Nagoya Math. J., **244** (2021), 136–157
DOI 10.1017/nmj.2020.9

MASS GROWTH OF OBJECTS AND CATEGORICAL ENTROPY

AKISHI IKEDA

Abstract. In the pioneering work by Dimitrov–Haiden–Katzarkov–Kontsevich, they introduced various categorical analogies from the classical theory of dynamical systems. In particular, they defined the entropy of an endofunctor on a triangulated category with a split generator. In the connection between the categorical theory and the classical theory, a stability condition on a triangulated category plays the role of a measured foliation so that one can measure the "volume" of objects, called the mass, via the stability condition. The aim of this paper is to establish fundamental properties of the growth rate of mass of objects under the mapping by the endofunctor and to clarify the relationship between it and the entropy. We also show that they coincide under a certain condition.

§1. Introduction

In the pioneering work [11], Dimitrov–Haiden–Katzarkov–Kontsevich introduced various categorical analogies of classical theory from dynamical systems. In particular, they defined the entropy of an endofunctor on a triangulated category with a split generator. One of their motivations comes from the connection between the theory of stability conditions on triangulated categories and the Teichmüller theory of surfaces [10, 12]. In this connection, a stability condition on a triangulated category corresponds to a measured foliation (a quadratic differential) on a surface, and the mass of stable objects corresponds to the length of geodesics. Thus the mass of objects plays the role of "volume" in some sense. In [11], they also suggested that there is a connection between the growth rate of mass of objects under the mapping by an endofunctor and the entropy of the endofunctor. In this paper, we establish fundamental properties of the mass growth and clarify the relationship between it and the entropy. We also show that they coincide under a certain condition. The result in this paper is motivated by the well-known classical work "Volume growth and entropy" by Yomdin [22] on classical dynamical systems.

1.1 Fundamental properties of mass growth

First, we introduce the mass growth with respect to endofunctors. Let $\mathcal{D}$ be a triangulated category and $K(\mathcal{D})$ be its Grothendieck group. A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ [8] is a pair of a linear map $Z\colon K(\mathcal{D}) \to \mathbb{C}$ and a family of full subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$ satisfying some axioms (see Definition 2.8). A nonzero object in $\mathcal{P}(\phi)$ is called a semistable object of phase ϕ. One of the axioms implies that any nonzero object $E \in \mathcal{D}$ can

Received April 15, 2019. Revised April 10, 2020. Accepted May 2, 2020.
2010 Mathematics subject classification. Primary 18E30; Secondary 37B40.
This work is supported by World Premier International Research Center Initiative (WPI initiative), MEXT, Japan, JSPS KAKENHI Grant Number JP16K17588, 16H06337, and JSPS bilateral Japan-Russia Research Cooperative Program. This paper was written while the author was visiting Perimeter Institute for Theoretical Physics by JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Research, Innovation and Science.

be decomposed into semistable objects with decreasing phases, that is, there is a sequence of exact triangles, called a Harder–Narasimhan filtration,

$$\begin{array}{ccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \to \cdots \to & E_{m-1} & \longrightarrow & E_m = E \\ & \nwarrow \quad \swarrow & & \nwarrow \quad \swarrow & & & & \nwarrow \quad \swarrow & \\ & A_1 & & A_2 & & & & A_m & \end{array}$$

with $A_i \in \mathcal{P}(\phi_i)$ and $\phi_1 > \phi_2 > \cdots > \phi_m$. Through the Harder–Narasimhan filtration, the mass of E with a parameter $t \in \mathbb{R}$ (see Definition 3.1) is defined by

$$m_{\sigma,t}(E) := \sum_{i=1}^{m} |Z(A_i)| e^{\phi_i t}.$$

Thus, a given stability condition defines the "volume" of objects in some sense. Actually in the connection between spaces of stability conditions and Teichmüller spaces, the mass of stable objects gives the length of corresponding geodesics [10, 12, 14, 15]. For an endofunctor $F\colon \mathcal{D} \to \mathcal{D}$, we want to consider the growth rate of mass of objects under the mapping by F. Therefore, we introduce the following quantity. The mass growth with respect to F is the function $h_{\sigma,t}(F)\colon \mathbb{R} \to [-\infty, \infty]$ defined by

$$h_{\sigma,t}(F) := \sup_{E \in \mathcal{D}} \left\{ \limsup_{n \to \infty} \frac{1}{n} \log(m_{\sigma,t}(F^n E)) \right\}.$$

(As conventions, set $m_{\sigma,t}(0) = 0$ and $\log 0 = -\infty$.) Fundamental properties of $h_{\sigma,t}(F)$ are stated as the main result of this paper. We also recall the space of stability conditions to consider the behavior of $h_{\sigma,t}(F)$ under the deformation of σ. In [8], it was shown that the set of stability conditions $\mathrm{Stab}(\mathcal{D})$ has a natural topology and in addition, $\mathrm{Stab}(\mathcal{D})$ becomes a complex manifold.

Next, we recall the entropy of endofunctors from [11]. Let $\mathcal{D}$ be a triangulated category with a split generator and $F\colon \mathcal{D} \to \mathcal{D}$ be an endofunctor. In [11], they introduced the function $h_t(F)\colon \mathbb{R} \to [-\infty, \infty)$, called the entropy of F (see Definition 2.4), and showed various fundamental properties of $h_t(F)$. In addition, they investigated the relationship between the entropy $h_t(F)$ and the mass growth $h_{\sigma,t}(F)$ (see [11, Section 4.5]). Our result is the following.

THEOREM 1.1. (Theorem 3.5 and Proposition 3.10) *Let $\mathcal{D}$ be a triangulated category, $F\colon \mathcal{D} \to \mathcal{D}$ be an endofunctor, and σ be a stability condition on $\mathcal{D}$. Assume that $\mathcal{D}$ has a split generator G. Then the mass growth $h_{\sigma,t}(F)$ satisfies the following.*

(1) *If a stability condition τ lies in the same connected component as σ in the space of stability conditions* $\mathrm{Stab}(\mathcal{D})$, *then*

$$h_{\sigma,t}(F) = h_{\tau,t}(F).$$

(2) *The mass growth of the generator G determines $h_{\sigma,t}(F)$, that is,*

$$h_{\sigma,t}(F) = \limsup_{n \to \infty} \frac{1}{n} \log(m_{\sigma,t}(F^n G)).$$

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

(3) *An inequality*

$$h_{\sigma,t}(F) \leqslant h_t(F) < \infty$$

holds.

In the case $t=0$, this result was stated in [11, Section 4.5] by using the triangle inequality for mass (see Proposition 3.3). However, there are no literature to prove the triangle inequality for mass even if $t=0$. Therefore, we give a detailed proof of it with a parameter t in Section 3.2, which is the most technical part of this paper.

1.2 Lower bound by the spectral radius

We consider the lower bound of the mass growth when $t=0$. Since $F\colon \mathcal{D} \to \mathcal{D}$ preserves exact triangles, F induces a linear transformation $[F]\colon K(\mathcal{D}) \to K(\mathcal{D})$. We extend $[F]$ on $K(\mathcal{D}) \otimes \mathbb{C}$ naturally. The spectral radius of $[F]$ is defined by

$$\rho([F]) := \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } [F] \text{ on } K(\mathcal{D}) \otimes \mathbb{C}\}.$$

THEOREM 1.2. (Proposition 3.11) *In the case $t=0$, we have an inequality*

$$\log \rho([F]) \leqslant h_{\sigma,0}(F) \leqslant h_0(F)$$

for any stability condition $\sigma \in \mathrm{Stab}(\mathcal{D})$.

As known results, if $\mathcal{D}$ is saturated, then it was shown in [11, Theorem 2.9] that for a linear map $\mathrm{HH}_*(F)\colon \mathrm{HH}_*(\mathcal{D}) \to \mathrm{HH}_*(\mathcal{D})$ induced on the Hochschild homology of $\mathcal{D}$, the inequality $\log \rho(\mathrm{HH}_*(F)) \leqslant h_0(F)$ holds under some condition for eigenvalues of $\mathrm{HH}_*(F)$. They also conjectured that the inequality holds without that condition. Our result, Theorem 1.2, holds without any conditions for $[F]$. However, we use the existence of stability conditions on $\mathcal{D}$. For many examples in [11, 17, 18], it was shown that the equality $\log \rho([F]) = h_0(F)$ holds. Kikuta–Takahashi gave a certain conjecture on the equality in [18, Conjecture 5.3].

1.3 Equality between mass growth and entropy

The remaining important question is to ask when the equality $h_{\sigma,t}(F) = h_t(F)$ holds. In the following, we give a sufficient condition for the equality. For a stability condition $\sigma = (Z, \mathcal{P})$, we can associate an abelian category, called the heart of $\mathcal{P}$, as the extension-closed subcategory generated by objects in $\mathcal{P}(\phi)$ for $\phi \in (0,1]$. Denote it by $\mathcal{P}((0,1])$. A stability condition $\sigma = (Z, \mathcal{P})$ is called algebraic if the heart $\mathcal{P}((0,1])$ is a finite length abelian category with finitely many simple objects (see Definitions 2.7 and 2.11).

THEOREM 1.3. (Theorem 3.14) *Let $F\colon \mathcal{D} \to \mathcal{D}$ be an endofunctor. If a connected component $\mathrm{Stab}^\circ(\mathcal{D}) \subset \mathrm{Stab}(D)$ contains an algebraic stability condition, then $\mathcal{D}$ has a split generator G and for any $\sigma \in \mathrm{Stab}^\circ(\mathcal{D})$ we have*

$$h_t(F) = h_{\sigma,t}(F) = \lim_{n\to\infty} \frac{1}{n} \log(m_{\sigma,t}(F^n G)).$$

Note that in the above theorem, the stability condition σ is not necessarily an algebraic stability condition.

We see a typical example, which satisfies the condition in Theorem 1.3 from Section 4.1. Let $A = \bigoplus_k A^k$ be a dg-algebra such that $H^0(A)$ is a finite-dimensional algebra and $H^k(A) = 0$ for $k > 0$. Denote by $\mathcal{D}_{fd}(A)$ the derived category of dg-modules over A with

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

finite-dimensional total cohomology, that is, $\sum_k \dim H^k(M) < \infty$. Then there is a bounded t-structure whose heart is isomorphic to the abelian category of finite-dimensional modules over $H^0(A)$. As a result, we can construct algebraic stability conditions on $\mathcal{D}_{fd}(A)$. Thus, in the context of representation theory, Theorem 1.3 works well. As an application, we compute the entropy of spherical twists in Section 4.2.

On the other hand, for derived categories coming from algebraic geometry, we cannot find algebraic hearts in general. Only in special cases, for example in the case that the derived category has a full strong exceptional collection, the work by Bondal [7] enables us to find algebraic hearts. It is an important problem to answer whether the equality $h_{\sigma,t}(F) = h_t(F)$ holds without the existence of algebraic stability conditions.

1.4 Categorical theory versus classical theory

We compare our result with the well-known classical result "Volume growth and entropy" by Yomdin [22]. Let M be a compact smooth manifold and $f\colon M \to M$ be a smooth map. The map f induces a linear map $f_*\colon H_*(M;\mathbb{R}) \to H_*(M;\mathbb{R})$ on the homology group $H_*(M;\mathbb{R})$. For the map f, we can define the topological entropy $h_{\mathrm{top}}(f)$ [1] and the inequality $\log \rho(f_*) \leqslant h_{\mathrm{top}}(f)$ was conjectured in [21]. In [22], Yomdin introduced the volume growth $v(f)$ by using a Riemannian metric on M and showed that

$$\log \rho(f_*) \leqslant v(f) \leqslant h_{\mathrm{top}}(f).$$

Our result Theorem 1.2 looks like the categorical analogue of this classical result. On the other hand, the difference between categorical theory and classical theory is that the categorical entropy $h_t(F)$ and the mass growth $h_{\sigma,t}(F)$ have the parameter t, which measures the growth rate of degree shifts in a triangulated category. This point is an essentially new feature of categorical theory.

Notations. We work over a field $\mathbb{K}$. All triangulated categories in this paper are $\mathbb{K}$-linear and their Grothendieck groups are free of finite rank, that is, $K(\mathcal{D}) \cong \mathbb{Z}^n$ for some n. An endofunctor $F\colon \mathcal{D} \to \mathcal{D}$ refers to an exact endofunctor, that is, F preserves all exact triangles and commutes with degree shifts. The natural logarithm is extended to $\log\colon [0,\infty) \to [-\infty,\infty)$ by setting $\log 0 := -\infty$.

§2. Preliminaries

In this section, we prepare basic terminologies mainly from [8, 11].

2.1 Complexity and entropy

First, we recall the notion of complexity and entropy from [11, Section 2].

Let $\mathcal{D}$ be a triangulated category. A triangulated subcategory is called *thick* if it is closed under taking direct summands. For an object $E \in \mathcal{D}$, we denote by $\langle E\rangle \subset \mathcal{D}$ the smallest thick triangulated subcategory containing E. An object $G \in \mathcal{D}$ is called a *split generator* if $\langle G\rangle = \mathcal{D}$. This implies that for any object $E \in \mathcal{D}$, there is some object $E' \in \mathcal{D}$ such that we have a sequence of exact triangles

$$\begin{array}{ccccccccccc}
0 = A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \to \cdots \to & A_{k-1} & \longrightarrow & A_k & = E \oplus E' \\
 & \nwarrow \swarrow & & \nwarrow \swarrow & & & & \nwarrow \swarrow & & \\
 & G[n_1] & & G[n_2] & & & & G[n_k] & &
\end{array}$$

with $n_i \in \mathbb{Z}$. We note that the object E' and the above sequence are not unique.

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

Definition 2.1. [11, Definition 2.1] Let E_1 and E_2 be objects in $\mathcal{D}$. The *complexity of E_2 relative to E_1* is the function $\delta_t(E_1, E_2)\colon \mathbb{R} \to [0, \infty]$ defined by

$$\delta_t(E_1, E_2) := \begin{cases} 0 & \text{if } E_2 \cong 0 \\ \inf\left\{ \sum_{i=1}^{k} e^{n_i t} \;\middle|\; \begin{array}{ccccc} 0 \longrightarrow A_1 & \cdots & A_{k-1} \longrightarrow E_2 \oplus E_2' \\ \nwarrow \quad \swarrow & & \nwarrow \quad \swarrow \\ E_1[n_1] & \cdots & E_1[n_k] \end{array} \right\} & \text{if } E_2 \in \langle E_1 \rangle \\ \infty & \text{if } E_2 \notin \langle E_1 \rangle. \end{cases}$$

By definition, we have an inequality $0 < \delta_t(G, E) < \infty$ for a split generator $G \in \mathcal{D}$ and a nonzero object $E \in \mathcal{D}$. We recall fundamental inequalities for complexity.

Proposition 2.2. [11, Proposition 2.3] *For $E_1, E_2, E_3 \in \mathcal{D}$,*

(1) $\delta_t(E_1, E_3) \leqslant \delta_t(E_1, E_2)\delta_t(E_2, E_3)$,
(2) $\delta_t(E_1, E_2 \oplus E_3) \leqslant \delta_t(E_1, E_2) + \delta_t(E_1, E_3)$,
(3) $\delta_t(F(E_1), F(E_2)) \leqslant \delta_t(E_1, E_2)$ *for an endofunctor* $F\colon \mathcal{D} \to \mathcal{D}$.

Similar to [11, Proposition 2.3], it is easy to check the following. This generalizes Proposition 2.2(2) to the nonsplit case.

Lemma 2.3. *For objects $D, E_1, E_2, E_3 \in \mathcal{D}$, if there is an exact triangle $E_1 \to E_2 \to E_3 \to E_1[1]$, then*

$$\delta_t(D, E_2) \leqslant \delta_t(D, E_1) + \delta_t(D, E_3).$$

Now, we introduce the notion of the entropy of endofunctors. The entropy of an endofunctor F measures the growth rate of complexity $\delta_t(G, F^n G)$ as $n \to \infty$.

Definition 2.4. [11, Definition 2.5] Let $\mathcal{D}$ be a triangulated category with a split generator G and let $F\colon \mathcal{D} \to \mathcal{D}$ be an endofunctor. The *entropy* of F is the function $h_t(F)\colon \mathbb{R} \to [-\infty, \infty)$ defined by

$$h_t(F) := \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G, F^n G).$$

By [11, Lemma 2.6], it follows that $h_t(F)$ is well defined and $h_t(F) < \infty$.

2.2 Bounded t-structures and the associated cohomology

Definition 2.5. [6] A *t-structure* on $\mathcal{D}$ is a full subcategory $\mathcal{F} \subset \mathcal{D}$ satisfying the following conditions:

(a) $\mathcal{F}[1] \subset \mathcal{F}$;
(b) define $\mathcal{F}^{\perp} := \{F \in \mathcal{D} \mid \mathrm{Hom}(D, F) = 0 \text{ for all } D \in \mathcal{F}\}$, then for every object $E \in \mathcal{D}$, there is an exact triangle $D \to E \to F \to D[1]$ in $\mathcal{D}$ with $D \in \mathcal{F}$ and $F \in \mathcal{F}^{\perp}$.

In addition, the t-structure $\mathcal{F} \subset \mathcal{D}$ is said to be *bounded* if $\mathcal{F}$ satisfies the condition

$$\mathcal{D} = \bigcup_{i,j\in\mathbb{Z}} \mathcal{F}^{\perp}[i] \cap \mathcal{F}[j].$$

For a t-structure $\mathcal{F} \subset \mathcal{D}$, we define the *heart* $\mathcal{H} \subset \mathcal{D}$ by

$$\mathcal{H} := \mathcal{F}^{\perp}[1] \cap \mathcal{F}.$$

It was proved in [6] that $\mathcal{H}$ becomes an abelian category. Bridgeland gave the characterization of the heart of a bounded t-structure as follows.

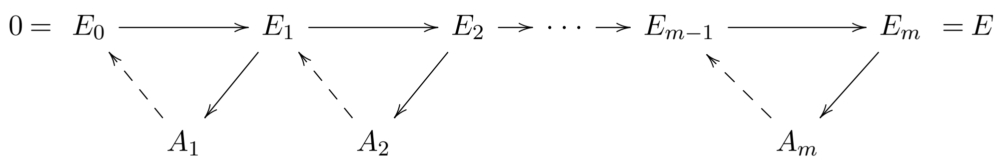
https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

LEMMA 2.6. [8, Lemma 3.2] *Let $\mathcal{H} \subset \mathcal{D}$ be a full additive subcategory. Then $\mathcal{H}$ is the heart of a bounded t-structure if and only if the following conditions hold:*

(a) *if $k_1 > k_2 \in \mathbb{Z}$ and $A_i \in \mathcal{H}[k_i]$ $(i = 1, 2)$, then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$;*
(b) *for $0 \neq E \in \mathcal{D}$, there is a finite sequence of integers*

$$k_1 > k_2 > \cdots > k_m$$

and a sequence of exact triangles



with $A_i \in \mathcal{H}[k_i]$ for all i.

The above filtration in the condition (b) defines the kth cohomology $H^k(E) \in \mathcal{H}$ of the object E by

$$H^k(E) := \begin{cases} A_i[-k_i] & \text{if } k = -k_i \\ 0 & \text{otherwise.} \end{cases}$$

This cohomology becomes a cohomological functor from $\mathcal{D}$ to $\mathcal{H}$, that is, if there is an exact triangle $D \to E \to F \to E[1]$, then we can obtain a long exact sequence

$$\cdots \to H^{k-1}(F) \to H^k(D) \to H^k(E) \to H^k(F) \to H^{k+1}(D) \to \cdots$$

in the abelian category $\mathcal{H}$. In the last of this section, we introduce a special class of bounded t-structures.

DEFINITION 2.7. We say that the heart of a bounded t-structure is *algebraic* if it is a finite length abelian category with finitely many isomorphism classes of.

If $\mathcal{D}$ has an algebraic heart $\mathcal{H}$ with simple objects $S_1, \ldots, S_n$, then it is easy to see that the direct sum $G := \bigoplus_{i=1}^n S_i$ becomes a split generator of $\mathcal{D}$.

2.3 Bridgeland stability conditions

In [8], Bridgeland introduced the notion of a stability condition on a triangulated category as follows.

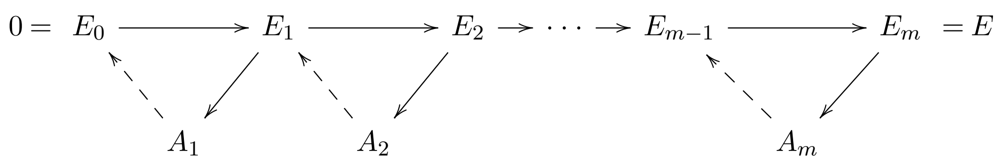
DEFINITION 2.8. Let $\mathcal{D}$ be a triangulated category and $K(\mathcal{D})$ be its Grothendieck group. A *stability condition* $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a group homomorphism $Z \colon K(\mathcal{D}) \to \mathbb{C}$, called a *central charge*, and a family of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$ satisfying the following conditions:

(a) if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$;
(b) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;
(c) if $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$ $(i = 1, 2)$, then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$;
(d) for $0 \neq E \in \mathcal{D}$, there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_m$$

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

and a sequence of exact triangles



with $A_i \in \mathcal{P}(\phi_i)$ for all i.

We write $\phi_\sigma^+(E) := \phi_1$ and $\phi_\sigma^-(E) := \phi_m$. Nonzero objects in $\mathcal{P}(\phi)$ are called *σ-semistable of phase ϕ in σ*. The sequence of exact triangles in (d) is called a *Harder–Narasimhan filtration* of E with semistable factors $A_1, \ldots, A_m$ of phases $\phi_1 > \cdots > \phi_m$.

In addition to the above axioms, we always assume that our stability conditions have the *support property* in [19]. Let $\| \cdot \|$ be some norm on $K(\mathcal{D}) \otimes \mathbb{R}$. A stability condition $\sigma = (Z, \mathcal{P})$ satisfies the support property if there is some constant $C > 0$ such that

$$\frac{|Z(E)|}{\|[E]\|} > C$$

for all σ-semistable objects $E \in \mathcal{D}$.

For an interval $I \subset \mathbb{R}$, we denote by $\mathcal{P}(I)$ the extension-closed subcategory generated by objects in $\mathcal{P}(\phi)$ for $\phi \in I$, namely

$$\mathcal{P}(I) := \{E \in \mathcal{D} \mid \phi_\sigma^\pm(E) \in I\} \cup \{0\}.$$

From a stability condition $(Z, \mathcal{P})$, we can construct a bounded t-structure $\mathcal{F} := \mathcal{P}((0, \infty))$ and its heart is given by $\mathcal{H} = \mathcal{P}((0, 1])$.

2.4 Algebraic stability conditions

In [8], Bridgeland gave the alternative description of a stability condition on $\mathcal{D}$ as a pair of a bounded t-structure and a central charge on its heart. By using this description, we construct algebraic stability conditions.

Definition 2.9. Let $\mathcal{H}$ be an abelian category and let $K(\mathcal{H})$ be its Grothendieck group. A central charge on $\mathcal{H}$ is a group homomorphism $Z \colon K(\mathcal{H}) \to \mathbb{C}$ such that for any nonzero object $0 \neq E \in \mathcal{H}$, the complex number $Z(E)$ lies in the semiclosed upper half-plane $\mathbb{H} := \{re^{i\pi\phi} \in \mathbb{C} \mid r \in \mathbb{R}_{>0}, \phi \in (0, 1]\}$.

For any nonzero object $E \in \mathcal{H}$, define the *phase of E* by

$$\phi(E) := \frac{1}{\pi} \arg Z(E) \in (0, 1].$$

An object $0 \neq E \in \mathcal{H}$ is called *Z-semistable* if every subobject $0 \neq A \subset E$ satisfies $\phi(A) \leqslant \phi(E)$. A *Harder–Narasimhan filtration* of $0 \neq E \in \mathcal{H}$ is the filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E,$$

whose extension factors $F_i := E_i / E_{i-1}$ are Z-semistable with decreasing phases

$$\phi(F_1) > \cdots > \phi(F_m).$$

A central charge Z is said to have the *Harder–Narasimhan property* if any nonzero object of $\mathcal{H}$ has a Harder–Narasimhan filtration. The following gives another definition of a stability condition.

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

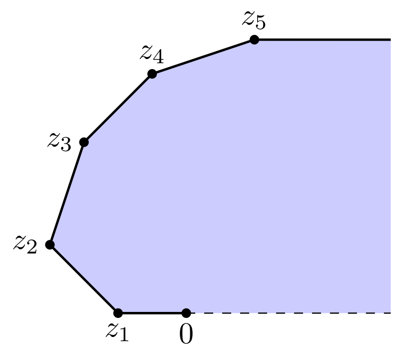


Figure 1.
Harder–Narasimhan polygon.

Proposition 2.10. [8, Proposition 5.3] *Giving a stability condition on $\mathcal{D}$ is equivalent to giving a heart $\mathcal{H}$ of a bounded structure on $\mathcal{D}$ and a central charge on $\mathcal{H}$ with the Harder–Narasimhan property.*

In Proposition 2.10, the pair $(Z, \mathcal{H})$ is constructed from a stability condition $(Z, \mathcal{P})$ by setting $\mathcal{H} := \mathcal{P}((0, 1])$.

Definition 2.11. A stability condition $(Z, \mathcal{P})$ is called *algebraic* if the corresponding heart $\mathcal{H} = \mathcal{P}((0, 1])$ is algebraic (for the definition of algebraic hearts, see Definition 2.7).

Algebraic stability conditions are constructed from algebraic hearts as follows. Let $\mathcal{H} \subset \mathcal{D}$ be an algebraic heart with simple objects $S_1, \ldots, S_n$. Then the Grothendieck group is given by $K(\mathcal{H}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$. Take $(z_1, \ldots, z_n) \in \mathbb{H}^n$ and define the central charge $Z\colon K(\mathcal{H}) \to \mathbb{C}$ by the linear extension of $Z(S_i) := z_i$. Then Z has the Harder–Narasimhan property by [8, Proposition 2.4]. Thus $(Z, \mathcal{H})$ becomes an algebraic stability condition on $\mathcal{D}$.

2.5 Harder–Narasimhan polygons

In this section, we discuss the Harder–Narasimhan polygon following [5]. This plays a key role in showing the triangle inequality for mass in Section 3.2. The following is based on [5, Section 3].

Definition 2.12. Let $\mathcal{H}$ be an abelian category and Z be a central charge on it. For an object $E \in \mathcal{H}$, the *Harder–Narasimhan polygon* $\mathrm{HN}^Z(E)$ of E is the convex hull of the subset $\{Z(A) \in \mathbb{C} \mid A \subset E\} \subset \mathbb{C}$ in the complex plane.

It is clear from the definition that if $F \subset E$, then $\mathrm{HN}^Z(F) \subset \mathrm{HN}^Z(E)$. The Harder–Narasimhan polygon $\mathrm{HN}^Z(E)$ is called *polyhedral on the left* if it has finitely many extremal points $0 = z_0, z_1, \ldots, z_k = Z(E)$ such that $\mathrm{HN}^Z(E)$ lies to the right of the path $z_0 z_1 \ldots z_k$. This implies that the intersection of $\mathrm{HN}^Z(E)$ and the closed half-plane to the left of the line through 0 and $Z(E)$ becomes a polygon with vertices $z_0, z_1, \ldots, z_k$ (see Figure 1).

Proposition 2.13. [5, Proposition 3.3] *The object E has a Harder–Narasimhan filtration if and only if $\mathrm{HN}^Z(E)$ is polyhedral on the left. In particular, if the Harder–Narasimhan filtration of E is given by*

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots E_k = E,$$

then extremal points of $\mathrm{HN}^Z(E)$ are given by $z_i = Z(E_i)$ for $i = 0, 1, \ldots, k$.

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

2.6 Topology on the space of stability conditions

In [8], Bridgeland introduced a natural topology on the space of stability conditions and showed that this space becomes a complex manifold. In the following, we recall his construction. Let $\mathrm{Stab}(\mathcal{D})$ be the set of stability conditions on a triangulated category $\mathcal{D}$ with the support property. For stability conditions $\sigma=(Z,\mathcal{P})$ and $\tau=(W,\mathcal{Q})$ in $\mathrm{Stab}(D)$, set

$$d(\mathcal{P},\mathcal{Q}):=\sup_{0\neq E\in\mathcal{D}}\{|\phi_\sigma^-(E)-\phi_\tau^-(E)|,|\phi_\sigma^+(E)-\phi_\tau^+(E)|\}\in[0,\infty]$$

and

$$\|Z-W\|_\sigma:=\sup\left\{\frac{|Z(E)-W(E)|}{|Z(E)|}\;\middle|\;E\text{ is }\sigma\text{-semistable}\right\}\in[0,\infty].$$

Define a subset $B_\epsilon(\sigma)\subset\mathrm{Stab}(\mathcal{D})$ by

$$B_\epsilon(\sigma):=\{\tau=(W,\mathcal{Q})\in\mathrm{Stab}(\mathcal{D})\mid d(\mathcal{P},\mathcal{Q})<\epsilon,\|Z-W\|_\sigma<\sin(\pi\epsilon)\}$$

for $0<\epsilon<\frac{1}{4}$.

In [8, Section 6], it was shown that the family of subsets

$$\{B_\epsilon(\sigma)\subset\mathrm{Stab}(\mathcal{D})\mid\sigma\in\mathrm{Stab}(\mathcal{D}),0<\epsilon<\tfrac{1}{4}\}$$

becomes an open basis of a topology on $\mathrm{Stab}(\mathcal{D})$. In [8], Bridgeland showed a crucial theorem.

THEOREM 2.14. [8, Theorem 1.2] *The projection map of central charges*

$$\pi\colon\mathrm{Stab}(\mathcal{D})\longrightarrow\mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}),\mathbb{C}),\quad(Z,\mathcal{P})\mapsto Z$$

is a local isomorphism of topological spaces. In particular, π induces a complex structure on $\mathrm{Stab}(\mathcal{D})$.

§3. Mass growth of objects and categorical entropy

3.1 Mass with a parameter and complexity

In this section, we introduce the mass growth of objects and show fundamental properties of it.

DEFINITION 3.1. [11, Section 4.5] Take a stability condition $\sigma=(Z,\mathcal{P})$ on $\mathcal{D}$. Let $E\in\mathcal{D}$ be a nonzero object with semistable factors $A_1,\ldots,A_m$ of phases $\phi_1>\cdots>\phi_m$. The *mass of E with a parameter* $t\in\mathbb{R}$ is the function $m_{\sigma,t}(E)\colon\mathbb{R}\to\mathbb{R}_{>0}$ defined by

$$m_{\sigma,t}(E):=\sum_{i=1}^m|Z(A_i)|e^{\phi_i t}.$$

When $t=0$, $m_{\sigma,0}(E)$ is called the *mass of E* and simply written as $m_\sigma(E):=m_{\sigma,0}(E)$. As a convention, set $m_{\sigma,t}(E):=0$ if $E\cong 0$.

In the following, if σ is clear in the context, we often drop it from the notation and write $m_t(E)$. Similar to the growth rate of complexity of a generator with respect to endofunctors, we consider the growth rate of mass of objects.

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

DEFINITION 3.2. [11, Section 4.5] Let σ be a stability condition on $\mathcal{D}$ and $F\colon \mathcal{D}\to\mathcal{D}$ be an endofunctor. The *mass growth with respect to* F is the function $h_{\sigma,t}(F)\colon \mathbb{R}\to[-\infty,\infty]$ defined by

$$h_{\sigma,t}(F) := \sup_{E\in\mathcal{D}}\left\{\limsup_{n\to\infty}\frac{1}{n}\log(m_{\sigma,t}(F^nE))\right\}.$$

In the rest of this section, we study fundamental properties of $h_{\sigma,t}(F)$. The triangle inequality for $m_{\sigma,t}$ plays an important role.

PROPOSITION 3.3. *For objects $D, E, F\in\mathcal{D}$, if there is an exact triangle $D\to E\to F\to D[1]$, then*

$$m_{\sigma,t}(E)\leqslant m_{\sigma,t}(D)+m_{\sigma,t}(F).$$

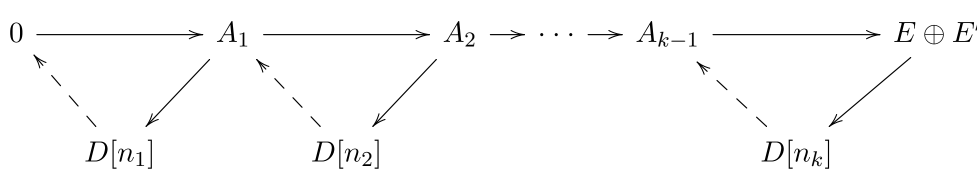
The proof of Proposition 3.3 is given in Section 3.2.

PROPOSITION 3.4. *Let σ be a stability condition on $\mathcal{D}$. Then*

$$m_{\sigma,t}(E)\leqslant m_{\sigma,t}(D)\delta_t(D,E)$$

for any objects $D, E\in\mathcal{D}$.

Proof. It is sufficient to show the case $E\in\langle D\rangle$. Then by the definition of complexity $\delta_t(D,E)$, for any $\epsilon>0$, there is a sequence of exact triangles



such that

$$\sum_{i=1}^{k} e^{n_i t} < \delta_t(D,E)+\epsilon.$$

Note that $m_{\sigma,t}$ satisfies $m_{\sigma,t}(D[n]) = m_{\sigma,t}(D)\cdot e^{nt}$ for $D\in\mathcal{D}$ and $n\in\mathbb{Z}$. By using the inequality in Proposition 3.3 repeatedly, we have

$$\begin{aligned} m_{\sigma,t}(E)\leqslant m_{\sigma,t}(E\oplus E') &\leqslant \sum_{i=1}^{k} m_{\sigma,t}(D[n_i]) \leqslant m_{\sigma,t}(D)\cdot\left(\sum_{i=1}^{k} e^{n_i t}\right)\\ &\leqslant m_{\sigma,t}(D)\delta_t(D,E)+\epsilon\cdot m_{\sigma,t}(D)\end{aligned}$$

for any $\epsilon>0$. This implies the result. ∎

Now, we show fundamental properties of the mass growth.

THEOREM 3.5. *Let $F\colon\mathcal{D}\to\mathcal{D}$ be an endofunctor and σ be a stability condition on $\mathcal{D}$. Assume that $\mathcal{D}$ has a split generator $G\in\mathcal{D}$. Then the mass growth $h_{\sigma,t}(F)$ satisfies the followings.*

(1) *$h_{\sigma,t}(F)$ is given by*

$$h_{\sigma,t}(F) = \limsup_{n\to\infty}\frac{1}{n}\log(m_{\sigma,t}(F^nG)).$$

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

(2) *We have an inequality*

$$h_{\sigma,t}(F) \leqslant h_t(F) < \infty,$$

where $h_t(F)$ is the entropy of F (see Definition 2.4).

Proof. By Proposition 2.2(3) and Proposition 3.4, we have

$$m_t(F^nE) \leqslant m_t(F^nG)\delta_t(F^nG, F^nE) \leqslant m_t(F^nG)\delta_t(G, E)$$

for any object $E \in \mathcal{D}$. Hence

$$\limsup_{n\to\infty} \frac{1}{n} \log m_t(F^nE) \leqslant \limsup_{n\to\infty} \frac{1}{n} \log m_t(F^nG)$$

and this inequality implies (1). Again by Proposition 3.4, we have

$$m_t(F^nG) \leqslant m_t(G)\delta_t(G, F^nG).$$

Hence

$$\limsup_{n\to\infty} \frac{1}{n} \log m_t(F^nG) \leqslant \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G, F^nG)$$

and this inequality implies (2). □

3.2 Triangle inequality for mass with a parameter

We prove Proposition 3.3. Recall the notation $\mathbb{H} = \{re^{i\pi\phi} \mid r > 0, \phi \in (0, 1]\}$. For a complex number $z \in \mathbb{H}$, define a function of $t \in \mathbb{R}$ by

$$g_t(z) := |z|e^{\phi(z)t},$$

where $\phi(z)$ is the phase of z given by $\phi(z) := (1/\pi) \arg z \in (0, 1]$. We start by showing the triangle inequality for $g_t(z)$.

Lemma 3.6. *For $z_1, z_2 \in \mathbb{H}$, an inequality*

$$g_t(z_1 + z_2) \leqslant g_t(z_1) + g_t(z_2)$$

holds.

Proof. Set $\phi_1 := \phi(z_1), \phi_2 := \phi(z_2)$ and $\phi_3 := \phi(z_1 + z_2)$. If $\phi_1 = \phi_2$, the result is trivial. Without loss of generality, we reduce to the case $\phi_1 > \phi_2$. By applying the law of sine for the triangle consisting of vertices $0, z_1, z_1 + z_2$ (see Figure 2), we obtain

$$|z_1 + z_2| = d\sin(\pi a + \pi b), \qquad |z_1| = d\sin \pi b, \qquad |z_2| = d\sin \pi a,$$

where $a = \phi_1 - \phi_3$, $b = \phi_3 - \phi_2$ and d is the diameter of the circumcircle of the triangle. By inputting these parameters, the inequality $g_t(z_1 + z_2) \leqslant g_t(z_1) + g_t(z_2)$ becomes

$$\sin(\pi a + \pi b) \leqslant e^{at} \sin \pi b + e^{-bt} \sin \pi a,$$

where $0 < a, b < 1$. Dividing by $\sin \pi a \sin \pi b$ and applying the addition formula, we have

$$\frac{e^{at} - \cos \pi a}{\sin \pi a} + \frac{e^{-bt} - \cos \pi b}{\sin \pi b} \geqslant 0.$$

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

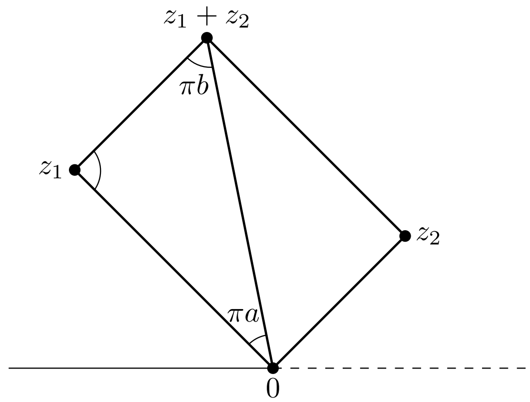


Figure 2.
Triangle consisting of vertices $0, z_1, z_1 + z_2$.

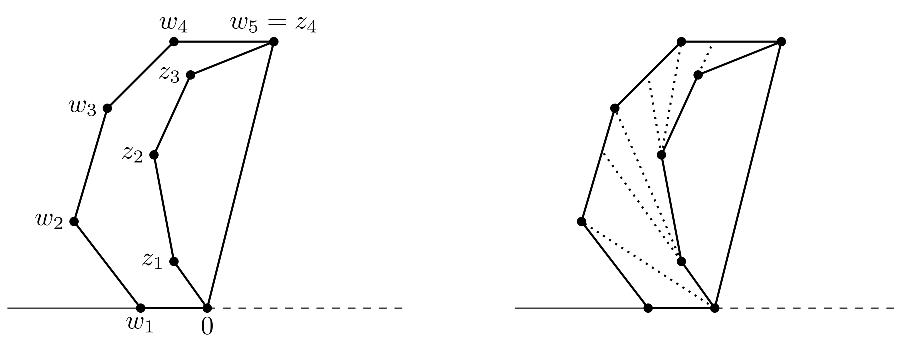


Figure 3.
Polygons and a triangulation of the encircled domain.

After setting $c = -b$, the above inequality is equivalent to $f(a) \geqslant f(c)$ for $-1 < c < 0 < a < 1$, where

$$f(x) = \frac{e^{xt} - \cos \pi x}{\sin \pi x}.$$

It is easy to check that $f(x)$ is increasing in the intervals $(-1, 0)$ and $(0, 1)$, and the limit of $f(x)$ at the zero is given by $\lim_{x \to \pm 0} f(x) = t/\pi$. ∎

The triangle inequality for $g_t(z)$ implies the following.

LEMMA 3.7. *Let $z_1, \ldots, z_k$ and $w_1, \ldots, w_l$ be the complex numbers in $\mathbb{H}$ with $z_k = w_l$ and set $z_0 = w_0 = 0$. If they satisfy the following conditions (see the left of Figure 3):*

(a) *$\phi(z_i - z_{i-1}) > \phi(z_{i+1} - z_i)$ and $\phi(w_j - w_{j-1}) > \phi(w_{j+1} - w_j)$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l$;*

(b) *the polygon $w_0 w_1 w_2 \ldots w_l w_0$ contains the polygon $z_0 z_1 z_2 \ldots z_k z_0$,*

then

$$\sum_{i=1}^{k} g_t(z_i - z_{i-1}) \leqslant \sum_{j=1}^{l} g_t(w_j - w_{j-1}).$$

Proof. By the condition (b), there is a unique domain encircled by two paths $z_0 z_1 z_2 \ldots z_k$ and $w_0 w_1 w_2 \ldots w_l$. By the convexity condition (a), we can triangulate this domain

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

as follows. First, we extend lines $z_i z_{i+1}$ for $i = 0, \dots, k-2$ to intersect $w_0 w_1 w_2 \dots w_l$. Then we obtain polygons which contain exactly one $z_i z_{i+1}$ for some i. Next, we triangulate the polygon containing $z_i z_{i+1}$ by drawing lines from z_i to w_j in the polygon. Then we can obtain the triangulated domain as in the right of Figure 3. Applying the triangle inequality for $g_t(z)$ (Lemma 3.2) repeatedly, we obtain the result. ∎

LEMMA 3.8. *Let $\sigma = (Z, \mathcal{P})$ be a stability condition and $\mathcal{H} = \mathcal{P}((0,1])$ be the associated heart. If there is a short exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathcal{H}$ and $C \in \mathcal{P}(1)$, then

$$m_t(A) \leqslant m_t(B) + e^{-t} m_t(C).$$

Proof. Let

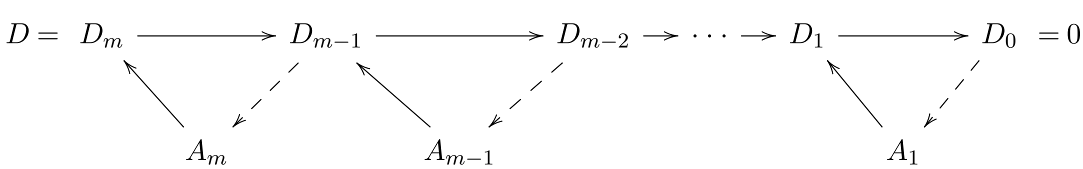
$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_k = A$$
$$0 = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_{l-1} = B$$

be Harder–Narasimhan filtrations of A and B. Set $z_i := Z(A_i)$ for $i = 0, 1, \dots, k$, $w_j := Z(B_j)$ for $j = 0, 1, \dots, l-1$ and $w_l := Z(B) - Z(C) = Z(A)$. Then by definition of the Harder–Narasimhan filtration, these complex numbers satisfy the condition (a) in Lemma 3.7. Consider the Harder–Narasimhan polygons $\mathrm{HN}^Z(A)$ and $\mathrm{HN}^Z(B)$ (see Definition 2.12). By Proposition 2.13, complex numbers $z_0, z_1, \dots, z_k$ and $w_0, w_1, \dots, w_{l-1}$ are extremal points of $\mathrm{HN}^Z(A)$ and $\mathrm{HN}^Z(B)$, respectively. Thus, the intersection of $\mathrm{HN}^Z(A)$ and the left of the line through 0 and $z_k = Z(A)$ is the polygon $z_0 z_1 z_2 \dots z_k z_0$ and the intersection of $\mathrm{HN}^Z(B)$ and the left of the line through 0 and $w_l = Z(A)$ is the polygon $w_0 w_1 w_2 \dots w_l w_0$. Since $\mathrm{HN}^Z(A) \subset \mathrm{HN}^Z(B)$, the polygon $w_0 w_1 w_2 \dots w_l w_0$ contains the polygon $z_0 z_1 z_2 \dots z_k z_0$ and this implies the condition (b) in Lemma 3.7. Since

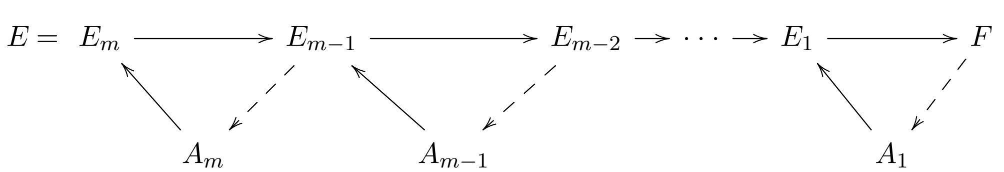
$$m_t(A) = \sum_{i=1}^{k} g_t(z_i - z_{i-1}), \qquad m_t(B) = \sum_{j=1}^{l-1} g_t(w_j - w_{j-1}), \qquad e^{-t} m_t(C) = g_t(w_l - w_{l-1}),$$

applying Lemma 3.7, we obtain the result. ∎

Proof of Proposition 3.3. Assume that there is an exact triangle $D \to E \to F \to D[1]$. From a Harder–Narasimhan filtration of D, we can construct the dual of it:



with $A_i \in \mathcal{P}(\phi_i)$ and $\phi_m > \phi_{m-1} > \cdots \phi_1$. Applying the octahedra axiom for the above sequence together with the exact triangle $D \to E \to F \to D[1]$, we can construct a sequence of exact triangles



https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

Since $m_t(D) = \sum_{i=1}^{m} |Z(A_i)|e^{t\phi_i}$ and A_i is semistable, the problem is reduced to the case that D is semistable. Without loss of generality, we can assume $D \in \mathcal{P}(1)$. By taking the cohomology associated with the heart $\mathcal{H} = \mathcal{P}((0, 1])$ (see Section 2.2), we have a long exact sequence

$$0 \to H^{-1}(E) \to H^{-1}(F) \to H^0(D) \to H^0(E) \to H^0(F) \to 0$$

and isomorphisms $H^i(E) \cong H^i(F)$ for $i \neq -1, 0$ in $\mathcal{H}$. If $1 > \phi^+(H^0(E))$, then the map $f\colon H^0(D) \to H^0(E)$ is zero. Hence, the long exact sequence splits into $0 \to H^{-1}(E) \to H^{-1}(F) \to H^0(D) \to 0$ and $H^0(E) \cong H^0(F)$. From Lemma 3.8, we have

$$m_t(H^{-1}(E))e^t \leqslant m_t(H^{-1}(F))e^t + m_t(D).$$

Thus, we obtain the result. If the map $f\colon H^0(D) \to H^0(E)$ is not zero, then the long exact sequence splits into two short exact sequences

$$0 \to H^{-1}(E) \to H^{-1}(F) \to \operatorname{Ker} f \to 0$$
$$0 \to \operatorname{Im} f \to H^0(E) \to H^0(F) \to 0.$$

Let $E_+ \in \mathcal{P}(1)$ be the semistable factor of E with phase one. Note that $m_t(D) = m_t(\operatorname{Ker} f) + m_t(\operatorname{Im} f)$ since $\operatorname{Ker} f \subset D \in \mathcal{P}(1)$ and $\operatorname{Im} f \subset E_+ \in \mathcal{P}(1)$. Again by Lemma 3.8, we have

$$m_t(H^{-1}(E))e^t \leqslant m_t(H^{-1}(F))e^t + m_t(\operatorname{Ker} f)$$

and it is easy to check that $m_t(H^0(E)) = m_t(\operatorname{Im} f) + m_t(H^0(F))$. ∎

3.3 Mass growth and deformation of stability conditions

The aim of this section is to show that for a stability condition σ and an endofunctor F, the mass growth $h_{\sigma,t}(F)$ is stable under the continuous deformation of σ. The following inequality was shown in [8, Proposition 8.1] when $t = 0$.

PROPOSITION 3.9. *Let $\sigma = (Z, \mathcal{P}) \in \operatorname{Stab}(\mathcal{D})$ be a stability condition on $\mathcal{D}$. If $\tau = (W, \mathcal{Q}) \in B_\epsilon(\sigma)$ with small enough $\epsilon > 0$, then there are functions $C_1, C_2\colon \mathbb{R} \to \mathbb{R}_{>0}$ such that*

$$C_1(t)m_{\tau,t}(E) < m_{\sigma,t}(E) < C_2(t)m_{\tau,t}(E)$$

for all $0 \neq E \in \mathcal{D}$.

Proof. We use an argument similar to the proof of [8, Proposition 8.1]. It is sufficient to show that for $\tau = (W, \mathcal{Q}) \in B_\epsilon(\sigma)$ with small enough $\epsilon > 0$, there are some constants $C > 1$ and $r > 0$ such that

$$m_{\tau,t}(E) < Ce^{r|t|}m_{\sigma,t}(E)$$

for any nonzero object $E \in \mathcal{D}$. We first consider the case $\phi^+_\sigma(E) - \phi^-_\sigma(E) < \eta$ for $0 < \eta < \frac{1}{4}$. In this case, it was shown in the proof of [8, Proposition 8.1] that there is a constant $C(\epsilon, \eta) > 1$ such that

$$m_\tau(E) \leqslant C(\epsilon, \eta)m_\sigma(E)$$

and $C(\epsilon, \eta) \to 1$ as $\max\{\epsilon, \eta\} \to 0$. Note that $\phi^+_\sigma(E) - \phi^-_\sigma(E) < \eta$ implies $\phi^\pm_\sigma(E) \in (\psi, \psi + \eta)$ for some $\psi \in \mathbb{R}$. Since $d(\mathcal{P}, \mathcal{Q}) < \epsilon$, we have $\phi^\pm_\tau(E) \in (\psi - \epsilon, \psi + \epsilon + \eta)$. By definition of $m_{\sigma,t}(E)$ and $m_{\tau,t}(E)$, it follows that

$$m_{\tau,t}(E) \leqslant m_\tau(E)\exp(\phi^+_\tau(E)|t|), \qquad m_\sigma(E)\exp(\phi^-_\sigma(E)|t|) \leqslant m_{\sigma,t}(E).$$

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

Since $\psi < \phi^-_\sigma(E)$ and $\phi^+_\tau(E) < \psi + \epsilon + \eta$, we have an inequality

$$m_{\tau,t}(E) \leqslant C(\epsilon, \eta) e^{(\epsilon+\eta)|t|} m_{\sigma,t}(E).$$

Next, we consider a general nonzero object E. Take a real number ϕ and a positive integer n. For $k \in \mathbb{Z}$, define intervals

$$I_k := [\phi + kn\epsilon, \phi + (k+1)n\epsilon), \qquad J_k := [\phi + kn\epsilon - \epsilon, \phi + (k+1)n\epsilon + \epsilon)$$

and let α_k and β_k be the truncation functors projecting into the subcategories $\mathcal{Q}(I_k)$ and $\mathcal{P}(J_k)$, respectively. Then, as in the proof of [8, Proposition 8.1], we have $\alpha_k \circ \beta_k = \alpha_k$ and therefore $m_{\tau,t}(\alpha_k(E)) \leqslant m_{\tau,t}(\beta_k(E))$. As a result, for small enough $n\epsilon$, we have

$$m_{\tau,t}(E) = \sum_k m_{\tau,t}(\alpha_k(E)) \leqslant \sum_k m_{\tau,t}(\beta_k(E)) < C(\epsilon, (n+2)\epsilon) e^{(n+3)\epsilon|t|} \sum_k m_{\sigma,t}(\beta_k(E)).$$

On the other hand, we can choose ϕ so that

$$\sum_k m_{\sigma,t}(\beta_k(E)) \leqslant \frac{n+2}{n} m_{\sigma,t}(E).$$

By taking the limits $\epsilon \to 0$ and $n \to \infty$ in keeping with $n\epsilon \to 0$, the result follows. ∎

From Proposition 3.9, we immediately have the following.

PROPOSITION 3.10. *Let $F\colon \mathcal{D} \to \mathcal{D}$ be an endofunctor, and σ and τ be stability conditions on $\mathcal{D}$. If σ and τ lie in the same connected component in* $\operatorname{Stab}(\mathcal{D})$*, then*

$$h_{\sigma,t}(F) = h_{\tau,t}(F).$$

Proof. Let $\sigma, \tau \in \operatorname{Stab}(\mathcal{D})$ be stability conditions such that $\tau \in B_\epsilon(\sigma)$ for small enough $\epsilon > 0$. Then Proposition 3.9 implies $h_{\sigma,t}(F) = h_{\tau,t}(F)$. Thus, $h_{\sigma,t}(F)$ is locally constant on the topological space $\operatorname{Stab}(\mathcal{D})$. ∎

3.4 Lower bound of the mass growth by the spectral radius

Let $F\colon \mathcal{D} \to \mathcal{D}$ be an endofunctor. Since F preserves exact triangles in $\mathcal{D}$, F induces a linear transformation $[F]\colon K(\mathcal{D}) \to K(\mathcal{D})$. We extend $[F]$ on $K(\mathcal{D}) \otimes \mathbb{C}$ naturally. The *spectral radius of* $[F]$ is defined by

$$\rho([F]) := \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } [F] \text{ on } K(\mathcal{D}) \otimes \mathbb{C}\}.$$

PROPOSITION 3.11. *For any stability condition $\sigma \in \operatorname{Stab}(\mathcal{D})$, we have an inequality*

$$\log \rho([F]) \leqslant h_{\sigma,0}(F).$$

Proof. We recall the assumption $K(\mathcal{D}) \cong \mathbb{Z}^{\oplus n}$. Set $K(\mathcal{D})_{\mathbb{C}} := K(\mathcal{D}) \otimes \mathbb{C}$. Let $A_1, \ldots, A_n \in \mathcal{D}$ be objects whose classes $[A_1], \ldots, [A_n]$ form a basis of $K(\mathcal{D})_{\mathbb{C}}$. Take an eigenvector

$$v = \sum_{i=1}^n a_i [A_i] \in K(\mathcal{D})_{\mathbb{C}} \quad (a_i \in \mathbb{C})$$

for the eigenvalue $\lambda \in \mathbb{C}$ of $[F]$ satisfying $|\lambda| = \rho([F])$. First, we consider the case that a stability condition $\sigma = (Z, \mathcal{P})$ satisfies $Z(v) \neq 0$. Note that the mass satisfies $|Z(E)| \leqslant m_\sigma(E)$

and $m_\sigma(E \oplus E') = m_\sigma(E) + m_\sigma(E')$ for any objects $E, E' \in \mathcal{D}$. Then

$$|\lambda|^k |Z(v)| = |Z(\lambda^k v)| = |Z([F]^k v)| \leqslant \sum_{i=1}^{n} |a_i| \cdot |Z(F^k A_i)| \leqslant \sum_{i=1}^{n} l_i m_\sigma(F^k A_i) = m_\sigma\left(F^k\left(\bigoplus_{i=1}^{n} A_i^{\oplus l_i}\right)\right),$$

where $l_1, \ldots, l_n$ are positive integers satisfying $|a_i| \leqslant l_i$. Since $|Z(v)| > 0$, we have

$$\log \rho([F]) = \lim_{k\to\infty} \frac{1}{k} \log(|\lambda|^k |Z(v)|) \leqslant \limsup_{k\to\infty} \frac{1}{k} \log(m_\sigma(F^k E)) \leqslant h_{\sigma,0}(F),$$

where $E = \bigoplus_{i=1}^{n} A_i^{\oplus l_i}$. Next, consider the case $Z(v) = 0$. Then by Theorem 2.14, we can deform $\sigma = (Z, \mathcal{P})$ to $\sigma' = (Z', \mathcal{P}')$ so that $Z'(v) \neq 0$. Again we have $\log \rho([F]) \leqslant h_{\sigma',0}(F)$ and Proposition 3.10 implies $h_{\sigma,0}(F) = h_{\sigma',0}(F)$. ∎

3.5 Mass growth via algebraic stability conditions

If a triangulated category has an algebraic stability condition, then we can show that the mass growth coincides with the entropy. Let $\mathcal{H} \subset \mathcal{D}$ be an algebraic heart with simple objects $S_1, \ldots, S_n$. Then the Grothendieck group is given by

$$K(\mathcal{D}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[S_i].$$

The class of an object $E \in \mathcal{H}$ is written as $[E] = \sum_{i=1}^{n} d_i [S_i]$ with $d_i \in \mathbb{Z}_{\geqslant 0}$. We define the *dimension of* E by $\dim E := \sum_{i=1}^{n} d_i \in \mathbb{Z}_{\geqslant 0}$. Then the dimension gives the upper bound of the complexity for objects in $\mathcal{H}$.

LEMMA 3.12. *Let $\mathcal{H} \subset \mathcal{D}$ be an algebraic heart with simple objects $S_1, \ldots, S_n$. Then for the split generator $G := \bigoplus_{i=1}^{n} S_i$, we have an inequality*

$$\delta_t(G, E) \leqslant \dim E.$$

Proof. Since $\mathcal{H}$ is a finite length abelian category, for any object $E \in \mathcal{H}$ there is a Jordan–Hölder filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l = E$$

of length $l = \dim E$ with $E_i/E_{i-1} \in \{S_1, \ldots, S_n\}$. As a result, we can construct a filtration

$$0 = E'_0 \subset E'_1 \subset E'_2 \subset \cdots \subset E'_l = E \oplus E'$$

of length $l = \dim E$ with $E'_i/E'_{i-1} = G$ and this implies the result. ∎

Following Section 2.4, we construct the special algebraic stability condition. For an algebraic heart $\mathcal{H} \subset \mathcal{D}$ with simple objects $S_1, \ldots, S_n$, define the central charge

$$Z_0 \colon K(\mathcal{D}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[S_i] \to \mathbb{C}$$

by $Z_0(S_i) := i$. Then the pair $\sigma_0 := (Z_0, \mathcal{H})$ becomes an algebraic stability condition. By definition, the mass of an object $E \in \mathcal{H}$ is given by

$$m_{\sigma_0,t}(E) = \dim E \cdot e^{(1/2)t}.$$

Together with Lemma 3.12, we obtain the following inequality.

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

PROPOSITION 3.13. *For any $E \in \mathcal{D}$, the generator $G = \bigoplus_{i=1}^{n} S_i$ and the algebraic stability condition $\sigma_0 = (Z_0, \mathcal{H})$, we have an inequality*

$$\delta_t(G, E) \leqslant e^{-(1/2)t} m_{\sigma_0,t}(E).$$

Proof. For an object $E \in \mathcal{D}$, we denote by $H^k(E) \in \mathcal{H}$ the cohomology associated with the heart $\mathcal{H}$ (see Section 2.2). By using Lemmas 2.3 and 3.12, we have

$$\delta_t(G, E) \leqslant \sum_k \delta_t(G, H^k(E)) e^{-kt} \leqslant \sum_k \dim H^k(E) e^{-kt}.$$

On the other hand, the definition of $m_{\sigma_0,t}(E)$ implies

$$m_{\sigma_0,t}(E) = \sum_k m_{\sigma_0,t}(H^k(E)) e^{-kt} = \sum_k \dim H^k(E) e^{(1/2)t} e^{-kt}.$$

Thus, we obtain the result. ∎

We show the main result of this section.

THEOREM 3.14. *Let $F \colon \mathcal{D} \to \mathcal{D}$ be an endofunctor. If a connected component $\mathrm{Stab}^\circ(\mathcal{D}) \subset \mathrm{Stab}(\mathcal{D})$ contains an algebraic stability condition, then $\mathcal{D}$ has a split generator G and for any $\sigma \in \mathrm{Stab}^\circ(\mathcal{D})$, we have*

$$h_t(F) = h_{\sigma,t}(F) = \lim_{n \to \infty} \frac{1}{n} \log(m_{\sigma,t}(F^n G)).$$

Proof. Let $\mathcal{H}$ be an algebraic heart with simple objects $S_1, \ldots, S_n$ and set $G = \bigoplus_{i=1}^{n} S_i$. Then G is a split generator of $\mathcal{D}$. Consider the special algebraic stability condition $\sigma_0 = (Z_0, \mathcal{H})$ which is constructed in this section. By Proposition 3.10, it is sufficient to show that

$$h_{\sigma_0,t}(F) = h_t(F).$$

By [11, Lemma 2.6], the limit

$$h_t(F) = \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, F^n G)$$

converges. On the other hand, by Proposition 3.4 and Proposition 3.13, we have

$$e^{(1/2)t} \delta_t(G, F^n G) \leqslant m_{\sigma_0,t}(F^n G) \leqslant m_{\sigma_0,t}(G) \delta_t(G, F^n G).$$

Hence the limit

$$\lim_{n \to \infty} \frac{1}{n} \log(m_{\sigma_0,t}(F^n G))$$

converges and coincides with $h_t(F)$. ∎

§4. Applications

4.1 Entropy on the derived categories of nonpositive dg-algebras

In this section, we discuss the entropy of endofunctors on the derived categories of nonpositive dg-algebras. In this case, we can describe the entropy as the growth rate of the Hilbert–Poincaré polynomial of a generator.

Let $A = \bigoplus_{k \in \mathbb{Z}} A^k$ be a dg-algebra over $\mathbb{K}$ satisfying the following conditions:

(a) $H^k(A) = 0$ for $i > 0$;
(b) $H^0(A)$ is a finite-dimensional algebra over $\mathbb{K}$.

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

Let $\mathcal{D}(A)$ be the derived category of dg-modules over A and $\mathcal{D}_{fd}(A)$ be the full subcategory of $\mathcal{D}(A)$ consisting of dg-modules with finite-dimensional total cohomology, that is,

$$\mathcal{D}_{fd}(A) := \left\{ M \in \mathcal{D}(A) \;\middle|\; \sum_k \dim H^k(M) < \infty \right\}.$$

Define the full subcategory $\mathcal{F} \subset \mathcal{D}_{fd}(A)$ by

$$\mathcal{F} := \{ M \in \mathcal{D}_{fd}(A) \mid H^k(M) = 0 \text{ for } k > 0 \}.$$

Then $\mathcal{F}$ becomes a bounded t-structure on $\mathcal{D}_{fd}(A)$. The heart $\mathcal{H}_s$ of $\mathcal{F}$ is called the *standard heart*. It is known that the 0th cohomology functor $H^0 \colon \mathcal{D}_{fd}(A) \to \text{mod-}\, H^0(A)$ induces an equivalence of abelian categories:

$$H^0 \colon \mathcal{H}_s \xrightarrow{\sim} \text{mod-}\, H^0(A),$$

where mod- $H^0(A)$ is the abelian category of finite-dimensional $H^0(A)$-modules. (For details, see [2, Section 2].) Since $H^0(A)$ is a finite-dimensional algebra, $\mathcal{H}_s$ is an algebraic heart. Thus, we can construct an algebraic stability condition on $\mathcal{D}_{fd}(A)$. Applying Theorem 3.14, we obtain the following.

Proposition 4.1. *Let* $\mathrm{Stab}^\circ(\mathcal{D}_{fd}(A))$ *be the connected component which contains stability conditions with the standard heart* $\mathcal{H}_s$. *Then for any stability condition* $\sigma \in \mathrm{Stab}^\circ(\mathcal{D}_{fd}(A))$ *and an endofunctor* $F \colon \mathcal{D}_{fd}(A) \to \mathcal{D}_{fd}(A)$, *we have*

$$h_t(F) = h_{\sigma,t}(F).$$

Next, we describe $h_t(F)$ by using the Hilbert–Poincaré polynomial.

Definition 4.2. For a dg-module $M \in \mathcal{D}_{fd}(A)$, define the *Hilbert–Poincaré polynomial of* M by

$$P_t(M) := \sum_{k \in \mathbb{Z}} \dim H^k(M) e^{-kt} \in \mathbb{Z}[e^t, e^{-t}].$$

As in Section 3.5, we construct the special stability condition $\sigma_0 = (Z_0, \mathcal{H}_s)$ by using the standard heart $\mathcal{H}_s$. Then by definition of σ_0, we have

$$m_{\sigma_0,t}(M) = e^{(1/2)t} P_t(M)$$

for any dg-module $M \in \mathcal{D}_{fd}(A)$. As a result, the entropy is described as follows.

Proposition 4.3. *Let* $F \colon \mathcal{D}_{fd}(A) \to \mathcal{D}_{fd}(A)$ *be an endofunctor and* $G \in \mathcal{D}_{fd}(A)$ *be a split generator. Then the entropy of* F *is given by*

$$h_t(F) = \lim_{n \to \infty} \frac{1}{n} \log P_t(F^n G).$$

4.2 Entropy of spherical twists

In this section, we compute the entropy of Seidel–Thomas spherical twists on the derived categories of Calabi–Yau algebras associated with acyclic quivers. Let Q be an acyclic quiver with vertices $\{1, \ldots, n\}$ and $\Gamma_N Q$ be the *Ginzburg* N*-Calabi–Yau dg-algebra associated with* Q for $N \geqslant 2$. (For the definition of $\Gamma_N Q$, see [13, Section 4.2] or [16, Section 6.2].)

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

Set $\mathcal{D}_Q^N := \mathcal{D}_{fd}(\Gamma_N Q)$. By [16, Theorem 6.3], the category $\mathcal{D}_Q^N$ is an N-*Calabi–Yau category*, that is, there is a natural isomorphism

$$\mathrm{Hom}(E, F) \xrightarrow{\sim} \mathrm{Hom}(F, E[N])^*$$

for $E, F \in \mathcal{D}_Q^N$. (Here V^* denotes the dual space of a $\mathbb{K}$-linear space V.) In the Calabi–Yau category, we can consider a certain class of objects, called spherical objects. An object $S \in \mathcal{D}_Q^N$ is called N-*spherical* if

$$\mathrm{Hom}(S, S[i]) = \begin{cases} \mathbb{K} & \text{if } i = 0, N \\ 0 & \text{otherwise.} \end{cases}$$

For a spherical object $S \in \mathcal{D}_Q^N$, Seidel–Thomas [20] defined an exact autoequivalence $\Phi_S \in \mathrm{Aut}(\mathcal{D}_Q^N)$, called a *spherical twist*, by the exact triangle

$$\mathrm{Hom}^\bullet(S, E) \otimes S \longrightarrow E \longrightarrow \Phi_S(E)$$

for any object $E \in \mathcal{D}_Q^N$. The inverse functor $\Phi_S^{-1} \in \mathrm{Aut}(\mathcal{D}_Q^N)$ is given by

$$\Phi_S^{-1}(E) \longrightarrow E \longrightarrow S \otimes \mathrm{Hom}^\bullet(E, S)^*.$$

The Ginzburg dg-algebra $\Gamma_N Q$ satisfies the conditions in Section 4.1 when $N \geqslant 2$. (In the case $N = 2$, we need some modification.) Hence, the category $\mathcal{D}_Q^N$ has the standard algebraic heart $\mathcal{H}_s \subset \mathcal{D}_Q^N$ generated by simple $\Gamma_N Q$-modules $S_1, \ldots, S_n$ corresponding to vertices $\{1, \ldots, n\}$ of Q. In addition, these objects $S_1, \ldots, S_n$ are N-spherical by [16, Lemma 4.4]. Thus, we can define spherical twists $\Phi_{S_1}, \ldots, \Phi_{S_n} \in \mathrm{Aut}(\mathcal{D}_Q^N)$. In the following, we compute the entropy of spherical twists $\Phi_{S_1}, \ldots, \Phi_{S_n}$ by using Proposition 4.3. For simplicity, write $\Phi_i := \Phi_{S_i}$.

LEMMA 4.4. *For a spherical twist $\Phi_i \in \mathrm{Aut}(\mathcal{D}_Q^N)$ and a spherical object $S_j \in \mathcal{D}_Q^N$, the Hilbert–Poincaré polynomial of $\Phi_i^k S_j$ ($k \geqslant 0$) is given by*

$$P_t(\Phi_i^k S_j) = \begin{cases} e^{k(1-N)t} & \text{if } i = j \\ 1 + q_{ij} \sum_{l=0}^{k-1} e^{l(1-N)t} & \text{if } q_{ij} > 0 \\ 1 + q_{ji} e^{(2-N)t} \sum_{l=0}^{k-1} e^{l(1-N)t} & \text{if } q_{ji} > 0 \\ 1 & \text{otherwise,} \end{cases}$$

where q_{ij} is the number of arrows from i to j in Q.

Proof. First, we note that

$$\dim \mathrm{Hom}(S_i, S_j[m]) = \begin{cases} 1 & \text{if } i = j \text{ and } m = 0, N \\ q_{ij} & \text{if } q_{ij} > 0 \text{ and } m = 1 \\ q_{ji} & \text{if } q_{ji} > 0 \text{ and } m = N - 1 \\ 0 & \text{otherwise.} \end{cases}$$

This follows by the definition of $S_1, \ldots, S_n$ for $m = 0$, by [4, Lemma 2.12] for $m = 1$, and the Serre duality for $m = N - 1, N$. By the definition of spherical twists, it is easy to see that $\Phi_i^k S_i = S_i[k(1-N)]$ and hence $P_t(\Phi_i^k S_i) = e^{k(1-N)t}$. If $i \neq j$ and $q_{ij} = q_{ji} = 0$, then

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

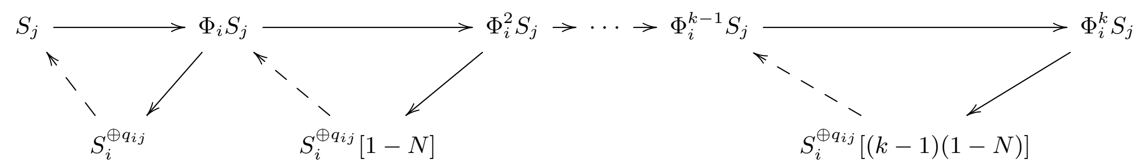
$\Phi_i^k S_j = S_j$ and hence $P_t(\Phi_i^k S_j) = 1$. Consider the case $q_{ij} > 0$. Since

$$\mathrm{Hom}^\bullet(S_i, S_j) \otimes S_i = \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}(S_i[m], S_j) \otimes S_i[m] \cong S_i^{\oplus q_{ij}}[-1],$$

we have an exact triangle

$$S_j \to \Phi_i S_j \to S_i^{\oplus q_{ij}} \to S_j[1].$$

Applying the spherical twist Φ_i for the above exact triangle repeatedly, we obtain a sequence of exact triangles



This implies the result in the case $q_{ij} > 0$ and a similar argument gives the result in the case $q_{ji} > 0$. ∎

PROPOSITION 4.5. *Let Q be a connected acyclic quiver and assume that Q is not a quiver with one vertex and no arrows. Then the entropy of spherical twists $\Phi_1, \dots, \Phi_n$ is given by*

$$h_t(\Phi_i) = \begin{cases} 0 & \text{if } t \geqslant 0 \\ (1-N)t & \text{if } t < 0. \end{cases}$$

Proof. We use the generator $G = \bigoplus_{j=1}^n S_j$. Then $P_t(\Phi_i^k G) = \sum_{j=1}^n P_t(\Phi_i^k S_j)$. Recall from Proposition 4.4 that

$$P_t(\Phi_i^k S_j) = 1 + q_{ij} \sum_{l=0}^{k-1} e^{l(1-N)t} = 1 + q_{ij} \frac{1-e^{k(1-N)t}}{1-e^{(1-N)t}}$$

in the case $q_{ij} > 0$ and

$$P_t(\Phi_i^k S_j) = 1 + q_{ji} e^{(2-N)t} \sum_{l=0}^{k-1} e^{l(1-N)t} = 1 + q_{ji} e^{(2-N)t} \frac{1-e^{k(1-N)t}}{1-e^{(1-N)t}}$$

in the case $q_{ji} > 0$. First, we consider the case $t > 0$. Then the above two terms converge to some positive real numbers as $k \to \infty$ since $(1-N)t < 0$. By the assumption on Q, the sum $\sum_{j=1}^n P_t(\Phi_i^k S_j)$ contains at least one of the above two. As a result, $\sum_{j=1}^n P_t(\Phi_i^k S_j)$ also converges to some positive real number as $k \to \infty$. Thus

$$h_t(\Phi_i) = \lim_{k\to\infty} \frac{1}{k} \log P_t(\Phi_i^k G) = 0$$

when $t > 0$. Next, consider the case $t < 0$. Similarly, we can show that $e^{-k(1-N)t} \sum_{j=1}^n P_t(\Phi_i^k S_j)$ converges to some positive real number as $k \to \infty$ since $-(1-N)t < 0$. Thus

$$\begin{aligned} h_t(\Phi_i) &= \lim_{k\to\infty} \frac{1}{k} \log P_t(\Phi_i^k G) = \lim_{k\to\infty} \frac{1}{k} \log e^{k(1-N)t} e^{-k(1-N)t} P_t(\Phi_i^k G) \\ &= (1-N)t + \lim_{k\to\infty} \frac{1}{k} \log e^{-k(1-N)t} P_t(\Phi_i^k G) = (1-N)t \end{aligned}$$

when $t < 0$. Finally, we can easily check that $h_t(\Phi_i) = 0$ when $t = 0$. ∎

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press

REMARK 4.6. The subgroup of autoequivalences generated by spherical twists

$$\mathrm{Sph}(\mathcal{D}_Q^N) := \langle \Phi_1, \ldots, \Phi_n \rangle \subset \mathrm{Aut}(\mathcal{D}_Q^N)$$

is called the Seidel–Thomas braid group. Here we only computed the entropy of generators $\Phi_1, \ldots, \Phi_n$. It is an important problem to compute the entropy $h_t(\Phi)$ for a general element $\Phi \in \mathrm{Sph}(\mathcal{D}_Q^N)$.

4.3 Lower bound of the entropy on the derived categories of surfaces

Let X be a smooth projective variety over $\mathbb{C}$ and denote by $\mathrm{D}^b(X)$ the bounded derived category of coherent sheaves on X. Define the Euler form $\chi\colon K(\mathrm{D}^b(X)) \times K(\mathrm{D}^b(X)) \to \mathbb{Z}$ by

$$\chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{D}^b(X)}(E, F[i]).$$

The *numerical Grothendieck group* $N(X)$ is the quotient of $K(\mathrm{D}^b(X))$ by the radical of the Euler form χ. Let $\mathrm{End}^{FM}(\mathrm{D}^b(X))$ be the semigroup consisting of Fourier–Mukai type endofunctors. Since these endofunctors preserve the radical of χ, they induce linear maps on $N(X)$, that is, the semigroup homomorphism

$$\mathrm{End}^{FM}(\mathrm{D}^b(X)) \to \mathrm{End}(N(X)), \quad F \mapsto [F]$$

is well defined (see [18, Section 5.1]). A stability condition $\sigma = (Z, \mathcal{P})$ is called *numerical* if $Z\colon K(\mathrm{D}^b(X)) \to \mathbb{C}$ factors through the numerical Grothendieck group $N(X)$.

In [3, 9], a numerical stability condition on $\mathrm{D}^b(X)$ was constructed when $\dim_{\mathbb{C}} X = 2$. Applying Theorem 3.5 and Proposition 3.11, we obtain the following lower bound of the entropy.

PROPOSITION 4.7. *Let X be a smooth projective surface over $\mathbb{C}$ and $F\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ be a Fourier–Mukai type endofunctor. Then*

$$\log \rho([F]) \leqslant h_0(F),$$

where $\rho([F])$ is the spectral radius of the induced linear map $[F]\colon N(X) \to N(X)$ and $h_0(F)$ is the entropy of F at $t = 0$.

REMARK 4.8. Let X be a smooth projective variety over $\mathbb{C}$. In [18], they conjectured that the equality $\log \rho([F]) = h_0(F)$ holds for any autoequivalence $F \in \mathrm{Aut}(\mathrm{D}^b(X))$. This conjecture was shown for a curve in [17] and for a variety with ample canonical bundle or ample anticanonical bundle in [18].

Acknowledgments. The author would like to thank Kohei Kikuta, Genki Ouchi, and Atsushi Takahashi for valuable discussions and comments.

REFERENCES

[1] R. Adler, A. Konheim and M. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.
[2] C. Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*, Ann. Inst. Fourier (Grenoble) **59**(6) (2009), 2525–2590.
[3] D. Arcara and A. Bertram, *Bridgeland-stable moduli spaces for K-trivial surfaces*, J. Eur. Math. Soc. (JEMS) **15**(1) (2013), 1–38; With an appendix by Max Lieblich.

[4] I. Assem, D. Simson and A. Skowronski, "*Elements of the representation theory of associative algebras*", in *Techniques of Representation Theory, Vol. 1*, London Mathematical Society Student Texts **65**, Cambridge University Press, Cambridge, 2006.
[5] A. Bayer, *A short proof of the deformation property of Bridgeland stability conditions*, preprint, 2016, arXiv:1606.02169.
[6] A. A. Beĭlinson, J. Bernstein and P. Deligne, "*Faisceaux pervers*", in *Analysis and Topology on Singular Spaces, I (Luminy, 1981)*, Astérisque **100**, Soc. Math. France, Paris, 1982, 5–171.
[7] A. Bondal, *Representations of associative algebras and coherent sheaves*, Izv. Akad. Nauk SSSR Ser. Mat. **53**(1) (1989), 25–41; translation in Math. USSR-Izv., **34**(1) (1990), 23–42.
[8] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. (2) **166**(2) (2007), 317–345.
[9] T. Bridgeland, *Stability conditions on K3 surfaces*, Duke Math. J. **141**(2) (2008), 241–291.
[10] T. Bridgeland and I. Smith, *Quadratic differentials as stability conditions*, Publ. Math. Inst. Hautes Études Sci. **121** (2015), 155–278.
[11] G. Dimitrov, F. Haiden, L. Katzarkov and M. Kontsevich, *Dynamical systems and categories*, Contemp. Math. **621** (2014), 133–170.
[12] D. Gaiotto, G. Moore and A. Neitzke., *Wall-crossing, Hitchin Systems, and the WKB Approximation*, Adv. Math. **234** (2013), 239–403.
[13] V. G. Ginzburg, *Calabi-Yau algebras*, preprint, 2006, arXiv:math/0612139.
[14] F. Haiden, L. Katzarkov and M. Kontsevich, *Flat surfaces and stability structures*, Publ. Math. Inst. Hautes Études Sci. **126** (2017), 247–318.
[15] A. Ikeda, *Stability conditions on* CY_N *categories associated to* A_n*-quivers and period maps*, Math. Ann. **367** (2017), 1–49.
[16] B. Keller, *Deformed Calabi–Yau completions*, J. Reine Angew. Math. **654** (2011), 125–180; With an appendix by Michel Van den Bergh.
[17] K. Kikuta, *On entropy for autoequivalences of the derived category of curves*, Adv. Math. **308** (2017), 699–712.
[18] K. Kikuta and A. Takahashi, *On the categorical entropy and topological entropy*, Int. Math. Res. Not. IMRN **2019**(2) (2019), 457–469.
[19] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*, preprint, 2008, arXiv:0811.2435.
[20] P. Seidel and R. P. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108**(1) (2001), 37–108.
[21] M. Shub, *Dynamical systems, filtrations and entropy*, Bull. Am. Math. Soc. **80** (1974), 27–41.
[22] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. **57** (1987), 285–300.

Department of Mathematics,
Graduate School of Science,
Osaka University,
Toyonaka, Osaka 560-0043, Japan
ikeda@math.sci.osaka-u.ac.jp

https://doi.org/10.1017/nmj.2020.9 Published online by Cambridge University Press