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Minimal rational curves on generalized Bott–Samelson varieties

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To the memory of C. S. Seshadri

Abstract

We investigate families of minimal rational curves on Schubert varieties, their Bott–Samelson desingularizations, and their generalizations constructed by Nicolas Perrin in the minuscule case. In particular, we describe the minimal families on small resolutions of minuscule Schubert varieties.

1. Introduction

Lines in flag varieties have been extensively investigated. In particular, for a homogeneous space $X = G/P$ where G is a semi-simple algebraic group and P a maximal parabolic subgroup, the lines in X passing through the base point x form a smooth projective variety $\mathcal{L}_x$ on which P acts with one or two orbits. When P is associated to a long simple root, $\mathcal{L}_x$ is the P-orbit of the Schubert line in X; moreover, the homogeneous space $\mathcal{L}_x$ is minuscule (see [CC98, Str02, LM03] for these results).

The variety $\mathcal{L}_x$ features prominently in work of Hwang and Mok establishing rigidity properties of X (see e.g. [HM02]). Its analogue for a smooth Schubert variety Y of X is an important ingredient in the study of the deformations of Y within X, by Hong *et al.* (see [HM13, Hon15, HK19]). But little seems to be known about lines in possibly singular Schubert varieties. The latter admit natural resolutions of singularities, the Bott–Samelson varieties and their generalizations introduced by Sankaran and Vanchinathan (see [SV94, SV95]), and by Perrin in [Per07]. For these generalized Bott–Samelson resolutions, the notion of lines (which depends on a projective embedding) may be replaced with the intrinsic notion of minimal rational curves. In loose terms, a family of rational curves on a projective variety X is minimal if the subfamily of curves through a general point $x \in X$ is non-empty and proper. The minimal families and the associated varieties of minimal rational tangents (consisting, in loose terms again, of the tangent directions at x of the curves through that point) play an important role in the geometry of X, see e.g. [Hwa14].

In this paper, we make the first steps in the investigation of lines in Schubert varieties, and minimal families in their generalized Bott–Samelson resolutions. Given a Schubert variety Y in $X = G/P$ with P maximal, it is easy to show that Y is covered by translates of the Schubert line (see Lemma 3.6 for details). If in addition P is associated to a long simple root, then

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the Chow variety of lines through a general point $y \in Y$ is a union of Schubert varieties in $\mathcal{L}_y$ (Proposition 3.7); it may be reducible (Example 3.9).

For Bott–Samelson resolutions, the minimal families turn out to be very restricted: there are only finitely many minimal rational curves through a general point (see Theorem 4.3 for details). This may be explained by the fact that Bott–Samelson resolutions are big (i.e. some fibers have a big dimension) and not canonical (indeed, the action of the connected automorphism group of the Schubert variety need not lift to the resolution; see [CKP15, § 7] for explicit examples).

By contrast, generalized Bott–Samelson resolutions include the small resolutions of minuscule Schubert varieties constructed by Zelevinsky (see [Zel83]), Sankaran and Vachinathan, and Perrin in full generality (see [Per07, Corollary 7.9]). These resolutions are obtained as towers of locally trivial fibrations, with fibers being minuscule homogeneous spaces. We describe their minimal families in terms of lines in these homogeneous spaces (Theorem 4.10). This relies on a structure result for minimal families on generalized Bott–Samelson resolutions (Proposition 4.7), and on two combinatorial properties of these resolutions (Propositions 5.7 and 5.13). Both properties were first proved in the companion article [BK19] via case-by-case arguments using reduced decompositions in Weyl groups. Then Perrin came up with uniform proofs based on the combinatorics of minuscule quivers developed in his papers [Per05, Per07, Per09]. Subsequently, we obtained somewhat shorter uniform proofs, which are presented here.

Our results are obtained over an algebraically closed field of arbitrary characteristic, whereas the setting of most earlier works is complex geometry. In particular, the theory of minimal rational curves seems to have been exclusively developed over $\mathbb{C}$ so far. Thus, we only rely on foundational material from [Kol99, Chapter II] (see also [Deb01, Chapter II]). The facts that we need are gathered in §§ 2.1 and 2.2.

Section 2.3 contains auxiliary results on almost homogeneous varieties, i.e. those on which an algebraic group acts with an open dense orbit; this class includes Schubert varieties, their (generalized) Bott–Samelson resolutions, and some naturally associated varieties. In this special setting, we obtain analogues of important general results on the existence and properties of free rational curves, which hold over $\mathbb{C}$ but generally fail in positive characteristics (see Remark 2.2). For this, we develop methods from [BF15, § 2].

In §§ 3.1 and 3.2, we set up notation and recall basic facts on flag varieties and their Schubert varieties. The minimal rational curves on the former are described in § 3.3, whereas § 3.4 explores the families of lines on the latter. An essential role is played by the curves which are stable by a maximal torus T of G. Our results are most complete for minuscule homogeneous spaces; these may be characterized by the condition that every T-stable curve is a line.

The minimal rational curves on Bott–Samelson desingularizations are considered in § 4.1. In § 4.2, we survey the construction of their generalizations, due to Perrin in [Per07, § 5]. The structure of their minimal families is investigated in § 4.3; again, T-stable curves form a key ingredient in all these developments. We illustrate our results on the simplest example of a singular Schubert variety: a quadratic cone of dimension 3 (Examples 3.9 and 4.12). Further examples, also involving exceptional groups, can be found in [BK19].

The final § 5 is devoted to combinatorial properties of generalized Bott–Samelson desingularizations. In § 5.2, we obtain an inequality involving certain positive roots; an equality of Weyl groups of isotropy groups is proved in § 5.3.

Our approach raises many open questions: for example, to parameterize the families of lines in a Schubert variety in combinatorial terms, and to characterize those families that contain lines consisting of smooth points. Also, it would be interesting to extend our results to the

setting of cominuscule homogeneous varieties, or to the horizontal Schubert varieties introduced in [KR17].

1.1 Notation and conventions

The ground field k is algebraically closed, of characteristic $p \geq 0$. By a *scheme*, we mean a separated k-scheme S; *points* of S are k-rational points unless otherwise stated. A *variety* is an integral scheme of finite type over k. A *curve* is a variety of dimension 1.

An *algebraic group* is a group scheme of finite type. Given an algebraic group G, a subgroup scheme H and a scheme Y equipped with an action of H, we denote by $G \times^H Y$ the quotient of $G \times Y$ by the H-action via $h \cdot (g, y) := (gh^{-1}, hy)$, if this quotient exists as a scheme. We then have a cartesian square

$$\begin{array}{ccc} G \times Y & \xrightarrow{\mathrm{pr}_1} & G \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ G \times^H Y & \xrightarrow{f} & G/H \end{array}$$

where pr_1 denotes the first projection and q, r are principal H-bundles. As a consequence, f is faithfully flat. Moreover, the G-action on $G \times Y$ via left multiplication on G descends to a unique action on $G \times^H Y$, and f is G-equivariant. We may view f as a homogeneous fibration with fiber Y.

By [MFK93, Proposition 7.1], the associated fiber bundle $G \times^H Y$ exists if Y admits an ample H-linearized line bundle. We will freely use the following observation: if X is a scheme equipped with an action of G and an equivariant morphism $\pi : X \to G/H$ with fiber Y at the base point of G/H, then there is a unique G-equivariant isomorphism $X \simeq G \times^H Y$ identifying π with f.

2. Rational curves on almost homogeneous varieties

2.1 Spaces of rational curves

Let X be a projective variety. The scheme of morphisms $\mathrm{Hom}(\mathbb{P}^1, X)$ is equipped with an action of $\mathrm{Aut}(\mathbb{P}^1)$ that stabilizes the open subscheme $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ consisting of morphisms which are birational to their image. Moreover, this action lifts uniquely to an action on the normalization

$$\eta : \mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X) \longrightarrow \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$$

By [Kol99, Theorem II.2.15], there is a natural commutative diagram of normal schemes

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X) & \xrightarrow{\lambda} & \mathrm{Univ}(X) \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X) & \xrightarrow{\kappa} & \mathrm{RatCurves}(X) \end{array} \tag{1}$$

where the horizontal arrows are principal $\mathrm{Aut}(\mathbb{P}^1)$-bundles. As a consequence, the above diagram is cartesian; moreover, ρ is a $\mathbb{P}^1$-bundle.

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In view of [Kol99, Definition II.2.11], there is another natural commutative diagram

$$\begin{array}{ccc} \mathrm{Univ}(X) & \xrightarrow{\delta} & \mathrm{Chow}(X) \times X \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathrm{RatCurves}(X) & \xrightarrow{\gamma} & \mathrm{Chow}(X) \end{array} \tag{2}$$

where $\mathrm{Chow}(X)$ denotes the Chow scheme. Moreover, γ is finite over its image, which is the locally closed subscheme of $\mathrm{Chow}(X)$ parameterizing irreducible, geometrically rational 1-cycles; also, δ is finite over its image, which is the universal Chow family over the above subscheme. By composing δ with the second projection, we obtain a morphism

$$\mu : \mathrm{Univ}(X) \longrightarrow X \tag{3}$$

such that the morphism $\rho \times \mu : \mathrm{Univ}(X) \to \mathrm{RatCurves}(X) \times X$ is finite.

The morphism (3) can be constructed alternatively as follows: composing the evaluation map

$$\mathbb{P}^1 \times \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X) \longrightarrow X, \quad (t, f) \longmapsto f(t) \tag{4}$$

with the map $\mathrm{id} \times \eta : \mathbb{P}^1 \times \mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X) \longrightarrow \mathbb{P}^1 \times \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$, we obtain a morphism

$$\varepsilon : \mathbb{P}^1 \times \mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X) \longrightarrow X.$$

One may check that ε is the composition of the quotient map

$$\lambda : \mathbb{P}^1 \times \mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X) \to \mathrm{Univ}(X)$$

with μ. In particular, for any $x \in X$, we have a principal $\mathrm{Aut}(\mathbb{P}^1)$-bundle $\varepsilon^{-1}(x) \to \mu^{-1}(x)$ between (scheme-theoretic) fibers. The first projection $\mathrm{pr}_1 : \varepsilon^{-1}(x) \to \mathbb{P}^1$ is $\mathrm{Aut}(\mathbb{P}^1)$-equivariant, and $\mathbb{P}^1$ may be identified with the homogeneous space $\mathrm{Aut}(\mathbb{P}^1)/\mathrm{Aut}(\mathbb{P}^1, 0)$. This identifies $\varepsilon^{-1}(x)$ with the associated fiber bundle $\mathrm{Aut}(\mathbb{P}^1) \times^{\mathrm{Aut}(\mathbb{P}^1,0)} \mathrm{pr}_1^{-1}(0)$. Also, note that

$$\mathrm{pr}_1^{-1}(0) \simeq \eta^{-1}(\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X; 0 \mapsto x))$$

equivariantly for $\mathrm{Aut}(\mathbb{P}^1, 0)$, where $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$ is the closed subscheme of $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ consisting of those morphisms f such that $f(0) = x$. Putting all of this together, we obtain a principal $\mathrm{Aut}(\mathbb{P}^1, 0)$-bundle

$$\eta^{-1}(\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X; 0 \mapsto x)) \to \mu^{-1}(x). \tag{5}$$

We may view $\mu^{-1}(x)$ as the space of rational curves on X through x.

Another space of rational curves on X through x is constructed in [Kol99, Theorem II.2.16]. More specifically, there is a natural commutative diagram of normal schemes

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & \longrightarrow & \mathrm{Univ}(x, X) \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow \\ \mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & \longrightarrow & \mathrm{RatCurves}(x, X) \end{array}$$

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where the horizontal arrows are principal $\mathrm{Aut}(\mathbb{P}^1, 0)$-bundles and the vertical arrows are $\mathbb{P}^1$-bundles. (Here $\mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$ denotes the normalization of $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$.) In general, the relation between $\mu^{-1}(x)$ and $\mathrm{RatCurves}(x, X)$ is not clear to us. But we will see that both share a common smooth open subscheme, consisting of the free curves through x; these may be defined as follows.

Let $f \in \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$, and C its image. We say that f is *free* if C is contained in the smooth locus X_{sm} and the vector bundle $f^*(T_{X_{\mathrm{sm}}})$ is generated by its global sections, where $T_{X_{\mathrm{sm}}}$ denotes the tangent bundle of X_{sm}.

Every free morphism f satisfies $H^1(\mathbb{P}^1, f^*(T_{X_{\mathrm{sm}}})) = 0$. Thus, the above notion of freeness coincides with that of [Kol99, Definition II.3.1] when X is smooth. By [Kol99, II.1, II.3], the free morphisms form a smooth open subscheme $\mathrm{Hom}_{\mathrm{fr}}(\mathbb{P}^1, X)$ of $\mathrm{Hom}(\mathbb{P}^1, X)$; its dimension at the point C is $-K_{X_{\mathrm{sm}}} \cdot C + \dim(X)$, where $K_{X_{\mathrm{sm}}}$ stands for the canonical class of the smooth locus. We denote by $\mathrm{RatCurves}_{\mathrm{fr}}(X)$ the corresponding smooth open subscheme of $\mathrm{RatCurves}(X)$.

We also have $H^1(\mathbb{P}^1, f^*(T_{X_{\mathrm{sm}}})(-1)) = 0$ when f is free. As a consequence, the free morphisms that send 0 to x form a smooth open subscheme $\mathrm{Hom}_{\mathrm{fr}}(\mathbb{P}^1, X; 0 \to x)$ of $\mathrm{Hom}(\mathbb{P}^1, X; 0 \to x)$, with dimension at C being $-K_{X_{\mathrm{sm}}} \cdot C$ (see [Kol99, Theorem II.1.7 and Proposition II.3.2] for these results).

Thus, any free morphism f yields a smooth point C of $\mathrm{RatCurves}(X)$. Since the evaluation map (4) is smooth along $\mathbb{P}^1 \times f$ (see [Kol99, Corollary II.3.5.4]), $\mu^{-1}(x)$ is smooth at C as well. As a consequence, $\mu^{-1}(x)$ and $\mathrm{RatCurves}(x, X)$ share a common smooth open subscheme $\mathrm{RatCurves}_{\mathrm{fr}}(x, X)$, the quotient of $\mathrm{Hom}_{\mathrm{fr}}(\mathbb{P}^1, X; 0 \to x)$ by $\mathrm{Aut}(\mathbb{P}^1, 0)$. The dimension of $\mathrm{RatCurves}_{\mathrm{fr}}(x, X)$ at C equals $-K_{X_{\mathrm{sm}}} \cdot C - 2$.

Consider again $f \in \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ with image C. We say that C is *embedded* if f is an immersion into X_{sm}. This is an open property in view of [Kol99, Lemma I.1.10.1], and hence the embedded free curves form an open subscheme $\mathrm{RatCurves}_{\mathrm{emfr}}(X)$ of $\mathrm{RatCurves}(X)$.

Lemma 2.1. *For any $x \in X_{\mathrm{sm}}$, the natural map $\rho_x : \mu^{-1}(x) \to \mathrm{RatCurves}(X)$ restricts to an immersion on the smooth open subscheme consisting of embedded free curves.*

Proof. By the above discussion, we may view ρ_x as the natural map

$$\pi : \mathrm{Hom}_{\mathrm{emfr}}(\mathbb{P}^1, X; 0 \to x)/\mathrm{Aut}(\mathbb{P}^1, 0) \longrightarrow \mathrm{Hom}_{\mathrm{emfr}}(\mathbb{P}^1, X)/\mathrm{Aut}(\mathbb{P}^1)$$

with an obvious notation.

We check that π is injective on k-rational points. For any f_1, f_2 in $\mathrm{Hom}_{\mathrm{emfr}}(\mathbb{P}^1, X; 0 \to x)$ such that $f_2 = f_1 \circ \varphi$ for some $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$, we have $x = f_2(0) = f_1(\varphi(0)) = f_1(0)$, and hence $\varphi(0) = 0$ as desired.

Next, we check that the differential of π at any k-rational point $f \in \mathrm{Hom}_{\mathrm{emfr}}(\mathbb{P}^1, X; 0 \to x)$ is injective. We have a commutative diagram (with an obvious notation again)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{LieAut}(\mathbb{P}^1, 0) & \longrightarrow & \mathrm{LieAut}(\mathbb{P}^1) & \longrightarrow & T_0\mathbb{P}^1 & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} & & \\
0 & \longrightarrow & H^0(\mathbb{P}^1, T_{\mathbb{P}^1}(-1)) & \longrightarrow & H^0(\mathbb{P}^1, T_{\mathbb{P}^1}) & \longrightarrow & T_0\mathbb{P}^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle df_0} & & \\
0 & \longrightarrow & H^0(\mathbb{P}^1, f^*(T_{X_{\mathrm{sm}}})(-1)) & \longrightarrow & H^0(\mathbb{P}^1, f^*(T_{X_{\mathrm{sm}}})) & \longrightarrow & T_xX & \longrightarrow & 0
\end{array}$$

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where df_0 is injective. So the induced map

$$H^0(\mathbb{P}^1, f^*(T_{X_{\mathrm{sm}}})(-1)))/\mathrm{LieAut}(\mathbb{P}^1, 0) \longrightarrow H^0(\mathbb{P}^1, f^*(T_{X_{\mathrm{sm}}}))/\mathrm{LieAut}(\mathbb{P}^1)$$

is injective as well, as desired. □

2.2 Families of rational curves

The normal scheme $\mathrm{RatCurves}(X)$ is the disjoint union of open and closed normal quasi-projective varieties. These (connected or irreducible) components are called the *families of rational curves on* X. Every such family $\mathcal{K}$ comes with a universal family $\mathcal{U} := \rho^{-1}(\mathcal{K}) \to \mathcal{K}$, where $\mathcal{U}$ is a component of $\mathrm{Univ}(X)$. For any $x \in X$, we denote the fiber of $\mu : \mathcal{U} \to X$ by $\mathcal{U}_x$, and let $\mathcal{K}_x := \rho(\mathcal{U}_x)$; then the induced morphism $\rho_x : \mathcal{U}_x \to \mathcal{K}_x$ is finite. The family $\mathcal{K}$ is *covering* if μ is dominant, i.e. $\mathcal{K}_x$ (or equivalently $\mathcal{U}_x$) is non-empty for a general point x. If in addition $\mathcal{K}_x$ (or equivalently $\mathcal{U}_x$) is projective for x general, then $\mathcal{K}$ is called a *family of minimal rational curves*, or just a *minimal family* for simplicity. Examples of minimal families include the covering families of lines in some projective embedding of X; also, note that lines contained in the smooth locus yield examples of embedded curves.

For any family of rational curves $\mathcal{K}$ as above, there exists a unique irreducible component Z of $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$ together with a principal $\mathrm{Aut}(\mathbb{P}^1)$-bundle $Z^{\mathrm{n}} \to \mathcal{K}$, where Z^{n} denotes the normalization. Let Z_{fr} denote the smooth open subscheme of Z consisting of free morphisms; then Z_{fr} is stable under $\mathrm{Aut}(\mathbb{P}^1)$, and hence we obtain a principal $\mathrm{Aut}(\mathbb{P}^1)$-bundle $Z_{\mathrm{fr}} \to \mathcal{K}_{\mathrm{fr}}$, where $\mathcal{K}_{\mathrm{fr}} \subset \mathcal{K}$ denotes the smooth open subscheme of free curves. We may view the points of $\mathcal{K}_{\mathrm{fr}}$ as (possibly singular) rational curves in X. For any such curve C, the fiber of ρ at C is a projective line, and the restriction of μ to this fiber yields the normalization map of C. Also, given $x \in X_{\mathrm{sm}}$, the morphism $\rho_x : \mathcal{U}_x \to \mathcal{K}_x$ restricts to an isomorphism on the open subscheme $\mathcal{K}_{\mathrm{emfr},x}$ consisting of embedded free curves (Lemma 2.1). Moreover, by assigning to each such curve its tangent direction at x, we obtain a morphism

$$\tau = \tau_{X,x} : \mathcal{K}_{\mathrm{emfr},x} \longrightarrow \mathbb{P}(T_x X), \tag{6}$$

where $\mathbb{P}(T_x X)$ denotes the projectivization of the tangent space. If $\mathrm{char}(k) = 0$ and the point x is general, then τ extends to a finite morphism from the normalization of $\mathcal{K}_x$ (see [Keb02, Theorem 3.4]).

Remark 2.2. Assume that X is smooth. If $\mathrm{char}(k) = 0$ then every covering family contains a free curve, as follows from [Kol99, Proposition II.3.10] together with generic smoothness. But this fails whenever $\mathrm{char}(k) = p > 0$, as shown by the following example adapted from [Kol99, Example V.1.4.3]. Consider the hypersurface X in $\mathbb{P}^2 \times \mathbb{P}^2$ with homogeneous equation

$$x_0 y_0^p + x_1 y_1^p + x_2 y_2^p = 0,$$

where x_0, x_1, x_2 (respectively y_0, y_1, y_2) are homogeneous coordinates on the first (respectively second) copy of $\mathbb{P}^2$. Then X is smooth, the geometric fibers of the first projection $\mathrm{pr}_1 : X \to \mathbb{P}^2$ are all non-reduced, and their reduced subschemes are lines. For the corresponding family of rational curves, $\mathcal{U}$ is the hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ with equation $x_0 y_0 + x_1 y_1 + x_2 y_2 = 0$, and $\mathcal{K} = \mathbb{P}^2$; in particular, $\mathcal{K}$ is minimal. Also, the morphism ρ is the first projection, and

$$\mu\big([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]\big) := \big([x_0^p : x_1^p : x_2^p], [y_0 : y_1 : y_2]\big).$$

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In particular, all the geometric fibers of $\mu : \mathcal{U} \to \mathcal{K}$ are fat points of multiplicity p, and hence $\mathcal{K}$ contains no free curve. Note that X is homogeneous under an appropriate action of $\mathrm{Aut}(\mathbb{P}^2)$, and the stabilizer of any $x \in X$ is not reduced (or equivalently, not smooth). So X is a variety of unseparated flags in the sense of [HL93]; one can show that any such variety admits a minimal family which contains no free curve.

We now discuss covariance properties under a morphism $\pi : X \to Y$, where Y is a projective variety. Let $\mathcal{K}$ be a family of rational curves on X, and Z the corresponding irreducible component of $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$. Assume that there exists $f \in Z$ such that the composition $\pi \circ f : \mathbb{P}^1 \to Y$ is free. Then $\pi \circ f$ is a smooth point of a unique irreducible component W of $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, Y)$, which defines in turn a family of rational curves $\mathcal{L}$ on Y. The composition of natural morphisms $Z^{\mathrm{n}} \to Z \to \mathrm{Hom}(\mathbb{P}^1, X) \to \mathrm{Hom}(\mathbb{P}^1, Y)$ is $\mathrm{Aut}(\mathbb{P}^1)$-equivariant and its image contains a smooth point of W. This yields a rational map

$$\pi_* : \mathcal{K} \dashrightarrow \mathcal{L},$$

which is defined on the open subset consisting of those free curves that are sent to free curves in Y.

In the opposite direction, given π and $\mathcal{K}$ as above, we say that π *contracts* some $C_0 \in \mathcal{K}$ if the composition $\rho^{-1}(C_0) \xrightarrow{\mu} X \xrightarrow{\pi} Y$ is constant. Then π contracts all $C \in \mathcal{K}$: indeed, choose an ample line bundle M on Y and let $L := \mu^*\pi^*(M)$. Then the degree of L on $\rho^{-1}(C)$ is independent of C (as follows e.g. from [Ful98, Proposition 10.3]), and this degree is 0 if and only if C is contracted by π.

Based on this observation, we now describe the minimal families in a product of varieties.

Lemma 2.3. *Let Y, Z be projective varieties, and $X := Y \times Z$ with projections $\pi : X \to Y$, $\varphi : X \to Z$.*

(i) *The pull-back morphism*

$$\pi^* : \mathrm{Hom}(\mathbb{P}^1, Y) \times Z \longrightarrow \mathrm{Hom}(\mathbb{P}^1, X), \quad (f, z) \longmapsto (t \mapsto (f(t), z))$$

induces a closed immersion $\mathrm{RatCurves}(Y) \times Z \to \mathrm{RatCurves}(X)$ *with image a union of components.*

(ii) *The morphism π^* sends covering (respectively minimal) families to covering (respectively minimal) families.*

(iii) *A family of rational curves $\mathcal{K}$ on X is the pull-back of a family on Y if and only if φ contracts some curve in $\mathcal{K}$.*

(iv) *Every family of minimal rational curves on X is the pull-back of a unique family of minimal rational curves on Y or Z.*

Proof. One may easily check that the 'constant' morphism

$$c : Z \longrightarrow \mathrm{Hom}(\mathbb{P}^1, Z), \quad z \longmapsto (t \mapsto z)$$

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is a closed immersion; moreover, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{P}^1, Y) \times Z & \longrightarrow & Z \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle c} \\ \mathrm{Hom}(\mathbb{P}^1, X) & \longrightarrow & \mathrm{Hom}(\mathbb{P}^1, Z) \end{array}$$

is cartesian, where the top horizontal arrow is the projection, and the bottom horizontal arrow is the composition with φ. Thus, π^* is a closed immersion as well. Also, π^* sends $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, Y) \times Z$ to $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$, and its image (considered with its reduced scheme structure) consists of those morphisms f such that $\deg \varphi^* f^*(M) = 0$ for a given ample line bundle M on Z. This readily yields the assertions (i), (ii) and (iii).

(iv) Let $\mathcal{K}$ be a minimal family on X, and $x \in X$ such that $\mathcal{U}_x$ is non-empty and projective. Choose $f \in \mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$ with image C such that C is also the image of a curve in $\mathcal{U}_x$. Write $x = (y, z)$ and $f = (g, h)$, where $g \in \mathrm{Hom}(\mathbb{P}^1, Y; 0 \mapsto y)$ and $h \in \mathrm{Hom}(\mathbb{P}^1, Z; 0 \mapsto z)$. We may view f as the composition

$$\mathbb{P}^1 \xrightarrow{\delta} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{g \times h} Y \times Z,$$

where δ denotes the diagonal embedding. If both g and h are non-constant, then $g \times h$ is finite. Since the image of δ degenerates in $\mathbb{P}^1 \times \mathbb{P}^1$ to $(\mathbb{P}^1 \times \{0\}) \cup (\{0\} \times \mathbb{P}^1)$, a reducible curve through $(0, 0)$, this yields a degeneration of C in X to a reducible curve through x. But this contradicts the minimality of $\mathcal{K}$. Thus, we may assume that g is constant; then (iii) implies that $\mathcal{K}$ is the pull-back of a minimal family on Y. □

2.3 Almost homogeneous varieties

We now assume that the projective variety X is equipped with an action of a connected algebraic group G. Then G acts on $\mathrm{Hom}(\mathbb{P}^1, X)$ via its action on X, which commutes with the action of $\mathrm{Aut}(\mathbb{P}^1)$ and stabilizes the open subscheme $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$. This yields actions of G on $\mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X)$, $\mathrm{RatCurves}(X)$, $\mathrm{Univ}(X)$ such that the diagram (1) is equivariant. Also, G acts on the Chow scheme and the diagram (2) is equivariant as well. Thus, so is the morphism (3). Since G is connected, every family of rational curves on X is stable by G.

Next, assume that there exists a point $x \in X$ such that the orbit $X^0 = G \cdot x$ is open in X, i.e. X is almost homogeneous under G. Then there exist covering families of rational curves on X, since G is a rational variety. We assume in addition that the (scheme-theoretic) stabilizer $H = G_x$ is smooth and connected.

Consider a family $\mathcal{K}$ of rational curves on X. Let $\mathcal{U}^0 := \mu^{-1}(X^0)$; this is an open G-stable subset of $\mathcal{U}$. Since $\rho : \mathcal{U} \to \mathcal{K}$ is flat, $\rho(\mathcal{U}^0) =: \mathcal{K}^0$ is a G-stable open subset of $\mathrm{RatCurves}(X)$; it consists of those curves in $\mathcal{K}$ that meet X^0. We also have a smaller G-stable open subset $\mathcal{K}(X^0)$, consisting of those curves that are contained in X^0. This yields a commutative diagram of G-varieties

$$\begin{array}{ccccc} \mathcal{U}(X^0) & \longrightarrow & \mathcal{U}^0 & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \rho} \\ \mathcal{K}(X^0) & \longrightarrow & \mathcal{K}^0 & \longrightarrow & \mathcal{K} \end{array} \tag{7}$$

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where the horizontal arrows are open immersions, the left and right vertical arrows are $\mathbb{P}^1$-bundles, and the middle vertical arrow is smooth.

Lemma 2.4. *With the preceding notation and assumptions, $\mathcal{K}$ is covering if and only if $\mathcal{U}_x$ is non-empty; equivalently, $\mathcal{K}_x$ is non-empty. Under these assumptions, $\mathcal{U}_x$ is a normal variety and $\mathcal{K}_x$ is a variety.*

Proof. The morphism μ restricts to a G-equivariant morphism $\mu^0 : \mathcal{U}^0 \longrightarrow X^0$ with fiber at x being $\mathcal{U}_x$. By identifying X^0 with the homogeneous space G/H, this yields a G-equivariant isomorphism $\mathcal{U}^0 \simeq G \times^H \mathcal{U}_x$, and in turn a cartesian square

$$\begin{array}{ccc} G \times \mathcal{U}_x & \longrightarrow & G \\ \downarrow & & \downarrow \\ \mathcal{U}^0 & \longrightarrow & G/H \end{array}$$

where the vertical arrows are principal H-bundles. Since $\mathcal{U}^0$ is a normal variety and H is smooth and connected, it follows that $\mathcal{U}_x$ is a normal variety as well. Moreover, $\mathcal{U}_x$ is non-empty if and only if so is $\mathcal{U}^0$, or equivalently $\mathcal{K}^0$; this completes the proof. □

Lemma 2.5. *Let $\mathcal{K}$ be a family of rational curves on X, containing a curve C which consists of smooth points and meets X^0.*

(i) *The curve C is free.*
(ii) *The family $\mathcal{K}$ is covering and $\mathcal{U}_x$ is a normal variety.*
(iii) *The variety $\mathcal{U}_x$ is smooth at C, of dimension $-K_{X_{\mathrm{sm}}} \cdot C - 2$.*
(iv) *The variety $\mathcal{U}_x$ is isomorphic to a component of* $\mathrm{RatCurves}(x, X)$.

Proof. (i) The smooth locus X_{sm} is G-stable and contains the open orbit $X^0 \simeq G/H$. Thus, the tangent bundle $T_{X_{\mathrm{sm}}}$ is equipped with a space of global sections: the image of the Lie algebra $\mathfrak{g}$ of G. Since H is smooth, this space generates T_{X^0}. Thus, the induced map $\mathcal{O}_{\mathbb{P}^1} \otimes \mathfrak{g} \to f^*(T_{X_{\mathrm{sm}}})$ is generically surjective, where $f : \mathbb{P}^1 \to C$ denotes the normalization. Using the fact that every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles, it follows easily that $f^*(T_{X_{\mathrm{sm}}})$ is globally generated.

(ii) This is a consequence of Lemma 2.4.

(iii) This follows from the properties of free morphisms recalled in § 2.1.

(iv) By (ii) and (5), there is a principal $\mathrm{Aut}(\mathbb{P}^1, 0)$-bundle $\eta^{-1}(Y) \to \mathcal{U}_x$ for some irreducible component Y of $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$; moreover, $\eta^{-1}(Y)$ is a normal variety. By the universal property of the normalization, this yields a finite morphism

$$\varphi : \eta^{-1}(Y) \to \mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X; 0 \mapsto x),$$

which restricts to an isomorphism on the open subset of free morphisms. Thus, φ yields an isomorphism to a component of $\mathrm{Hom}^{\mathrm{n}}_{\mathrm{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$. As φ is $\mathrm{Aut}(\mathbb{P}^1, 0)$-equivariant, it descends to the desired isomorphism. □

Next, we consider covariance properties of covering families, building on the observations after Remark 2.2. Let X be as above, and $\pi : X \to Y$ a surjective morphism, where Y is a

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projective variety. Assume that Y is equipped with a G-action such that π is equivariant (this assumption holds if $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ in view of Blanchard's lemma, see e.g. [BSU13, § 4.2]). Let $y := \pi(x)$ and $Y^0 := Gy$; then $Y^0 = \pi(X^0)$ is open in Y. Finally, let $\mathcal{K}$ be a covering family of rational curves on X.

Lemma 2.6. *Assume that there exists a curve $C \in \mathcal{K}^0$ such that $C \subset X_{\mathrm{sm}}$, $\pi|_C$ is birational onto its image D, and $D \subset Y_{\mathrm{sm}}$. Then $D \in \mathcal{L}$ for a unique covering family $\mathcal{L}$ of rational curves on Y. Moreover, π induces a G-equivariant rational map $\pi_* : \mathcal{K} \dashrightarrow \mathcal{L}$ which is defined at C and such that $\pi_*(C) = D$.*

Proof. The assumptions make sense, since every rational curve on X meeting X^0 and contained in X_{sm} is free (Lemma 2.5). The statement follows readily from the discussion at the end of § 2.2, with the exception of the equivariance of π_* which is easily checked. □

Remark 2.7. Under the assumptions of the above lemma, π_* restricts to an H-equivariant rational map $\pi_* = \pi_{*,x} : \mathcal{K}_x \dashrightarrow \mathcal{L}_y$. (Indeed, replacing C with the translate gC for some $g \in G$, we may assume that $x \in C$, and hence $y \in D$.) Moreover, we have a commutative diagram of rational maps

$$\begin{array}{ccc} \mathcal{K}_x & \dashrightarrow & \mathbb{P}(T_xX) \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle d\pi_x} \\ \mathcal{L}_y & \dashrightarrow & \mathbb{P}(T_yY) \end{array} \tag{8}$$

where the horizontal arrows arise from the tangent maps (6).

The assumptions of Lemma 2.6 hold if π is birational and $\mathcal{K}(X^0)$ is non-empty; then π_* restricts to an isomorphism $\mathcal{K}(X^0) \to \mathcal{L}(X^0)$, and hence is birational. The assumptions also hold when π is birational and X, Y are smooth; then π_* is an injective morphism.

Lemma 2.8. *Let $I := G_y \supset G_x = H$ and $F := \pi^{-1}(y)$; assume that I is smooth and connected. Then F is a projective variety equipped with an action of I and containing an open orbit $F^0 = Ix \simeq I/H$.*

If in addition π contracts some curve in $\mathcal{K}$, then $\mathcal{K}_x = \mathcal{L}_x$ for a unique covering family of rational curves $\mathcal{L}$ on F. Moreover, $\mathcal{K}^0 = G\mathcal{L}^0$, and $\mathcal{K}$ is minimal if and only if so is $\mathcal{L}$.

Proof. Note that $\pi^{-1}(Y^0)$ is a G-stable open subvariety of X; in particular, it contains the open orbit X^0. Moreover, π restricts to a G-equivariant morphism $\pi^{-1}(Y^0) \to Y^0 \simeq G/I$ with fiber F at y. This yields a G-equivariant isomorphism $\pi^{-1}(Y^0) \simeq G \times^I F$, and in turn the first assertion by arguing as in the proof of Lemma 2.4.

If π contracts some curve in $\mathcal{K}$, then it contracts all curves in this family, as seen at the end of § 2.2. Thus, every curve in $\mathcal{K}_x$ is a rational curve on F. The second assertion follows readily from this. □

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3. Minimal rational curves on flag varieties

3.1 Flag varieties

Let X be a projective variety, homogeneous under the action of a connected linear algebraic group G. Choose $x \in X$ and assume that the stabilizer G_x is smooth. Then G_x is a parabolic subgroup of G, and hence is connected. Moreover, replacing G with its largest semi-simple quotient and then with its simply-connected cover, we may and will assume that G is semi-simple and simply-connected. We identify X with the homogeneous space G/P, where $P := G_x$. The Lie algebras of $G, G_x, P, \ldots$ will be denoted by $\mathfrak{g}, \mathfrak{g}_x, \mathfrak{p}, \ldots$.

Choose a Borel subgroup $B \subset P$ and a maximal torus $T \subset B$. Let R denote the root system of (G,T), and R^+ the subset of roots of (B,T); then R^+ is a set of positive roots of R. Denote by R^- the corresponding set of negative roots, and by $S = \{\alpha_1, \ldots, \alpha_n\} \subset R^+$ the set of simple roots. The Weyl group $W = N_G(T)/T$ is generated by the associated simple reflections $s_1, \ldots, s_n$. For any $w \in W$, we denote by $\dot{w} \in N_G(T)$ a representative. Also, for any $\beta \in R$, we denote by $U_\beta \subset G$ the corresponding root subgroup. Let $G_\beta \subset G$ denote the subgroup generated by U_β and $U_{-\beta}$; then G_β is a closed semi-simple subgroup of rank 1, normalized by T. For any $w \in W$, the conjugation by $\dot{w}$ sends U_β to $U_{w(\beta)}$, and G_β to $G_{w(\beta)}$.

We also have the coroot system $R^\vee$ with simple roots $\alpha_1^\vee, \ldots, \alpha_n^\vee$; these form a basis of the cocharacter lattice $X_*(T)$. The dual basis of the character lattice $X^*(T)$ consists of the fundamental weights $\varpi_1, \ldots, \varpi_n$. More intrinsically, for any simple root α, we will denote by ϖ_α the fundamental weight with value 1 at $\alpha^\vee$, and 0 at all other simple coroots. Let $\rho := \varpi_1 + \cdots + \varpi_n$; then $\rho = (1/2)\sum_{\alpha \in R^+} \alpha$. The height of any $\beta \in R^\vee$ is $\mathrm{ht}(\beta) := \langle \rho, \beta \rangle$; this equals the sum of the coordinates of β in the basis of simple coroots.

Consider the Levi decomposition $P = R_u(P)L$, where L is a connected reductive subgroup of G containing T; then $B_L := B \cap L$ is a Borel subgroup of L. Denote by $R_L \subset R$ the root system of (L,T), with subset of positive roots $R_L^+ = R_L \cap R^+$ and subset of simple roots $I := R_L \cap S$. Then P is generated by B and the $\dot{s}_\alpha$, where $\alpha \in I$; we write $P = P_I$ and $L = L_I$.

The character group $X^*(P)$ is identified via restriction to the subgroup of $X^*(T)$ with basis the ϖ_α, where $\alpha \in S \setminus I$. Also, every $\lambda \in X^*(P)$ defines the associated line bundle $\mathcal{L}_{G/P}(\lambda)$ on G/P; moreover, $\mathcal{L}_{G/P}(\lambda)$ is ample if and only if λ has positive coordinates in the above basis. The assignment $\lambda \mapsto \mathcal{L}_{G/P}(\lambda)$ yields an isomorphism $X^*(P) \simeq \mathrm{Pic}(G/P)$. In particular, G/P has a smallest ample line bundle, namely $\mathcal{L}_{G/P}(\varpi)$, where $\varpi = \varpi_I := \sum_{\alpha \in S \setminus I} \varpi_\alpha$. Every ample line bundle on G/P is very ample, and hence defines a projective embedding $G/P \hookrightarrow \mathbb{P}(V(\lambda))$, where $V(\lambda) := H^0(G/P, \mathcal{L}_{G/P}(\lambda))^*$, and $\mathbb{P}(V)$ denotes the projective space of lines in a vector space V. This embedding is equivariant for the natural action of G on G/P, and its linear representation in $V(\lambda)$ (a highest weight module, see e.g. [Jan03, § II.2.13]). In particular, we have a 'smallest' projective embedding

$$G/P \hookrightarrow \mathbb{P}(V(\varpi)). \tag{9}$$

Also, recall that the canonical class of X satisfies

$$\mathcal{O}(K_X) \simeq \mathcal{L}_{G/P}(-2(\rho - \rho_I)). \tag{10}$$

The parabolic subgroup P is maximal if and only if I is the complement of a unique simple root α. We then write $P = P^\alpha$. More generally, we will use the notation $P^{S \setminus I}$ for P_I whenever this is convenient.

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3.2 Schubert varieties

We keep the notation and assumptions of the previous subsection. The Weyl group $W_L = N_L(T)/T$ is generated by the simple reflections s_α, where $\alpha \in I$; we also denote this group by W_I. Let W^I denote the subset of W consisting of those w such that $w(\alpha) \in R^+$ for all $\alpha \in I$; equivalently, $R_I^+ \subset w^{-1}(R^+)$. Then W^I is a set of representatives of the coset space W/W_I, consisting of the elements of minimal length in their right coset (for the length function ℓ on W relative to the generators $s_1, \ldots, s_n$). Note that $w \in W^I$ has length 1 if and only if $w = s_\alpha$ for some $\alpha \in S \setminus I$. On the other hand, the unique element of maximal length in W^I is $w_0 w_{0,I}$, where w_0 (respectively $w_{0,I}$) denotes the longest element of W (respectively W_I).

For any $w \in W$, the point $\dot{w}x \in G/P$ is independent of the choice of the representative $\dot{w}$; we thus denote this point by wx. Recall that the wx, where $w \in W^I$, are exactly the T-fixed points in G/P; moreover, G/P is the disjoint union of the B-orbits Bwx. The stabilizer B_{wx} is generated by T and the root subgroups U_β, where $\beta \in R^+ \cap w(R^+)$; in particular, B_{wx} is smooth and connected. The closure of Bwx in G/P is the Schubert variety $X(w)$; we have $\dim(X(w)) = \dim(Bwx) = \ell(w)$. In particular, the Schubert varieties of dimension 1 are exactly the $X(s_\alpha)$, where $\alpha \in S \setminus I$. Note that $X(s_\alpha) = P_\alpha x = L_\alpha x = G_\alpha x = \overline{U_{-\alpha} x}$ is a T-stable curve in G/P with fixed points $x, s_\alpha x$.

We now collect some basic properties of T-stable curves, which are essentially known (see [FW04, §§ 3 and 4]); we will provide proofs for completeness.

Lemma 3.1.

(i) *The T-stable curves in G/P through the T-fixed point wx are exactly the curves*

$$C_{w,\beta} := G_\beta wx = \overline{U_{-\beta} wx},$$

where $\beta \in w(R^+ \setminus R_I^+)$ (so that $X(s_\alpha) = C_{1,\alpha}$ for any $\alpha \in S \setminus I$).

(ii) *The T-fixed points in $C_{w,\beta}$ are exactly wx and $s_\beta wx$.*

(iii) *We have $C_{w,\beta} \simeq \mathbb{P}^1$.*

(iv) *For any $\lambda \in X^*(P)$, we have*

$$\mathcal{L}_{G/P}(\lambda) \cdot C_{w,\beta} = \langle \lambda, w^{-1}(\beta^\vee) \rangle.$$

(v) *We have*

$$-K_{G/P} \cdot C_{w,\beta} = 2\langle \rho - \rho_I, w^{-1}(\beta^\vee) \rangle = \mathrm{ht}(w^{-1}(\beta^\vee)) + \mathrm{ht}(w_{0,I} w^{-1}(\beta^\vee)).$$

In particular, $-K_{G/P} \cdot X(s_\alpha) = \mathrm{ht}(w_{0,I}(\alpha^\vee)) + 1$.

Proof. Using the action of $N_G(T)$ on G/P which yields a transitive action of W on T-fixed points, we may reduce to the case where $w = 1$.

By the Bruhat decomposition, we have a T-equivariant open immersion

$$\prod_{\beta \in R^+ \setminus R_I^+} U_{-\beta} \longrightarrow G/P, \quad (g_\beta) \longmapsto \left(\prod_\beta g_\beta \right) x,$$

where the product is taken in any order. Thus, x has an open T-stable neighborhood in G/P, isomorphic to an affine space on which T acts linearly with weights being the roots $-\beta$, where $\beta \in R^+ \setminus R_I^+$; moreover, each such weight has multiplicity 1. It follows that the T-stable curves

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in G/P through x are exactly the closures of the T-stable lines in this neighborhood, i.e. of the orbits $U_{-\beta}x$; moreover, the orbit maps $U_{-\beta} \to U_{-\beta}x$ are isomorphisms. Since x is fixed by $G_\beta \cap B$, a Borel subgroup of G_β, we have

$$C_{1,\beta} = \overline{U_{-\beta}x} = G_\beta x \simeq G_\beta/(B \cap G_\beta).$$

This implies (i), (ii) and (iii).

(iv) The restriction to $C_{1,\beta}$ of the G-linearized line bundle $\mathcal{L}_{G/P}(\lambda)$ is the G_β-linearized line bundle $\mathcal{L}_{G_\beta/(B\cap G_\beta)}(\lambda) =: \mathcal{L}$. Moreover, $T \cap G_\beta$ is a maximal torus of G_β, the image of the coroot $\beta^\vee : \mathbb{G}_m \to T$; the scheme-theoretic center of $G_\beta \simeq \mathrm{SL}(2)$ is the image of the 2-torsion subgroup scheme $\mu_2 \subset \mathbb{G}_m$ under $\beta^\vee$. Also, $\beta^\vee$ acts on the fiber of $\mathcal{L}$ at x (respectively $s_\beta x$) via the weight $\langle \lambda, \beta^\vee \rangle$ (respectively $\langle s_\beta(\lambda), \beta^\vee \rangle = -\langle \lambda, \beta^\vee \rangle$). Identifying $C_{1,\beta}$ with $\mathbb{P}^1$, it follows easily that $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1}(n)$, where $n := \langle \lambda, \beta^\vee \rangle$.

(v) The first equality follows readily from (iv) in view of the isomorphism (10). As $\rho - w_{0,I}(\rho) = 2\rho_I$, we obtain $2(\rho - \rho_I) = \rho + w_{0,I}(\rho)$; this implies the second equality. □

By Lemma 3.1, every one-dimensional Schubert variety $X(s_\alpha)$ satisfies

$$\mathcal{L}_{G/P}(\varpi) \cdot X(s_\alpha) = \langle \varpi, \alpha^\vee \rangle = 1,$$

that is, $X(s_\alpha)$ is a line in $\mathbb{P}(V(\varpi))$. We thus say that $X(s_\alpha)$ is a *Schubert line*.

Also, note that every T-stable curve in G/P is a line if and only if we have $\langle \varpi, \beta^\vee \rangle \leq 1$ for all $\beta \in R^+$, i.e. the dominant weight ϖ is *minuscule*. Then the parabolic subgroup P is also called minuscule; it is maximal if G is simple. The projective homogeneous space G/P is called minuscule as well. Moreover, the weights of T in $V(\varpi)$ are exactly the $w(\varpi)$, where $w \in W$; as a consequence, the G-module $V(\varpi)$ is simple.

3.3 Lines on flag varieties

We still keep the notation and assumptions of § 3.1, and consider a minimal family $\mathcal{K}$ on $X = G/P$.

Lemma 3.2.

(i) *The family $\mathcal{K}$ consists of free curves.*
(ii) *The subfamily $\mathcal{K}_x$ is a smooth projective variety containing a unique Schubert line $X(s_\alpha)$, where $\alpha \in S \setminus I$. Moreover,* $\dim(\mathcal{K}_x) = \mathrm{ht}(w_{0,I}(\alpha^\vee)) - 1$.
(iii) *The subfamily $\mathcal{K}_x$ consists of the lines in the orbit $P_{I\cup\{\alpha\}}x$ through x.*
(iv) *The family $\mathcal{K}$ consists of those lines in G/P that are contracted by the natural morphism*

$$\pi_\alpha : G/P = G/P_I \longrightarrow G/P_{I\cup\{\alpha\}}.$$

Proof. (i) This follows from Lemma 2.5.

(ii) The scheme $\mathcal{K}_x$ is projective by assumption. It is smooth in view of (i) and § 2.2, and irreducible by Lemma 2.4. Moreover, $\mathcal{K}_x$ is equipped with an action of $P = G_x$. By Borel's fixed point theorem, it follows that $\mathcal{K}_x$ contains a B-fixed point, and hence a Schubert line $X(s_\alpha)$. The assertion on the dimension follows by combining Lemmas 2.5 and 3.1(v).

The morphism π_α contracts the Schubert line $X(s_\alpha)$ and sends any other Schubert line isomorphically to its image. As a consequence, every curve in $\mathcal{K}$ is contracted by π_α, and $\mathcal{K}_x$ contains no other Schubert line.

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(iii) and (iv) By (i), the lines in G/P form a disjoint union of minimal families. Thus, $\mathcal{K}$ is the unique family of lines such that $\mathcal{K}_x$ contains $X(s_\alpha)$. This yields the two assertions by using Lemma 2.8. □

With the notation of Lemma 3.2, we have

$$P_{I\cup\{\alpha\}}x \simeq P_{I\cup\{\alpha\}}/P_I \simeq L_{I\cup\{\alpha\}}/(P_I \cap L_{I\cup\{\alpha\}}).$$

Moreover, the scheme-theoretic intersection $P_I \cap L_{I\cup\{\alpha\}}$ is smooth (see [Bor91, Corollary 13.21]), and hence is a maximal parabolic subgroup of the connected reductive group $L_{I\cup\{\alpha\}}$. So this lemma reduces the description of minimal families on G/P to the case where P is maximal; then there is a unique such family, and it consists of the lines in $G/P \subset \mathbb{P}(V(\varpi_\alpha))$, where $P = P^\alpha$. We may further assume that G is simple; if in addition the simple root α is long, then we have the following result, which is known over the field of complex numbers (see [HM02, Proposition 1] and [LM03, Theorem 4.8]).

Proposition 3.3. *Let $P = P^\alpha$, where α is a long simple root. Denote by $\mathcal{L}$ the family of lines in G/P, and by $\mathcal{L}_x$ the subfamily of lines through x.*

(i) *The subfamily $\mathcal{L}_x$ is the L-orbit of the Schubert line $X(s_\alpha)$.*
(ii) *The tangent map (6) yields an immersion of $\mathcal{L}_x$ into $\mathbb{P}(T_xX)$.*
(iii) *The (scheme-theoretic) stabilizer of $X(s_\alpha)$ in L is the parabolic subgroup $L \cap P_{\alpha^\perp}$, where $\alpha^\perp := \{\beta \in S | \langle\beta, \alpha^\vee\rangle = 0\}$.*

Proof. By § 2.1 and Lemma 3.2(i), we have $\dim(\mathcal{L}_x) = -K_X \cdot X(s_\alpha) - 2$. Using Lemma 3.1(v), this yields

$$\dim(\mathcal{L}_x) = 2\langle\rho - \rho_I, \alpha^\vee\rangle - 2 = -2\langle\rho_I, \alpha^\vee\rangle = -\sum_{\beta\in R_I^+}\langle\beta, \alpha^\vee\rangle.$$

Note that $0 \leq -\langle\beta, \alpha^\vee\rangle \leq 1$ for all $\beta \in R_I^+$, since α is a long simple root and differs from all the simple roots occurring in β. As a consequence,

$$\dim(\mathcal{L}_x) = \#(R_I^+ \setminus R_{I\cap\alpha^\perp}^+). \tag{11}$$

Next, observe that the tangent map yields a morphism $\tau : \mathcal{L}_x \to \mathbb{P}(T_xX)$, since $\mathcal{L}_x$ consists of embedded free curves. Also, $T_xX \simeq \mathfrak{g}/\mathfrak{p}$ and this identifies $\tau(X(s_\alpha))$ with $[\mathfrak{g}_{-\alpha}]$, the image of the root subspace $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}$ in $\mathbb{P}(\mathfrak{g}/\mathfrak{p})$. Since τ is P-equivariant, we have the inclusion of stabilizers $L_{X(s_\alpha)} \subset L_{[\mathfrak{g}_{-\alpha}]}$. We now show that

$$L_{X(s_\alpha)} = L_{[\mathfrak{g}_{-\alpha}]} = L \cap P_{\alpha^\perp}. \tag{12}$$

The Lie algebra $\mathfrak{l}_{[\mathfrak{g}_{-\alpha}]}$ of $L_{[\mathfrak{g}_{-\alpha}]}$ is a subalgebra of $\mathfrak{l}$ containing the Borel subalgebra $\mathfrak{b} \cap \mathfrak{l}$, and hence is generated by $\mathfrak{b} \cap \mathfrak{l}$ and the $\mathfrak{g}_{-\beta}$, where $\beta \in I$ and $\mathfrak{g}_{-\beta}$ stabilizes $[\mathfrak{g}_{-\alpha}]$. The latter condition is equivalent to $[\mathfrak{g}_{-\beta}, \mathfrak{g}_{-\alpha}] \subset \mathfrak{g}_{-\alpha} + \mathfrak{p}$. But $[\mathfrak{g}_{-\beta}, \mathfrak{g}_{-\alpha}] = \mathfrak{g}_{-\alpha-\beta}$ by a result of Chevalley (see e.g. [Hum72, § 25.2]), and $\mathfrak{g}_{-\alpha-\beta} = 0$ if $-\alpha - \beta$ is not a root. In any case, $-\alpha - \beta$ is not a root of P. Thus, the Lie algebra $\mathfrak{l}_{[\mathfrak{g}_{-\alpha}]}$ is generated by $\mathfrak{b} \cap \mathfrak{l}$ and the $\mathfrak{g}_{-\beta}$, where $\beta \in I$ and $[\mathfrak{g}_{-\beta}, \mathfrak{g}_{-\alpha}] = 0$; equivalently, $\beta \in \alpha^\perp$. In other terms, $\mathfrak{l}_{[\mathfrak{g}_{-\alpha}]} = \mathfrak{l} \cap \mathfrak{p}_{\alpha^\perp}$. On the other hand, $L \cap P_{\alpha^\perp}$ stabilizes $X(s_\alpha)$ and hence $[\mathfrak{g}_{-\alpha}]$. Thus,

$$L \cap P_{\alpha^\perp} \subset L_{X(s_\alpha)} \subset L_{[\mathfrak{g}_{-\alpha}]}$$

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and equality holds for the corresponding Lie algebras. Since $L \cap P_{\alpha^\perp}$ is smooth, this easily implies the equalities (12). In turn, this yields (iii) and also the inequalities

$$\dim(\mathcal{L}_x) \geq \dim(LX(s_\alpha)) = \dim(L_I/(L_I \cap P_{\alpha^\perp})) = \#(R_I^+ \setminus R_{I\cap\alpha^\perp}^+).$$

By (11), it follows that $\mathcal{L}_x = LX(s_\alpha)$, proving (i). Finally, (ii) follows from (i) and (12). □

Remark 3.4. We still assume that G is simple and $P = P^\alpha$.

(i) Assume in addition that G is *simply-laced*, i.e. all the roots have the same length. Then α is long and $-\alpha$ is a minuscule weight of L; we denote by $V_L(-\alpha)$ the corresponding highest weight module. By Proposition 3.3 and its proof, $\mathcal{L}_x$ is a minuscule homogeneous space under L; moreover, the tangent map is an immersion onto the closed L-orbit in $\mathbb{P}(V_L(-\alpha)) \subset \mathbb{P}(T_xX)$. One can show that the T-weights of $V_L(-\alpha)$ are exactly the roots $-\beta$, where $\beta \in R^+$ and $\beta = \alpha + \sum_{\alpha_i \in S, \alpha_i \neq \alpha} n_i\alpha_i$ for some non-negative integers n_i; these form a unique orbit of W_L. Moreover, P is minuscule if and only if $V_L(-\alpha) = T_xX$; equivalently, W_L acts transitively on $R^- \setminus R_L$.

On the other hand, if G is not simply-laced, then there exists a short root α such that the variety of lines through x in G/P^α is not homogeneous under P^α; see the main theorem of [CC98] for a more specific result, valid over an arbitrary field, and [Str02, LM03] for further developments.

(ii) When G is simply-laced, the minuscule homogeneous spaces G/P and their varieties of lines are exactly those in the following table:

Type of G	α	X	Type of L	$\mathcal{L}_x$
A_n	α_m	$\mathbb{G}(m, n+1)$	$\mathrm{A}_{m-1} \times \mathrm{A}_{n-m}$	$(\mathbb{P}^{m-1})^* \times \mathbb{P}^{n-m}$
D_n	α_1	$\mathbb{Q}^{2n-2}$	D_{n-1}	$\mathbb{Q}^{2n-4}$
D_n	α_{n-1}, α_n	$\mathbb{S}^{n(n-1)/2}$	A_{n-1}	$\mathbb{G}(2, n)$
E_6	α_1, α_6	$\mathbb{X}^{16}$	D_5	$\mathbb{S}^{10}$
E_7	α_7	$\mathbb{X}^{27}$	E_6	$\mathbb{X}^{16}$

Here $\mathbb{G}(m, n+1)$ denotes the Grassmannian of m-dimensional linear subspaces of k^{n+1}, and $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ the n-dimensional smooth quadric. Also, $\mathbb{S}^{n(n-1)/2}$ stands for the spinor variety of the corresponding dimension, and $\mathbb{X}^{16}, \mathbb{X}^{27}$ are two exceptional varieties of the corresponding dimensions again. The simple roots are ordered as in [Bou08, Chapter VI], and the list of minuscule weights is taken from [Bou08, § 4 and Exercise 15].

When G is not simply-laced, one obtains in addition the pairs (B_n, α_n) and (C_n, α_1). The associated minuscule varieties are isomorphic to those of the pairs $(\mathrm{D}_{n+1}, \alpha_n)$ and $(\mathrm{A}_{2n-1}, \alpha_1)$ respectively, that is, $\mathbb{S}^{n(n+1)/2}$, respectively $\mathbb{P}^{2n-1}$. Moreover, this identifies the Schubert varieties of the former pairs to those of the latter ones. Thus, we may assume that G is simply-laced when studying Schubert varieties in minuscule G-homogeneous spaces.

3.4 Lines on Schubert varieties

We keep the notations and assumptions of §§ 3.1 and 3.2, and start with the following observation.

Lemma 3.5. *The T-stable curves in $X(w)$ through the base point wx are exactly the $C_{w,\beta}$, where $\beta \in w(R^+) \cap R^-$.*

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Proof. The Bruhat decomposition yields a T-equivariant open immersion

$$\prod_{\beta\in R^+\cap w(R^-)} U_\beta \longrightarrow X(w), \quad (g_\beta)\longmapsto \left(\prod_\beta g_\beta\right) wx$$

with image Bwx, where the product is taken in any order. The assertion follows from this by arguing as in the proof of Lemma 3.1. □

Our next result implies that every Schubert variety is covered by translates of Schubert lines (see also [HM13, Proposition 3.1]).

LEMMA 3.6. *The following are equivalent for $w \in W^I$ and $\alpha \in S \setminus I$:*

(i) $w(\alpha) \in R^-$;
(ii) $X(w)$ *is covered by G-translates of the Schubert line* $X(s_\alpha)$.

Proof. (i) ⇒ (ii) The translate $wX(s_\alpha)$ is a T-stable curve with T-fixed points $wx, ws_\alpha x$. By Lemma 3.1 or a direct argument, it follows that $wX(s_\alpha) = C_{w,w(\alpha)}$. So $wX(s_\alpha) \subset X(w)$ in view of Lemma 3.5. We conclude that the translates $bwX(s_\alpha)$, where $b \in B$, cover $X(w)$.

(ii) ⇒ (i) By assumption, there exists $g \in G$ such that $gX(s_\alpha)$ meets Bwx and is contained in $X(w)$; equivalently, $Bwx \cap gX(s_\alpha)$ is dense in $gX(s_\alpha)$. The Bruhat decomposition yields that $g = b\dot{v}b'$ for some $b, b' \in B$ and $v \in W$; then $Bwx \cap vX(s_\alpha)$ is also dense in $vX(s_\alpha)$. Since $Bwx \cap vX(s_\alpha)$ is closed in Bwx and stable by T, it contains wx. Thus, $vX(s_\alpha)$ is a T-stable curve through wx in $X(w)$; in particular, we have either $wx = vx$ or $wx = vs_\alpha x$. Replacing v with vs_α, we may assume that $v(\alpha) \in R^-$; then $vs_\alpha < v$ for the Bruhat order in W, and hence in W/W_I. So we must have $wx = vx$, i.e. $w = vu$ for some $u \in W_I$. Then $wu^{-1}(\alpha) \in R^-$; as $w(R_I^+) \subset R^+$, it follows that $w(\alpha) \in R^-$ as well. □

Next, assume that $P = P^\alpha$ where α is a long simple root. Let $\mathcal{K}$ be a family of lines on $X(w)$, and $\mathcal{K}_{wx}$ the subfamily of lines through wx. It will be convenient to consider the translate $\dot{w}^{-1}\mathcal{K}$, a family of lines in $w^{-1}X(w)$ through the base point x. Recall from Proposition 3.3 that the family $\mathcal{L}_x$ of lines in G/P through x is the minuscule variety $LX(s_\alpha) \simeq L/(L \cap P_{\alpha^\perp})$.

PROPOSITION 3.7. *With the preceding notation and assumptions, $w^{-1}\mathcal{K}_{wx}$ is a Schubert subvariety of $\mathcal{L}_x$. Moreover, the Schubert subvarieties obtained in this way are exactly the $\overline{B_L vX(s_\alpha)}$ where $v \in W_L$, $wv(\alpha) \in R^-$, and v is maximal for this property.*

Proof. By Lemma 2.4, $\mathcal{K}_{wx}$ is a projective variety equipped with an action of $B \cap wPw^{-1}$. Thus, $w^{-1}\mathcal{K}_{wx}$ is a projective variety equipped with an action of $w^{-1}Bw \cap P$, and hence of B_L. Moreover, there is a quasi-finite equivariant morphism

$$\gamma : w^{-1}\mathcal{K}_{wx} \longrightarrow \mathcal{L}_x,$$

since we may view $\mathcal{L}_x$ as the Chow variety of lines in G/P. Since $\mathcal{L}_x$ is a flag variety under L, it has only finitely many B_L-orbits by the Bruhat decomposition; also, the B_L-isotropy groups are smooth and connected. As a consequence, $w^{-1}\mathcal{K}_{wx}$ contains an open orbit of B_L; moreover, the stabilizer of a point C of this orbit is contained in the stabilizer $(B_L)_{\gamma(C)}$, and both have the same dimension. Since $(B_L)_{\gamma(C)}$ is smooth and connected, both stabilizers must be equal and hence γ

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is birational. Using the normality of Schubert varieties, it follows that γ is an isomorphism. This proves the first assertion.

For the second assertion, recall that every B_L-orbit in $\mathcal{L}_x$ contains a unique T-fixed point; moreover, these fixed points are exactly the $vX(s_\alpha)$, where $v \in W_L$. Also, $vX(s_\alpha) = C_{1,v(\alpha)}$ as $v(\alpha) \in R^+$. By Lemma 3.5, it follows that $vX(s_\alpha) \subset w^{-1}X(w)$ if and only if $wv(\alpha) \in R^-$. □

Remark 3.8. Let $P = P^\alpha$ as above and assume in addition that P is minuscule. Then every minimal family $\mathcal{K}$ in $X(w)$ consists of lines. Indeed, $\mathcal{K}_{wx}$ is a projective variety equipped with a T-action, and hence contains a T-fixed point; also, every T-stable curve is a line.

Further, note that the smooth locus of $X(w)$ is covered by lines. Indeed, we may choose $\beta \in S$ such that $s_\beta w < w$ for the Bruhat order in W; equivalently, $w^{-1}(\beta) \in R^-$. Then $X(w)$ is stable by the minimal parabolic subgroup P_β; moreover, we have $C_{w,-\beta} \subset P_\beta wx \subset X(w)_{\mathrm{sm}}$ and hence the B-translates of $C_{w,-\beta}$ cover $X(w)_{\mathrm{sm}}$.

Given β as above, we have $w^{-1}(\beta) \in R^- \setminus R_L$. In view of Remark 3.4, it follows that there exists $v \in W_L$ such that $w^{-1}(\beta) = -v(\alpha)$; equivalently, $wv(\alpha) = -\beta$. Then $C_{w,-\beta} = wvX(s_\alpha)$; this realizes $C_{w,-\beta}$ as a translate of the Schubert line.

Example 3.9. We illustrate the results of this subsection in the case where $G := \mathrm{SL}(4)$ (with simple roots $\alpha_1, \alpha_2, \alpha_3$), and $I := \{\alpha_1, \alpha_3\}$; then the parabolic subgroup $P = P_I = P^{\alpha_2}$ is minuscule. Let $w := s_1 s_3 s_2$; then $w \in W^I$ and $\ell(w) = 3$. The T-stable lines through wx in $X(w)$ are exactly

$$C_1 := C_{w,-\alpha_1}, \quad C_2 := C_{w,-\alpha_1-\alpha_2-\alpha_3}, \quad C_3 := C_{w,-\alpha_3}.$$

Moreover, $C_1 = G_{\alpha_1} wx$, $C_2 = s_1 s_2 G_{\alpha_3} wx$ and $C_3 = G_{\alpha_3} wx$. Since $s_1 w, s_3 w < w$, we see that C_1, C_3 are contained in the smooth locus of $X(w)$. But C_2 contains x, which is the unique singular point of $X(w)$.

The family $\mathcal{L}_x$ of lines in G/P through x satisfies $\mathcal{L}_x = LX(s_2) \simeq \mathbb{P}^1 \times \mathbb{P}^1$; this isomorphism identifies $X(s_2)$ with (∞, ∞). Moreover, the Chow variety of lines in $w^{-1}X(w)$ through x is identified with $(\mathbb{P}^1 \times \{\infty\}) \cup (\{\infty\} \times \mathbb{P}^1)$, the union of two lines meeting at the point corresponding to $X(s_2) = w^{-1}C_2$. These lines are the B_L-orbit closures of $w^{-1}C_1, w^{-1}C_3$. As a consequence, there are exactly two families of minimal rational curves on $X(w)$; those through wx are the w-translates of the above lines.

These results can also be obtained by direct geometric arguments, since $X = \mathbb{G}(2,4)$ is embedded in $\mathbb{P}(V(\varpi_2)) = \mathbb{P}(\Lambda^2 k^4) \simeq \mathbb{P}^5$ as a quadric; moreover, $X(w)$ is the intersection of X with a tangent hyperplane. Thus, $X(w)$ is the projective cone over $\mathbb{Q}^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with vertex x. This cone contains two families of planes (the projective cones over the two families of lines in $\mathbb{P}^1 \times \mathbb{P}^1$) and the lines in these planes form the two minimal families.

4. Generalized Bott–Samelson varieties

4.1 Bott–Samelson desingularizations

We keep the notation of §§ 3.1 and 3.2. Let $w \in W^I$. If $w \neq 1$, then there exists a decomposition $w = s_{i_1} w'$, where s_{i_1} is a simple reflection, $w' \in W$, and $\ell(w) = \ell(w') + 1$. It follows that the minimal parabolic subgroup P_{i_1} stabilizes $X(w)$, and $w' \in W^I$. Consider the Schubert variety $X(w') \subset G/P$ and the associated fiber bundle $P_{i_1} \times^B X(w')$. This is a projective variety equipped

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with an action of P_{i_1} and an equivariant morphism

$$f_{i_1,w'} : P_{i_1} \times^B X(w') \longrightarrow P_{i_1}/B \simeq \mathbb{P}^1,$$

which is a locally trivial fibration (for the Zariski topology) with fiber $X(w')$. We also have a P_{i_1}-equivariant morphism

$$\pi_{i_1,w'} : P_{i_1} \times^B X(w') \longrightarrow X(w),$$

which restricts to an isomorphism above the open orbit Bwx.

The above 'one-step construction' can be iterated: given a reduced decomposition

$$\tilde{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_\ell})$$

(i.e. a sequence of simple reflections such that $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ and $\ell(w) = \ell$), this yields a projective variety

$$\tilde{X}(\tilde{w}) = P_{i_1} \times^B P_{i_2} \times^B \cdots \times^B P_{i_\ell}/B$$

of dimension ℓ, equipped with an action of P_{i_1} and two equivariant morphisms,

$$f : \tilde{X}(\tilde{w}) \longrightarrow P_{i_1}/B,$$

a locally trivial fibration with fiber $\tilde{X}(\tilde{w}')$, where $\tilde{w}' := (s_{i_2}, \ldots, s_{i_\ell})$, and

$$\pi : \tilde{X}(\tilde{w}) \longrightarrow X(w),$$

which restricts to an isomorphism above the open orbit Bwx. Also, note that $\tilde{X}(\tilde{w})$ is smooth, and hence π is a desingularization of $X(w)$.

More generally, for $1 \leq j \leq \ell$, we have a fibration

$$f_j : \tilde{X}(\tilde{w}) \longrightarrow \tilde{X}(s_{i_1}, \ldots, s_{i_j})$$

with fiber $\tilde{X}(s_{i_{j+1}}, \ldots, s_{i_\ell})$. In particular, $f_{\ell-1}$ is a $\mathbb{P}^1$-bundle; also, $f_1 = f$.

The Bott–Samelson variety $\tilde{X}(\tilde{w})$ is equipped with a base point $\tilde{x}$, defined as the image of $(\dot{s}_{i_1}, \dot{s}_{i_2}, \ldots, \dot{s}_{i_\ell}) \in P_{i_1} \times P_{i_2} \times \cdots \times P_{i_\ell}$. Moreover, $\pi(\tilde{x}) = wx$ and $B_{\tilde{x}} = B_{wx}$. In particular, $\tilde{x}$ is fixed by T. Denoting by $\tilde{x}_j$ the base point of $\tilde{X}(s_{i_1}, \ldots, s_{i_j})$, we have $f_j(\tilde{x}) = \tilde{x}_j$. Further, the fiber of f_j at $\tilde{x}_j$ is T-equivariantly isomorphic to $\tilde{X}(s_{i_{j+1}}, \ldots, s_{i_\ell})$ on which the T-action is twisted by the Weyl group element $s_{i_1} \cdots s_{i_j}$.

We now recall the description of line bundles on $\tilde{X}(\tilde{w})$ obtained in [LT04]. For $1 \leq j \leq \ell$, consider the natural morphism

$$\pi_j : \tilde{X}(s_{i_1}, \ldots, s_{i_j}) \longrightarrow G/B$$

with image the Schubert variety $X(s_{i_1} \cdots s_{i_j})$, and let

$$\mathcal{L}_j := f_j^* \pi_j^* \mathcal{L}_{G/B}(\varpi_{i_j}).$$

Then the isomorphism classes of $\mathcal{L}_1, \ldots, \mathcal{L}_\ell$ form a basis of $\mathrm{Pic}(\tilde{X}(\tilde{w}))$. Further, for any integers $n_1, \ldots, n_\ell$, the line bundle $\mathcal{L}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{L}_\ell^{\otimes n_\ell}$ is ample if and only if $n_1, \ldots, n_\ell > 0$; also, every ample line bundle on $\tilde{X}(\tilde{w})$ is very ample. In particular, $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_\ell$ is the smallest very ample line bundle on $\tilde{X}(\tilde{w})$.

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Next, we describe the T-stable curves through $\tilde{x}$ in $\tilde{X}(\tilde{w})$. Let

$$\beta_1 := \alpha_{i_1}, \beta_2 := s_{i_1}(\alpha_{i_2}), \ldots, \beta_\ell := s_{i_1} \cdots s_{i_{\ell-1}}(\alpha_{i_\ell}).$$

Then we have

$$R^+ \cap w(R^-) = \{\beta_1, \beta_2, \ldots, \beta_\ell\}. \tag{13}$$

We may now state a version of Lemma 3.1 for Bott–Samelson varieties.

Lemma 4.1. *Keep the above notation.*

(i) *The T-stable curves in $\tilde{X}(\tilde{w})$ through $\tilde{x}$ are exactly the $\tilde{C}_j := \overline{U_{\beta_j}\tilde{x}}$, where $1 \le j \le \ell$.*
(ii) *The morphism π restricts to isomorphisms $\tilde{C}_j \to C_{w,-\beta_j}$ for all such j.*
(iii) *For $1 \le j, k \le \ell$, we have $\mathcal{L}_k \cdot \tilde{C}_j = 0$ if $j > k$. If $j \le k$ then $\mathcal{L}_k \cdot \tilde{C}_j$ is the coefficient of $\alpha_{i_k}^\vee$ in $s_{i_k} \cdots s_{i_{j+1}}(\alpha_{i_j}^\vee)$, viewed as a linear combination of simple coroots.*
(iv) *We have $-K_{\tilde{X}(\tilde{w})} \cdot \tilde{C}_j = \mathrm{ht}(s_{i_\ell} \cdots s_{i_{j+1}}(\alpha_{i_j}^\vee)) + 1$.*

Proof. (i) This follows from Lemmas 3.1 and 3.5, since π sends $B\tilde{x}$ (an open T-stable neighborhood of $\tilde{x}$ in $\tilde{X}(\tilde{w})$) isomorphically to Bwx.

For (ii), note that π restricts to a birational morphism $\tilde{C}_j \to C_{w,-\beta_j} \simeq \mathbb{P}^1$.

(iii) If $j > k$ then $\tilde{C}_j$ is contracted by f_k. This implies the first assertion by using the projection formula.

If $j \le k$ then f_k restricts to an isomorphism of $\tilde{C}_j$ onto the jth T-stable curve in $\tilde{X}(s_{i_1}, \ldots, s_{i_k})$ through $\tilde{x}_k$. Further, the latter curve is sent isomorphically by π_k to the T-stable curve $C_{s_{i_1} \cdots s_{i_k}, -\beta_j}$ in G/B. Using Lemma 3.1, it follows that

$$\mathcal{L}_k \cdot \tilde{C}_j = \langle \varpi_{i_k}, -s_{i_k} \cdots s_{i_1}(\beta_j^\vee) \rangle = \langle \varpi_{i_k}, s_{i_k} \cdots s_{i_{j+1}}(\alpha_j^\vee) \rangle.$$

This yields the second assertion.

(iv) We first determine $-K_{\tilde{X}(\tilde{w})} \cdot \tilde{C}_1$. Note that $G_{\alpha_{i_1}} \subset P_{i_1}$ acts on $\tilde{X}(\tilde{w})$ and we have $\tilde{C}_1 = G_{\alpha_{i_1}}\tilde{x}$; also, the tangent space of $\tilde{X}(\tilde{w})$ at $\tilde{x}$ is a direct sum of T-stable lines with weights $\beta_1, \ldots, \beta_\ell$. Arguing as in the proof of Lemma 3.1(iv), it follows that

$$-K_{\tilde{X}(\tilde{w})} \cdot \tilde{C}_1 = \langle \beta_1 + \cdots + \beta_\ell, \alpha_{i_1}^\vee \rangle.$$

But $\beta_1 + \cdots + \beta_\ell = \rho - w(\rho)$ in view of (13), and hence

$$-K_{\tilde{X}(\tilde{w})} \cdot \tilde{C}_1 = \langle \rho - w(\rho), \alpha_{i_1}^\vee \rangle = 1 + \mathrm{ht}(-w^{-1}(\alpha_{i_1}^\vee)).$$

This yields the assertion, since $-w^{-1}(\alpha_{i_1}) = s_{i_\ell} \cdots s_{i_2}(\alpha_{i_1})$.

Next, we determine $-K_{\tilde{X}(\tilde{w})} \cdot \tilde{C}_j$, where $j \ge 2$. Then $\tilde{C}_j$ is contracted by f, i.e. we have $\tilde{C}_j \subset F := f^{-1}f(\tilde{x})$. Since f is a locally trivial fibration, it follows that $-K_{\tilde{X}(\tilde{w})} \cdot \tilde{C}_j = -K_F \cdot \tilde{C}_j$. Recall that F is T-equivariantly isomorphic to the Bott–Samelson variety $\tilde{X}(\tilde{w}')$ on which the T-action is twisted by s_{i_1}, and this isomorphism sends $\tilde{x}$ to the base point of $\tilde{X}(\tilde{w}')$. Using an easy induction argument, this completes the proof. □

Remark 4.2. By Lemma 4.1, we have $\mathcal{L}_k \cdot \tilde{C}_\ell = 0$ for $1 \le k < \ell$, and $\mathcal{L}_\ell \cdot \tilde{C}_\ell = 1$. Thus, $\tilde{C}_\ell$ is a line in the smallest projective embedding of $\tilde{X}(\tilde{w})$.

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Also, $\mathcal{L}_j \cdot \tilde{C}_j = 1$ for $1 \leq j \leq \ell$; as a consequence, $\tilde{C}_j$ is a line if and only if $\mathcal{L}_k \cdot \tilde{C}_j = 0$ for all $j < k$. By an easy argument, this is equivalent to the assertion that s_{i_j} commutes with $s_{i_{j+1}}, \ldots, s_{i_\ell}$. Then there is an isomorphism of resolutions of $X(w)$

$$\tilde{X}(\tilde{w}) = \tilde{X}(s_{i_1}, \ldots, s_{i_\ell}) \simeq \tilde{X}(s_{i_1}, \ldots, s_{i_{j-1}}, s_{i_{j+1}}, \ldots, s_{i_\ell}, s_{i_j})$$

which identifies $\tilde{C}_j$ with the line in the right-hand side constructed as above.

Next, we determine the minimal rational curves in $\tilde{X}(\tilde{w})$ (these include of course the lines discussed above).

Theorem 4.3. *Every minimal family $\mathcal{K}$ on $\tilde{X}(\tilde{w})$ satisfies $\mathcal{K}_{\tilde{x}} = \{\tilde{C}_j\}$ for some $1 \leq j \leq \ell$. Moreover, the minimal rational curves in $\tilde{X}(\tilde{w})$ through $\tilde{x}$ are exactly those $\tilde{C}_j$ such that the root $s_{i_\ell} \cdots s_{i_{j+1}}(\alpha_{i_j})$ is simple.*

Proof. We argue as in the proof of Lemma 3.2. By Lemma 2.4, $\mathcal{K}_{\tilde{x}}$ is a projective variety; moreover, $\mathcal{K}$ consists of free curves in view of Lemma 2.5. Since $\mathcal{K}_{\tilde{x}}$ is equipped with an action of T, it contains a T-fixed point, say $\tilde{C}_j$.

If $j > 1$ then $\tilde{C}_j$ is contracted by f. In view of Lemma 2.8, it follows that $\mathcal{K}_{\tilde{x}} = \mathcal{L}_{\tilde{x}}$ for a unique minimal family $\mathcal{L}$ on the fiber of f at $\tilde{x}$. Since this fiber is a translate of a smaller Bott–Samelson variety, we may conclude by induction on ℓ.

Thus, we may assume that $j = 1$; then $\tilde{C}_1$ is the unique T-fixed point of $\mathcal{K}_{\tilde{x}}$. Also, $\mathcal{K}_{\tilde{x}}$ admits an ample T-linearized line bundle: indeed, it is equipped with a finite T-equivariant morphism to some Chow variety of $\tilde{X}(\tilde{w})$, which in turn is equipped with a finite T-equivariant morphism to the projectivization of a T-module in view of its construction in [Kol99, § I.3]. As a consequence, $\mathcal{K}_{\tilde{x}}$ admits a T-equivariant immersion in the projectivization of a T-module. Using [Bor91, Proposition 13.5], it follows that $\mathcal{K}_{\tilde{x}}$ consists of the unique curve $\tilde{C}_1$.

On the other hand, by Lemma 2.5, $\tilde{C}_1$ lies in a unique family of rational curves $\mathcal{L}$ on $\tilde{X}(\tilde{w})$; moreover, $\mathcal{L}$ is covering and satisfies

$$\dim(\mathcal{L}_{\tilde{x}}) = -K_{\tilde{X}(\tilde{w})} \cdot \tilde{C}_1 - 2.$$

In view of Lemma 4.1(iii), this vanishes if and only if the root $s_{i_\ell} \cdots s_{i_2}(\alpha_{i_1})$ is simple. □

Remark 4.4. (i) Since $s_{i_\ell} \cdots s_{i_{j+1}}(\alpha_{i_j}) = -w^{-1}(\beta_j)$, the simple roots α_k which can be obtained as $s_{i_\ell} \cdots s_{i_{j+1}}(\alpha_{i_j})$ for some j are exactly those such that $w(\alpha_k) \in R^-$. Then π sends $\tilde{C}_j$ to $wX(s_k)$, a translate of a Schubert line. This relates the minimal rational curves in $\tilde{X}(\tilde{w})$ through $\tilde{x}$ to the lines in $X(w)$ through wx constructed in Lemma 3.6. In particular, we may take $j = \ell$, ie., $\alpha_k = \alpha_{i_\ell}$; this just gives back the line $\tilde{C}_\ell$ (Remark 4.2).

If $P = P^\alpha$ is maximal, then we must have $\alpha = \alpha_k$ and $j = \ell$. Thus, $\tilde{X}(\tilde{w})$ has a unique minimal family, consisting of the fibers of the $\mathbb{P}^1$-bundle $f_{\ell-1} : \tilde{X}(\tilde{w}) \to \tilde{X}(s_{i_1}, \ldots, s_{i_{\ell-1}})$.

(ii) The condition that $s_{i_\ell} \cdots s_{i_{j+1}}(\alpha_{i_j})$ is a simple root, say α_k, turns out to be equivalent to the exchange condition

$$s_{i_1} s_{i_2} \cdots s_{i_\ell} = s_{i_1} \cdots \widehat{s_{i_j}} \cdots s_{i_\ell} s_k,$$

where both sides are reduced decompositions of w.

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4.2 Their generalizations à la Perrin

Let $w \in W$. Recall that the set of simple roots α such that s_α occurs in a reduced decomposition of w is independent of the reduced decomposition, and called the *support* of w. We denote this set by $\mathrm{Supp}(w)$. The subgroup of G generated by the $U_{\pm\alpha}$, where $\alpha \in \mathrm{Supp}(w)$, will be denoted by G_w; this is the derived subgroup of the Levi subgroup $L_{\mathrm{Supp}(w)}$, and hence is a semi-simple subgroup of G, normalized by T and containing a representative of w.

Denote by P^w be the largest parabolic subgroup of G such that $P^w \supset B$ and $w \in W^{P^w}$; then $P^w = P_{I^w}$, where $I^w := \{\alpha \in S | w(\alpha) \in R^+\}$. Consider the associated Schubert variety $X(w) \subset G/P^w$, and denote by P_w the closed reduced subgroup of G consisting of those g such that $gX(w) = X(w)$. Then P_w is a parabolic subgroup of G containing B, and hence $P_w = P_{I_w}$, where $I_w := \{\alpha \in S \mid s_\alpha w \leq w\}$; here $\leq$ denotes the Bruhat order on $W^{I^w} = W/W_{I^w}$. Note that $P_w \cap G_w$ is a parabolic subgroup of G_w, and we have

$$X(w) = \overline{P_w w P^w}/P^w \simeq \overline{(P_w \cap G_w)w(P^w \cap G_w)}/(P^w \cap G_w) \subset G_w/(P^w \cap G_w). \tag{14}$$

We say that $w \in W$ is minuscule if so is G/P^w; then one may readily check that $G_w/(P^w \cap G_w)$ is minuscule as well. Also, note that G_w is simply-laced if so is G. For any minuscule $w \in W$, we have $X(w)_{\mathrm{sm}} = (P_w \cap G_w)wx$ by [BP99, Proposition 3.3]. In particular, $X(w)$ is smooth if and only if it is homogeneous under $P_w \cap G_w$.

Next, let $w = w_1 w'$, where $w_1, w' \in W$ satisfy $P^{w_1} \cap G_{w_1} \subset P_{w'}$; equivalently, we have the inclusion $I^{w_1} \cap \mathrm{Supp}(w_1) \subset I_{w'}$. We may then define

$$\tilde{X}(w_1, w') := \overline{(P_{w_1} \cap G_{w_1})w_1(P^{w_1} \cap G_{w_1})} \times^{P^{w_1} \cap G_{w_1}} X(w').$$

This is a projective variety equipped with an action of $P_{w_1} \cap G_{w_1}$ and an equivariant morphism

$$f_{w_1,w'} : \tilde{X}(w_1, w') \longrightarrow X(w_1),$$

which is a Zariski locally trivial fibration with fiber $X(w')$. If in addition $\ell(w) = \ell(w_1) + \ell(w')$, then $w' \in W^{P^w}$ and hence $P^{w'} \supset P^w$. Thus, if P^w is maximal and $w' \neq 1$, then $P^{w'} = P^w$. Under these assumptions, we obtain another equivariant morphism

$$\pi_{w_1,w'} : \tilde{X}(w_1, w') \longrightarrow G/P^w.$$

One may check that $\pi_{w_1,w'}$ is birational to its image $X(w)$; it restricts to an isomorphism above the open orbit Bwx. Also, note that

$$P_{w_1} \cap G_{w_1} \subset P_w \cap G_w. \tag{15}$$

Remark 4.5. Assume that w_1 is a simple reflection s_α. Then P^{w_1} is the maximal parabolic subgroup $P^\alpha = P_{S\setminus\{\alpha\}}$, and $X(w_1)$ is the Schubert line in G/P^α. Moreover, we have $G_{w_1} = G_\alpha$ and $P_{w_1} = P_{\{\alpha\} \cup \alpha^\perp}$. So $P^{w_1} \cap G_{w_1} = B \cap G_\alpha$ is a Borel subgroup of G_α. Thus, we have

$$\tilde{X}(s_\alpha, w') = G_\alpha \times^{B \cap G_\alpha} X(w') \simeq P_\alpha \times^B X(w'),$$

with fibration $f_{s_\alpha,w'}$ over $P_\alpha/B \simeq \mathbb{P}^1$. If in addition $\ell(w) = \ell(w') + 1$, then $\pi_{s_\alpha,w'}$ yields a birational morphism to $X(w)$. Therefore, the above 'one-step construction' generalizes that of Bott–Samelson varieties.

This construction can be iterated, under certain additional assumptions that are discussed in detail in [Per07, § 5.2]. We now present some notions and results from [Per07, § 5.2]: a finite

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sequence $\hat{w} = (w_1, \ldots, w_m)$ of elements of W is called a *generalized reduced decomposition of* w, if we have $w = w_1 \cdots w_m$ and $\ell(w) = \ell(w_1) + \cdots + \ell(w_m)$. Such a decomposition is called *good* if in addition w is minuscule and we have

$$I^{w_i} \cap \mathrm{Supp}(w_i) \subset I_{w_{i+1}\cdots w_m} \subset w_i^{\perp} \cup \mathrm{Supp}(w_i) \quad (1 \leq i \leq m-1),$$

where $w_i^{\perp}$ denotes the set of simple roots α such that s_α commutes with w_i. Under these assumptions, $(w_{i+1}, \ldots, w_m)$ is a good generalized reduced decomposition of $w_{i+1} \cdots w_m$ for $i = 1, \ldots, m-1$. Moreover, $P^{w_i} \cap G_{w_i} \subset P_{w_{i+1}\cdots w_m}$ for all such i.

Given a good generalized reduced decomposition $\hat{w}$ of w, we obtain a projective variety $\hat{X}(\hat{w})$ equipped with an action of P_w, a locally trivial fibration

$$\hat{f} : \hat{X}(\hat{w}) \longrightarrow X(w_1)$$

with fiber $\hat{X}(w_2, \ldots, w_m)$, and a birational morphism

$$\hat{\pi} : \hat{X}(\hat{w}) \longrightarrow X(w).$$

Also, $\hat{X}(\hat{w})$ has a base point $\hat{x}$ such that $\hat{\pi}(\hat{x}) = wx$ and $\hat{f}(\hat{x}) = w_1 x_1$, where x_1 denotes the base point of G/P^{w_1}.

More generally, for $1 \leq i \leq m$, we have a fibration

$$\hat{f}_i : \hat{X}(\hat{w}) \longrightarrow \hat{X}(w_1, \ldots, w_i)$$

with fiber $\hat{X}(w_{i+1}, \ldots, w_m)$. Further, $\hat{f}_i(\hat{x}) = \hat{x}_i$ with an obvious notation, and $\hat{f}_1 = \hat{f}$.

By [Per07, § 5.1], the morphism $\hat{f}$ is P_w-equivariant. As a consequence, we have the equality of stabilizers $P_{w,\hat{x}} = P_{w,wx} = P_w \cap wP^w w^{-1}$. Thus, $P_{w,\hat{x}}$ contains the maximal torus T. Also, note that $P_{w,\hat{x}}$ is smooth and connected, in view of the following result.

Lemma 4.6. *Let P, Q be two parabolic subgroups of G containing the maximal torus T.*

(i) *The (scheme-theoretic) intersection $P \cap Q$ is smooth and connected.*
(ii) *Denote by L (respectively M) the Levi subgroup of P (respectively Q) containing T. Then $P \cap Q$ has a Levi decomposition with Levi subgroup $L \cap M$.*

Proof. (i) The smoothness of $P \cap Q$ follows from [Bor91, Corollary 13.21], and the connectedness from [Bor91, Proposition 14.22].

(ii) This is a consequence of [DM91, Proposition 2.1]. □

4.3 Structure of minimal families

We still consider a good generalized reduced decomposition $\hat{w} = (w_1, \ldots, w_m)$ of a minuscule element $w \in W$, and set $P := P^w$. Also, we choose reduced decompositions

$$\tilde{w}_i = (s_{i,1}, \ldots, s_{i,\ell_i})$$

of w_i for $1 \leq i \leq m$. This yields a reduced decomposition of w by concatenation, and hence a Bott–Samelson variety $\tilde{X}(\tilde{w})$. By using [Per07, § 5.3], we obtain a commutative diagram of

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pointed varieties

$$\begin{array}{ccccc} (\tilde{X}(\tilde{w}),\tilde{x}) & \xrightarrow{\tilde{\pi}} & (\hat{X}(\hat{w}),\hat{x}) & \xrightarrow{\hat{\pi}} & (X(w),wx) \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \hat{f}} & & \\ (\tilde{X}(\tilde{w}_1),\tilde{x}_1) & \xrightarrow{\tilde{\pi}_1} & (X(w_1),w_1x_1) & & \end{array}$$

where the horizontal arrows induce local isomorphisms at the corresponding base points, and the vertical arrows are locally trivial fibrations; moreover, the composition $\hat{\pi}\circ\tilde{\pi}$ is the Bott–Samelson resolution $\pi : \tilde{X}(\tilde{w}) \to X(w)$.

By arguing as in the proof of Lemma 4.1, one checks that $\tilde{\pi}$ and $\hat{\pi}$ induce isomorphisms on T-stable curves through the respective base points. Thus, we may index the T-stable curves through $\hat{x}$ in $\hat{X}(\hat{w})$ as $\hat{C}_{i,j}$, where $1 \le i \le \ell_j$ and $1 \le j \le m$. Note that $\hat{f}$ contracts all the $\hat{C}_{i,j}$ with $j \ge 2$, and sends each $\hat{C}_{i,1}$ isomorphically to $C_{w_1,-\beta_i}$; in particular, $\hat{f}$ yields a bijection from $\{\hat{C}_{1,1},\dots,\hat{C}_{\ell_1,1}\}$ to the set of T-stable curves through w_1x_1 in $X(w_1)$. Also, every minimal family on $\hat{X}(\hat{w})$ contains some $\hat{C}_{i,j}$, as follows from Borel's fixed point theorem.

We now assume that $\hat{X}(\hat{w})$ is smooth; equivalently, $X(w_i)$ is smooth for $i = 1,\dots,m$. Then each $\hat{C}_{i,j}$ is an embedded free rational curve (Lemma 2.5). Let $\mathcal{K} = \mathcal{K}_{i,j}$ be the family of rational curves on $\hat{X}(\hat{w})$ that contains $\hat{C}_{i,j}$; then $\mathcal{K}$ is covering in view of Lemma 2.4. If $j \ge 2$ then by Lemma 2.8, there exists a unique covering family $\mathcal{L}$ of rational curves on $\hat{Y} := \hat{X}(w_2,\dots,w_m)$ such that $\mathcal{K}_{\hat{x}} = \mathcal{L}_{\hat{y}}$, where $\hat{y}$ denotes the base point of $\hat{Y}$; the above isomorphism is T-equivariant, where the T-action on $\hat{Y}$ is twisted by w_1. Arguing by induction on m, we may thus reduce to the case where $j = 1$.

Assume in addition that G is simply-laced and $w_1,\dots,w_m$ are minuscule; then each $X(w_i)$ is a smooth Schubert variety in the minuscule homogeneous space G/P^{w_i}, and hence is a minuscule homogeneous space as well (see [BP99, Proposition 3.3]). By combining Lemma 2.6, Remark 2.7 and Proposition 3.3, we obtain two $P_{w,\hat{x}}$-equivariant rational maps

$$\hat{\pi}_* : \mathcal{K}_{\hat{x}} \dashrightarrow \mathcal{L}(w)_{wx}, \quad \hat{f}_* : \mathcal{K}_{\hat{x}} \dashrightarrow \mathcal{L}(w_1)_{w_1x_1}, \tag{16}$$

where $\mathcal{L}(w)$ is a family of lines in $X(w)$, and $\mathcal{L}(w_1)$ the family of all lines in $X(w_1)$.

We may now obtain a qualitative analogue of the description of minimal families in minuscule varieties (Proposition 3.3 and Remark 3.4).

Proposition 4.7. *Assume that G is simply-laced. Let $w \in W$ be a minuscule element, and $\hat{w} = (w_1,\dots,w_m)$ a good generalized reduced decomposition of w, where $w_1,\dots,w_m$ are minuscule and $X(w_1),\dots,X(w_m)$ are smooth. Let $\mathcal{K}$ be a minimal family on $\hat{X}(\hat{w})$.*

(i) *The subfamily $\mathcal{K}_{\hat{x}}$ consists of embedded free curves.*
(ii) *The rational maps (16) are immersions.*
(iii) *The subfamily $\mathcal{K}_{\hat{x}}$ is a minuscule homogeneous space.*
(iv) *The tangent map (6) yields an immersion of $\mathcal{K}_{\hat{x}}$ into $\mathbb{P}(T_{\hat{x}}\hat{X}(\hat{w}))$.*

Proof. (i) By Lemma 2.5, every curve in $\mathcal{K}_{\hat{x}}$ is free. Also, recall that being embedded is an open property, invariant under the T-action. Since every T-fixed curve in $\mathcal{K}_{\hat{x}}$ is embedded, this yields the assertion.

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(ii) In view of Lemma 2.6, $\hat{f}_*$ is defined at any T-fixed point, and hence everywhere by the above argument. Also, $\hat{C}_{i,1}$ is the unique T-fixed point of its fiber under $\hat{f}_*$. By arguing as in the proof of Theorem 4.3, it follows that this fiber consists of a unique point. Thus, all the fibers of $\hat{f}_*$ are finite by upper semi-continuity of the dimensions of fibers. So $\hat{f}_*$ is a finite morphism.

Viewing $\hat{\pi}$ as a morphism to G/P and adapting the arguments of the above paragraph, we see that $\hat{\pi}_*$ is a finite morphism as well. Its image is contained in the variety $\mathcal{L}_{wx}$ of lines in G/P through wx, which is a minuscule homogeneous space under wLw^{-1} (Remark 3.4). We now use the equivariance of $\hat{\pi}_*$ under $P_{w,\hat{x}} = P_{w,wx} = P_w \cap wPw^{-1}$, and hence under wB_Lw^{-1}, a Borel subgroup of wLw^{-1}. The image of $\hat{\pi}_*$ is a Schubert subvariety of $\mathcal{L}_{wx}$ with respect to this Borel subgroup. By arguing as in the proof of Proposition 3.7, it follows that $\hat{\pi}_*$ is an immersion. Likewise, $\hat{f}_*$ is an immersion as well.

(iii) Consider the universal family $\rho : \mathcal{U} \to \mathcal{K}$. Then $\mathcal{U}_{\hat{x}}$ is smooth by (i) and Lemma 2.5. Using (i) again and Lemma 2.1, it follows that $\mathcal{K}_{\hat{x}}$ is smooth as well. So $\mathcal{K}_{\hat{x}}$ is isomorphic to a smooth Schubert variety in the minuscule homogeneous space $\mathcal{L}_{wx}$. This implies the statement in view of [BP99, Proposition 3.3].

(iv) By (i) and (8), we have a commutative diagram

$$\begin{array}{ccc} \mathcal{K}_{\hat{x}} & \longrightarrow & \mathbb{P}(T_{\hat{x}}\hat{X}(\hat{w})) \\ {\scriptstyle \hat{\pi}_*}\downarrow & & \downarrow{\scriptstyle d\hat{\pi}_{\hat{x}}} \\ \mathcal{L}(w)_{wx} & \longrightarrow & \mathbb{P}(T_{wx}X(w)) \end{array}$$

where the horizontal arrows are the tangent maps. Moreover, $\hat{\pi}_*$ is an immersion, $d\hat{\pi}_{\hat{x}}$ is an isomorphism, and the bottom horizontal arrow is an immersion as well by Proposition 3.3. This yields the assertion. □

Next, we obtain a more quantitative version of Proposition 4.7 under additional assumptions. We will need the following observation.

Lemma 4.8. *If $X(w)$ is smooth, then $G_w \subset P_w$.*

Proof. By [BP99, Proposition 3.3] and the smoothness assumption, $X(w)$ is a unique orbit of P_w. Since $x \in X(w)$, it follows that $X(w) = P_w x$. Thus, we have $w = uv$ for some $u \in W_{I_w}$ and $v \in W_{I^w}$. Using a reduced decomposition of w and the fact that $w \in W^{I^w}$, it follows that $w \in W_{I_w}$. Therefore, $\mathrm{Supp}(w) \subset I_w$. This completes the proof. □

Since $X(w_1)$ is smooth, we have $G_{w_1} \subset P_{w_1}$ by the above lemma. In view of (15), it follows that G_{w_1} is a subgroup of $P_w \cap G_w$; also, $\hat{f} : \hat{X}(\hat{w}) \to X(w_1)$ is clearly equivariant under this subgroup, and sends $\hat{x}$ to w_1x_1. Thus, we have the inclusion of stabilizers

$$G_{w_1,\hat{x}} \subset G_{w_1,w_1x_1}. \tag{17}$$

Moreover, $G_{w_1,\hat{x}} = G_{w_1} \cap P^w_{wx} = G_{w_1} \cap wP^w w^{-1}$ is the intersection of two parabolic subgroups of G_{w_1}; both contain $T_1 := T \cap G_{w_1}$ as a maximal torus. By Lemma 4.6, it follows that $G_{w_1,\hat{x}}$ is smooth, connected, and admits a Levi subgroup containing T_1; all these properties also hold for G_{w_1,w_1x_1}. We may now state our result.

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Proposition 4.9. *Let G be a simply-laced semi-simple algebraic group, $w \in W$ a minuscule element, and $\hat{w} = (w_1, \ldots, w_m)$ a good generalized reduced decomposition of w. Assume that $w_1, \ldots, w_m$ are minuscule, $X(w_1), \ldots, X(w_m)$ are smooth, and the inclusion (17) induces an equality of Levi subgroups containing T_1. Let $\mathcal{K}$ be a family of rational curves on $\hat{X}(\hat{w})$ containing a T-stable curve $\hat{C}$ which is not contracted by $\hat{f}$. Then every $\alpha \in S$ such that $w_1^{-1}(\alpha) \in R^-$ satisfies $w^{-1}(\alpha) \in R^-$ and*

$$\mathrm{ht}(-w^{-1}(\alpha)) \geq \mathrm{ht}(-w_1^{-1}(\alpha)). \tag{18}$$

Moreover, equality holds for some α as above if and only if $\mathcal{K}$ is minimal, and then the morphisms $\hat{\pi}_* : \mathcal{K}_{\hat{x}} \to \mathcal{L}(w)_{wx}$, $\hat{f}_* : \mathcal{K}_{\hat{x}} \to \mathcal{L}(w_1)_{w_1 x_1}$ are isomorphisms.

Proof. We may choose a reduced decomposition $(s_{1,1}, \ldots, s_{1,\ell_1})$ of w_1 such that $\alpha = \alpha_{1,1}$. Then $\hat{C} = \hat{C}_{i,1}$ where $1 \leq i \leq \ell_1$.

Denote by $G_1 \supset T_1$ the common Levi subgroup to $(P_{w_1} \cap G_{w_1})_{\hat{x}}$ and $(P_{w_1} \cap G_{w_1})_{w_1 x_1}$. Since $X(w_1)$ is a minuscule variety under $P_{w_1} \cap G_{w_1}$, it follows from Proposition 3.3 and Remark 3.4(i) that $\mathcal{L}(w_1)_{w_1 x_1}$ is a minuscule homogeneous space under G_1. Therefore, the T-stable curves through $w_1 x_1$ in $X(w_1)$ form a unique orbit of the Weyl group W_1 of (G_1, T_1). So the same holds for the T-stable curves $\hat{C}_{1,1}, \ldots, \hat{C}_{\ell_1,1}$. Since $\mathcal{K}_{\hat{x}}$ is stable under G_1, it contains all the latter curves.

As a consequence, we have

$$\dim(\mathcal{K}_{\hat{x}}) = -K_{\hat{X}(\hat{w})} \cdot \hat{C}_{i,1} - 2 = -K_{\hat{X}(\hat{w})} \cdot \hat{C}_{1,1} - 2.$$

We now determine $-K_{\hat{X}(\hat{w})} \cdot \hat{C}_{1,1}$. Note that the morphism $\tilde{\pi} : \tilde{X}(\tilde{w}) \to \hat{X}(\hat{w})$ restricts to an isomorphism over the open orbit of the minimal parabolic subgroup P_α (a subgroup of P_w) in $\hat{X}(\hat{w})$; in particular, $\tilde{\pi}$ restricts to an isomorphism from $\tilde{C}_1 = G_\alpha \tilde{x}$ to $\hat{C}_{1,1}$. This yields the equality $-K_{\hat{X}(\hat{w})} \cdot \hat{C}_{1,1} = -K_{\tilde{X}(\tilde{w})} \cdot \tilde{C}_1$. By using Lemma 4.1(iii), this yields in turn

$$\dim(\mathcal{K}_{\hat{x}}) = \mathrm{ht}(-w^{-1}(\alpha^\vee)) - 1.$$

The rational map $\hat{f}_*$ is G_1-equivariant, and hence dominant as $\mathcal{L}(w_1)_{w_1 x_1}$ consists of a unique orbit of G_1. Thus,

$$\dim(\mathcal{K}_{\hat{x}}) \geq \dim(\mathcal{L}(w_1)_{w_1 x_1}).$$

But we obtain

$$\dim(\mathcal{L}(w_1)_{w_1 x_1}) = \mathrm{ht}(-w_1^{-1}(\alpha^\vee)) - 1$$

by arguing as above with the morphism $\tilde{\pi}_1 : \tilde{X}(\tilde{w}_1) \to X(w_1)$. This yields the inequality (18).

If the family $\mathcal{K}$ is minimal, then $\hat{f}_*$ is an immersion (Proposition 4.7). It follows that equality holds in (18) and $\hat{f}_*$ is surjective. Also, $\hat{\pi}_*$ is an immersion (Proposition 4.7 again), and is birational in view of Remark 2.7; thus, $\hat{\pi}_*$ is an isomorphism.

Conversely, assume that equality holds in (18); then the dominant rational map $\hat{f}_*$ is generically finite. Using G_1-equivariance and homogeneity of $\mathcal{L}(w_1)_{w_1 x_1}$ once more, it follows that $\hat{f}_*$ is an isomorphism. In particular, $\mathcal{K}_{\hat{x}}$ is projective, i.e. $\mathcal{K}$ is minimal. □

The assumptions of the above proposition hold for the generalized Bott–Samelson resolutions obtained by Perrin's Construction 1 (see [Per07, § 5.4]; these yield all small resolutions of $X(w)$ in view of [Per07, Corollary 7.9]). More specifically, any such decomposition is good and consists of minuscule elements by [Per07, § 5.4] again. Also, the inclusion (17) induces an equality of Weyl

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groups relative to T_1 (see Proposition 5.13, or [BK19, Theorems 4.1, 4.8 and 4.14]), and hence of Levi subgroups. Using Proposition 5.7, or [BK19, Propositions 4.7, 4.13 and 4.15], we now obtain a complete description of minimal families in this setting.

Theorem 4.10. *Let G be a simply-laced semi-simple algebraic group, $w \in W$ a minuscule element, and $\hat{w} = (w_1, \ldots, w_m)$ a generalized reduced decomposition of w obtained by Perrin's Construction 1. Assume that $X(w_1), \ldots, X(w_m)$ are smooth, and consider a family $\mathcal{K}$ of minimal rational curves on $\hat{X}(\hat{w})$. Then there exist an integer $1 \leq i \leq m$ and an isomorphism $\hat{X}(w_i, \ldots, w_m) \simeq Y \times Z$, where Y is a minuscule homogeneous space, such that $\mathcal{K}_{\hat{x}}$ consists of the lines in Y through y. Here $\hat{X}(w_i, \ldots, w_m)$ is identified with the fiber at $\hat{x}$ of the fibration $\hat{f}_{i-1} : \hat{X}(\hat{w}) \to \hat{X}(w_1, \ldots, w_{i-1})$, and y denotes the image of $\hat{x}$ under the projection to Y.*

Proof. We argue as in the proofs of Lemma 3.2 and Theorem 4.3. By Borel's fixed point theorem, $\mathcal{K}_{\hat{x}}$ contains a T-stable curve, i.e. some curve $\hat{C}_{i,j}$. If $j \geq 2$ then $\hat{C}_{i,j}$ is contracted by $\hat{f}$. Using Lemma 2.8, it follows that $\mathcal{K}_{\hat{x}} = \mathcal{L}_{\hat{x}}$ for a unique minimal family $\mathcal{L}$ on the fiber of $\hat{f}$ at $\hat{x}$. As this fiber is isomorphic to $\hat{X}(w_2, \ldots, w_m)$, we conclude by induction on m.

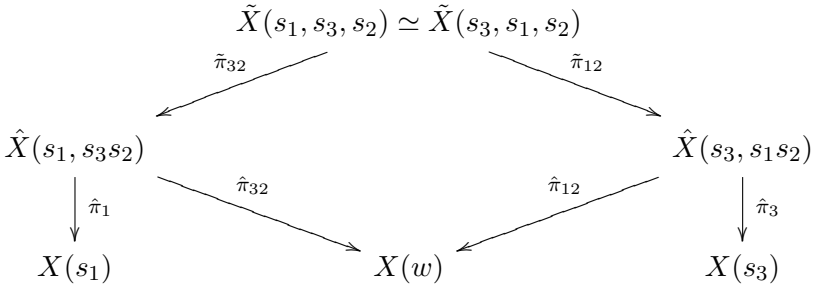
Thus, we may assume that $\mathcal{K}_{\hat{x}}$ contains a T-stable curve of the form $\hat{C}_{i,1}$. Also, decomposing G into a direct product of simple groups yields a decomposition of $\hat{X}(\hat{w})$ into a product of the associated generalized Bott–Samelson resolutions. So we may further assume that G is simple by using Lemma 2.3.

By Lemma 5.3, there exists a unique $\alpha \in S$ such that $w_1^{-1}(\alpha) \in R^-$. Using Proposition 4.9, we obtain that $w^{-1}(\alpha) \in R^-$ and $\mathrm{ht}(-w^{-1}(\alpha)) = \mathrm{ht}(-w_1^{-1}(\alpha))$. If $w \neq w_1$ then we have $w_1^{-1}(\alpha) > w^{-1}(\alpha)$ by Proposition 5.7, which contradicts the above equality of heights. Thus, $w = w_1$; then $\hat{X}(\hat{w}) = X(w_1)$ is just a minuscule homogeneous space, and we finish the proof by Proposition 3.3. □

Remark 4.11. The description of line bundles on Bott–Samelson varieties extends to their generalized versions obtained by Construction 1 (see [Per07, § 6.1]), and Lemma 4.1 can also be extended to this setting. In particular, $\tilde{X}(\tilde{w})$ admits a smallest very ample line bundle, and the T-stable curves $\tilde{C}_{i,m}$ are lines in the corresponding projective embedding. In fact, all lines are obtained by a variant of this construction, as follows from Theorem 4.10.

Also, the canonical class of $\tilde{X}(\tilde{w})$ is described in combinatorial terms in [Per07, § 6.2]. This yields a formula for the dimension of the family of rational curves on $\tilde{X}(\tilde{w})$ containing a given T-stable curve. But we do not know how to deduce the above theorem directly from this formula.

Example 4.12. As in Example 3.9, we illustrate the above results in the case where $G = \mathrm{SL}(4)$ and $w = s_1 s_3 s_2 = s_3 s_1 s_2$. The Schubert variety $X(w)$ admits three generalized Bott–Samelson desingularizations, displayed in the following commutative diagram,



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where $X(s_1) \simeq \mathbb{P}^1 \simeq X(s_3)$ and $\hat{\pi}_1$ (respectively $\hat{\pi}_3$) is a locally trivial fibration with fiber $X(s_3s_2)$ (respectively $X(s_1s_2)$); both fibers are isomorphic to $\mathbb{P}^2$. Moreover, $\tilde{X}(s_1, s_3, s_2)$ admits a unique minimal family, consisting of the fibers of the natural morphism $\tilde{X}(s_1, s_3, s_2) \to \tilde{X}(s_1, s_3) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Likewise, $\hat{X}(s_1, s_3s_2)$ (respectively $\hat{X}(s_3, s_1s_2)$) admits a unique minimal family, consisting of the lines in the fibers of $\hat{\pi}_1$ (respectively $\hat{\pi}_3$). The two latter minimal families are sent isomorphically to the two families of lines in $X(w)$.

In geometric terms, $\hat{\pi}_{32}$ (respectively $\hat{\pi}_{12}$) is the blowing-up of $X(w)$ along its Weil divisor $X(s_3s_2)$ (respectively $X(s_1s_2)$), and the composition $\pi = \hat{\pi}_{32} \circ \tilde{\pi}_{32} = \hat{\pi}_{12} \circ \tilde{\pi}_{12}$ is the blowing-up at the singular point x. The above diagram gives back the Atiyah flop (see [Ati58]).

5. Some combinatorial results on generalized Bott–Samelson varieties

5.1 A sequence of roots

Throughout this section, we consider a simple, simply-laced and simply-connected algebraic group G, a minuscule parabolic subgroup $P = P_I = P^\alpha$, and a Weyl group element $w \in W^I$.

We identify roots and coroots via a W-invariant scalar product $\langle , \rangle$. For any $v \in W$, we set $R^+(v) := \{\gamma \in R^+ \mid v(\gamma) \in R^-\}$; then $|R^+(v)| = \ell(v)$.

Let $\hat{w} = (w_1, w_2, \ldots, w_m)$ be a generalized reduced decomposition of w obtained by Construction 1 of [Per05, § 5.4]. Choose a reduced decomposition $\tilde{w} = (s_{\beta_1}, \ldots, s_{\beta_r})$ of w that refines the above generalized reduced decomposition. We will freely use the associated quiver Q_w, as defined in [Per05, §§ 2.1 and 4.2]. In particular, this quiver has vertices $1, \ldots, r$, colored by simple roots via the map $\beta : j \mapsto \beta_j$. The set of peaks $\mathrm{Peaks}(Q_w)$ is equipped with an ordering $i_1 \preceq i_2 \preceq \ldots \preceq i_m$ that defines the above generalized reduced decomposition. We illustrate this on two examples which will be reconsidered repeatedly.

Example 5.1. Let $G = \mathrm{SL}_5$ (a simple group of type A_4) and $P = P^{\alpha_2}$. Then $X = G/P$ is the Grassmannian $\mathbb{G}(2, 5)$. Take

$$w = s_2s_1s_4s_3s_2$$

so that $X(w)$ is the Schubert divisor in X. We have $\mathrm{Peaks}(Q_w) = \{1, 3\}$ and $\beta(\mathrm{Peaks}(Q_w)) = \{\alpha_2, \alpha_4\}$. The generalized reduced decomposition associated with the standard ordering $\alpha_2 \preceq \alpha_4$ of $\mathrm{Peaks}(Q_w)$ has $w_1 = s_2s_1$ and $w_2 = s_4s_3s_2$. For the reverse ordering, $\alpha_4 \preceq' \alpha_2$, we have $w'_1 = s_4$ and $w'_2 = s_2s_1s_3s_2$.

Example 5.2. Let $G = \mathrm{SL}_9$ (type A_8) and $P = P^{\alpha_4}$; then $X = G/P = \mathbb{G}(4, 9)$. Take

$$w = s_3s_2s_1s_5s_4s_3s_2s_6s_5s_4s_3s_8s_7s_6s_5s_4,$$

then $X(w)$ has dimension 16. We have $\mathrm{Peaks}(Q_w) = \{1, 4, 12\}$ and $\beta(\mathrm{Peaks}(Q_w)) = \{\alpha_3, \alpha_5, \alpha_8\}$. The standard ordering $\alpha_3 \preceq \alpha_5 \preceq \alpha_8$ of $\mathrm{Peaks}(Q_w)$ has $w_1 = s_3s_2s_1$, $w_2 = s_5s_4s_3s_2s_6s_5s_4s_3$, and $w_3 = s_8s_7s_6s_5s_4$. For the ordering $\alpha_8 \preceq' \alpha_3 \preceq' \alpha_5$, we have $w'_1 = s_8$, $w'_2 = s_3s_2s_1$, and $w'_3 = s_5s_4s_3s_2s_6s_5s_4s_3s_7s_6s_5s_4$.

Lemma 5.3. *With the above notation, we have* $R^+(w_1^{-1}) \cap S = \{\beta_1\}$.

Proof. By [Per07, Proposition 5.13], Q_{w_1} has only one peak, namely, i_1. On the other hand, we have $\beta(\mathrm{Peaks}(Q_{w_1})) = R^+(w_1^{-1}) \cap S$. This yields the assertion. □

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By construction, there exists an increasing sequence $l_1 = 1 \leq l_2 < l_3 < \cdots < l_{m+1} = r$ of positive integers such that $w_1 = s_{\beta_{l_1}} \cdots s_{\beta_{l_2}}$ and $w_j = s_{\beta_{l_j+1}} \cdots s_{\beta_{l_{j+1}}}$ for all $2 \leq j \leq m$.

Let $v_i := s_{\beta_i} s_{\beta_{i-1}} \cdots s_{\beta_1}$ $(1 \leq i \leq r)$. Let $\gamma_i := v_i(\beta_1)$ $(1 \leq i \leq r)$; then γ_i is a negative root. In the rest of this subsection, we prove the following.

Proposition 5.4. *$\gamma_{i+1} \leq \gamma_i$ for all $1 \leq i \leq r-1$.*

We first obtain two preliminary results.

Lemma 5.5. *Let $\gamma \in R^+$. Let $v \in W$ be a minimal element such that $v(\gamma) = \alpha_0$, the highest root. Then for any $\mu \in R^+(v)$, we have $\langle \mu, \gamma \rangle = -1$.*

Proof. By induction on $\ell(v)$. If $\ell(v) = 1$, then we have $v = s_i$ for some i and $\gamma = s_i(\alpha_0) = \alpha_0 - \alpha_i$, since G is simply-laced. So, $\langle \alpha_i, \gamma \rangle = -1$.

Assume that $\ell(v) \geq 2$. Choose an integer i such that $\ell(vs_i) = \ell(v) - 1$. Since v is minimal such that $v(\gamma) = \alpha_0$, we have $\langle \alpha_i, \gamma \rangle = -1$. On the other hand by induction, for any $\mu \in R^+(v) \setminus \{\alpha_i\}$, we have $\langle \mu, \gamma \rangle = \langle s_i(\mu), s_i(\gamma) \rangle = -1$. □

Lemma 5.6. *For any $\mu \in R^+(w^{-1})$, we have $\langle \mu, \beta_1 \rangle \geq 0$.*

Proof. We have $-w^{-1}(\mu), -w^{-1}(\beta_1) \in R^+(w)$, and hence, $\alpha \leq -w^{-1}(\mu)$ and $\alpha \leq -w^{-1}(\beta_1)$. Now, if $\langle \mu, \beta_1 \rangle \leq -1$, then $\mu + \beta_1$ is a root and $\langle \varpi_\alpha, -w^{-1}(\mu + \beta_1) \rangle \geq \langle \varpi_\alpha, 2\alpha \rangle = 2$, contradicting the fact that ϖ_α is minuscule. Thus, we have $\langle \mu, \beta_1 \rangle \geq 0$. □

We may now prove Proposition 5.4.

Proof of Proposition 5.4. Let $v \in W$ be a minimal element such that $v(\beta_1) = \alpha_0$. Then $R^+(w^{-1}) \cap R^+(v) = \emptyset$ by Lemmas 5.5 and 5.6. Therefore, we have $\ell(w^{-1}v^{-1}) = \ell(w) + \ell(v)$. In particular, we have $\ell(v_i v^{-1}) = \ell(v) + i$ for all $1 \leq i \leq r$. Therefore, $(v_i v^{-1})^{-1}(\beta_{i+1})$ is a positive root. Since α_0 is dominant, we have

$$\langle v_i v^{-1}(\alpha_0), \beta_{i+1} \rangle = \langle \alpha_0, (v_i v^{-1})^{-1}(\beta_{i+1}) \rangle \geq 0.$$

Therefore, for all $1 \leq i \leq r-1$, we have

$$\gamma_{i+1} = v_{i+1}(\beta_1) = v_{i+1}v^{-1}(\alpha_0) = v_i v^{-1}(\alpha_0) - \langle v_i v^{-1}(\alpha_0), \beta_{i+1} \rangle \beta_{i+1} \leq v_i v^{-1}(\alpha_0) = \gamma_i.$$

This completes the proof. □

5.2 Root inequality

We keep the notation of § 5.1, and prove the following.

Proposition 5.7. *Assume that $w \neq w_1$. Then we have $w_1^{-1}(\beta_1) > w^{-1}(\beta_1)$.*

Example 5.8. With the notation of Example 5.1, we have $w_1^{-1}(\alpha_2) = -\alpha_1 - \alpha_2$ and $w^{-1}(\alpha_2) = -\alpha_1 - \alpha_2 - \alpha_3$. Therefore, we have indeed $w_1^{-1}(\alpha_2) > w^{-1}(\alpha_2)$. Also, $(w_1')^{-1}(\alpha_4) = -\alpha_4$ and $w^{-1}(\alpha_4) = -\alpha_2 - \alpha_3 - \alpha_4$, so that $(w_1')^{-1}(\alpha_4) > w^{-1}(\alpha_4)$.

In the setting of Example 5.2, we obtain $w_1^{-1}(\alpha_3) = -\alpha_1 - \alpha_2 - \alpha_3 > -\sum_{i=1}^{6} \alpha_i = w^{-1}(\alpha_3)$ and $(w_1')^{-1}(\alpha_8) = -\alpha_8 > -\sum_{i=4}^{8} \alpha_i = w^{-1}(\alpha_8)$.

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Proof of Proposition 5.7. We begin the proof with a preliminary result. □

Lemma 5.9. *We have* $\mathrm{Supp}(w_1) = \mathrm{Supp}(w_1^{-1}(\beta_1))$.

Proof. Recall that $v_i = s_{\beta_i} \cdots s_{\beta_1}$. Let $l = l_2$, then $v_l^{-1} = w_1$. Thus, it suffices to show that $\mathrm{Supp}(v_i) = \mathrm{Supp}(v_i(\beta_1))$ for all $1 \leq i \leq l$. As the inclusion $\mathrm{Supp}(v_i(\beta_1)) \subset \mathrm{Supp}(v_i)$ is obvious, it suffices in turn to prove the opposite inclusion.

We argue by induction on i, $1 \leq i \leq l$. If $i = 1$, then $\mathrm{Supp}(v_1) = \{\beta_1\} = \mathrm{Supp}(v_1(\beta_1))$. Let $1 \leq i \leq l-1$. By induction, we may assume that $\mathrm{Supp}(v_j) = \mathrm{Supp}(v_j(\beta_1))$ for all $1 \leq j \leq i$. If $s_{\beta_{i+1}} \leq v_i$, we are done by Proposition 5.4. Otherwise, $v_{i+1}(\beta_1) = v_i(\beta_1) - \langle v_i(\beta_1), \beta_{i+1}\rangle \beta_{i+1}$. Since $R^+(w_1^{-1}) \cap S = \{\beta_1\}$ and $s_{\beta_{i+1}} \not\leq v_i$, we have $\langle v_i(\beta_1), \beta_{i+1}\rangle \geq 0$. Further, if $\langle v_i(\beta_1), \beta_{i+1}\rangle = 0$, then we have $s_{\beta_{i+1}} v_i = v_i s_{\beta_{i+1}}$. Hence, $v_{i+1}(\beta_{i+1})$ is a negative root. As a consequence, we have $\beta_{i+1} \in R^+(w_1^{-1}) \cap S$, forcing $\beta_{i+1} = \beta_1$. This contradicts the fact that $s_{\beta_{i+1}} \not\leq v_i$. Hence, we have $\langle v_i(\beta_1), \beta_{i+1}\rangle \geq 1$. Thus, we obtain $\mathrm{Supp}(v_{i+1}) = \mathrm{Supp}(v_{i+1}(\beta_1))$. □

We may now prove Proposition 5.7. Since $R^+(w) \cap S = \{\alpha\}$, there is an integer $2 \leq t \leq m$ such that w_1 does not commute with w_t. Let $2 \leq t_0 \leq m$ be the least integer such that w_1 does not commute with w_{t_0}.

Let $s = \ell_{t_0} + 1$ and $e = \ell_{t_0+1}$. Then we have $w_{t_0} = s_{\beta_s} s_{\beta_{s+1}} \cdots s_{\beta_e}$. By [Per07, Definition 2.3] and [Per07, Definition 4.4(i)], it follows that for any $i < s$, $\langle \beta_i, \beta_s\rangle \neq 0$ implies $i \preceq s$. This is a contradiction to s being a peak. Therefore, we have $\langle \beta_i, \beta_s\rangle = 0$ for all $i < s$. So s_{β_s} commutes with w_i for all $1 \leq i \leq t_0 - 1$. In particular, s_{β_s} commutes with w_1.

Let $s+1 \leq k \leq e$ be the least integer such that s_{β_k} does not commute with w_1. Since s_{β_j} commutes with w_1 for all $s \leq j \leq k-1$ and since any two reduced decompositions of w differ only by commuting relations (see [Ste01, Proposition 2.1]), s_{β_j} commutes with s_{β_f} for all $1 \leq f \leq l$ and for all $s \leq j \leq k-1$. In particular, we have $s_{\beta_k} \not\leq w_1$. Since w_i commutes with w_1 for all $2 \leq i \leq t_0 - 1$, and s_{β_j} commutes with $v_l = w_1^{-1}$ for all $s \leq j \leq k-1$, we have $v_k(\beta_1) = s_{\beta_k}(w_1^{-1}(\beta_1))$.

On the other hand, we have

$$v_k(\beta_1) = s_{\beta_k}(w_1^{-1}(\beta_1)) = w_1^{-1}(\beta_1) - \langle w_1{-}1(\beta_1), \beta_k\rangle \beta_k.$$

By Lemma 5.9, we have $\mathrm{Supp}(w_1) = \mathrm{Supp}(w_1^{-1}(\beta_1))$. Since s_{β_k} does not commute with w_1, and $s_{\beta_k} \not\leq w_1$, we have $\langle w_1^{-1}(\beta_1), \beta_k\rangle \geq 1$. Thus, we have $v_k(\beta_1) < w_1^{-1}(\beta_1)$. By Proposition 5.4 again, we have $w^{-1}(\beta_1) \leq v_k(\beta_1)$. So, we are done.

5.3 Equality of Weyl groups

We still keep the notation of § 5.1, and recall from Lemma 5.3 that $R^+(w_1^{-1}) \cap S = \{\beta_1\}$. Also, note that we have

$$\mathrm{Peaks}(Q_w) = \{1, l_2 + 1, l_3 + 1, \ldots, l_m + 1\}.$$

By Construction 1, we have $\mathrm{Peaks}(Q_w) \cap Q_w(\{l_j + 1\}) = \{l_j + 1\}$. On the other hand, we have $\beta(\mathrm{Peaks}(Q_w)) = R^+(w^{-1}) \cap S$. Thus, we have $R^+(w_j^{-1}) \cap S = \{\beta_{l_j+1}\}$. Let $w' := w_2 w_3 \cdots w_m$. As $\mathrm{Peaks}(Q_{w'}) = \{l_2 + 1, l_3 + 1, \ldots, l_m + 1\}$, we have $R^+((w')^{-1}) \cap S = \{\beta_{l_2+1}, \beta_{l_3+1}, \ldots, \beta_{l_m+1}\}$.

Let T_{w_1} be the neutral component of $T \cap G_{w_1}$. Note that T_{w_1} is a maximal torus of G_{w_1}. Further, we have an isomorphism of Weyl groups $W(G_{w_1}, T_{w_1}) \simeq W(L_{\mathrm{Supp}(w_1)}, T)$. This identifies $W(G_{w_1}, T_{w_1})$ with a subgroup of W.

Lemma 5.10. *For any* $u \in W(G_{w_1}, T_{w_1})$, *we have* $\ell(uw') = \ell(u) + \ell(w')$.

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Proof. Since $\ell(v) = |R^+(v)|$ for all $v \in W$, it suffices to show that $R^+((w')^{-1}) \cap R^+(u) = \emptyset$.

Let $\gamma \in R^+((w')^{-1})$. Since $R^+((w')^{-1}) \cap S = \{\beta_{l_2+1}, \beta_{l_3+1}, \ldots, \beta_{l_m+1}\}$, there exists an integer $2 \leq j \leq m$ such that $\beta_{l_j+1} \leq \gamma$. By [Per07, Definition 2.3] and [Per07, Definition 4.4(i)], it follows that if there is an integer $1 \leq f \leq l_2$ such that $\langle \beta_f, \beta_{l_j+1} \rangle \neq 0$, then we have $f \preceq 1 + l_j$. This is a contradiction to $1 + l_j$ being a peak of Q_w. Therefore, $s_{\beta_{1+l_j}}$ commutes with s_{β_f} for all $1 \leq f \leq l_2$. In particular, we have $\beta_{l_j+1} \notin \mathrm{Supp}(w_1)$, and $s_\mu(\beta_{l_j+1}) = \beta_{l_j+1}$ for all $\mu \in \mathrm{Supp}(w_1)$. Further, since $u \in W(G_{w_1}, T_{w_1})$, we have $\mathrm{Supp}(u) \subset \mathrm{Supp}(w_1)$. Therefore, we have $u(\beta_{l_j+1}) = \beta_{l_j+1}$. In particular, the coefficient of β_{l_j+1} in the expression of $u(\gamma)$ is equal to the coefficient of β_{l_j+1} in the expression of γ, and it is positive since $\beta_{l_j+1} \leq \gamma$.

Thus, we have $\gamma \notin R^+(u)$ as desired. □

Example 5.11. With the notation of Example 5.1, the standard ordering of $\mathrm{Peaks}(Q_w)$ has $w_1 = s_2 s_1$ and $w' = w_2 = s_4 s_3 s_2$. Note that $R^+((w')^{-1}) \subset \{\beta \in R^+ \mid \alpha_4 \leq \beta\}$ and $s_4 \not\leq w_1$. Therefore, for any $u \in W(G_{w_1}, T_{w_1})$, we have indeed $\ell(uw') = \ell(u) + \ell(w')$.

For the reverse ordering of $\mathrm{Peaks}(Q_w)$, we have $w'_1 = s_4$, and $w' = w'_2 = s_2 s_1 s_3 s_2$. Clearly, for any $u \in W(G_{w'_1}, T_{w'_1})$, we have again $\ell(uw') = \ell(u) + \ell(w')$.

Example 5.12. In the setting of Example 5.2, for the standard ordering $\alpha_3 \preceq \alpha_5 \preceq \alpha_8$ of $\mathrm{Peaks}(Q_w)$, we have $w_1 = s_3 s_2 s_1$ and $w' = s_5 s_4 s_3 s_2 s_6 s_5 s_4 s_3 s_8 s_7 s_6 s_5 s_4$. As a consequence, $R^+((w')^{-1}) \subset \{\beta | \alpha_5 \leq \beta\} \cup \{\alpha_8\}$. So, for any $u \in W(G_{w_1}, T_{w_1})$, we obtain $\ell(uw') = \ell(u) + \ell(w')$ as asserted.

For the ordering $\alpha_8 \preceq' \alpha_3 \preceq' \alpha_5$, we have $w'_1 = s_8$ and $w' = s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4$. It readily follows that $\ell(uw') = \ell(u) + \ell(w')$ for any $u \in W(G_{w'_1}, T_{w'_1})$.

Proposition 5.13. *We have* $W(G_{w_1,\hat{x}}, T_{w_1}) = W(G_{w_1,w_1x_1}, T_{w_1})$.

Proof. Since $\hat{f} : \hat{X}(\hat{w}) \to X(w_1)$ is G_{w_1}-equivariant and $\hat{f}(\hat{x}) = w_1 x_1$, we have

$$W(G_{w_1,\hat{x}}, T_{w_1}) \subset W(G_{w_1,w_1x_1}, T_{w_1}).$$

Also, since $\hat{\pi} : \hat{X}(\hat{w}) \to X(w)$ is a local isomorphism at $\hat{x}$ and $\hat{\pi}(\hat{x}) = wx$, we have

$$W(G_{w_1,\hat{x}}, T_{w_1}) = W(G_{w_1,wx}, T_{w_1}).$$

Therefore, it suffices to prove that

$$W(G_{w_1,w_1x_1}, T_{w_1}) \subset W(G_{w_1,wx}, T_{w_1}).$$

Let $v \in W(G_{w_1,w_1x_1}, T_{w_1})$. Then $vw_1x_1 = w_1x_1$ and hence there exists $\tau \in W(P^{w_1}, T)$ such that $vw_1 = w_1\tau$. Thus, we have $w_1 \leq vw_1$ in $W(G_{w_1}, T_{w_1})$. Note that both v and w_1 are in $W(G_{w_1}, T_{w_1})$. Therefore, by Lemma 5.10, we have $\ell(vw_1w') = \ell(vw_1) + \ell(w')$. So, we have $w = w_1w' \leq vw_1w' = vw$. Since $w \in W^I$, it follows that $w \leq v'$ in W^I, where v' denotes the minimal representative of vw in W^I.

On the other hand, we have $G_{w_1} \subset G_w \cap P_w$. Therefore, $v \in W(G_w \cap P_w, T_w)$. Thus, we have $vX(w) = X(w)$. Therefore, vw is in a coset uW_I with $u \in W^I$ such that $u \leq w$. In particular, we have $v' \leq w$ in W^I. Thus, we obtain $v' = w$. Therefore, we have $vwx = wx$ as desired. □

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