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DIAGONALS OF SEPARATELY ABSOLUTELY CONTINUOUS MAPPINGS COINCIDE WITH THE SUMS OF ABSOLUTELY CONVERGENT SERIES OF CONTINUOUS FUNCTIONS

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Abstract We prove that, for an interval $X \subseteq \mathbb{R}$ and a normed space Z, diagonals of separately absolutely continuous mappings $f\colon X^2 \to Z$ are exactly mappings $g\colon X \to Z$, which are the sums of absolutely convergent series of continuous functions.

Keywords: absolutely continuous function; semi-continuity; Baire-one mapping

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1. Introduction

Let $f\colon X^2 \to Z$ be a mapping. We call a mapping $g\colon X \to Z$, $g(x) = f(x,x)$, *the diagonal of* f.

Investigations focusing on diagonals of separately continuous functions $f\colon X^2 \to \mathbb{R}$ started in the classical work of Baire [**1**]. He showed that diagonals of separately continuous functions of two real variables are exactly Baire-one functions, i.e. pointwise limits of continuous functions. His result was generalized by Lebesgue and Hahn for real-valued functions of several real variables (see [**5**, **8**, **9**]).

Since the second half the 20th century, Baire classification of separately continuous mappings and their analogues has been intensively studied by many mathematicians (see [**2**, **3**, **12**, **13**, **16**, **18**]). The inverse problem on the construction of separately continuous functions with a given diagonal was solved in [**11**]. In [**15**] it was shown that for any topological space X and a function $g\colon X \to \mathbb{R}$ of the $(n-1)$th Baire class there exists a separately continuous function $f\colon X^n \to \mathbb{R}$ with the diagonal g.

In [**10**] the diagonal variant of the problem of Eidelheit from the famous 'Scottish book' on a composition of absolutely continuous functions was investigated. It was proved that there exists a separately absolutely continuous function $f\colon [0,1]^2 \to \mathbb{R}$ such that its partial derivatives f'_x and f'_y in the degree p are integrable on $[0,1]^2$ for every $p > 1$, and such that its diagonal g is not absolutely continuous.

The following problem naturally arises.

Problem 1.1. *Find necessary and sufficient conditions on a function* $g\colon [0,1] \to \mathbb{R}$ *under which there is a separately absolutely continuous function* $f\colon [0,1]^2 \to \mathbb{R}$ *with the diagonal* g.

In this paper we prove that for any interval $X \subseteq \mathbb{R}$ and a normed space Z, diagonals of separately absolute continuous mappings $f\colon X^2 \to Z$ are exactly mappings $g\colon X \to Z$, which are the sums of absolutely convergent series of continuous functions.

2. Preliminaries

For topological spaces X, Y and Z, a mapping $f\colon X \times Y \to Z$ that is continuous with respect to every variable is called *separately continuous*.

Let X and Y be topological spaces. A mapping $f\colon X \to Y$ is *a mapping of the first Baire class*, or *a Baire-one mapping*, if there exists a sequence $(f_n)_{n=1}^{\infty}$ of continuous mappings $f_n\colon X \to Y$ that pointwise converges to f on X.

For a metric space X, we denote the metric on this space by $|\cdot - \cdot|_X$.

Let $X \subseteq \mathbb{R}$ be an interval and let Z be a metric space. A mapping $f\colon X \to Z$ is called *absolutely continuous* if for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for every collection $a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_n < b_n$ of elements $a_1, b_1, \ldots, a_n, b_n \in X$ with $\sum_{k=1}^{n}(b_k - a_k) < \delta$, the inequality $\sum_{k=1}^{n}|f(b_k) - f(a_k)|_Z < \varepsilon$ holds. Let, moreover, Y be an interval. A mapping $f\colon X \times Y \to Z$ that is absolutely continuous with respect to each variable is called *separately absolutely continuous*.

A mapping $f\colon X \to Z$ has *bounded variation on an interval* X if there exists $C > 0$ such that for any collection $a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_n < b_n$ of elements $a_1, b_1, \ldots, a_n, b_n \in X$, the inequality $\sum_{k=1}^{n}|f(b_k) - f(a_k)|_Z \leqslant C$ holds. Moreover, for an interval $X = [a,b]$, the least upper bound of all values $\sum_{k=1}^{n}|f(b_k) - f(a_k)|_Z$ is called *the variation of* f *on* $[a,b]$.

Let $X \subseteq \mathbb{R}$ be an interval, let Z be a metric space, let $f\colon X \to Z$ be a mapping and let $x_0 \in X$. We say that f *has finite variation at* x_0 if there exists a segment $[a,b] \subseteq X$ such that $[a,b]$ is a neighbourhood of x_0 in X and f has finite variation on $[a,b]$.

Let X and Z be metric spaces and let $A \subseteq X$. A mapping $f\colon X \to Z$ is *Lipschitz on a set* A *with a constant* $C \geqslant 0$ if $|f(x) - f(y)|_Z \leqslant C|x - y|_X$ for any $x, y \in A$. A mapping $f\colon X \to Z$ is called *Lipschitz on a set* A if there exists $C \geqslant 0$ such that f is Lipschitz on A with the constant C. A mapping $f\colon X \to Z$ is called *Lipschitz (with a constant* C*)* if it is Lipschitz (with a constant C) on X.

A function $p\colon X \to \mathbb{R}$ defined on a vector space X over a field $\mathbb{K}$ is said to be *a pseudo-norm* if for any $x, y \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leqslant 1$ the following conditions hold: $p(x) \geqslant 0$, moreover, $p(x) = 0$ if and only if $x = 0$; $p(\lambda x) \leqslant p(x)$ and $p(x+y) \leqslant p(x) + p(y)$. It is well known that for any metrizable topological vector space X there exists a metric ρ on X that generates a topological structure on X and $\rho(x,y) = p(x-y)$ for some pseudo-norm p on X.

Let X be a topological space and let Z be a metric space. A mapping $f\colon X \to Z$ is said to be *an absolute Baire-one mapping* or *a mapping of the first absolute Baire class* if there exists a sequence $(f_n)_{n=1}^{\infty}$ of continuous functions $f_n\colon X \to Z$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{and} \quad \sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)|_Z < \infty$$

for every $x \in X$. Note that for a normed space Z a mapping $f\colon X \to Z$ is an absolute Baire-one mapping if and only if f is the sum of an absolutely convergent series of continuous functions. It is well known (see [**17**] and [**6**, Chapter 41 § 5, Chapter 42 § 2]) that a real-valued function is the sum of an absolutely convergent series of continuous functions if and only if it is the sum of a lower semi-continuous and an upper semi-continuous function. Properties of real-valued absolute Baire-one functions have been studied by many mathematicians (see, for example, [**7**, **14**]).

3. Necessary conditions on diagonals of separately absolutely continuous mappings

Proposition 3.1. *Let $X = \mathbb{R}$, let Z be a metric topological vector space with the metric generated by a pseudo-norm p, let $f\colon X^2 \to Z$ be a continuous mapping with respect to the first variable and let $\alpha\colon X \to (0, +\infty)$ be a continuous function. Then there exist functions $\beta, \gamma\colon X \to (\frac{1}{2}\alpha(x), \alpha(x))$ and a continuous mapping $g\colon X \to Z$ such that*

$$p(g(x) - f(x, x + \beta(x))) \leqslant p(f(x, x + \beta(x)) - f(x, x + \gamma(x)))$$

for every $x \in X$.

Proof. For every $x \in X$ we write $y_x = x + \frac{3}{4}\alpha(x)$ and, taking into account that α is continuous at x, we choose an open neighbourhood U_x of x in X such that $t + \frac{1}{2}\alpha(t) < y_x < t + \alpha(t)$ for every $t \in U_x$.

Let $(V_i\colon i \in I)$ be an open locally finite refinement of $(U_x\colon x \in X)$ such that for every $x \in X$ the set $I_x = \{i \in I\colon x \in V_i\}$ contains at most two elements. Let $(\varphi_i\colon i \in I)$ be a partition of unity on X such that $V_i = \varphi_i^{(-1)}((0, 1])$ for every $i \in I$. For every $i \in I$ we choose $x_i \in X$ such that $V_i \subseteq U_{x_i}$ and let $y_i = y_{x_i}$. For every $x \in X$ let

$$g(x) = \sum_{i\in I} \varphi_i(x) f(x, y_i).$$

Clearly, g is continuous. Moreover, according to the choice of U_x the following condition holds:

$$t + \tfrac{1}{2}\alpha(t) < y_i < t + \alpha(t) \quad \text{for every } t \in V_i. \tag{a}$$

For every $x \in X$ we pick $i_x \in I_x$. Moreover, we take $j_x \in I_x \setminus \{i_x\}$ if $|I_x| = 2$, and $j_x = i_x$ if $|I_x| = 1$. Let $\beta(x) = y_{i_x} - x$ and let $\gamma(x) = y_{j_x} - x$. Since $x \in V_{i_x}$ and $x \in V_{j_x}$, by (a) we have $\frac{1}{2}\alpha(x) < \beta(x) < \alpha(x)$ and $\frac{1}{2}\alpha(x) < \gamma(x) < \alpha(x)$.

Let $|I_x| = 1$, i.e. $I_x = \{i\}$. Then, $i_x = i$ and $g(x) = f(x, y_i) = f(x, x + \beta(x))$. Now assume that $|I_x| = 2$. Then $I_x = \{i_x, j_x\}$ and

$$\begin{aligned} p(g(x) - f(x, x + \beta(x))) &= p\biggl(\sum_{i \in I_x} \varphi_i(x) f(x, y_i) - f(x, y_{i_x})\biggr) \\ &= p(\varphi_{j_x}(x) f(x, y_{j_x}) - \varphi_{j_x}(x) f(x, y_{i_x})) \\ &\leqslant p(f(x, y_{j_x}) - f(x, y_{i_x})) \\ &= p(f(x, x + \beta(x)) - f(x, x + \gamma(x))). \end{aligned}$$

□

Theorem 3.2. *Let $X = \mathbb{R}$, let Z be a metric linear space with the metric generated by a pseudo-norm p, and let $f\colon X^2 \to Z$ be a mapping that is continuous with respect to the first variable and has a finite variation and is continuous with respect to the second variable at every point of the diagonal $\Delta = \{(x, x)\colon x \in X\}$. Then $g(x) = f(x, x)$ is an absolute Baire-one mapping.*

Proof. For every $n \in \mathbb{N}$ we apply Proposition 3.1 to f and to $\alpha_n\colon X \to (0, +\infty)$, where $\alpha_n(x) = 1/2^n$. We then obtain sequences $(g_n)_{n=1}^{\infty}$, $(\beta_n)_{n=1}^{\infty}$ and $(\gamma_n)_{n=1}^{\infty}$ of continuous mappings $g_n\colon X \to Z$ and functions $\beta_n\colon X \to (1/2^{n+1}, 1/2^n)$ and $\gamma_n\colon X \to (1/2^{n+1}, 1/2^n)$ such that

$$p(g_n(x) - f(x, x + \beta_n(x))) \leqslant p(f(x, x + \beta_n(x)) - f(x, x + \gamma_n(x)))$$

for any $n \in \mathbb{N}$ and $x \in X$.

We show that the sequence $(g_n)_{n=1}^{\infty}$ is as required. Fix $x \in X$ and let $z_n = g_n(x)$, $u_n = f(x, x + \beta_n(x))$ and $v_n = f(x, x + \gamma_n(x))$ for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \beta_n(x) = \lim_{n\to\infty} \gamma_n(x) = 0$ and f is continuous with respect to the second variable at (x, x),

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = f(x, x) \quad \text{and} \quad \lim_{n\to\infty} p(u_n - v_n) = 0.$$

Now, taking into account that $p(z_n - u_n) \leqslant p(u_n - v_n)$ for every $n \in \mathbb{N}$, we obtain that $\lim_{n\to\infty} p(z_n - u_n) = 0$ and

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} z_n = \lim_{n\to\infty} (z_n - u_n) + \lim_{n\to\infty} u_n = f(x, x) = g(x).$$

We remark that for every $n \in \mathbb{N}$ the points $x + \beta_n(x)$ and $x + \gamma_n(x)$ belong to the interval $I_n = (x + 1/2^{n+1}, x + 1/2_n)$ and $I_n \cap I_m = \emptyset$ for all distinct n and m. Moreover, $\lim_{n\to\infty}(x + \beta_n(x)) = \lim_{n\to\infty}(x + \gamma_n(x)) = x$ and the mapping $f^x\colon X \to Z$, $f^x(t) = f(x, t)$, has a finite variation at x. Hence,

$$\sum_{n=1}^{\infty} p(f^x(x + \beta_n(x)) - f^x(x + \gamma_n(x))) = \sum_{n=1}^{\infty} p(u_n - v_n) = C_1 < \infty$$

and

$$\sum_{n=1}^{\infty} p(f^x(x+\beta_{n+1}(x)) - f^x(x+\beta_n(x))) = \sum_{n=1}^{\infty} p(u_{n+1}-u_n) = C_2 < \infty.$$

Taking into account that $p(z_n - u_n) \leqslant p(u_n - v_n)$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} p(g_{n+1}(x) - g_n(x)) &= \sum_{n=1}^{\infty} p(z_{n+1} - z_n) \\
&\leqslant \sum_{n=1}^{\infty} (p(z_{n+1} - u_{n+1}) + p(u_{n+1} - u_n) + p(z_n - u_n)) \\
&\leqslant \sum_{n=2}^{\infty} p(u_n - v_n) + \sum_{n=1}^{\infty} p(u_{n+1} - u_n) + \sum_{n=1}^{\infty} p(u_n - v_n) \\
&\leqslant 2C_1 + C_2 \\
&< \infty.
\end{aligned}$$

Thus, $\sum_{n=1}^{\infty}|g_{n+1}(x) - g_n(x)|_Z < \infty$ and g is an absolute Baire-one mapping. □

Corollary 3.3. *Let $X \subseteq \mathbb{R}$ be an interval, let Z be a metric linear space with the metric generated by a pseudo-norm, and let $f\colon X^2 \to Z$ be a separately continuous mapping, which has finite variation with respect to the second variable at every point of the diagonal $\Delta = \{(x,x)\colon x \in X\}$. Then the mapping $g\colon X \to Z$, $g(x) = f(x,x)$, is an absolute Baire-one mapping.*

Proof. (a) Let $X = (a,b)$, where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$. Consider a homeomorphism $\varphi\colon (a,b) \to \mathbb{R}$ such that φ and φ^{-1} are locally Lipschitz. Then the mapping $\tilde{f}\colon \mathbb{R} \to Z$, $\tilde{f}(t) = f(\varphi^{-1}(t))$, satisfies the conditions of Theorem 3.2. Therefore, there exists a sequence $(\tilde{g}_n)_{n=1}^{\infty}$ of continuous functions $\tilde{g}_n\colon \mathbb{R} \to Z$ such that $\lim_{n\to\infty} \tilde{g}_n(t) = \tilde{f}(t,t)$ and $\sum_{n=1}^{\infty} |\tilde{g}_{n+1}(t) - \tilde{g}_n(t)|_Z < \infty$ for every $t \in \mathbb{R}$. Then the sequence $(g_n)_{n=1}^{\infty}$ of continuous functions $g_n\colon X \to Z$, $g_n(x) = \tilde{g}_n(\varphi(x))$, pointwise converges to g on X and $\sum_{n=1}^{\infty}|g_{n+1}(x) - g_n(x)|_Z < \infty$ for every $x \in X$.

(b) Now let $X = [a,b)$. According to (a) there is a sequence $(h_n)_{n=1}^{\infty}$ of continuous functions $h_n\colon (a,b) \to Z$ such that $\lim_{n\to\infty} h_n(x) = g(x)$ and $\sum_{n=1}^{\infty}|h_{n+1}(x) - h_n(x)| < \infty$ for every $x \in (a,b)$. We choose a convergent to a monotone sequence $(a_n)_{n=1}^{\infty}$ of real numbers $a_n \in (a,b)$ and for every $n \in \mathbb{N}$ let $g_n\colon X \to Z$ be a continuous mapping such that $g_n(a) = g(a)$ and $g_n(x) = h_n(x)$ for every $x \in [a_n,b)$. Then $(g_n)_{n=1}^{\infty}$ pointwise converges to g on X and $\sum_{n=1}^{\infty}|g_{n+1}(x) - g_n(x)| < \infty$ for every $x \in X$.

(c) For the case in which $X = (a,b]$ or $X = [a,b]$ we argue analogously to the proof of (b). □

4. The characterization of diagonals of separately absolutely continuous functions

Proposition 4.1. *Let X be a metric space, let Z be a normed space, let $f\colon X \to Z$ be a continuous mapping and let $\varepsilon > 0$. There then exists a locally Lipschitz mapping $g\colon X \to Z$ such that $\|f(x) - g(x)\| \leqslant \varepsilon$ for every $x \in X$.*

Proof. For every $x \in X$ we choose an open neighbourhood U_x of x in X such that $\operatorname{diam} f(U_x) < \varepsilon$. Using [**4**, Theorem 4.4.1], we choose an open locally finite refinement $\mathcal{U}$ of $(U_x\colon x \in X)$. Thus, $\mathcal{U}$ is a locally finite cover of X by open non-empty sets U with $\operatorname{diam} f(U) < \varepsilon$. For every $U \in \mathcal{U}$ and $x \in X$, by $\psi_U(x)$ we denote the distance from x to $X \setminus U$. Let $\varphi_U = \psi_U / \sum_{V\in\mathcal{U}} \psi_V$ for every $U \in \mathcal{U}$. Note that $(\varphi_U\colon U \in \mathcal{U})$ is a partition of unity on X, all the functions $\varphi_U\colon X \to [0,1]$ are locally Lipschitz and $\operatorname{supp} \varphi_U = U$. Moreover, for every $U \in \mathcal{U}$ we take $z_U \in f(U)$. Consider the function $g\colon X \to Z$, $g(x) = \sum_{U\in\mathcal{U}} \varphi_U(x) z_U$. It is easy to see that $\|f(x) - g(x)\| \leqslant \varepsilon$ for every $x \in X$. Since $\mathcal{U}$ is locally finite and the φ_U are locally Lipschitz, g is locally Lipschitz. □

The following example shows that the analogue of Proposition 4.1 for locally convex F-spaces Z is not valid.

Example 4.2. Let $0 < p < 1$, let $X = \mathbb{R}$ be a space with the Euclid metric, let $Z = \mathbb{R}$ be a space with the metric $|x-y|_Z = |x-y|^p$, let $f(x) = x$ for every $x \in X$ and let $\varepsilon > 0$. Since every locally Lipschitz mapping $g\colon X \to Z$ is a constant, $\sup\{|f(x) - g(x)|_p\colon x \in X\} > \varepsilon$ for any locally Lipschitz mapping $g\colon X \to Z$.

Proposition 4.3. *Let $g\colon [a,b] \to \mathbb{R}$ and $h\colon [a,b] \to \mathbb{R}$ be Lipschitz functions with a constant K, let $\varphi\colon [a,b] \to [0,1]$ be a Lipschitz function with a constant L and let $|g(x) - h(x)| \leqslant M$ for every $x \in [a,b]$. Then the function*

$$f(x) = \varphi(x)g(x) + (1 - \varphi(x))h(x)$$

is Lipschitz with the constant $C = K + LM$ on $[a,b]$.

Proof. Let $x_1, x_2 \in [a,b]$. Then we have

$$\begin{aligned} |f(x_2) - f(x_1)| &= |(\varphi(x_2) - \varphi(x_1))(g(x_2) - h(x_2)) + \varphi(x_1)(g(x_2) - g(x_1)) \\ &\qquad + (1 - \varphi(x_1))(h(x_2) - h(x_1))| \\ &\leqslant LM|x_2 - x_1| + \varphi(x_1)K|x_2 - x_1| + (1 - \varphi(x_1))K|x_2 - x_1| \\ &= (K + LM)|x_2 - x_1|. \end{aligned}$$

□

Proposition 4.4. *Let Z be a metric space, let $f\colon [a,b] \to \mathbb{Z}$ be a continuous mapping and let $(a_n)_{n=1}^{\infty}$ be a strictly increasing sequence such that $a_1 = a$, $\lim_{n\to\infty} a_n = b$. Let, for every $n \in \mathbb{N}$, the mapping f be Lipschitz with a constant C_n on $[a_n, a_{n+1}]$ and, moreover, let the series $\sum_{n=1}^{\infty} C_n(a_{n+1} - a_n)$ be convergent. Then f is absolutely continuous on $[a,b]$.*

Proof. Fix $\varepsilon > 0$ and choose $m \in \mathbb{N}$ such that $\sum_{n>m} C_n(a_{n+1} - a_n) < \varepsilon/2$. Set $\delta = \varepsilon/2 \max\{C_n : 1 \leqslant n \leqslant m\}$. Let

$$a \leqslant u_1 < v_1 \leqslant u_2 < v_2 \leqslant \cdots \leqslant u_i < v_i \leqslant a_m \leqslant u_{i+1} < v_{i+1} \leqslant \cdots \leqslant u_{i+j} < v_{i+j} \leqslant b$$

with $\sum_{k=1}^{i+j}(v_k - u_k) < \delta$. Then

$$\begin{aligned}\sum_{k=1}^{i+j}|f(v_k) - f(u_k)|_Z &= \sum_{k=1}^{i}|f(v_k) - f(u_k)|_Z + \sum_{k=i+1}^{i+j}|f(v_k) - f(u_k)|_Z \\ &\leqslant \max\{C_n : 1 \leqslant n \leqslant m\} \cdot \sum_{k=1}^{i}(v_k - u_k) + \sum_{n>m} C_n(a_{n+1} - a_n) \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \\ &= \varepsilon.\end{aligned}$$

Now let

$$a \leqslant u_1 < v_1 \leqslant \cdots \leqslant u_i < v_i \leqslant b$$

with $\sum_{k=1}^{i}(v_k - u_k) < \delta$ and $k \leqslant i$ such that $a_m \in (u_k, v_k)$. Using the above-obtained estimation, we have

$$\begin{aligned}\sum_{j=1}^{i}|f(v_j) - f(u_j)|_Z = \sum_{j=1}^{k-1}|f(v_j) - f(u_j)|_Z + |f(a_m) - f(u_k)| + |f(v_k) - f(a_m)| \\ + \sum_{j=k+1}^{i}|f(v_j) - f(u_j)|_Z \\ \leqslant \varepsilon.\end{aligned}$$

□

Theorem 4.5. *Let $X \subseteq \mathbb{R}$ be an interval, let Z be a normed space and let $g : X \to Z$ be an absolute Baire-one mapping. There then exists a separately absolutely continuous mapping $f : X^2 \to Z$ such that $g(x) = f(x, x)$ for every $x \in X$.*

Proof. Let $(I_n)_{n=1}^{\infty}$ be an increasing sequence of segments $I_n = [a_n, b_n]$ such that $X = \bigcup_{n=1}^{\infty} I_n$. Let $g_0(x) = 0$ for every $x \in X$. Using the definition of absolute Baire-one class and Proposition 4.1, we choose a sequence $(\tilde{g}_n)_{n=1}^{\infty}$ of locally Lipschitz mappings $\tilde{g}_n : X \to Z$ such that $\lim_{n\to\infty} \tilde{g}_n(x) = g(x)$ and $\sum_{n=1}^{\infty} \|\tilde{g}_{n+1}(x) - \tilde{g}_n(x)\| < \infty$ for every $x \in X$. Now, for arbitrary $n \in \mathbb{N}$ and $x \in X$, let

$$g_n(x) = \begin{cases}\tilde{g}_n(x), & x \in I_n, \\ \tilde{g}_n(a_n), & x \in X \cap (-\infty, a_n), \\ \tilde{g}_n(b_n), & x \in X \cap (b_n, +\infty).\end{cases}$$

We then obtain the sequence $(g_n)_{n=1}^{\infty}$ of Lipschitz mappings $g_n : X \to Z$ such that $\lim_{n\to\infty} g_n(x) = g(x)$ and $\sum_{n=1}^{\infty} \|g_{n+1}(x) - g_n(x)\| < \infty$ for every $x \in X$.

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Let $(K_n)_{n=1}^{\infty}$ be a sequence of real numbers $K_n > 0$ such that the functions g_{n-1} and g_n are Lipschitz with the constant K_n. We choose a strictly decreasing sequence $(\delta_n)_{n=1}^{\infty}$ of real numbers $\delta_n > 0$ such that $\lim_{n\to\infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} K_n \delta_n < \infty$.

For every $n \in \mathbb{N}$ let $F_n = \{(x, y) \in X^2 : |x-y| \leqslant \delta_n\}$ and $G_n = \{(x, y) \in X^2 : |x-y| < \delta_n\}$. We consider the function $\varphi_n : \mathbb{R} \to [0, 1]$ defined by

$$\varphi_n(t) = \begin{cases} 1, & |t| > \delta_n, \\ \dfrac{t - \delta_{n+1}}{\delta_n - \delta_{n+1}}, & \delta_{n+1} \leqslant |t| \leqslant \delta_n, \\ 0, & |t| < \delta_{n+1}. \end{cases}$$

We remark that every function φ_n is Lipschitz with the constant $L_n = 1/(\delta_n - \delta_{n+1})$.

We consider the following function $f : X^2 \to Z$:

$$f(x, y) = \begin{cases} g_0(x), & |x - y| > \delta_1, \\ \varphi_n(x - y)g_{n-1}(x) + (1 - \varphi_n(x - y))g_n(x), & (x, y) \in F_n \setminus F_{n+1}, \\ g(x), & x = y \in X. \end{cases}$$

Let us show that f has the desired properties. Since for every $n \in \mathbb{N}$ with $|x - y| = \delta_{n+1}$ we have $\varphi_n(x - y) = 0$ and $\varphi_{n+1}(x - y) = 1$,

$$f(x, y) = g_n(x) = \varphi_n(x - y)g_{n-1}(x) + (1 - \varphi_n(x - y))g_n(x).$$

Therefore,

$$f(x, y) = \varphi_n(x - y)g_{n-1}(x) + (1 - \varphi_n(x - y))g_n(x)$$

for every $(x, y) \in F_n \setminus G_{n+1}$. The continuity of g_n and φ_n implies that f is jointly continuous on $X^2 \setminus \Delta$, where $\Delta = \{(x, x) : x \in X\}$. Moreover, the equality $\lim_{n\to\infty} g_n(x) = g(x)$ for every $x \in X$ implies that f is continuous with respect to the first and second variables at all points of Δ. Thus, f is separately continuous. Moreover, since all g_n and all φ_n are Lipschitz, f is Lipschitz on $X^2 \setminus G_n$ for every $n \in \mathbb{N}$.

We fix $x_0 \in X$ and show that the mapping $f_{x_0} : X \to Z$, $f_{x_0}(x) = f(x, x_0)$, is absolutely continuous. It is sufficient to prove that the mapping f_{x_0} is absolutely continuous on the intervals $X_1 = X \cap (-\infty, x_0]$ and $X_2 = X \cap [x_0, +\infty)$. Let us consider the case when the interval X_1 is non-degenerated, i.e. $X_1 \cap (-\infty, x_0) \neq \emptyset$. Since $X = \bigcup_{n=1}^{\infty} I_n$ and since $(I_n)_{n=1}^{\infty}$ increases and $\lim_{n\to\infty} \delta_n = 0$, there exists a number $m \in \mathbb{N}$ such that $[x_0 - \delta_m, x_0] \subseteq I_m$. Observe that $f_{x_0}(x) = 0$ for every $x \in X \cap (-\infty, x_0 - \delta_1]$ and f_{x_0} is Lipschitz on $X_1 \cap [x_0 - \delta_1, x_0 - \delta_m]$. Therefore, the absolute continuity of f_{x_0} of X_1 is equivalent to the absolute continuity of f_{x_0} on $[x_0 - \delta_m, x_0]$.

Fix $n \geqslant m$. Notice that g_{n-1} and g_n are Lipschitz with the constant K_n, and φ_n is Lipschitz with the constant L_n. Moreover, for every $x \in [x_0 - \delta_n, x_0 - \delta_{n+1}]$ we have

$$\begin{aligned} \|g_n(x) - g_{n-1}(x)\| &\leqslant \|g_n(x_0) - g_{n-1}(x_0)\| + \|g_n(x) - g_n(x_0)\| + \|g_{n-1}(x) - g_{n-1}(x_0)\| \\ &\leqslant \|g_n(x_0) - g_{n-1}(x_0)\| + 2K_n\delta_n \\ &= M_n. \end{aligned}$$

Hence, f_{x_0} is Lipschitz on $[x_0 - \delta_n, x_0 - \delta_{n+1}]$ with the constant

$$C_n = K_n + L_n M_n = K_n + \frac{1}{\delta_n - \delta_{n+1}}(\|g_n(x_0) - g_{n-1}(x_0)\| + 2K_n\delta_n)$$

by Proposition 4.3. Now we have

$$\sum_{n\geqslant m} C_n(\delta_n - \delta_{n+1}) \leqslant 3\sum_{n\geqslant m} K_n\delta_n + \sum_{n\geqslant m} \|g_n(x_0) - g_{n-1}(x_0)\| < \infty.$$

By Proposition 4.4, f_{x_0} is absolutely continuous in $[x_0 - \delta_m, x_0]$ and, consequently, on X_1.

The absolute continuity of f_{x_0} on X_2 and the absolute continuity of f with respect to the second variable are proved similarly. □

Theorems 3.2 and 4.5 imply the following result.

Theorem 4.6. *Let $X \subseteq \mathbb{R}$ be an interval, let Z be a normed space and let $g\colon X \to Z$. Then the following conditions are equivalent.*

(i) *There exists a separately absolutely continuous mapping $f\colon X^2 \to Z$ with the diagonal g.*

(ii) *There exists a mapping $f\colon X^2 \to Z$ with the diagonal g that is continuous with respect to the first variable and has a finite variation and is continuous with respect to the second variable at every point of the diagonal $\Delta = \{(x,x)\colon x \in X\}$.*

(iii) *g is an absolute Baire-one mapping.*

5. Open problems

Question 5.1. *Let $X = [0,1]$, let Z be a metric space and let $f\colon X^2 \to Z$ be a separately absolutely continuous mapping. Is $g(x) = f(x,x)$ an absolute Baire-one mapping?*

Question 5.2. *Let $X = [0,1]$, let Z be a metric topological vector space and let $g\colon X \to Z$ be an absolute Baire-one mapping. Does there exist a separately absolutely continuous mapping $f\colon X^2 \to Z$ such that $g(x) = f(x,x)$ for every $x \in X$?*

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