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A GENERALISATION OF MACKEY'S THEOREM AND THE UNIFORM BOUNDEDNESS PRINCIPLE

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We construct a locally convex topology which is stronger than the Mackey topology but still has the same bounded sets as the Mackey topology. We use this topology to give a locally convex version of the Uniform Boundedness Principle which is valid without any completeness or barrelledness assumptions.

A well known theorem of Mackey ([**2**, 20.11.7]) states that a subset of a locally convex topological vector space is bounded if and only if it is weakly bounded. In particular, this means that if two vector spaces X and X' are in duality, then the weak topology and the Mackey topology of X have the same bounded sets. In this note we construct a locally convex topology on X which is always stronger than the Mackey topology but which still has the same bounded sets as the Mackey (or the weak) topology. We give an example which shows that this topology which always has the same bounded sets as the Mackey topology can be strictly stronger than the Mackey topology and, therefore, can fail to be compatible with the duality between X and X'. As an application of this result, we give several versions of the Uniform Boundedness Principle for locally convex spaces which involve no completeness or barrelledness assumptions.

For the remainder of this note, let X and X' be two vector spaces in duality with respect to the bilinear pairing $<,>$. The weak (Mackey, strong) topology on X will be denoted by $\sigma(X, X')(\tau(X, X'), \beta(X, X'))$. If (E, τ) is a topological vector space, a sequence $\{x_k\}$ in E is said to be $\tau - \mathcal{K}$ convergent to 0 if and only if every subsequence of $\{x_k\}$ has a subsequence $\{x_{n_k}\}$ such that the subseries $\sum x_{n_k}$ is τ-convergent to an element $x \in E$ ([**1**, Section 3]). If the topology τ is understood, we use the term $\mathcal{K}$ convergent. Note that a sequence which is $\tau - \mathcal{K}$ convergent to 0 is τ-convergent to 0, and, for example, if E is a complete metrisable space, then a sequence $\{x_k\}$ in E is convergent to 0 if and only if it is $\mathcal{K}$ convergent to 0 ([**1**, 3.2]). However, a sequence can converge to 0 and fail to be $\mathcal{K}$ convergent to 0; for example, let c_{00} be the vector space of all real-valued sequences which are eventually 0 equipped with the sup norm. If e_k is the sequence which has a 1 in the kth coordinate and 0 elsewhere, then the sequence $\{e_k/k\}$ is norm convergent to 0 but is not norm - $\mathcal{K}$ convergent to 0. By analogy with

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the Hyer criterion for boundedness in a topologial vector space ([**2**, 15.6.3]), we say that a subset $A \subseteq E$ is $\tau - \mathcal{K}$ bounded if whenever $\{x_k\} \subseteq A$ and $\{t_k\}$ is a sequence of positive scalars which converges to 0, then $\{t_k x_k\}$ is $\tau - \mathcal{K}$ convergent to 0 ([**1**, Section 3]). Thus, a $\tau - \mathcal{K}$ bounded subset is always τ bounded but not conversely. However, in a complete metrisable space a subset is bounded if and only if it is $\mathcal{K}$ bounded.

The locally convex topology on X which we consider is the topology of uniform convergence on $\sigma(X', X) - \mathcal{K}$ bounded subsets of X'. Since any $\sigma(X', X) - \mathcal{K}$ bounded subset of X' is $\sigma(X', X)$ bounded, this topology is weaker than the strong topology of X ([**2**, 21.2]). We present an example below (Example 10) which shows that this topology can be strictly weaker than the strong topology. We now show that this topology is stronger than the Mackey topology.

PROPOSITION 1. *If $B \subseteq X'$ is absolutely convex and $\sigma(X', X)$ compact, then B is $\sigma(X', X) - \mathcal{K}$ bounded.*

PROOF: Let $\{x'_k\} \subseteq B$ and let $\{t_k\}$ be a sequence of scalars which converges to 0. Given any subsequence of $\{t_k\}$ pick a further subsequence $\{t_{n_k}\}$ such that $\sum_{k=1}^{\infty} |t_{n_k}| \leqslant 1$. Since B is absolutely convex, $s_n = \sum_{k=1}^{n} t_{n_k} x'_{n_k} \in B$ for each n, and since $\{x'_k\}$ is $\sigma(X', X)$ bounded, the sequence $\{s_n\}$ is $\sigma(X', X)$ Cauchy

$$[\forall x \in X, \sum_{k=1}^{\infty} \left| t_{n_k} \langle x'_{n_k}, x \rangle \right| < \infty].$$

But, B is $\sigma(X', X)$ compact and, therefore, $\sigma(X', X)$ is complete so there exists $x' \in B$ such that $\{s_n\}$ is $\sigma(X', X)$ convergent to x'. Hence $\{t_k x'_k\}$ is $\sigma(X', X) - \mathcal{K}$ convergent to 0, and B is $\sigma(X', X) - \mathcal{K}$ bounded.

The Mackey topology on X, $\tau(X, X')$, is the topology of uniform convergence on absolutely convex, $\sigma(X', X)$ compact subsets of X' ([**2**, 21.4]). If we let $\mathcal{K}(X, X')$ be the topology of uniform convergence on $\sigma(X', X) - \mathcal{K}$ bounded subsets of X', it follows from Proposition 1 that $\mathcal{K}(X, X')$ is stronger than $\tau(X, X')$. We present an example below (Example 9) which shows that $\mathcal{K}(X, X')$ can be strictly stronger than the Mackey topology and, therefore, in general is not compatible with the duality between X and X' ([**2**, 21.4.2]). Despite this fact, we next show that a subset of X is $\sigma(X, X')(\tau(X, X'))$ bounded if and only if it is $\mathcal{K}(X, X')$ bounded. For the proof of this result we use the Antosik–Mikusinski Diagonal Theorem ([**1**, Section 2]). For the sake of completeness, we give a statement of the scalar version of this theorem which we use below. ∎

THEOREM 2. *(Antosik–Mikusinski) Let $M = [t_{ij}]$ be an infinite matrix of scalars. Suppose that the columns of M converge to 0 and that every increasing sequence of*

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positive integers $\{m_j\}$ *has a subsequence* $\{n_j\}$ *such that the subseries* $\sum_{j=1}^{\infty} t_{in_j} = t_i$ *converge and* $\lim t_i = 0$. *Then* $\lim t_{ii} = 0$.

For a proof of this theorem in a more general setting see [**1**, 2.2].

THEOREM 3. *If* $A \subseteq X$ *is* $\sigma(X, X')$ *bounded, then* A *is uniformly bounded on* $\sigma(X', X) - \mathcal{K}$ *bounded subsets of* X'.

PROOF: Let $B \subseteq X'$ be $\sigma(X', X) - \mathcal{K}$ bounded. It suffices to show that $\{\langle x'_j, x_j \rangle\}$ is bounded whenever $\{x_j\} \subseteq A$ and $\{x'_j\} \subseteq B$. Let $t_j > 0$, $\lim t_j = 0$. Consider the matrix $M = [\langle \sqrt{t_j} x'_j, \sqrt{t_i} x_i \rangle]$. Since $\{x_i\}$ is $\sigma(X, X')$ bounded, the columns of M converge to 0. If $\{m_j\}$ is an increasing sequence of positive integers, then since $\{\sqrt{t_j} x'_j\}$ is $\sigma(X', X) - \mathcal{K}$ convergent to 0, there is a subsequence $\{n_j\}$ of $\{m_j\}$ such that the series $\sum_{j=1}^{\infty} \sqrt{t_{n_j}} x'_{n_j}$ is $\sigma(X', X)$ convergent to some $x' \in X'$. Thus, $\sum_{j=1}^{\infty} \langle \sqrt{t_{n_j}} x'_{n_j}, \sqrt{t_i} x_i \rangle = \langle x', \sqrt{t_i} x_i \rangle \to 0$, and from the Diagonal Theorem, it follows that $t_i \langle x'_i, x_i \rangle \to 0$ so $\{\langle x'_i, x_i \rangle\}$ is bounded. ∎

It follows from Theorem 3 that a subset $A \subseteq X$ is $\sigma(X, X')(\tau(X, X'))$ bounded if and only if it is $\mathcal{K}(X, X')$ bounded, and since $\mathcal{K}(X, X')$ is stronger than $\tau(X, X')$, Mackey's Theorem ([**2**, 20.11.7]) is a special case of Theorem 3. Thus, if X is a bornological space, then the topology of X is equal to $\mathcal{K}(X, X')$. It is also of interest to contrast the proof of Theorem 3 with the proof of Mackey's Theorem given in [**2**, 20.11.7].

As an application of Theorem 3 we give a generalisation of the classical Uniform Boundedness Principle for locally convex spaces which involves no completeness or barrelledness assumptions. Let E and F be locally convex Hausdorff spaces and let $T_i \colon E \to F$ be a sequence of continuous linear operators. The adjoint of T_i, $T'_i \colon F' \to E'$, is defined by $\langle T'_i y', x \rangle = \langle y', T_i x \rangle$.

THEOREM 4. *Suppose that* $\{T_i x : i \in \mathbb{N}\}$ *is bounded in* F *for each* $x \in E$. *If* $B \subseteq F'$ *is* $\beta(F', F)$ *bounded, then* $\{T'_i y' : i \in \mathbb{N}, y' \in B\}$ *is uniformly bounded on* $\sigma(E, E') - \mathcal{K}$ *bounded subsets of* E.

PROOF: Since $\{T_i x : i \in \mathbb{N}\}$ is bounded in F for each $x \in E$, $\{\langle T'_i y', x \rangle : y' \in B, i \in \mathbb{N}\}$ is bounded, that is, $\{T'_i y' : y' \in B, i \in \mathbb{N}\}$ is $\sigma(E', E)$ bounded. The result now follows from Theorem 3. ∎

We now show that Theorem 4 gives a generalisation of the Uniform Boundedness Principle (UBP). For normed spaces this asserts that a sequence of continuous linear operators from a Banach space into a normed space which is pointwise bounded is

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actually uniformly bounded on bounded subsets of the domain space ([**1**, 4.1], [**2**, 15.3]). Without a second category assumption on the domain space, this result is no longer valid ([**1**, Section 4]). In order to give versions of the UBP without completeness-type assumptions on the domain space, we seek a family of bounded subsets of the domain space which has the property that a pointwise bounded sequence of continuous linear operators is uniformly bounded on each member of the family. Such results for metric linear spaces are given in [**1**, Section 4]. Using Theorem 4, we can derive such a UBP result for locally convex spaces.

Corollary 5. *If $T_i: E \to F$ is pointwise bounded on E, then $\{T_i\}$ is uniformly bounded on $\sigma(E, E') - \mathcal{K}$ bounded subsets of E.*

Proof: Let $A \subseteq E$ be $\sigma(E, E') - \mathcal{K}$ bounded. Then $\{T_i x : x \in A, i \in \mathbb{N}\}$ is bounded since for each $y' \in F'$, $\{T_i' y' : i \in \mathbb{N}\}$ is uniformly bounded on A by Theorem 4. □

A topological vector space (X, τ) is called a $\mathcal{K}$-space if every sequence which is τ-convergent to 0 is $\tau - \mathcal{K}$ convergent to 0 ([**1**, Section 3]). A complete metric linear space is a $\mathcal{K}$-space but there are $\mathcal{K}$-spaces which are not complete ([**1**, Section 3]). In a $\mathcal{K}$-space every τ-bounded set is $\tau - \mathcal{K}$ bounded. From Corollary 5, we have

Corollary 6. *If E is a $\mathcal{K}$-space for any topology τ which is compatible with the duality between E and E', then any pointwise bounded sequence of continuous linear operators $T_i: E \to F$ is uniformly bounded on bounded subsets of E.*

Proof: If $A \subseteq E$ is bounded, then A is τ-bounded and, hence $\tau - \mathcal{K}$ bounded and $\sigma(E, E') - \mathcal{K}$ bounded. The result follows from Corollary 5. □

For the case when E is metrisable, this generalises Corollary 4.5 of [**1**]. In the locally convex version of the UBP for barrelled spaces, the conclusion is that a pointwise bounded sequence of continuous linear operators is equicontinuous. The following Proposition gives sufficient conditions which guarantee equicontinuity from the conclusion of Corollary 6.

Proposition 7. *Let $T_i: E \to F$ be linear and continuous. Consider*

(a) *$\{T_i\}$ is equicontinuous;*

(b) *$\{T_i\}$ is uniformly bounded on bounded subsets of E.*

Always (a) implies (b), and if E is bornological, then (b) implies (a).

Proof: That (a) implies (b) is easily checked. Assume that (b) holds and let V be an absolutely convex neighbourhood of 0 in F. Set $U = \bigcap_{i=1}^{\infty} T_i^{-1}(V)$. Then U is absolutely convex, and if $A \subseteq E$ is bounded, $\{T_i x : x \in A, i \in \mathbb{N}\} = B$ is bounded so

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there exists $t > 0$ such that $tB \subseteq V$ or $tA \subseteq U$. Therefore, U absorbs bounded sets and is a neighbourhood of 0 in E. $\square$

It follows from Corollary 6 and Proposition 7, that if E is a bornological $\mathcal{K}$-space, then any pointwise bounded sequence of continuous linear operators on E is equicontinuous. Since any regular inductive limit of $\mathcal{K}$-spaces (Frechet spaces) is a $\mathcal{K}$-space, Proposition 7 gives a generalisation of the UBP in [**1**, 4.5] to locally convex spaces.

We close by giving several examples. First we present an example which shows that, in general, (b) does not imply (a).

Example 8. Let c_0 be the space of all real-valued sequences which converge to 0 equipped with the weak topology $\sigma(c_0, \ell^1)$. Note that a subset of c_0 is $\sigma(c_0, \ell^1)$ bounded if and only if it is bounded in the sup-norm, $\| \|_\infty$. Thus, a subset $B \subseteq \ell^1$ is uniformly bounded on $\sigma(c_0, \ell^1)$ bounded subsets if and only if it is bounded in ℓ^1-norm. If e_n is the sequence with a 1 in the nth coordinate and 0 elsewhere, then $\{e_n\} \subseteq \ell^1$ is uniformly bounded on $\sigma(c_0, \ell^1)$ bounded subsets but is not $\sigma(c_0, \ell^1)$ equicontinuous since $\{e_n\}$ converges to 0 in $\sigma(c_0, \ell^1)$ but $\langle e_n, e_n\rangle = 1$ does not converge to 0.

As noted above, we always have $\tau(X, X') \subseteq \mathcal{K}(X, X') \subseteq \beta(X, X')$. In the two examples below, we show that both of these containments can be proper.

Example 9. Let $X = \ell^\infty$, $X' = \ell^1$. It is easy to check that the family of $\sigma(\ell^1, \ell^\infty)-\mathcal{K}$ bounded subsets of ℓ^1 is exactly the family of norm-bounded subsets of ℓ^1. Thus, the topology $\mathcal{K}(\ell^\infty, \ell^1)$ on ℓ^∞ is just the norm topology of ℓ^∞. Since the dual of ℓ^∞ under the norm topology is $BV(\mathbb{N})$ ([**2**, 31.1]), this shows that $\mathcal{K}(\ell^\infty, \ell^1)$ is strictly stronger than the Mackey topology $\tau(\ell^\infty, \ell^1)$.

Example 10. Let $X = X' = c_{00}$. Then X and X' are in duality under the pairing $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$, where $x = \{x_i\}$, $y = \{y_i\} \in c_{00}$. A subset $M \subseteq c_{00}$ is $\sigma(c_{00}, c_{00})$ bounded or $\mathcal{K}(c_{00}, c_{00})$ bounded if and only if the elements of M are coordinatewise bounded, and M is strongly bounded if and only if there is n_0 such that $x_i = 0$ for $x = \{x_i\} \in M$ and $i \geqslant n_0$ ([**2**, 21.11]). Thus, the strong topology is strictly stronger than $\mathcal{K}(c_{00}, c_{00})$ in this case.

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