OU bridge, with α_y and μ_y as in the annual model. The values are almost the same as those shown in Table 2, so we do not show them.

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2.2.7. We calculate the same statistics for the sideways deviations and show them in Table 3. For eight months out of eleven, the means are negative, and for four of these are significantly so. But the kurtosis for almost all months is very large, and the skewness for several also is; we do not show them, but they are similar to those of the forwards ratios. The *M*-ratios are reasonably close to unity, so although some individual values are very large, the overall tendency is not too different from a BB.

2.2.8. We calculate the same statistics for the 33 years when there are upwards deviations, the 29 years when there are downwards ones, and the 28 years when there are medium ones. We have 62 years contributing to reversed deviations. We show the means in Table 4. If the data followed OU bridges more closely than BBs, the means of the upwards deviations would tend to be positive; in fact there are three positive and eight negative means; the means of the downwards deviations would tend to be negative; in fact there are seven positive and four negative means; the means of the reversed deviations would all tend to be positive; in fact all 11 are negative.

Table 3. Statistics for sideways deviations, Df_j, of *YL*, from June 1924 to June 2014.

Month	Mean of Ds_j	s.d. of Ds_j	*T*-ratio	Multiple	s.d. ratio (rs_j)	*M*-ratio
June	0	0	0	0	0	0
July	0.0021	0.0431	0.46	0.8658	0.9574	0.90
August	–0.0039	0.0704	–0.52	1.4130	1.2910	1.09
September	0.0060	0.0868	0.65	1.7434	1.5000	1.16
October	–0.0001	0.0943	–0.01	1.8930	1.6330	1.16
November	0.0018	0.0987	0.17	1.9811	1.7078	1.16
December	–0.0080	0.0985	–0.77	1.9781	1.7321	1.14
January	–0.0218	0.0804	–2.57	1.6143	1.7078	0.95
February	–0.0168	0.0777	–2.05	1.5596	1.6330	0.96
March	–0.0120	0.0697	–1.63	1.3988	1.5000	0.93
April	–0.0256	0.0607	–4.00	1.2186	1.2910	0.94
May	–0.0102	0.0455	–2.13	0.9145	0.9574	0.96
June	0	0	0	0	0	0

Table 4. Means for upwards, downwards, medium and reversed deviations of *YL*, from June 1924 to June 2014.

Month	Upwards	Downwards	Medium	Reversed
June	0	0	0	0
July	0.0028	0.0126	–0.0097	–0.0044
August	0.0012	0.0063	–0.0205	–0.0023
September	0.0117	0.0274	–0.0229	–0.0066
October	–0.0071	0.0383	–0.0316	–0.0217
November	–0.0117	0.0368	–0.0186	–0.0235
December	–0.0227	0.0311	–0.0311	–0.0266
January	–0.0257	–0.0031	–0.0366	–0.0122
February	–0.0187	0.0028	–0.0349	–0.0112
March	–0.0094	–0.0054	–0.0218	–0.0025
April	–0.0180	–0.0178	–0.0425	–0.0013
May	–0.0129	–0.0014	–0.0161	–0.0062
June	0	0	0	0

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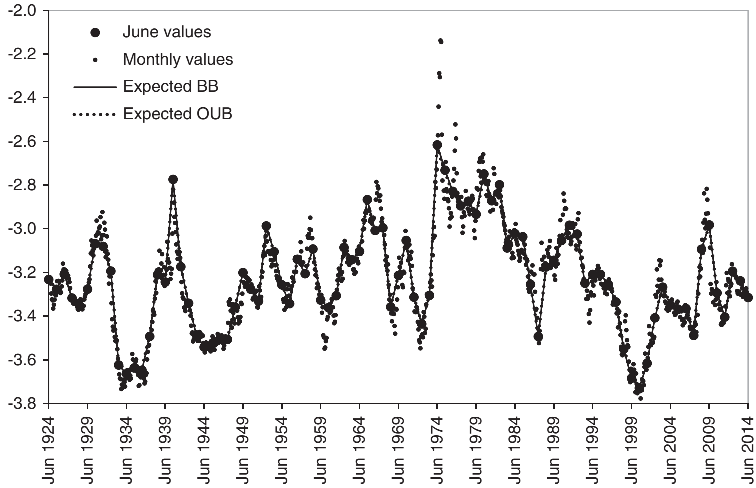

Figure 2. Monthly values of YL, from June 1924 to June 2014, with expected values assuming Brownian bridges (BBs) and Ornstein–Uhlenbeck (OU) bridges.

2.2.9. As further evidence, we show in Figure 2 the monthly values of YL from June 1924 to June 2014, marked with small dots, and with the June values marked with large dots, along with the expected values assuming BBs, with straight solid lines, and the expected values assuming OU bridges, with dotted lines. It is almost impossible to distinguish the dotted lines, even on a larger scale than shown in print, except in one or two places, where the values are well away from the middle, such as 1933–1934 and 1934–1935, or 1974–1975 and 1975–1976. In each case, the scatter of the actual values is so wide that any separation of a BB model and an OU model is impossible.

2.2.10. We conclude that it would, with this evidence, be perverse to suggest either that OU bridges fitted the yield data better than BBs, or vice versa, although the evidence of the annual data is very strongly in favour of an AR(1) model, as one can see by how often the actual data crosses the mean, which is taken as $\ln(YMU) + YW.QMU = -3.21676$.

2.2.11. Just as for QL and WL there is some evidence that the value of σ_m in each year is connected with the change in YL over the year. We put $YLD(t) = YL(t+1) - YL(t)$ and we investigate the same eight regression possibilities as for QL, regressing σ_m and σ_m^2 against $\text{Abs}(YLD(t))$, $YLD(t)^2$, $\text{Abs}(YLD(t) - YSC)$ and $(YLD(t) - YSC)^2$, where YSC is the calculated mean value of $YLD(t) = -0.00092$, which is very close to 0. The correlation coefficients are shown in Table 5.

2.2.12. The correlation coefficients are smaller than those for QL, and not very different from those for WL, and are rather similar for σ_m against any of the four options. The value of YSC is so small that there is little difference between the results for the first four regressions (with $YSC = 0$) and the last four. The correlation coefficients are very similar, as also are the values of A and B for corresponding regressions; the T-ratios for B for the σ_m regressions are all about 2.6 so adequately significantly different from zero. Since the theoretical long-term mean of YLD (as for all the autoregressive series) is 0, we prefer to use $YSC = 0$. We show the same chart in Figure 3 as for the previous series, with the actual values of $\sigma_m(t)$ plotted against those for $YLD(t)$ as dots, and two lines for the expected values when regressed against $\text{Abs}(YLD(t))$ (formula 1) and against $YLD(t)^2$ (formula 3). These two are quite similar in their effect as well as in the correlation coefficients.

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Table 5. Correlation coefficients from regressions.

	1924–2014	
	σ_m	σ_m^2
Abs($YLD(t)$)	0.2601	0.1844
$YLD(t)^2$	0.2649	0.1899
Abs($YLD(t) - YSC$)	0.2614	0.1849
$(YLD(t) - YSC)^2$	0.2660	0.1906

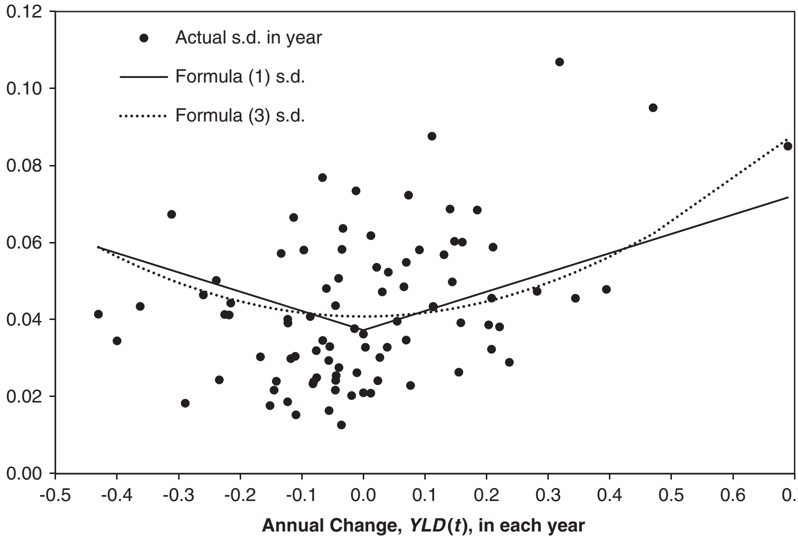

Figure 3. Standard deviations, and estimates of σ_m in each year, related to $YLD(t)$ in each year, from 1924 to 2014.

2.2.13. The quadratic formula gives a slightly higher correlation coefficient

$$\sigma_m(t) = YSM(t) = YSA + YSB \times YLD(t)^2$$

where YSA = 0.040788 and YSB = 0.097117. We include this variation of σ_m.

2.2.14. We next consider autoregression within the series of Df_j values or of the standardised values Df_jZ. There is almost no correlation from one year to the next in Df_jZ for individual months, unlike the situation for QL. There is a small autocorrelation within the two series for lags of 2, 5, 6 and 7 months, the (absolutely) largest correlation coefficient of –0.16 being for DF_j for a two months lag, but we consider that this can be ignored.

2.2.15. The remaining possible connection is a cross-correlation between the values of Df_jZ for YL, and the corresponding values for QL or WL. Most of the coefficients, simultaneous or for small lags, are very small. The biggest are a simultaneous correlation between the changes in the Df_j values for YL and for WL of 0.1556 and for the corresponding Df_jZ values of 0.1193. As we have noted in section 2.1.4, it is not likely that there would be direct simultaneous correlation between changes in YL over the month, and changes in QL over the month, and direct correlation with changes in WL seems less likely. We note the correlations, but they are still quite small and we ignore them.

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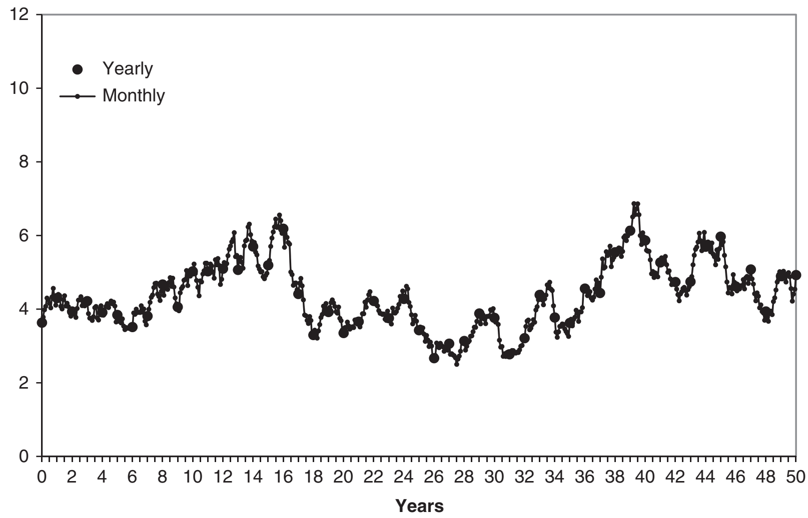

Figure 4. Stochastic yearly and monthly simulation for 50 years of $Y(t)\%$.

2.3. Simulations

2.3.1. All this analysis leaves us with a slightly simpler model for YL than for QL or WL. YL is modelled over the year with a BB, with $\sigma_m(t)$ modelled with the quadratic formula in section 2.2.13 and no other complications.

2.3.2. We show simulated values of $Y(t)\%$ monthly, for 50 years, in Figure 4, simulated as described above, and put onto the same vertical scale as Figure 2. When we compare these with the graph of actual values of $Y(t)\%$ in Figure 2, they look reasonably similar, though the extreme peak in 1974–1975 is not repeated.

3. The Share Dividend Index

3.1. The dividend index model

3.1.1. The dividend index model resembles the Wages model, in that it involves the dividend index, $D(t)$, its logarithm $DL(t)$ and the annual rate of dividend increase, $K(t) = \ln D(t) - \ln D(t-1)$. $K(t)$ is dependent on the series for the annual rate of inflation, $I(t)$, with a long moving average structure. There are small influences from the dividend yield series and the residual for the dividend index in the previous year $DE(t-1)$. The basic formulae for $K(t)$ and $D(t)$ are as follows:

$$DM(t) = DD.I(t) + (1 - DD).DM(t-1)$$

$$DI(t) = DW.DM(t) + DX.I(t)$$

$$K(t) = DI(t) + DMU + DY.YE(t-1) + DB.DE(t-1) + DE(t)$$

$$DL(t) = DL(t-1) + K(t)$$

$$D(t) = \exp(DL(t))$$

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The parameters suggested in Part 1 (Wilkie *et al.*, 2011) were as follows:

$DW = 0.43;\ DD = 0.16;\ DX = 1 - DW = 0.57;\ DMU = 0.011;\ DY = -0.22;\ DB = 0.43;\ DSD = 0.07$

3.1.2. The relationship between, $K(t)$, $DL(t)$ and $D(t)$ exactly parallels that between $I(t)$, $QL(t)$ and $Q(t)$, as did $J(t)$, $WL(t)$ and $W(t)$, so we can again conclude that we should apply bridging to $DL(t)$ and not to $K(t)$, deriving the simulated monthly values of $K(t)$ from the simulated values of $DL(t)$.

3.1.3. As for J and WL, we can partition K and DL into two parts, one depending on $I(t)$ and the other not

$$K1(t) = DI(t) = DW.DM(t) + DX.I(t)$$

$$K2(t) = DMU + DY.YE(t-1) + DB.DE(t-1) + DE(t)$$

We can apply the partitioning to $DL(t)$ as

$$DL1(t) = DL1(t-1) + K1(t)$$

$$DL2(t) = DL2(t-1) + K2(t)$$

We can then investigate both DL and $DL2$. Although $DL2(t)$ includes an influence from $YE(t-1)$ and from $DE(t-1)$ we can take these into account by investigating autocorrelations and cross-correlations of the resulting Df_j or Df_jZ series. However, in practice the calculation of $DL2$, being based on an accumulation of values of $K2$, has the same problems as that of $WL2$, and the results are not satisfactory.

3.2. The dividend index data

3.2.1. The dividend index data comes from the same sources as the dividend yield data, and its values have been calculated by multiplying the given share price index by the given dividend yield. Those who construct indices must calculate a dividend index in order to calculate the dividend yield, but this index is not normally published. There is a problem with constructing a satisfactory dividend index as well as a satisfactory price index. When stocks enter or leave a share index, or change their nominal share issue, the index can be “chain-linked” so that there is no discontinuity in the price index. But normally there is a discontinuity in the corresponding dividend index. When we have changed from one index to another as in 1928, 1929 and 1962, we have had to adjust the yield to give us a continuous dividend index, but we can make no adjustments to the ongoing FTSE-Actuaries All-Share Index, used since 1962. However, we assume that the movements of stocks in and out of the index are not so very large as to affect the whole index significantly, especially since the index contains many stocks, none overwhelmingly large (the same is not true of individual sectors within the indices, but we do not use these). We show in Figure 5 values of the dividend index, $D(t)$ (multiplied by 2.5 to fit the scale better) from December 1922 to December 2014, and in Figure 6 $K(t)$, the annual change in $\ln D(t)$, over the same period.

3.2.2. Like the dividend yield index the dividend index was linearly interpolated between year-end values in 1923, so we start our analysis in June 1924 and carry it through to June 2014. In Figure 7, we show the monthly changes in $DL(t)$ from 1923 to the end of 2014. One can see a few constant values at the start of the series, which indicate the linear interpolation. Otherwise the scatter does not

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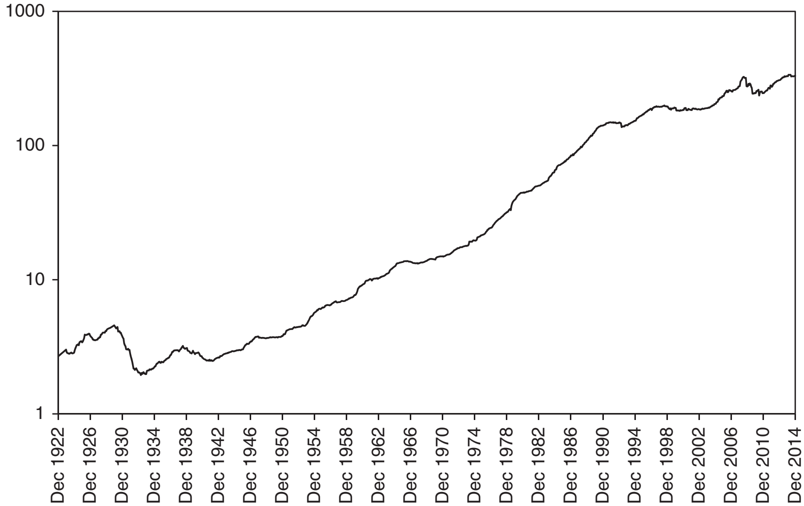

Figure 5. Values of $2.5 \times D(t)$, the dividend index, monthly, from December 1922 to December 2014.

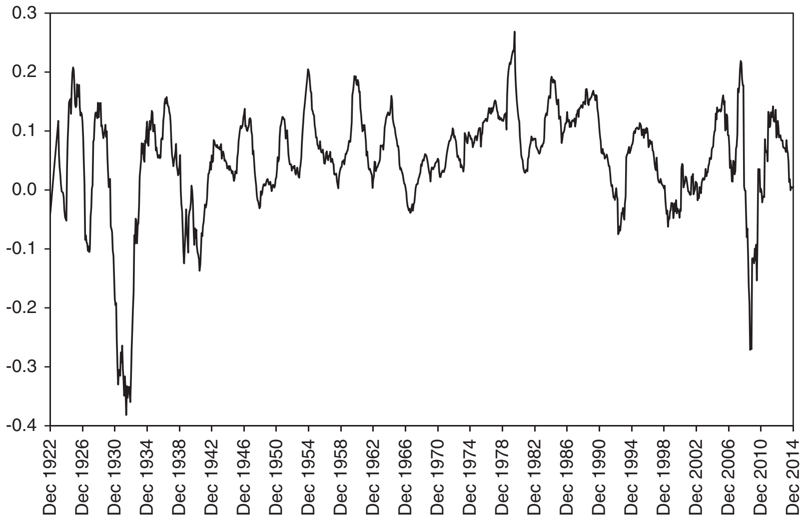

Figure 6. Values of $K(t)$, monthly, from December 1922 to December 2014.

indicate any problems with the source data. However, we can identify some rather large decreases and rather fewer large increases.

3.2.3. We assume BBs rather than OU bridges throughout. We start by comparing the estimated values of σ_m for DL and for $DL2$. That for the former is 0.01337 and for the latter is much bigger, at 0.03936. Allowing explicitly for $DL1$ does not help us at all. This is also not surprising. Dividends are declared by companies often twice a year, as interim and final, but in either order within the calendar year, because they have different year-ends. Many companies have accounting year-ends on 31 December, so declare a final dividend in about March of the following year, with an interim in

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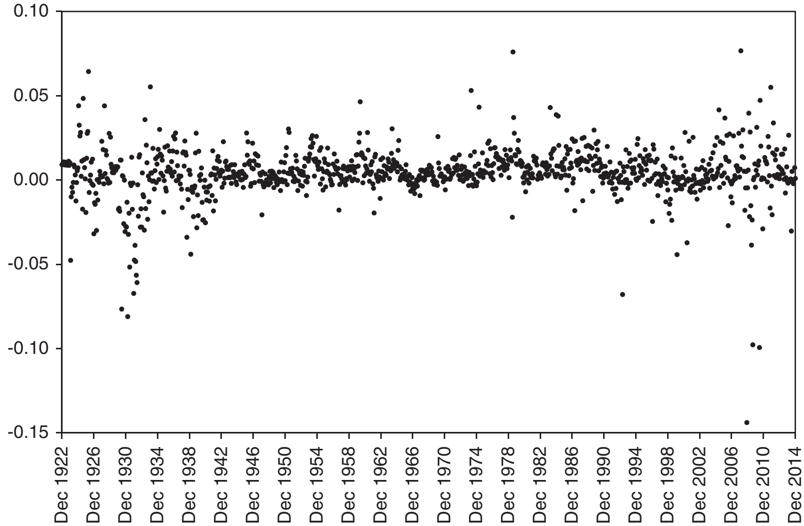

Figure 7. Monthly changes in $\ln D(t)$ from 1923 to 2014.

the autumn. But many other have their year-ends on other quarters, and quite a number of companies pay dividends quarterly, which is common practice in the United States. Then some companies make losses and perhaps pay no dividends at times. A few, as a matter of policy, pay no dividends during a period when they are growing rapidly and have opportunities to use the shareholders money effectively in the business. Yet, others feel they have sufficient capital and return money to shareholders by share buybacks, capital distributions, or special dividends. In principle special dividends are not included in the FTSE index dividend yield, though they are allowed for in the "xd adjustment", which shows actual cash attributed on the xd date for each stock. The yield index should allow for dividends as they are announced. They may go "ex dividend" some weeks later, and be actually paid some weeks later than that, but these last two steps should not affect the index.

3.2.4. All these practical aspects of the dividend index mean that it may alter in rather a lumpy way, as bigger companies increase or cut their dividends. But in detail it is not connected with the retail or consumer price indices. Consumer prices have their effect slowly and indirectly. If there is high general inflation, companies may be able to increase their own prices, and may make higher profits and pay higher dividends, all with quite a long time lag. This is all reflected in the annual model, and we would not expect any simultaneous correlation.

3.2.5. In Table 6 we show statistics for the forwards deviations, first unadjusted, and then standardised by dividing each value of Df_j by the estimated value of σ_m for the same year calculated from the values of Df_j in that year. We consider first the unadjusted values. These show quite large positive deviations in February, March and May, the period when we might expect companies to announce increases to final dividends, with quite large negative deviations in October, December and January, periods when announcements are scarcer. However, the kurtosis in every month is very large, going up to 36.03 for November, and often the skewness is large and negative, but never large and positive. In general this may reflect the actions of companies, whose directors may well increase dividends cautiously from year to year, but when dividends must be reduced, feel that it is better to do that in a big cut, which may allow gentle increases from a lower level in subsequent years. But very specifically this identifies the months when there were isolated big changes. The biggest negative values of Dfj

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Table 6. Statistics for forwards deviations, Df_j, of DL, from June 1924 to June 2014; unadjusted and standardised.

	Unadjusted				Standardised			
Month	Mean of Df_j	T-ratio	Skewness	Kurtosis	Mean of DfZ_j	T-ratio	Skewness	Kurtosis
July	0.00092	0.72	1.23	7.32	0.0065	0.07	0.36	3.04
August	−0.00057	−0.41	−2.35	19.31	−0.0824	−0.88	0.32	2.86
September	−0.00062	−0.57	0.55	6.05	−0.1164	−1.24	−0.15	3.11
October	−0.00198	−2.29	0.88	6.29	−0.2826	−4.09	0.96	4.66
November	−0.00081	−0.45	−4.52	36.03	−0.0342	−0.35	−0.33	3.48
December	−0.00249	−2.45	−0.70	5.53	−0.2022	−2.51	0.39	2.82
January	−0.00316	−2.73	1.21	8.59	−0.3964	−4.20	0.40	4.04
February	0.00325	2.15	0.84	7.09	0.2514	2.18	−0.11	3.08
March	0.00334	2.92	−1.49	11.39	0.3807	3.87	−0.16	3.45
April	0.00155	0.99	0.01	8.14	0.1677	1.46	−0.03	3.64
May	0.00329	2.71	−0.03	6.25	0.3654	3.49	0.06	3.26
June	−0.00272	−1.75	−3.19	17.41	−0.0575	−0.56	−0.48	3.66

occur in November 2008, −0.1303; August 2009, −0.0850; and June 2010, −0.0866, all within the most recent crisis period. The large drops in 1931 and 1932, visible in Figure 5, do not provide such large values of Df_j, because they occurred in years when the overall drop was large, and Df_j measures movements relative to the mean for the June to June year, so the drop in March 1931 gives a Df_j value of −0.0556, and that in June 1930, which was a smaller decrease in $DL(t)$, gives a Df_j of −0.0714. When we note that the overall value of σ_m is 0.01337, we see how large these drops are. The data are very fat-tailed and far from normally distributed.

3.2.6. The effect of the irregular monthly pattern in the forwards deviations follows through to the sideways deviations, which are mostly significantly negative; but again the kurtosis is so large that conclusions based on significance tests based on normality are suspect. We can take account of some of this irregular seasonal pattern by including a correlation between the same months of successive years, as we did for QL and as we shall investigate, but we shall not make any other adjustment for it.

3.2.7. We next consider the variation in the values of σ_m year by year. They are very variable and as before vary with the change in DL over the year, that is, $DLD(t) = K(t+1)$. We show the correlation coefficients of the regressions in Table 7. The quadratic and absolute models centred on DSC show high correlation coefficients with the best being that for σ_m and the absolute value (formula 5):

$$\sigma_m(t) = DSM(t) = DSA + DSB \times \text{Abs}(DLD(t) - DSC)$$

with DSA = 0.005893, DSB = 0.075657 and DSC = 0.052764. The T-ratios of A and B are very large (6.93 and 8.49, respectively). We show this regression in Figure 8 as formula 5. There is a big bunch of apparently uncorrelated points in the middle, but the extreme values indicate that the regression is useful.

3.2.8. We now look at the correlation between the standardised forwards deviations in the same months of successive years. For DL the correlations are weaker than for either QL or WL, with an

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Table 7. Correlation coefficients from regressions.

	1924–2014	
	σ_m	σ_m^2
Abs($DLD(t)$)	0.4835	0.4001
$DLD(t)^2$	0.4794	0.3963
Abs($DLD(t) - DSC$)	0.6712	0.6048
$(DLD(t) - DSC)^2$	0.5640	0.5245

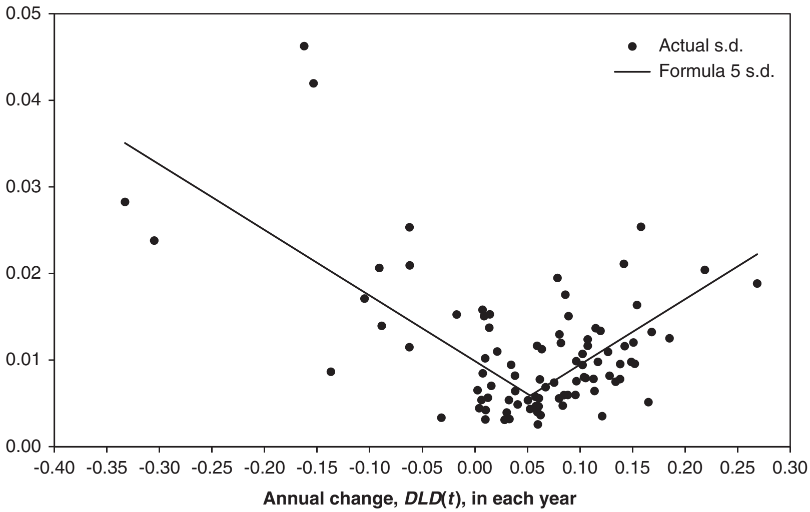

Figure 8. Standard deviations, and estimates of σ_m in each year, related to $DLD(t)$ in each year, from 1924 to 2014.

average correlation coefficient of 0.1782, though this varies very much from month to month. We might have expected a higher value, but the high variation in average forwards deviation from month to month may be caused more by exceptionally large values than by a consistent seasonal variation, in spite of the plausibility of there being correlation because of the annual cycle of company announcements.

3.2.9. There is, however, another correlation well worth taking into account. There is simultaneous correlation between the standardised deviations between those for YL and those for DL, with a coefficient of 0.2580. This is not huge, but still very significant. It is, further, not surprising. Since the dividend yield is calculated by dividing the dividend index by the share price index, a jump in the dividend index automatically causes a corresponding jump in the yield, unless the price also changes correspondingly. We are looking at changes over a month, and a lot of other things may influence prices and hence yields in that time, so the correlation is reasonably a lot less than unity. But it is included in our model. We find no other larger cross-correlations with earlier series. As with correlation coefficients in successive years, provided the values of the correlation coefficients in corresponding months are the same, the Zs and the Z^*s have the same correlation coefficient.

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3.3. Simulations

3.3.1. We now have a model for the dividend index. We use a BB for DL, with $\sigma_m(t)$ varying according to the formula in section 3.2.7 and with correlation of the Z_j values with those of YL_j. We show one simulation for 50 years for K in Figure 9 and for D in Figure 10. We make the vertical scale of Figure 9 the same as that of Figure 6. The annual values do not fluctuate as much in the simulations as in the actual data, but the bridging appears to us to be comparable. In so far as one can observe anything from these figures, Figure 10 seems locally similar to Figure 5.

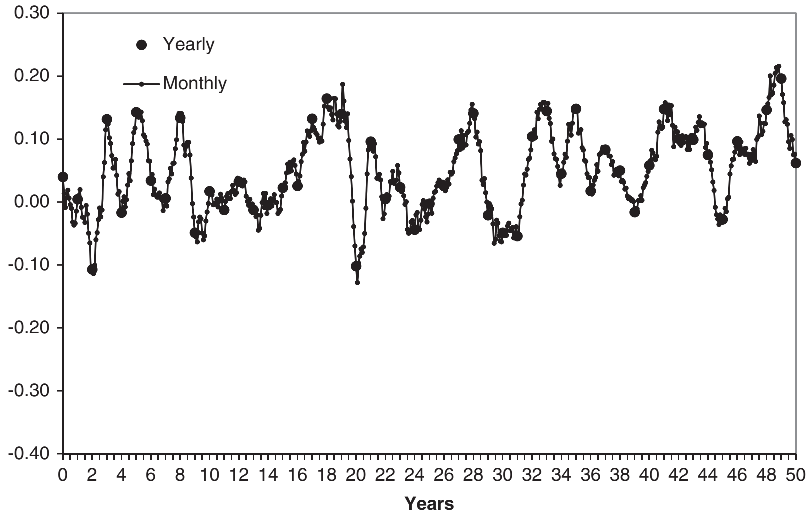

Figure 9. Stochastic yearly and monthly simulation for 50 years of $K(t)$, with initial conditions as at June 2014.

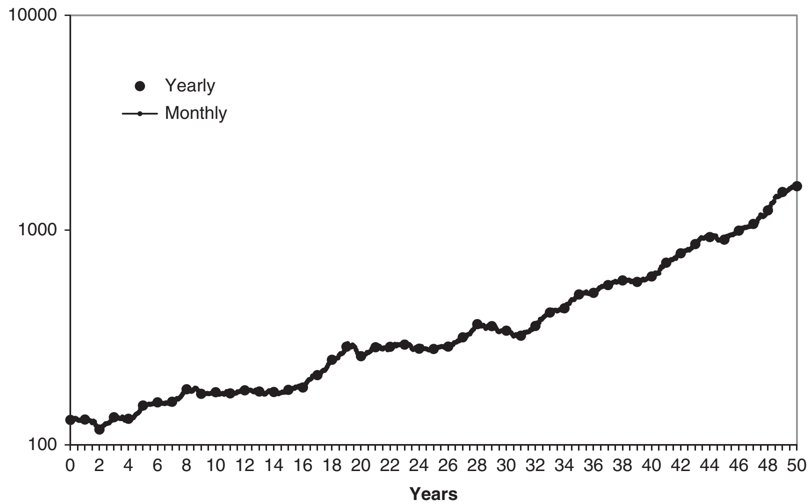

Figure 10. Stochastic yearly and monthly simulation for 50 years of $D(t)$, with initial conditions as at June 2014.

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3.4. The share price index

3.4.1. Part of the basic source data for the share indices is the prices index, $P(t)$, from which the dividend index is derived as $D(t) = P(t) \times Y(t)$. In Figure 11 we show the actual values of $P(t)$, along with the dividend index, $D(t)$, multiplied by 25 so as to get comparable scales, and in Figure 12 we show the annual changes in $\ln P(t)$, which we denote as $PLD(t)$.

3.4.2. Now that we have models for simulating YL and DL we can construct simulations for the price index, PL, since $PL(t) = DL(t) - YL(t)$ and $P(t) = \exp(PL(t))$. We show in Figure 13 a simulation of $P(t)$

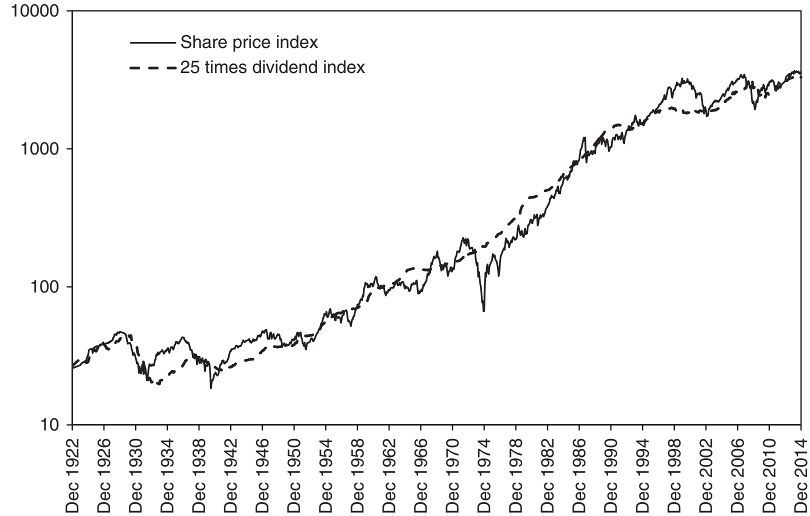

Figure 11. Values of the share prices index, $P(t)$, monthly, along with $25 \times D$(t), from December 1922 to December 2014.

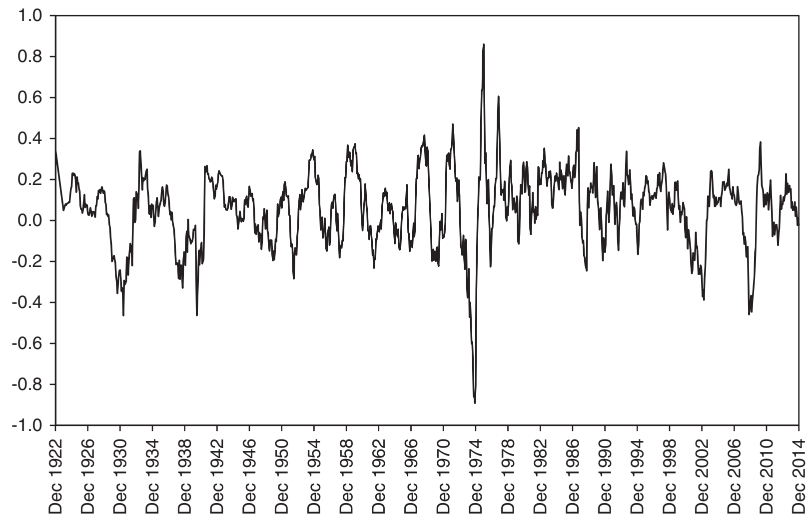

Figure 12. Values of annual changes in logarithm of share prices index, $PLD(t)$, from December 1922 to December 2014.

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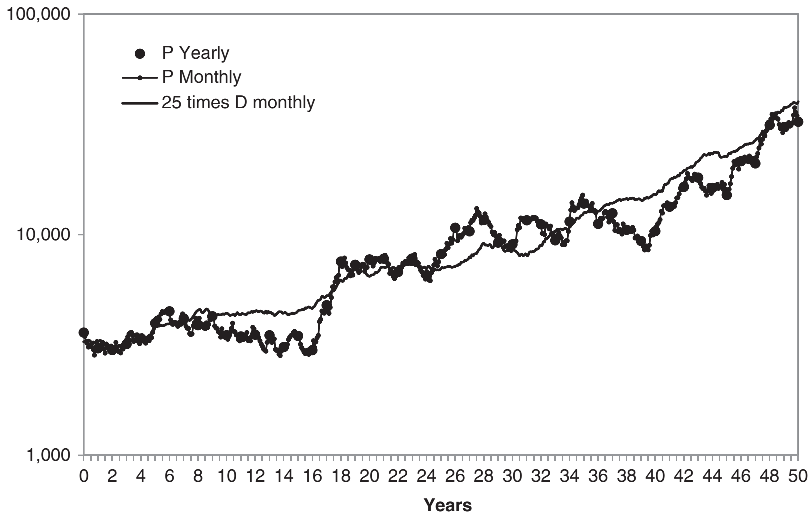

Figure 13. Stochastic yearly and monthly simulation for 50 years of $P(t)$, along with 25 times simulated $D(t)$, with initial conditions as at June 2014.

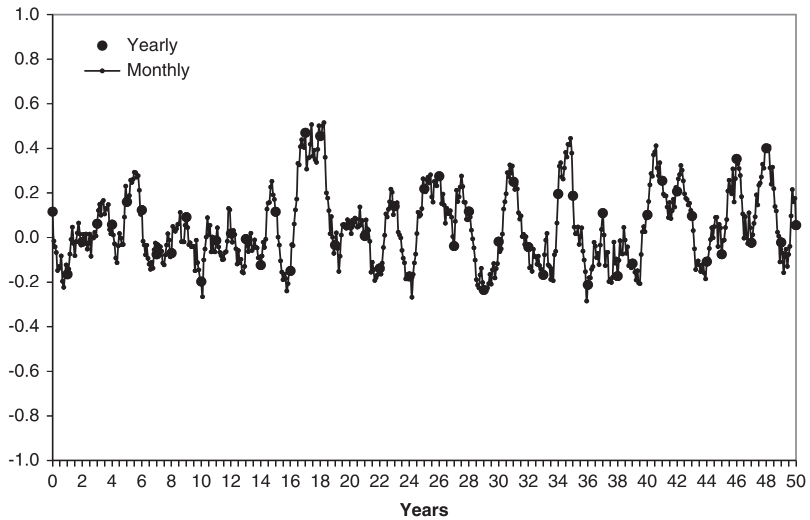

Figure 14. Stochastic yearly and monthly simulation for 50 years of $PLD(t)$, with initial conditions as at June 2014.

for 50 years, based on the simulated values for Y and D in Figures 4 and 10, on which we include the simulated values of $D(t)$ multiplied by 25. We can compare this with the graph of the actual values of $P(t)$ and $25D(t)$ in Figure 11. To complete the picture we show in Figure 14 the simulated values of $PLD(t)$, on the same scale as Figure 12, which we can compare with the actual values in Figure 12. The results look plausible, though the extreme values seen in Figure 12 do not appear in the simulated series. This may be because we have used normally distributed random innovations in the simulations, whereas we know that the actual data are distributed in a more fat-tailed way. We hope to return to this in later part of this series.

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4 Long-Term Interest Rates

4.1. The long-term interest rate ("Consols") model

4.1.1. The model for long-term interest rates, which we still describe as the "Consols" yield although for many years we have not based our data on the yield on the UK government's $2\frac{1}{2}\%$ Consolidated Stock (which was wholly redeemed in July 2015), is, in principle:

$$CM(t) = CD.I(t) + (1 - CD).CM(t-1)$$

$$CN(t) = CA.CN(t-1) + CY.YE(t) + CE(t)$$

$$CRL(t) = \ln CR(t) = \ln CMU + CN(t)$$

$$C(t) = CM(t) + CR(t)$$

with parameters suggested in Part 1 (Wilkie *et al.*, 2011):

$$CD = 0.045;\; CMU = 2.23\,\%\,;\; CA = 0.92;\; CY = 0.37;\; CSD = 0.255$$

Two different constraints are then introduced, one of which is applied to past data, where we know the values of $C(t)$ but wish to ensure that the values of $CR(t)$ are positive. For this we put

$$CM(t) = \text{Min}(CD.I(t) + (1 - CD).CM(t-1), C(t) - CMIN)$$

so that we may reduce $CM(t)$ to ensure that $CR(t) \geq CMIN$.

The other is applied to future simulations, in which we simulate $CRL(t)$, ensuring that $CR(t)$ is positive, but then put

$$CM(t) = \text{Max}(CD.I(t) + (1 - CD).CM(t-1), CMIN - CR(t))$$

so that we may increase $CM(t)$ to ensure that $C(t) \geq CMIN$.

We could use different values of $CMIN$ in these two situations, but in practice put $CMIN = 0.005 = \frac{1}{2}\%$ in both cases.

4.1.2. $C(t)$ too can be decomposed into two parts, which we have already denoted $CM(t)$ and $CR(t)$. However, $CM(t)$ is not observable in the market; we have constructed it as a proxy for "the market's" estimate of future inflation, for which, as noted in Part 2, "implied inflation", the difference between the yields on conventional and index-linked stocks, might be an alternative for more recent years. Thus, although the annual values of $CM(t)$ are given by the model, we can define $CM(t)$ for intervening months in different ways, and we investigate some of these. We could then simulate bridges for $CRL(t)$, calculate $CR(t)$, add $CM(t)$ and apply the second constraint to ensure that $C(t) \geq CMIN$. An alternative is to simulate bridging for $CL(t)$ directly, thus ensuring that $C(t)>0$. For both methods an OU bridge might be natural, but, as we show, there is very little difference between an OU bridge and a BB for these, especially since the value of the autoregressive parameter CA is large.

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4.2. The "Consols" yield data

4.2.1. We have available data at monthly intervals from 1923 to 2014 for this series, first from the yield on $2^1/_2\%$ Consols, then the yield on irredeemables from the *Financial Times* Actuaries Fixed Interest Indices (now the FTSE Actuaries UK gilts Indices), and most recently the 45-year yield from those indices. Apart from the possibility that in the early years, up to about 1930, the "monthly" figures might be "monthly averages" (in practice probably the mid-point of the range of daily closing yields) rather than end month yields, there are no problems with these figures. We plot them in Figure 15.

4.2.2. Our formulae for $CM(t)$ for annual intervals is

$$CM(t) = CD.I(t) + (1 - CD).CM(t-1)$$

subject to the constraint

$$CM(t) = \mathrm{Min}(CD.I(t) + (1 - CD).CM(t-1), C(t) - CMIN)$$

An obvious way to calculate $CM(t)$ for intervening months is to use the same formula, using values of $I(t)$ ending in the same month in previous years. This is what we have quoted in Table 8 of Part 2, as being appropriate for annual simulations starting in the given months. But the values, which contain values of $I(t)$ for past years with exponentially decaying weights, are perhaps more erratic than they should be if the purpose is to indicate "the market's" estimate of future inflation. We denote these monthly values as $CM1$.

4.2.3. A second way of calculating monthly values would be to slide from $CM(t)$ to $CM(t+1)$, a year later, gradually. $CM(t+1) = CD.I(t+1) + (1 - CD).CM(t)$, and we can separate this into two components denoted, $CM2A$ and $CM2B$. For year t, month m, we define

$$CM2A(t, j) = CD.(\ln(Q(t, j)) - \ln(Q(t, 0)))$$

Figure 15. Values of long-term interest rates, $C(t)$, monthly from December 1922 to December 2014.

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Table 8. Values of $C\%$, $CM1$, $CM2$ and $CM3$, and $CR1$, $CR2$ and $CR3$, from June 2013 to June 2014.

Month end	$C\%$	$CM1$	$CM2$	$CM3$	$CR1$	$CR2$	$CR3$
June 2013	3.54	0.02656	0.02656	0.02656	0.00884	0.00884	0.00884
July 2013	3.47	0.02536	0.02646	0.02656	0.00934	0.00824	0.00814
August 2013	3.51	0.02610	0.02659	0.02656	0.00900	0.00851	0.00854
September 2013	3.46	0.02661	0.02665	0.02656	0.00799	0.00795	0.00804
October 2013	3.41	0.02673	0.02655	0.02656	0.00737	0.00755	0.00754
November 2013	3.50	0.02591	0.02649	0.02655	0.00909	0.00851	0.00845
December 2013	3.57	0.02612	0.02662	0.02655	0.00958	0.00908	0.00915
January 2014	3.46	0.02605	0.02638	0.02655	0.00855	0.00822	0.00805
February 2014	3.46	0.02780	0.02656	0.02655	0.00680	0.00804	0.00805
March 2014	3.49	0.02659	0.02657	0.02655	0.00831	0.00833	0.00835
April 2014	3.40	0.02546	0.02663	0.02655	0.00854	0.00737	0.00745
May 2014	3.33	0.02527	0.02657	0.02654	0.00803	0.00673	0.00676
June 2014	3.37	0.02656	0.02656	0.02656	0.00884	0.00884	0.00884

and

$$CM2B(t, j) = (1 - CD)^{j/12}.CM(t)$$

Then we put $CM2(t, j) = CM2A(t, j) + CM2B(t, j)$ and we reach $CM2(t, 12) = CM(t+1)$.

4.2.4. A third way is by simple linear interpolation

$$CM3(t, j) = CM(t) + (CM(t+1) - CM(t)).j/12$$

4.2.5. In each of these cases we need to consider the constraint. For $CM1(t, j)$ we can apply the constraint as for annual values, carrying forwards the adjusted value of $CM1(t, j)$ to the next year. For $CM2(t, j)$ we can apply the constraint to that single value, letting the two components $CM2A$ and $CM2B$ continue unaltered. For $CM3(t, j)$ we can apply the constraint, and then interpolate linearly between the adjusted value of $CM3(t, j)$ and the final "target" value, $CM(t+1)$, over the remaining months. In some years we may have several such adjustments. In Table 8 we give an example of these different values of CM, over the year from June 2013 to June 2014, along with $C\%$ and the corresponding values of CR, denoted $CR1$, $CR2$ and $CR3$. In no case does the constraint apply.

4.2.6. Calculations show that, for this short period, the standard deviation of the values for the 11 intervening months, is, for $CM1$ 0.0067, for $CM2$ 0.0008 and for $CM3$ 0.0001, so (rather obviously) $CM3$ is the smoothest. The standard deviation for $CR1$, $CR2$ and $CR3$ are 0.00083, 0.00047 and 0.00046, respectively. Since the standard deviation for C alone is 0.00046, the effect of $CM1$ is to add to the variability of $CR1$ rather than to reduce it, whereas the other two formulations leave the variability almost unchanged.

4.2.7. In Figure 16 we show the values of C (as a fraction, not a percentage), and $CM1$, $CM2$ and $CM3$. C is shown as a dashed line at the top of the group, $CM3$ is shown by a solid line; it is mainly short straight lines between June values. $CM2$ is shown by a dashed line, which is almost indistinguishable from $CM3$. $CM1$ is shown by a dotted line, also mostly indistinguishable from $CM3$. But in December 1998 all three move down to equal an amount $CMIN = 0.005$ below the reduced

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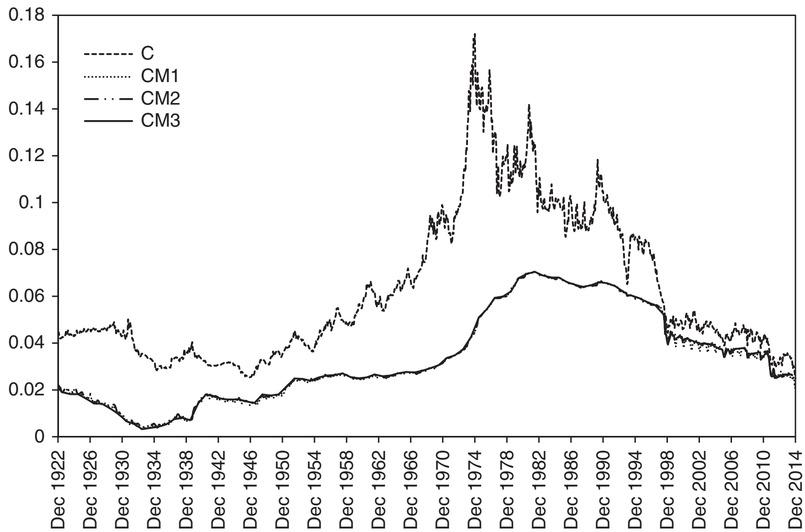

Figure 16. The Consols yield, C, $CM1$, $CM2$ and $CM3$, from 1922 to 2014.

Table 9. Estimates of σ_m for $CL(t)$, $CRL1(t)$, $CRL2(t)$ and $CRL3(t)$, 1923–2014.

	Brownian bridge	OU bridge
$CL(t)$	0.0337	0.0336
$CRL1(t)$	0.1188	0.1184
$CRL2(t)$	0.1074	0.1070
$CRL3(t)$	0.1132	0.1128

values of C; the sequence from October 1998 in percentage terms is 5.14, 4.72, 4.55. 4.44. 4.76. 4.72, so C is low again in January 1999, but goes back up a bit in February 1999 and thereafter. $CM2$ and $CM3$ are reduced temporarily, but the reduced value of $CM1$ is carried forward into subsequent years, decaying only slowly, so that the "hiccup" that can be seen in the graph continues for several years. This is not very satisfactory. It is an instructive example of how one can see from a graph features that are not easily observed in a table of numbers.

4.2.8. Rather than work directly with C and CR, we analyse the bridging properties of $CL(t) = \ln(C(t))$ and $CRL(t) = \ln(CR(t))$, both as BBs and as OU bridges. The initial estimates of σ_m for each of what are now eight possibilities are shown in Table 9. These numbers are not strictly comparable as between CL and CRL because of the non-linear relationship between them, but we see that the values of σ_m for CL are much lower than for CRL. Of the three versions of CRL, $CRL2$ gives rather lower values than the others. There is very little difference between the values of σ_m for the BB and OU bridge.

4.2.9. An important influence on the value of σ_m for the CRL bridges is the value of $CMIN$. In the 1,093 months from June 1923 to June 2014 there were 63 months in which $CR1(t) = 0.005$, 64 months for $CR2(t)$ and 40 months for $CR3(t)$; in the case of $CR1(t)$ these included every month in the year from June 1998 to June 1999. The lowest value of $CRL(t)$ is $\ln(0.005) = -5.2983$.

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Since our purpose is solely to keep $CR(t) > 0$ we could use a much smaller value but if we change to $CMIN = 0.001$ with $\ln(0.001) = -6.9077$, then we get slightly fewer values at the minimum, but very greatly increased values of σ_m, around 0.22–0.25. This is uncomfortable.

4.2.10. Experiments with simulations, using $CMIN = 0.005$ (but in none of the simulated years did it apply), and using the same innovations, showed very little difference between the simulated values of $C(t)$ with any of the eight methods. The furthest out were those using $CM1$, both BB and OU. Excluding these two, the largest difference, in only one simulation over only 10 years, was 0.096%, less than 10 basis points, and that was between $CRL2$ with an OU bridge and CL with a BB. The difference between corresponding OU bridges and BBs was in no case greater then 0.0016, less than 2 basis points. This inclines us towards recommending simulating with the simplest model, with CL and a BB. Simulating bridging for CL keeps the value of $C(t)$ necessarily positive and saves us having to choose between different versions of CM, and different values of $CMIN$. The strong effect of inflation on interest rates is taken into account in the annual model, and does not need to be considered separately for the bridging model.

4.3. Further analysis

4.3.1. We now consider the analysis of $CL(t)$ more closely. We assume a BB. Statistics for the forwards deviations (not standardised) are shown in Table 10, and for the sideways ones in Table 11. Among the forwards deviations only October shows a significantly low mean (just), and June a significantly high one. But the first few months show negative means, and this results in the means for the sideways deviations being all negative. In both cases the kurtosis is high in many months. There are particularly large forwards deviations in January 2009 and September 1931 and a particularly large negative one in September 1977, but apart from the first of these we cannot identify any particular economic reasons for these.

4.3.2. The means of the 55 upwards deviations are all negative, as are those of the 32 downwards ones and the 4 medium ones. This does not suggest that an OU bridge is any more appropriate. Indeed when we analyse the data as an OU bridge with $\alpha_y = 0.92$, the statistics are quite similar. But with such a large α_y this is not surprising. As with so many of our series, the residuals are fatter tailed than normal, and tests based on an assumption of normality are unreliable.

Table 10. Statistics for forwards deviations, Df_j, of CL, from June 1923 to June 2014.

Month	Mean of Ds_j	T-ratio	Skewness	Kurtosis
July	−0.0030	−1.12	0.01	2.69
August	−0.0060	−1.92	−0.92	4.87
September	−0.0013	−0.31	−0.43	8.18
October	−0.0076	−2.02	0.20	4.86
November	0.0013	0.37	0.24	5.04
December	0.0005	0.18	−0.66	4.77
January	−0.0041	−1.02	0.52	7.51
February	0.0026	0.78	0.17	4.73
March	0.0019	0.54	−0.56	3.95
April	0.0033	1.17	0.64	4.97
May	0.0039	1.14	−0.38	4.15
June	0.0085	3.30	0.28	3.62

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Table 11. Statistics for sideways deviations, Ds_j, of CL, from June 1923 to June 2014.

Month	Mean of Df_j	T-ratio	Skewness	Kurtosis
July	−0.0030	−1.12	0.01	2.69
August	−0.0091	−2.21	0.00	3.13
September	−0.0104	−1.74	0.47	5.63
October	−0.0180	−3.09	0.23	4.63
November	−0.0167	−2.81	0.48	5.57
December	−0.0162	−2.51	−0.50	6.11
January	−0.0203	−3.46	−0.78	6.31
February	−0.0177	−3.69	−0.45	4.87
March	−0.0158	−3.53	−0.08	3.08
April	−0.0125	−3.24	0.08	4.94
May	−0.0085	−3.30	−0.28	3.62

Table 12. Correlation coefficients from regressions.

	1923–2014	
	σ_m	σ_m^2
Abs($CLD(t)$)	0.2076	0.1868
$CLD(t)^2$	0.2308	0.2196
Abs($CLD(t) - CSC$)	0.2008	0.1791
$(CLD(t) - CSC)^2$	0.2268	0.2148

4.3.3. We investigate the variation of the monthly estimate of σ_m in each year with the overall change in $CL(t)$ in the year. We put $CLD(t) = CL(t+1) - CL(t)$ and consider the same eight possible regressions as before. The constant CSC is the calculated mean value of $CLD(t)$ which is quite small at −0.0025. The long-term expected value of this is 0, so the first four regressions are also relevant. The correlation coefficients are shown in Table 12. The coefficients for σ_m are larger than those for σ_m^2; those for $CSC = 0$ are a little larger (but very little larger) than those for $CSC = -0.0025$, and those for the squared function are a little larger than those for the absolute function. We therefore use formula (3) and put

$$\sigma_m(t) = CSM(t) = CSA + CSB \times CLD(t)^2$$

with $CSA = 0.028392$ and $CSB = 0.165597$. We note that the T-ratio of CSB is 2.36, so it is reasonably, but not extremely, significant.

4.3.4. Correlation of the standardised forwards deviations in corresponding months in successive years is very small, the overall correlation coefficients being 0.0573, and the largest, that for December, being only 0.16. When we look at the forwards deviations as a single series there is some autocorrelation, just significant, with negative partial autocorrelation coefficients at all lags up to about 18 months. But we showed in section 2.2.6 that we expect an autocorrelation within the year of −1/11 = −0.09 so this result is only to be expected.

4.3.5. The one cross-correlation of interest is that between the residuals of $CL(t)$ and $YL(t)$ in the same month, which is 0.2558 for the standardised and 0.2610 for the unstandardised residuals. This is not surprising. To some extent the equity and gilt markets can move together in the short term, and

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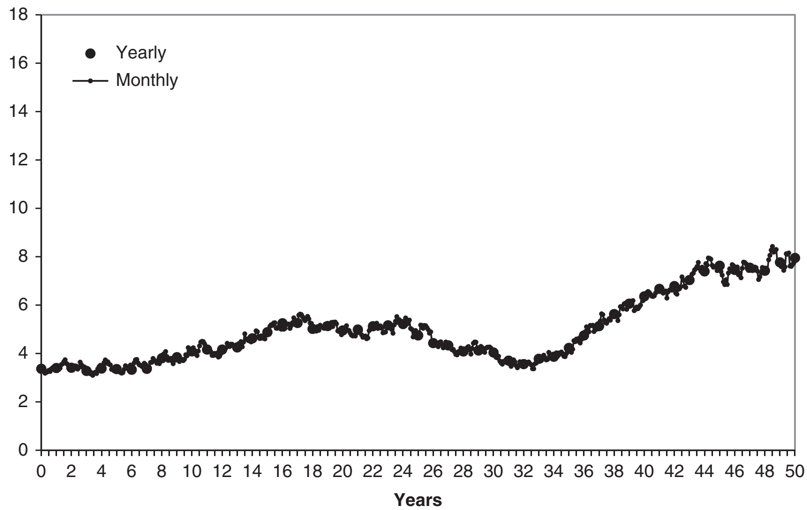

Figure 17. Stochastic yearly and monthly simulation for 50 years of $C(t)$%, with initial conditions as at June 2014.

this is reflected in our model by changes in the yields. We therefore take this into account, using a value of 0.2558. There is, however, no apparent significant correlation monthly between changes in CL and changes in QL, WL or DL.

4.4. Simulations

4.4.1. We end up therefore with a similar model to those for many of our series. We use a BB for $CL(t)$, with a σ_m varying according to formula (3) with

$$\sigma_m(t) = CSA + CSB \times CLD(t)^2$$

and with CSA = 0.028392 and CSB = 0.165597; with a further correlation coefficient with the monthly innovations of $YL(t)$ of 0.2558. One simulation on this basis for 50 years is shown in Figure 17, with the same vertical scale as Figure 15. We can compare this with the actual values of $C(t)$ % shown in Figure 15. The rates do not rise so high as they did in 1974, nor are they as low as they have fallen since June 2014, but these, if anything, are criticisms of our annual model, and we consider that the monthly variation looks reasonable.

5. Short-Term Interest Rates

5.1. The short-term ("Base Rate") interest rate model

5.1.1. The model for short-term interest rates, for which we use the Bank of England's Base Rate and its predecessors, is

$$BD(t) = \ln C(t) - \ln B(t)$$

$$BD(t) = BMU + BA.(BD(t-1) - BMU) + BE(t)$$

$$B(t) = C(t).\exp\{-\mathrm{BD(t)}\}$$

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In 2009, the change in $B(t)$ down to $^1/_2\%$ was so large, and the corresponding value of $BD(t)$ was so large, that we used, for that year only, an "intervention" term, $BInt(t)$ putting

$$BD(t) = BMU + BA.(BD(t-1) - BMU) + BI.BInt(t) + BE(t)$$

But this was exceptional and is not part of the normal model. Parameters suggested in Part 1 (Wilkie *et al.*, 2011) are as follows:

$$BMU = 0.17;\ \ BA = 0.73;\ \ BSD = 0.3$$

5.2. The "Base Rate" data

5.2.1. Base Rate changes on particular days, and remains constant between changes. Although we have available complete daily data, we have also recorded it at the end of each month, and we use these end-of-month values. Up to 1972 the values were always multiples of $^1/_2\%$, but in October of that year the rate was $7^1/_4\%$ for a few days, and in December it was $7^3/_4\%$ for a few days; the first $^1/_4\%$ we have recorded is for January 1973. In recent years changes have occurred or not occurred at monthly intervals, but in earlier years there might have been several changes in one month. For long periods it has remained unchanged; from June 1932 to November 1951 it was at 2%, except for a few weeks in August to October 1939, and from March 2009 to August 2016 it was at $^1/_2\%$. It is the sort of series that would lend itself to being analysed as a Poisson process with discrete jumps, but we have in our annual model pretended that it has a continuous distribution, so we keep to that here.

5.2.2. The value clearly goes up and down along with long-term interest rates. In Figure 18 we plot both $C(t)$ and $B(t)$ at monthly intervals from December 1922 to December 2014. Note that $B(t)$ always jumps discretely at one or more days in the month, so the slightly sloping lines joining the dots should be vertical. We express the relation between $B(t)$ and $C(t)$ through $BD(t)$, the difference between their logarithms, or equivalently the logarithm of their ratio, and we plot the values of $BD(t)$ in Figure 19. Up to 2009 $BD(t)$ clearly is mean reverting, wandering up and down around some middle position, but in 2009 it jumped to a different level altogether.

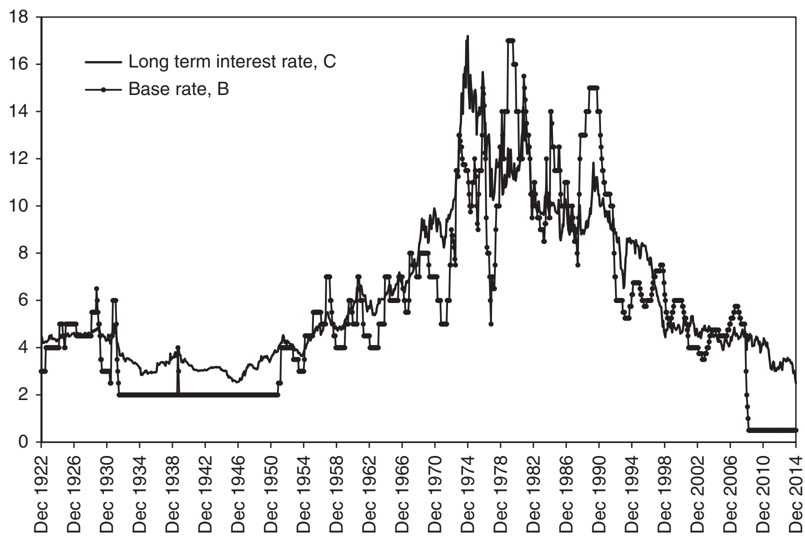

Figure 18. Values of the long-term interest rate, $C(t)$, and bank Base Rate $B(t)$, monthly from December 1922 to December 2014.

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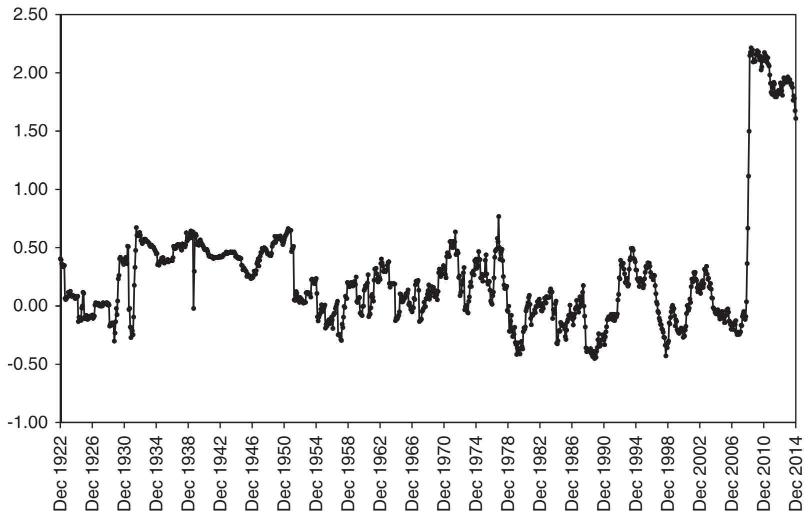

Figure 19. Values of $BD(t) = \ln(C(t)/B(t))$, monthly from December 1922 to December 2014.

5.2.3. We have the choice of modelling the monthly bridges either on the logarithm of $B(t)$, which we can call $BL(t)$, or on $BD(t)$, deriving monthly values of $B(t)$ from those of $C(t)$ and $BD(t)$. We use $BL(t)$ because that ensures that the values of $B(t)$ remains positive, but this is perhaps not now a necessary constraint, since some other countries have introduced negative bank rates (Switzerland and Denmark, for example). We could, if we wished, round the simulated future values of $B(t)$ to $\frac{1}{4}\%$ units. We could use BBs or OU bridges in either case. However, a problem with analysing $BL(t)$ is that in many years there is no change in its value at all, and the estimated value of σ_m is therefore 0. This occurs in 27 of the 91 tears. When there are changes, they are in discrete jumps, slightly concealed because of taking logarithms. This very much distorts the figures, which are nothing like normally distributed. On the other hand, in these same periods, changes in $BD(t)$ are caused entirely by changes in $C(t)$ so we may seem to be repeating our analysis of that variable.

5.2.4. In Table 13 we show statistics for the forwards deviations for both BL and BD assuming BBs. Because of their construction, deviations for BL and BD have opposite signs, so we expect the means to have sign reversed, and this occurs for 9 out of the 12 months. There are large T-ratios for three months, but the skewness and kurtosis are so large for every month that any significance test would be invalid. The statistics for the forwards deviations for an OU model are very similar, and we do not show them.

5.2.5. The sideways deviations show even larger kurtosis, and we can learn little from them. We determine whether a deviation is "upwards", "downwards" or "medium" by the relation with the assumed mean if there were an OU bridge. For BL, the 52 years with upwards deviations show positive means in every month; but for downwards deviations so do 8 out of the 11 months, and so does every month for the medium deviations. For BD the numbers are almost the same, but the signs are reversed, negative means prevailing.

5.2.6. There is nothing from the available statistics that allows us to express any firm preference for bridging on BL or BD, or on using a Brownian or an OU bridge. Since the simulated results of using

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Table 13. Statistics for forwards deviations, Df_j, of *BL* and *BD*, from June 1923 to June 2014.

	BL				*BD*			
Month	Mean of *Dfj*	*T*-ratio	Skewness	Kurtosis	Mean of *Dfj*	*T*-ratio	Skewness	Kurtosis
July	0.0240	2.40	3.90	20.17	–0.0270	–2.77	–3.25	15.79
August	0.0056	0.64	5.90	49.57	–0.0117	–1.37	–5.51	43.49
September	0.0042	0.53	1.17	11.11	–0.0055	–0.77	–0.06	10.79
October	–0.0061	–0.90	–2.22	19.28	–0.0015	–0.23	1.62	11.61
November	0.0073	0.93	1.90	9.95	–0.0061	–0.83	–1.60	8.18
December	–0.0029	–0.65	0.86	18.18	0.0035	0.71	–1.03	7.70
January	0.0000	0.00	2.75	18.93	–0.0041	–0.62	–0.77	12.61
February	0.0018	0.29	0.86	8.71	0.0008	0.12	–0.46	4.94
March	–0.0227	–2.38	–0.95	16.67	0.0246	2.80	0.16	17.90
April	–0.0106	–2.16	0.62	8.11	0.0139	2.48	0.29	4.94
May	–0.0056	–1.10	0.14	8.34	0.0096	1.80	0.11	4.01
June	0.0050	0.86	0.28	3.62	0.0036	0.58	–0.91	7.41

OU bridges are very close to those with BBs, as is usual with our series, we continue to consider only BBs. The simulated values for *BL* and *BD* are quite similar (for *BD* we reverse the signs of the innovations used for *BL*), but either shows reasonable looking results.

5.2.7. We consider the regressions of σ_m and σ_m^2 on the usual functions of the annual changes, defined as $BLD(t) = BL(t+1) - BL(t)$ and $BDD(t) = BD(t+1) - BD(t)$. The values are shown in Table 14. The larger correlation coefficients are certainly worth taking into account. For *BL* the largest is for σ_m on the absolute difference from the mean $BSC = -0.01969$, but this is only marginally higher than that when $BSC = 0$; for *BD* it is for is σ_m^2 on the squared difference from the mean, a different $BSC = 0.01714$, with again a very small difference when $BSC = 0$. Since the long-term expected values of both *BLD* and *BDD* are zero we prefer these forms.

5.2.8. In Figure 20 we show the actual values of σ_m for *BL* estimated from the data in each year along with estimates on two regression formulae: (the formula numbers are as in section 8.5 of Part 3A)

$$(\text{Formula 1})\ \sigma_m = BSM(t) = BSA + BSB \times \text{Abs}(BLD(t))$$

where $BSA = 0.029183$ and $BSB = 0.112781$, and

$$(\text{Formula 2})\ \sigma_m = BSM(t) = \sqrt{BSA + BSB \times \text{Abs}(BLD(t))}$$

where $BSA = 0.001616$ and $BSB = 0.021683$.

In Figure 21 we show corresponding values for *BD* along with estimates on two regression formulae:

$$(\text{Formula 4})\ \sigma_m = BSM(t) = \sqrt{BSA + BSB \times BDD(t)^2}$$

where $BSA = 0.004317$ and $BSB = 0.010021$, and

$$(\text{Formula 2})\ \sigma_m = BSM(t) = BSA + BSB \times \text{Abs}(BDD(t))$$

where $BSA = 0.001852$ and $BSB = 0.019851$.

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Table 14. Correlation coefficients from regressions for BL and for BD, 1923–2014.

BL			BD		
	σ_m	σ_m^2		σ_m	σ_m^2
Abs($BLD(t)$)	0.5625	0.5036	Abs($BDD(t)$)	0.4866	0.5092
$BLD(t)^2$	0.4207	0.4910	$BDD(t)^2$	0.4431	0.5312
Abs($BLD(t) - BSC$)	0.5634	0.5076	Abs($BDD(t) - BSC$)	0.4926	0.5135
$(BLD(t) - BSC)^2$	0.4237	0.4925	$(BDD(t) - BSC)^2$	0.4448	0.5323

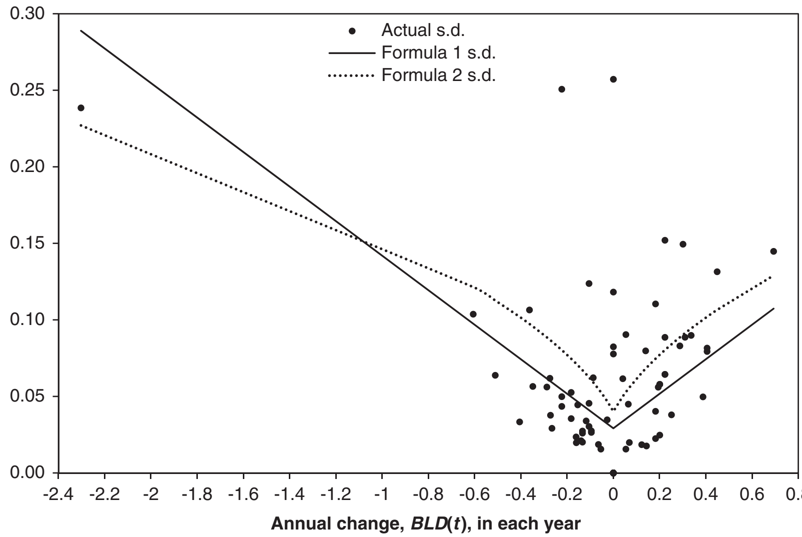

Figure 20. Standard deviations, and two estimates of σ_m in each year, related to $BLD(t)$ in each year, 1923–2014.

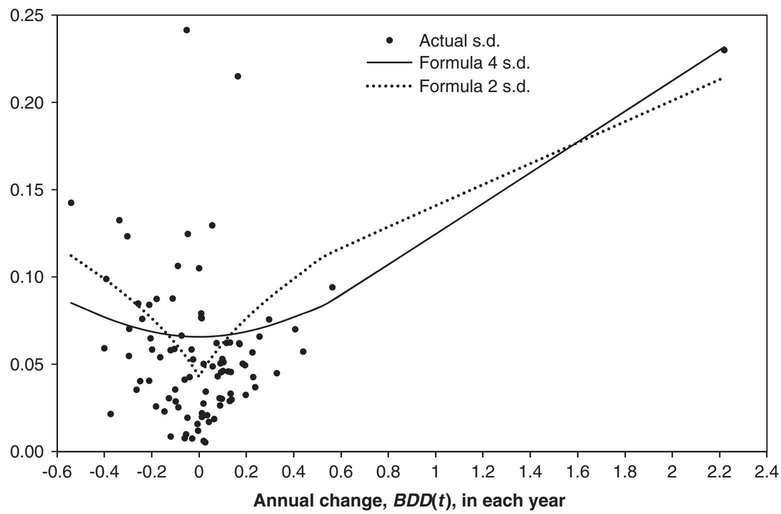

Figure 21. Standard deviations, and two estimates of σ_m in each year, related to $BDD(t)$ in each year, 1923–2014.

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5.2.9. We can see how both are dominated by the extreme value for 2009, on the left for *BL* and on the right for *BD*. Nevertheless, there is a sort of V shape in the position of the other points, which must be partly caused by the discrete nature of the data. If the change in *B* in the year is small, and changes are restricted to ¼% or ½% points, they cannot be evenly spaced across the months, so the variance cannot be very small. On the other hand, in the graph for *BL* there are 27 points piled up on (0, 0).

5.2.10. We look also at possible auto- and cross-correlations. For neither variable is there any significant correlation from year to year in corresponding months. But simultaneous cross-correlations can be observed. For *BL* the correlation between standardised residuals, the values of DfZ_j, is moderate for both *YL*, at 0.1796, and *CL* at 0.1476, and negligible elsewhere. For *BD* the correlation with *YL* is small, but that with *CL* is large at 0.5286. This is not surprising, since *CL* is one of the constituents of *BD*, and it would be appropriate to take it into account.

5.3. Simulations

5.3.1. We now have two possible models, one based on bridging for *BL*, the other on bridging for *BD*. BBs are appropriate for both; there is no strong evidence in favour of an OU bridge in either case. Both show significant variation in σ_m from year to year depending on the change in *BL* or *BD* over the year according to the formulae in section 5.2.8; we prefer formulae (1) and (4), respectively. Neither shows correlation over the residuals from year to year, but *BD* shows a strong simultaneous correlation with the residuals of *CL*.

5.3.2. We show one simulation for 10 years, with values of $B(t)$ simulated with bridging both *BL* and *BD* and then rounded to the nearest ¼%. We use the same innovations for both sets of simulations, but with the sign reversed, since *BD* and *BL* are negatively related. We also show values of $C(t)$ for the corresponding simulation.

5.3.3. Visually, the results seem quite similar, with many months showing no changes in $B(t)$, and many months showing the same values for the two sets of simulations. However, simulations using *BD* seem to "jiggle" up and down a little more than those using *BL*. We can compare the results numerically. In the 10 years there are 110 intermediate months (excluding the annual values which are necessarily equal), and in 14 of these the values of $B(t)$ using *BL* and *BD* show different values. There are 120 months when a change could occur, and there are changes in 29 of these using *BL*, and 43 using *BD*. The sum of the absolute value of the monthly changes is 7.5% using *BL* and 11.5% using *BD*. In both cases most changes are of ¼%, but using *BL* there is one change of ½% and using *BD* there are three. Figures for simulations over 50 years, which we do not show fully, are comparable. Neither bridging method is incompatible with the real data, but neither shows the long periods of stability that can be seen in Figure 22. On balance we prefer to use bridging over *BL*.

5.3.4. In Figure 23 we show the simulations continued for a total of 50 years using bridging over *BL*. The overall pattern is similar. We note that nowhere in this simulation do the values rise to anything like the levels actually seen in the 1970s and 1980s, nor does $C(t)$ drop as far as it actually has since June 2014. But for all our simulations we are using $QMU = 0.043$, the value found from the data in 2009, rather than the lower value of say 0.025 that we suggested then might be more appropriate. Further, in both the annual and the monthly models we use normally distributed innovations, rather than the fatter-tailed ones that would correspond better with the experiences. But that is for another part of our investigations.

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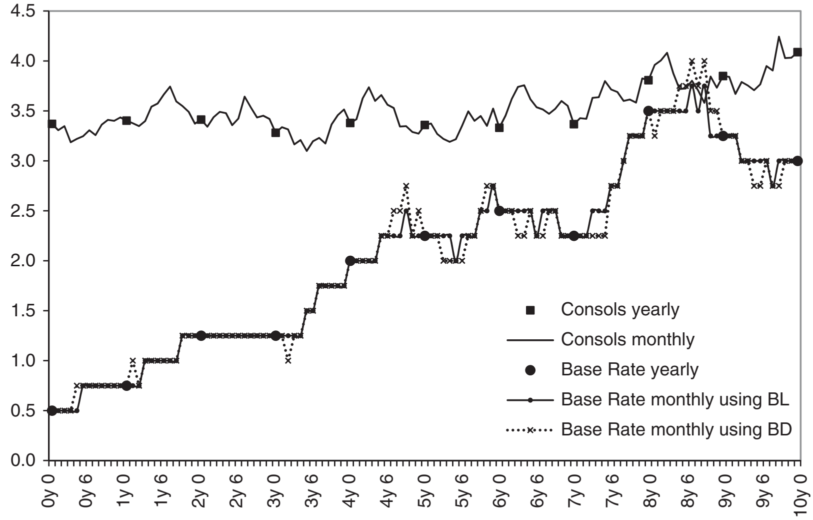

Figure 22. Stochastic yearly and monthly simulation for 10 years of $B(t)$%, bridging both with BL and with BD, and showing also values of $C(t)$%, with initial conditions as at June 2014.

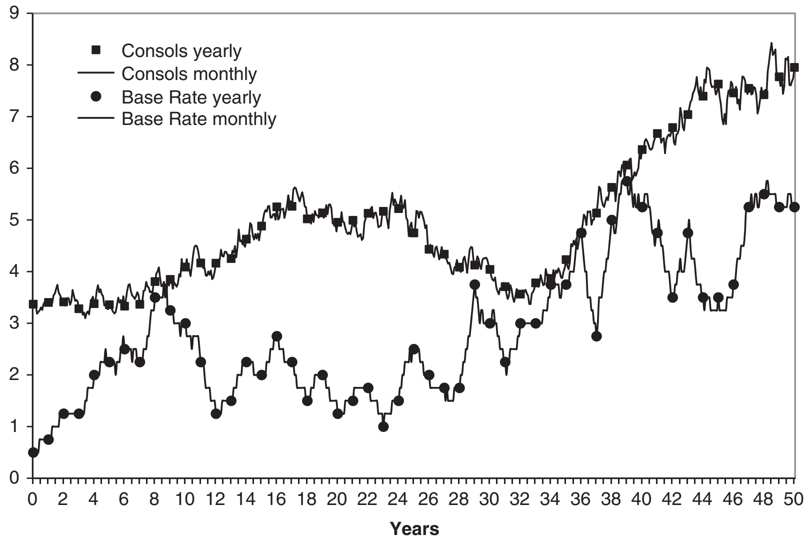

Figure 23. Stochastic yearly and monthly simulation for 50 years of $B(t)$%, bridging with BL, and showing also values of $C(t)$%, with initial conditions as at June 2014.

6. Real Yields on Index-Linked Stocks

6.1. The model for "real" yields on index-linked stocks

6.1.1. The model for "real" yields that we suggested in 2009, altered from that suggested in Wilkie (1995), is

$$R(t) = RMU + RA.(R(t-1) - RMU) + RBC.CE(t) + RE(t)$$

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with parameters suggested in Part 1 (Wilkie *et al*., 2011):

$$RMU = 3\%,\ RA = 0.95,\ RBC = 0.008,\ RSD = 0.003$$

6.1.2. This is a simple AR(1) model for $R(t)$, with the only dependence on other variables being a simultaneous correlation with the innovations for $C(t)$. An OU bridge would be the obvious choice, but the suggested value of RA of 0.95 is so high that an OU bridge and a BB would be almost indistinguishable.

6.2. The real yield data

6.2.1. We have used throughout the yields from the FTSE-Actuaries index-linked indices for stocks over 5 years, with an assumption of 5% inflation, which are available from May 1981 when the first index-linked stocks in the United Kingdom were issued. In Figure 24 we plot the values of $R(t)$ at monthly intervals from May 1981 to December 2014. The values rose and fell again in the first years, as if centred somewhere around 3%, but from 2001 there has been a steady decline, with the rates going negative in 2011 and staying below zero most months since then. But it is economically most implausible that they will continue downwards indefinitely. There are specific institutional arrangements in the United Kingdom at present which almost force pension funds and insurance companies to bid for the limited supply of government index-linked stock, but these may ease sometime. In any case, our current problem is not with the overall annual model, which does suggest reversal to a mean of 3%, but with stochastic bridging within that model.

6.2.2. We have only 33 full years to analyse, from June 1981 to June 2014. We calculate forwards and sideways deviations in the usual way, assuming both BBs, and OU bridges. The numbers are very similar. We show statistics for forwards and sideways deviations for BBs in Table 15. We multiply the means by 100 to improve the presentation. The values of the kurtosis statistic for most months are very high, though less high than for some of our series. This makes the T-ratios unreliable as a test of whether the means are equal to zero. The mean is apparently well below zero for August, just significantly above zero for April, but not unusual for any other months. The sideways deviations consequently show means well below zero for December through to April.

Figure 24. Values of real yield, $R(t)$%, monthly from May 1981 to December 2014.

Table 15. Statistics for forwards and sideways deviations, Df_j and Ds_j, of $R(t)$, from June 1981 to June 2014.

	Forwards deviations				Sideways deviations			
Month	100 times mean of Df_j	T-ratio	Skewness	Kurtosis	100 times mean of Ds_j	T-ratio	Skewness	Kurtosis
July	0.0371	1.51	0.23	3.06	0.0371	1.51	0.23	3.06
August	−0.0778	−3.29	0.34	2.77	−0.0407	−1.23	0.65	3.67
September	0.0159	0.58	−0.89	6.13	−0.0248	−0.81	0.24	3.68
October	−0.0278	−0.80	0.35	4.80	−0.0526	−1.11	0.57	3.56
November	−0.0363	−1.38	−0.92	4.73	−0.0889	−1.99	1.14	6.12
December	−0.0305	−1.01	−1.71	7.64	−0.1194	−2.88	−0.71	4.15
January	0.0028	0.14	−0.05	2.89	−0.1166	−2.70	−0.73	3.90
February	−0.0178	−0.74	−0.25	3.43	−0.1343	−3.36	−0.09	2.79
March	0.0149	0.57	−1.20	6.55	−0.1194	−3.48	−0.10	2.64
April	0.0449	2.04	−1.72	8.49	−0.0744	−2.58	−0.59	3.40
May	0.0440	1.85	0.56	3.57	−0.0304	−1.80	−0.49	2.63
June	0.0304	1.80	0.49	2.63				

Table 16. Correlation coefficients from regressions for R with Brownian bridges, 1981–2014.

	σ_m	σ_m^2
Abs($RD(t)$)	−0.1250	−0.1209
$RD(t)^2$	−0.0974	−0.0724
Abs($RD(t) - RSC$)	−0.0644	−0.0725
$(RD(t) - RSC)^2$	−0.0637	−0.0543

6.2.3. The overall estimate of σ_m is 0.001569, which corresponds with an annual σ_y of 0.0054. This is much larger than the value of the estimated annual standard deviation RSD, of 0.003.

6.2.4. There are 16 years with upwards deviations, and 9 out of 11 months show negative means for these. There are 15 years with downwards deviations, and 10 out of 11 months show negative means. There are only two with medium deviations. This does not give strong evidence in favour of OU bridges.

6.2.5. We calculate the usual eight regressions of the values of σ_m and σ_m^2 on the annual changes of $R(t)$, which we denote $RD(t) = R(t+1) - R(t)$. We show the correlation coefficients for BBs in Table 16. The overall observed mean value of $RD(t)$ is −0.00848, whereas the long-term expected value assuming that an annual AR(1) is correct would be 0. All the correlation coefficients are negative; none is significant at all. The largest value of the T-ratio of RSB is only 0.7, so we assume a constant value of σ_m of 0.001569.

6.2.6. We look also at correlations. The correlation coefficients of the standardised forwards deviations with those of the corresponding month in the previous year are all quite small, so we can ignore these. The cross-correlations with other series, however, are not negligible. The standardised residuals show correlation coefficients of 0.2398 with YL, 0.6367 with CL and a much smaller 0.1116 with BL. Over the same period of 33 years the cross-correlation coefficient of the

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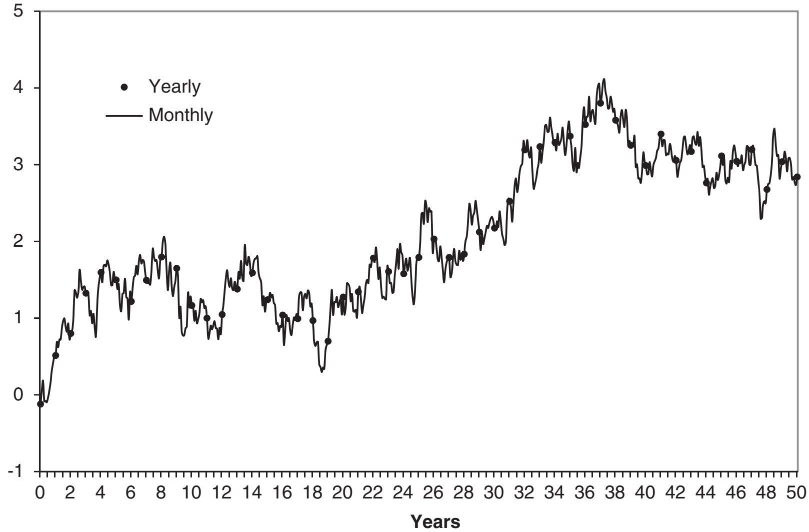

Figure 25. Stochastic yearly and monthly simulation for 50 years of $R(t)$%, with initial conditions as at June 2014.

standardised residuals of *YL* and *CL* is 0.1607 and over the whole period is 0.26. The extra information by taking into account the correlation between *R* and *YL* is small and we can ignore it, leaving us the large coefficient of 0.6367 between *R* and *CL*.

6.3 Simulations

6.3.1. We now have quite a simple model for bridging *R*, a BB, with a constant value of σ_m and a simultaneous cross-correlation with the innovations of *CL*. We show in Figure 25 one simulation for 50 years. Because we retain our assumed value of *RMU* of 3% the annual values creep up, rather slowly, towards that level again. Whether this is reasonable or not is not discussed here. Here, the important point is that, given the annual values, the monthly values seem to behave reasonably similarly to the actual values shown in Figure 24.

7. Conclusions

7.1. Summary

7.1.1. At the end of the third subpart of this long paper we have now completed our tour of the variables in the latest version of the “Wilkie model”. We have explored the monthly data in each case, treating each as possible BBs, and, where appropriate, treating them also as possible OU bridges. Each data series has its own particular features which needed separate consideration.

7.1.2. As a result of our analysis, we have found that it is desirable to bridge integrated (summation) series (*QL*, *WL* and *DL*) at the highest level, even if the annual model considers the annual steps (*I*, *J* and *K*). In every case, we have found that the evidence for an OU treatment is no better than for a BB treatment, so we have chosen the slightly simpler model in each case. We have bridged on the logarithms of all except the *R* series, in order to avoid intermediate values becoming negative.

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Where the annual model is built up by construction from another series (as *WL* can be split into *WL*1 and *WL*2, and *YL*, *DL*, *CL* and *BL* are similarly decomposable) we find that bridging on the resulting top level series (*WL*, *YL*, etc.), is more satisfactory. The annual model takes care of the more complicated interactions.

7.1.3. In most cases we have found that the estimated value of the monthly standard deviation, σ_m, within each year, depends on the size of the jump within that year, a large jump in either direction implying a larger value of σ_m. We have used three out of eight possible formula relating either $\sigma_m(t)$, or $\sigma_m(t)^2$ to either the square or the absolute value of the extent to which the annual jump differs from an average figure or from zero. Any of the formulae would give reasonably similar results, but we saw no reason to standardise on any one of them. Only for *R* did the monthly standard deviation seem to be reasonably constant from year to year.

7.1.4. Two series, retail prices and wages, show correlations between the standardised residuals in the same month from one year to the next. This reflects some seasonal variation, which, however, may change from time to time. These variables are not traded on exchanges. All the other series except share dividends are traded, and none shows this sort of year-to-year influence at this level (though there is plenty in the annual series). But several show simultaneous cross-correlations, of which only one is important for any one series.

7.1.5. All this gives us BBs with a lot of distinctive features, but simulations of each, based on the analysis of the actual data, seem to resemble the source series fairly reasonably.

7.2. The real world

7.2.1. We have assumed a simulation model in which each year's end-point is simulated first, and then the monthly values are simulated conditional on the end-point. Whether all years are simulated first before any monthly values are filled, or each year is simulated fully before moving on to the next is not important. However, the real world is not like this. At the beginning of a year the end-point is not known; all that happens is that each month's data emerges as it occurs, in a natural filtration. This is modified slightly, in that the values of exchange traded variables are known at the close of business on the last working day of the month, whereas the value of the retail price index for the month, estimated about the second week of the month, is not published till about the middle of the following month. The delays are larger for the wages index, for which the final values are not known for about three months.

7.2.2. The statistical model for any variable in any period is that, for the value at the end of the period, there is some expected value and some variance and some distribution, which may well not be normal. We postpone discussion of this last point till a later paper. But the mean and variance may well both vary with time, and may depend on the filtration at time t, $\mathcal{F}_t$, which includes all the values of the relevant variables at or before time t (subject to caveats about the publication date of certain series). For a pure random walk, the mean and variance are constant. If the series is modelled as an AR(1) model, the mean varies with $\mathcal{F}_t$; if it is modelled with an ARCH (autoregressive conditional heteroscedastic) model, the variance varies with $\mathcal{F}_t$ (or may be modelled by another random series with stochastic volatility; we find this usually an unnecessary complication). Our monthly models show means and variances that vary with the annual jump in the underlying annual series (which is not known in advance) but we can imagine some monthly model that takes into account at least the expected value of the annual series. This is worth investigating.

7.2.3. Such a divergence between short-term and long-term modelling occurs in other fields. In weather forecasting the daily forecasts take into account recent observations over a wide area and use ever more complex models to produce reasonable forecasts for a few days ahead. But the overall seasonal variation of the climate from winter to summer and back again through the year has a different sort of model, within which the daily weather variation must fit. We do not suggest that there is any such annual seasonality in our economic models, except where we have noted it, and we do not wish to draw analogies too far. There are many other examples of such divergences between the small scale and the large scale.

7.2.4. A further aspect of our modelling is that we have pinned it all to June values. If we were to fit our annual model using values at the end of January, February, etc., through to December, we would have twelve different annual models; if we then investigated the monthly values as bridges over each year from January to January, etc., we would have twelve different bridging models. We have not yet done this, and it remains to be reported on. We would not expect the models to be substantially different, but almost certainly the parameter values would be numerically different.

7.3. Implications for financial economics

7.3.1. Our results have considerable importance for financial economics generally. There have been two, rather conflicting, schools of thought. The older one, advanced by many among whom we single out Eugena Fama, a Nobel prize winner in 2013, proposes an efficient market hypothesis, with random walks or geometric Brownian motion the underlying models. The other, among whom we single out Robert Shiller, also a Nobel prize winner in 2013, proposes much more mean reversion, suggesting that at times the market can be out of balance, and get either too low or too high.

7.3.2. The "Wilkie model" falls clearly into the latter school, and we stick with this for the long-term model. But our investigations into series at monthly intervals show that, over a single year, or perhaps longer, the movements of all the relevant series are indistinguishable from random walks. We see that the means and the variances may change from year to year, but over a single year they can be treated as constant. It seems likely that over shorter periods, weekly or daily, the same sort of results would be found. All our monthly series show fat-tailed residuals, so are not consistent with pure Brownian motion, with normally distributed innovations, and it is also quite possible that further investigations might display stochastic, or at least varying, volatility.

7.3.3. We conclude finally that the long term, over multiple years, can best be modelled by different sorts of mean-reverting models, in agreement with Schiller, but that the shorter term, a year or less, can best be modelled by random walks, in agreement with Fama. This seems to reconcile the conflict, and we believe that is an important result for financial economics. But this is not the place for a full discussion of this matter.

7.4. A caution

7.4.1. If anyone bases any stochastic model on past data, and uses it to produce simulated futures, there is always a risk that the future model may prove different from the past. There may be changes in economic regimes, as has happened with exchange rates during the 20th century, with three or four different successive regimes, or with more revolutionary events, as in the experience of investments in Russia after 1917. It is uncertain at present whether we are moving into a period of comparative stability for developed countries, with low growth, low inflation and low interest rates; the Japanese

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economy may have reached this position some years ago. In the longer term climate change may have unforeseen effects.

7.4.2. All these points apply to the annual model, and to the values of the parameters in that, especially the means. However, we would expect that, provided the economy continues in not too different a way from the past, our monthly interpolation models would be reasonable. But to return to the exchange rate example: in the Bretton Woods period, exchange rates were constrained to move in very narrow bands, of about 2% each side of a fixed rate, and large changes only occurred with devaluations or revaluations, large step changes at infrequent intervals. We have not covered exchange rates in this paper, but we would not be surprised that their monthly fluctuations in the present floating rates regime behaved in a similar sort of way to the monthly data we have looked at here, within a longer term model that takes account of a drift towards purchasing power parity.

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Appendix: Summary of Results

A1.1. In this Appendix, we repeat the final results for each of the monthly models for our different series, including also those discussed in Part 3B.

A1.2. Our preferred method is to use BBs on the following variables: QL, WL, YL, DL, CL, BL and R. In each case, the mean change per month is one-twelfth of the annual change; the annual changes from t to $t+1$ are denoted, respectively, by $QLD(t) = I(t+1)$, $WLD(t) = J(t+1)$, $YLD(t)$, $DLD(t) = K(t+1)$, $CLD(t)$, $BLD(t)$ and $RD(t)$.

A1.3. The standard deviation for each bridge for each year, $\sigma_m(t)$, may be constant or may vary with the values of the annual change. We use a more specific notation for each $\sigma_m(t)$ (e.g. $QSM(t)$), and the values of A, B and C (e.g. QSA, QSB and QSC). The formula numbers given are those in section 8.5 of Part 3A.

For the logarithm of the retail prices index, QL, we have

$$\text{(Formula 5) } QSM(t) = QSA + QSB.\text{Abs}(QLD(t) - QSC)$$

with $QSA = 0.004204$, $QSB = 0.055403$, $QSC = 0.041041$, also with QR (the correlation between $Z(t, j)$ and $Z(t-1, j) = 0.5469$.

For the logarithm of the wages index, WL, we have

$$(\text{Formula 1})\ WSM(t) = WSA + WSB.\text{Abs}(WLD(t))$$

with WSA = 0.002586, WSB = 0.010719, also with WR (the correlation between $Z(t, j)$ and $Z(t-1, j)$ = 0.2047.

For the logarithm of the share dividend yield, YL, we have

$$(\text{Formula 3})\ YSM(t) = YSA + YSB.YLD(t)^2$$

with YSA = 0.040788, YSB = 0.097117, with no extra correlations.

For the logarithm of the share dividend index, DL, we have

$$(\text{Formula 5})\ DSM(t) = DSA + DSB.\text{Abs}(DLD(t) - DSC)$$

with DSA = 0.005893, DSB = 0.075657, DSC = 0.052764, also with DYR (the correlation between Zs for DL and YL) = 0.2580.

For the logarithm of the long-term interest rate, CL, we have

$$(\text{Formula 3})\ CSM(t) = CSA + CSB.CLD(t)^2$$

with CSA = 0.028392, CSB = 0.165597, also with CYR (the correlation between Zs for CL and YL) = 0.2558.

For the logarithm of the short-term interest rate, BL, we have

$$(\text{Formula 1})\ BSM(t) = BSA + BSB.\text{Abs}(BLD(t))$$

with BSA = 0.029183, BSB = 0.112781, with no extra correlations.

For the real yield on index-linked stocks, R, we have

$$(\text{Constant})\ RSM(t) = RSA = 0.001569$$

also with RCR (the correlation between Zs for R and CL) = 0.6367.