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POLES OF L-FUNCTIONS AND THETA LIFTINGS FOR ORTHOGONAL GROUPS

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In memory of our teacher Ilya Piatetski-Shapiro

Abstract In this paper, we prove that the first occurrence of global theta liftings from any orthogonal group to either symplectic groups or metaplectic groups can be characterized completely in terms of the location of poles of certain Eisenstein series. This extends the work of Kudla and Rallis and the work of Moeglin to all orthogonal groups. As applications, we obtain results about basic structures of cuspidal automorphic representations and the domain of holomorphy of twisted standard L-functions.

Keywords: orthogonal groups; theta liftings; automorphic forms; L-functions; periods

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1. Introduction

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. Let O_m be the orthogonal group associated to a non-degenerate quadratic k-vector space X of dimension m. Let χ be a quadratic character of $\mathbb{A}^\times/k^\times$. By means of the spinor norm, one may view χ as an automorphic character of $\mathrm{O}_m(\mathbb{A})$. We denote by $\mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ the set of irreducible cuspidal automorphic representations of $\mathrm{O}_m(\mathbb{A})$, which occur as irreducible subspaces in the space of cuspidal automorphic functions on $\mathrm{O}_m(\mathbb{A})$. For any $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$, we study in this paper the location of poles of the partial twisted standard L-functions $L^S(s, \sigma\otimes\chi)$, where S is a finite set of local places of k including all archimedean local places.

By the doubling method of Piatetski-Shapiro and Rallis [**5**, Part I], the partial twisted standard L-function $L^S(s, \sigma\otimes\chi)$ has at most simple poles at $\mathrm{Re}(s) > \frac{1}{2}$, and the possible poles may occur at $s = \frac{1}{2}m - j > 0$ [**5**, **17**]. It is a program of Kudla and Rallis to determine the location of the poles of $L^S(s, \sigma\otimes\chi)$ in terms of the non-vanishing of the relevant theta liftings via the regularized Siegel–Weil formula [**18**, **25**].

We recall some basic facts from the theory of the theta correspondence. Let Sp_{2l} be the symplectic group of k-rank l. Then $(\mathrm{O}_m, \mathrm{Sp}_{2l})$ forms a reductive dual pair in Sp_{2lm}, in the sense of Howe [**9**]. We denote by $\mathrm{Mp}_{2l}(\mathbb{A})$ the metaplectic double cover $\widetilde{\mathrm{Sp}}_{2l}(\mathbb{A})$ of $\mathrm{Sp}_{2l}(\mathbb{A})$ if $m = 2n+1$ or the $\mathbb{A}$-rational points $\mathrm{Sp}_{2l}(\mathbb{A})$ of Sp_{2l} if $m = 2n$. For a non-trivial character ψ of $\mathbb{A}/k$, there exists the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}_{2lm}(\mathbb{A})$, which is realized in the Schrödinger model $\mathcal{S}(\mathbb{A}^{ml})$, where $\mathcal{S}(\mathbb{A}^{ml})$ is the space of $\mathbb{C}$-valued Schwartz–Bruhat functions on $\mathbb{A}^{ml}$.

For $\varphi \in \mathcal{S}(\mathbb{A}^{ml})$, we form the theta function

$$\theta_{\psi,\varphi}(x) := \sum_{\xi \in k^{ml}} \omega_\psi(x)(\varphi)(\xi),$$

on $\widetilde{\mathrm{Sp}}_{2ml}(\mathbb{A})$. There is a natural homomorphism

$$\mathrm{O}_m(\mathbb{A}) \times \mathrm{Mp}_{2l}(\mathbb{A}) \to \widetilde{\mathrm{Sp}}_{2ml}(\mathbb{A})$$

with the kernel $C_2 = \{\pm 1\}$, and the centre of $\mathrm{O}_m(\mathbb{A})$ diagonally embedded. We pull the Weil representation ω_ψ back to $\mathrm{O}_m(\mathbb{A}) \times \mathrm{Mp}_{2l}(\mathbb{A})$. The details may be found in [**15**], for instance.

For a $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$, the integral

$$\theta^{2l}_{\psi,m}(g; \phi_\sigma, \varphi) := \int_{\mathrm{O}_m(k)\backslash \mathrm{O}_m(\mathbb{A})} \phi_\sigma(h)\theta_{\psi,\varphi}(g,h)\,\mathrm{d}h, \tag{1.1}$$

with $\phi_\sigma \in V_\sigma$, defines an automorphic function on $\mathrm{Mp}_{2l}(\mathbb{A})$. We denote by $\theta^{2l}_{\psi,m}(\sigma)$ the space of the automorphic representation generated by all $\theta^{2l}_{\psi,m}(g; \phi_\sigma, \varphi)$, as φ and ϕ_σ vary, and call $\theta^{2l}_{\psi,m}(\sigma)$ the ψ-theta lifting of σ to $\mathrm{Mp}_{2l}(\mathbb{A})$.

A basic problem in the theory of the theta correspondence is to determine when the ψ-theta lifting $\theta^{2l}_{\psi,m}(\sigma)$ is non-zero. By the Rallis theta tower property [**24**], if for any $l_1 < l_0$, the ψ-theta lifting of σ, $\theta^{2l_1}_{\psi,m}(\sigma)$ to $\mathrm{Mp}_{2l_1}(\mathbb{A})$ is zero, then the ψ-theta lifting of σ, $\theta^{2l_0}_{\psi,m}(\sigma)$ is a cuspidal automorphic representation of $\mathrm{Mp}_{2l_0}(\mathbb{A})$; and if $\theta^{2l_0}_{\psi,m}(\sigma)$ is non-zero and cuspidal, then for any $l_2 > l_0$, the space $\theta^{2l_2}_{\psi,m}(\sigma)$ consists of automorphic functions on $\mathrm{Mp}_{2l_2}(\mathbb{A})$, which are no longer cuspidal. In fact, by the work of Moeglin [**20, 21**], one knows that they are orthogonal to the space of cuspidal automorphic forms on $\mathrm{Mp}_{2l_2}(\mathbb{A})$. In this case, the integer l_0 is called the *first occurrence* of σ in the Witt tower of $\mathrm{Mp}_{2l}(\mathbb{A})$, and is denoted by $\mathrm{FO}_\psi(\sigma) := 2l_0$. Furthermore, for a given $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$, it is proved in [**24**] that the first occurrence $\mathrm{FO}_\psi(\sigma)$ is a positive integer, which is at most $2m$. It is interesting and important to characterize the first occurrence $\mathrm{FO}_\psi(\sigma)$ in terms of basic properties of σ. We remark that for any $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$, if the first occurrence $\mathrm{FO}_\psi(\sigma) = 2l_0$, then the ψ-theta lifting of σ, $\theta^{2l_0}_{\psi,m}(\sigma)$ is a non-zero irreducible cuspidal automorphic representation of $\mathrm{Mp}_{2l_0}(\mathbb{A})$. This is proved in [**21**] when m is even and in [**15**] when m is odd.

The same problem for the first occurrence can be formulated for irreducible cuspidal automorphic representations of $\mathrm{Mp}_{2l}(\mathbb{A})$.

In [**18**], Kudla and Rallis illustrated their theory for the symplectic groups Sp_{2l}. It is clear that their arguments work for O_m and for Mp_{2l} provided that the relevant theories

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for O_m and for Mp_{2l} are established, respectively. In this paper, we prove the following orthogonal group analogue of Theorem 7.2.5 of [**18**] as a consequence of one of the main results of this paper (Theorem 1.3).

Note that because of the disconnectedness of O_m, there are automorphic sign characters of $\mathrm{O}_m(\mathbb{A})$ which are trivial on $\mathrm{O}_m(k)$. Such characters can be constructed as follows. Consider the adelic group of the group $\mathrm{Z}_2 := \{\pm 1\}$ and consider any character ϵ of $\mathrm{Z}_2(\mathbb{A})$ which takes the value 1 at all local places v of k, except an even number of local places, where it takes value -1. such characters of $\mathrm{Z}_2(\mathbb{A})$ induce automorphic sign characters of $\mathrm{O}_m(\mathbb{A})$. For any automorphic sign character ϵ and any $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$, $L(s, \sigma_v) = L^S(s, \sigma_v \otimes \epsilon_v)$, for all places v. On the other hand, σ and $\sigma \otimes \epsilon$ may have different first occurrences when they are lifted by ψ-theta correspondence to the tower $\mathrm{Mp}_{2l}(\mathbb{A})$.

It is natural to introduce the notion of the lowest occurrence $\mathrm{LO}_\psi(\sigma)$ of σ, with respect to all twists by automorphic sign characters of $\mathrm{O}_m(\mathbb{A})$, in the tower $\mathrm{Mp}_{2l}(\mathbb{A})$ via ψ-theta correspondence. We define it by

$$\mathrm{LO}_\psi(\sigma) := \min_{\epsilon}\{\mathrm{FO}_\psi(\sigma \otimes \epsilon)\}, \tag{1.2}$$

where ϵ runs through all automorphic sign characters of $\mathrm{O}_m(\mathbb{A})$.

We prove the following theorem, which is the orthogonal group analogue of Theorem 7.2.5 of [**18**]. Here, as pointed to us by Moeglin, the choice of a twist by a sign character is analogous to the choice of a Hasse invariant in Theorem 7.2.5 of [**18**].

Theorem 1.1. *Let O_m be the orthogonal group attached to a quadratic k-vector space of dimension m, and let χ be a quadratic character of $k^\times\backslash\mathbb{A}^\times$ and $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$.*

(1) *If the partial L-function $L^S(s, \sigma \otimes \chi)$ has a pole at $s_0 = \frac{1}{2}m - j > 0$, or if m is odd and the partial L-function $L^S(s, \sigma \otimes \chi)$ does not vanish at $s = \frac{1}{2}$, i.e. $j = 2[\frac{1}{2}m]$, then there is an automorphic sign character ϵ of $\mathrm{O}_m(\mathbb{A})$, such that the ψ-theta lifting of $(\sigma \otimes \chi) \otimes \epsilon$ to $\mathrm{Mp}_{2j}(\mathbb{A})$, does not vanish, that is, $\mathrm{LO}_\psi(\sigma \otimes \chi) \leqslant 2j$.*

(2) *If the lowest occurrence of $\sigma \otimes \chi$, $\mathrm{LO}_\psi(\sigma \otimes \chi)$ is $2j_0 < m$, then the partial L-function $L^S(s, \sigma \otimes \chi)$ is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}m - j_0$.*

(3) *If (in (2)) $2j_0 \geqslant m$, then the partial L-function $L^S(s, \sigma \otimes \chi)$ is holomorphic for $\mathrm{Re}(s) \geqslant \frac{1}{2}$.*

This theorem will be proved in § 4, by using Theorem 3.1, which is a preliminary step towards the main theorem in this paper (Theorem 1.3 below). We remark that in Theorem 1.1 (2), it is not known if the partial L-function $L^S(s, \sigma \otimes \chi)$ has a pole at $s = s_0 = \frac{1}{2}m - j_0$. We state it as a conjecture.

Conjecture 1.2. *Let O_m be the orthogonal group attached to a quadratic k-vector space of dimension m, and let χ be a quadratic character of $k^\times\backslash\mathbb{A}^\times$ and $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$. The partial L-function $L^S(s, \sigma \otimes \chi)$ has a pole at $s_0 = \frac{1}{2}m - j_0 > 0$ and is holomorphic for $\mathrm{Re}(s) > s_0$, or $L^S(\frac{1}{2}, \sigma \otimes \chi) \neq 0$ when $m = 2n+1$ and $j_0 = n$, if and only if the lowest occurrence $\mathrm{LO}_\psi(\sigma \otimes \chi) = 2j_0$.*

It is clear that new ideas are needed in order to determine the intrinsic relation between the precise location of the possible poles of $L^S(s, \sigma\otimes\chi)$ and the first occurrence of $\sigma\otimes\chi$, $\mathrm{FO}_\psi(\sigma\otimes\chi)$, or more naturally, the lowest occurrence $\mathrm{LO}_\psi(\sigma\otimes\chi)$ as defined in (1.2).

It was Moeglin's idea [**20, 21**] to replace the partial L-function $L^S(s, \sigma\otimes\chi)$ by the Eisenstein series $E(g;\phi_{\sigma\otimes\chi}, s)$ on $\mathrm{O}_{m+2}(\mathbb{A})$ attached to the cuspidal datum $(Q_1, \chi\otimes\sigma)$, where the Levi part of Q_1 is $\mathrm{GL}_1\times\mathrm{O}_m$. The precise definition of $E(g;\phi_{\sigma\otimes\chi}, s)$ is given in §§ 2 and 4. The key point in using this Eisenstein series is that the constant term of $E(g;\phi_{\sigma\otimes\chi}, s)$ along the maximal parabolic subgroup Q_1 is closely related to $L^S(s, \sigma\otimes\chi)$. We refer to §§ 2 and 4 for more detailed discussion.

We denote by $\mathcal{P}(\sigma\otimes\chi, Q_1)$ the set of positive poles of the Eisenstein series $E(g;\phi_{\sigma\otimes\chi}, s)$. As in [**20**] and [**21**], for m even, one can prove that for a given cuspidal datum $(Q_1, \chi\otimes\sigma)$, if the set $\mathcal{P}(\sigma\otimes\chi, Q_1)$ is non-empty, then the maximal member in the set $\mathcal{P}(\sigma\otimes\chi, Q_1)$ must be of form $\frac{1}{2}m - j > 0$. In this case, there is an automorphic sign character ϵ, such that the ψ-theta lifting of $(\sigma\otimes\chi)\otimes\epsilon$ to $\mathrm{Mp}_{2j}(\mathbb{A})$ is non-zero. Hence the lowest occurrence $\mathrm{LO}_\psi(\sigma\otimes\chi)\leqslant 2j$.

One of the main results in this paper is to show, that for any orthogonal group O_m and for any $\sigma\in\mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$, if the set $\mathcal{P}(\sigma\otimes\chi, Q_1)$ is non-empty, then the maximal member $s_0 = \frac{1}{2}m - j_0 > 0$ yields the exact information on the lowest occurrence $\mathrm{LO}_\psi(\sigma\otimes\chi)$. More precisely, we have the following theorem.

Theorem 1.3. *Let O_m be the orthogonal group attached to a quadratic k-vector space of dimension m, and let χ be a quadratic character of $k^\times\backslash\mathbb{A}^\times$ and $\sigma\in\mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$. Assume that the set $\mathcal{P}(\sigma\otimes\chi, Q_1)$ as defined above is non-empty. Then the maximal member in $\mathcal{P}(\sigma\otimes\chi, Q_1)$ is of the form $s_0 = \frac{1}{2}m - j_0 > 0$, where j_0 is a positive integer, if and only if the lowest occurrence $\mathrm{LO}_\psi(\sigma\otimes\chi)$ is $2j_0$.*

This theorem is a combination of Theorem 3.1 and Theorem 5.1, which are proved in § 3 and § 6, respectively. Theorem 1.3 will be proved in § 5. Theorem 3.1 will be proved by following the same arguments as in [**20**] and [**21**], for m is even. Theorem 5.1 determines the existence of the pole of $E(g;\phi_{\sigma\otimes\chi}, s)$ in terms of the first occurrence $\mathrm{FO}_\psi(\sigma\otimes\chi)$.

It should be mentioned that Theorem 5.1 contains also a result for $s_0 = \frac{1}{2}m - j_0 < 0$. This is new and is a result of the application of the Arthur truncation method in the content of theta correspondence. More precisely, we apply the Arthur truncation to the Eisenstein series $E(g;\phi_{\sigma\otimes\chi}, s)$ and show that if the first occurrence $\mathrm{FO}_\psi(\sigma\otimes\chi) = 2j < m$, a certain period of the truncation of the Eisenstein series $E(g;\phi_{\sigma\otimes\chi}, s)$ has a pole at $s = \frac{1}{2}m - j$. In particular, the Eisenstein series $E(g;\phi_{\sigma\otimes\chi}, s)$ has a pole at $s = \frac{1}{2}m - j$. When $\frac{1}{2}m - j > 0$, we prove that all the integrals are absolutely convergent. However, when $\frac{1}{2}m - j < 0$, we impose the condition that $6 < m$ and $\frac{1}{2}m < j < m-2$ to avoid the technical complication of proving absolute convergence of certain integrals in this case. See Theorem 5.1 and its proof in § 6 for details.

Based on this, when $\frac{1}{2}m - j > 0$, we apply again the Arthur truncation to the residue of the Eisenstein series $E(g;\phi_{\sigma\otimes\chi}, s)$ at $s = \frac{1}{2}m - j$ and show that such periods converges absolutely and can be expressed in terms of the similar periods of the cuspidal datum $\sigma\otimes\chi$ (Theorem 6.6). In particular, if the first occurrence $\mathrm{FO}_\psi(\sigma\otimes\chi) = 2j < m$, then the residue at $s = \frac{1}{2}m - j$ of the Eisenstein series $E(g;\phi_{\sigma\otimes\chi}, s)$ has a non-zero period over a certain (smaller) orthogonal group (Theorem 5.3). This is also a new feature that the Arthur truncation method is applied in the content of theta correspondence.

We prove Theorems 5.1, 5.3 and 6.6 in § 6.

In case the Eisenstein series is holomorphic for $\mathrm{Re}(s) > 0$, i.e. the set $\mathcal{P}(\sigma \otimes \chi, Q_1)$ is empty, we have the following conjecture.

Conjecture 1.4. *The set* $\mathcal{P}(\sigma \otimes \chi, Q_1)$ *is empty if and only if the lowest occurrence* $\mathrm{LO}_\psi(\sigma \otimes \chi)$ *is equal to* $2[\frac{1}{2}m] + 2$.

In one direction, when $m = 2n + 1$ is odd, if the set $\mathcal{P}(\sigma \otimes \chi, Q_1)$ is empty, then the L-function $L^S(s, \sigma \otimes \chi)$ is holomorphic for $\mathrm{Re}(s) \geqslant \frac{1}{2}$. We expect, in this case, that the lowest occurrence $\mathrm{LO}_\psi(\sigma \otimes \chi)$ is equal to $2n$ if and only if $L^S(\frac{1}{2}, \sigma \otimes \chi) \neq 0$.

When O_m is a k-split even orthogonal group and σ is an irreducible, automorphic, cuspidal representation of $\mathrm{SO}_m(\mathbb{A})$, and assumed to be generic, i.e. has a non-zero Whittaker–Fourier coefficient, it is proved in [**6**] that $L^S(s, \sigma \otimes \chi)$ is holomorphic for $\mathrm{Re}(s) > 1$. Furthermore, it is proved in [**6**] that $L^S(s, \sigma \otimes \chi)$ has a pole at $s = 1$, if and only if the first occurrence of $\sigma \otimes \chi$, $\mathrm{FO}_\psi(\sigma \otimes \chi)$ is $m - 2$; otherwise, the first occurrence of $\sigma \otimes \chi$, $\mathrm{FO}_\psi(\sigma \otimes \chi)$ is m. Here, the notion of first occurrence for the pair $(\mathrm{SO}_m, \mathrm{Sp}_{2n})$ is defined in the same way. When O_m is a k-split odd orthogonal group and, as above, $\sigma \in \mathcal{A}^c(\mathrm{SO}_m / k)$ is generic, it follows from the Langlands functorial transfer from SO_m to GL_{m-1} [**4**] that $L^S(s, \sigma \otimes \chi)$ is holomorphic for $\mathrm{Re}(s) \geqslant \frac{1}{2}$. It is proved in [**14**] and [**15**] that $L^S(s, \sigma \otimes \chi, \frac{1}{2}) \neq 0$, if and only if the first occurrence of $\sigma \otimes \chi$, $\mathrm{FO}_\psi(\sigma \otimes \chi)$ is $m - 1$; otherwise, the first occurrence of $\sigma \otimes \chi$, $\mathrm{FO}_\psi(\sigma \otimes \chi)$ is $m + 1$. In this case, the non-vanishing of $L^S(s, \sigma \otimes \chi)$ at $s = \frac{1}{2}$ is equivalent to the existence of the pole of $E(g; \phi_{\sigma\otimes\chi,s})$ at $s = \frac{1}{2}$. Hence both Conjectures 1.2 and 1.4 hold for irreducible generic cuspidal automorphic representations σ of k-split orthogonal groups SO_m. See § 7 for more details.

Combining the above results with those in [**13**], [**14**] and [**15**], we can deduce the following interesting consequences, which may be deduced from the Arthur conjecture (or Arthur's Theorem assuming the availability of various types of the fundamental lemmas) on the basic structures of the discrete spectrum of automorphic forms [**3**].

Theorem 1.5. *Let* SO_m *be the special orthogonal group attached to a quadratic* k*-vector space of dimension* m*, and let* χ *be a quadratic character of* $k^\times \backslash \mathbb{A}^\times$ *and* $\sigma = \otimes_v \sigma_v \in \mathcal{A}^c(\mathrm{SO}_m / k)$*. If there is a local place* v*, such that* $\mathrm{SO}_m(k_v)$ *is* k_v*-quasi-split and the* v*-local component* σ_v *is (locally) generic, then the partial L-function* $L^S(s, \sigma \otimes \chi)$ *is holomorphic for* $\mathrm{Re}(s) > s_0$*, where* $s_0 = 1$ *if* m *is even and* $s_0 = \frac{1}{2}$ *if* m *is odd.*

Next, we consider a special case when SO_m is the k-split odd orthogonal group ($m = 2n + 1$), and σ is an irreducible, automorphic, cuspidal representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$, with Bessel model of special type. Recall that the Whittaker–Fourier coefficients are the Fourier coefficients attached to the regular nilpotent orbit, while the Bessel model of special type may be viewed as certain Fourier coefficients attached to the subregular nilpotent orbit, which have a non-trivial period along a certain subgroup, isomorphic some SO_2. See § 7 and [**14**] for details.

Theorem 1.6. *Let* SO_{2n+1} *be the* k*-split odd special orthogonal group. Assume that* $\sigma = \otimes_v \sigma_v \in \mathcal{A}^{\mathrm{c}}(\mathrm{SO}_{2n+1}/k)$ *has a non-zero (global) Bessel model of special type. If there is a place* v*, such that the* v*-local component* σ_v *is (locally) generic, then* σ *is nearly equivalent to an irreducible generic (globally) cuspidal automorphic representation* σ' *of* $\mathrm{SO}_{2n+1}(\mathbb{A})$*.*

We will give the proofs for Theorem 1.5 and Theorem 1.6 in § 7, when v is a finite place of k. When v is an archimedean place of k, the proofs are similar, and we omit them. We remark that Theorem 1.6 was proved in [**14**] with an additional requirement on the place v. Theorem 7.1 provides a domain of holomorphy of the L-functions in terms of the first occurrence in the theory of local Howe correspondence. As consequences of Theorem 7.1 and Theorem 1.3, we prove Theorem 1.5 and Theorem 1.6.

We expect that the first occurrence of irreducible admissible representations in the theory of the local Howe correspondence could be expressed in terms of generalized Gelfand–Graev models, which are generalizations of the Whittaker models used in Theorem 1.5; and also the determination of the domain of holomorphy of L-functions could have a deep impact on the estimate of the Satake parameters at the unramified places. These topics will be included in our forthcoming work. In connection with the CAP conjecture [**14**], the following theorem can be proved as applications of Theorem 3.1 combined with [**14**] and [**15**].

Theorem 1.7. *Let* G *be the* k*-split group* SO_{2n+1}*. Assume that* $\sigma \in \mathcal{A}^{\mathrm{c}}(G/k)$ *has a non-zero Fourier coefficient attached to the subregular nilpotent orbit of the complex Lie algebra of* G*. Then the following hold.*

(1) *The partial* L*-function* $L^S(s, \sigma \otimes \chi)$ *is holomorphic for* $\mathrm{Re}(s) > \frac{3}{2}$*, for all quadratic characters* χ *of* $k^\times \backslash \mathbb{A}^\times$*.*

(2) *If the partial* L*-function* $L^S(s, \sigma \otimes \chi)$ *has a pole at* $s = \frac{3}{2}$*, for some quadratic character* χ *of* $k^\times \backslash \mathbb{A}^\times$*, then* σ *is a CAP representation. More precisely,* σ *is either a CAP representation with respect to the generic cuspidal datum* $(P_1; \chi|\cdot|^{1/2} \otimes \sigma_{n-1})$ *where* P_1 *is the standard parabolic subgroup, whose Levi part is isomorphic to* $\mathrm{GL}_1 \times \mathrm{SO}_{2n-1}$*, and* σ_{n-1} *is an irreducible generic cuspidal automorphic representation of* $\mathrm{SO}_{2n-1}(\mathbb{A})$*, or a CAP representation with respect to the generic cuspidal data*
$$(P_{1,1}; \chi|\cdot|^{1/2} \otimes \chi\chi'|\cdot|^{1/2} \otimes \sigma_{n-2}),$$
where $P_{1,1}$ *is the standard parabolic subgroup, whose Levi part is isomorphic to* $\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{SO}_{2n-3}$*,* χ' *is a certain quadratic character attached to the Fourier coefficient, and* σ_{n-2} *is an irreducible generic cuspidal automorphic representation of* $\mathrm{SO}_{2n-3}(\mathbb{A})$*.*

Under the assumption that σ has a non-zero Bessel model of special type, Theorem 1.7 was proved in [**14**] and [**15**]. We think that the formulation of Theorem 1.7 serves as a model to understand the CAP conjecture for general cuspidal automorphic representations of general classical groups. We will give a sketch of the proof in § 7.3. More complete theory will be included in our forthcoming work [**8**].

2. Poles of certain Eisenstein series

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. Let (X, b) be a non-degenerate quadratic k-vector space of dimension m, where b is the corresponding symmetric form. Denote its k-Witt index by r and hence $r \in \{0, 1, \ldots, [\frac{1}{2}m]\}$. Without loss of generality, we assume that $m > 2$ in this paper.

For any non-negative integer a, denote by

$$\mathcal{H}_a = \ell_a^+ \oplus \ell_a^- \tag{2.1}$$

the polarization of the $2a$-dimensional quadratic k-vector space $\mathcal{H}_a$, which is the direct sum of a-copies of the hyperbolic plane. Then we define

$$X_a := X \perp (\ell_a^+ \oplus \ell_a^-). \tag{2.2}$$

This is a non-degenerate quadratic k-vector space of dimension $m + 2a$, with k-Witt index $a + r$. Take $Q_a = M_a N_a$ to be the maximal parabolic k-subgroup of the orthogonal group $\mathrm{O}(X_a)$, which preserves ℓ_a^+; its Levi subgroup is

$$M_a \cong \mathrm{GL}(\ell_a^+) \times \mathrm{O}(X) = \mathrm{GL}_a \times \mathrm{O}_m\,. \tag{2.3}$$

The elements of M_a will be denoted by $m(x, h)$, with $x \in \mathrm{GL}_a$ and $h \in \mathrm{O}_m$.

Let $K_a = \prod_v K_{a,v}$ be a (good) maximal compact subgroup of $\mathrm{O}(X_a)(\mathbb{A})$, such that the Iwasawa decomposition

$$\mathrm{O}(X_a)(\mathbb{A}) = Q_a(\mathbb{A})K_a \tag{2.4}$$

holds. For instance, $K_{a,v} = \mathrm{O}_{m+2a}(\mathcal{O}_v)$, the v-integral points of O_{m+2a}, when O_{m+2a} is k_v-split. Then the Langlands decomposition of $\mathrm{O}_{m+2a}(\mathbb{A})$ is

$$\mathrm{O}_{m+2a}(\mathbb{A}) = N_a(\mathbb{A})M_a^1 A_a^+ K_a, \tag{2.5}$$

where A_a is the (split) centre of M_a. The unique reduced root in $\Phi^+(Q_a, A_a)$ can be identified with the simple root α_a. As normalized in [**26**], we denote

$$\tilde{\alpha}_a := \langle \rho_{Q_a}, \alpha_a \rangle^{-1} \rho_{Q_a}, \tag{2.6}$$

where ρ_{Q_a} is half of the sum of all positive root in N_a and $\langle \cdot\,, \cdot \rangle$ is the usual Killing–Cartan form for the root system of O_{m+2a}. We let

$$\mathfrak{a}_{M_a} = \mathrm{Hom}_{\mathbb{R}}(X(M_a), \mathbb{R}), \qquad \mathfrak{a}_{M_a}^* = X(M_a) \otimes \mathbb{R}, \tag{2.7}$$

where $X(M_a)$ denotes the group of all rational characters of M_a. Since Q_a is maximal, $\mathfrak{a}_{M_a}^*$ is one-dimensional. We identify $\mathbb{C}$ with $\mathfrak{a}_{M_a,\mathbb{C}}^*$ via $s \mapsto s\tilde{\alpha}_a$.

Let $\mathrm{H}_a : M_a(\mathbb{A}) \mapsto \mathfrak{a}_{M_a}$ be the map defined as follows, for any $\chi \in \mathfrak{a}_{M_a}^*$,

$$\mathrm{H}_a(m)(\chi) = \prod_v |\chi(m_v)|_v \tag{2.8}$$

for $m \in M_a(\mathbb{A})$. It follows that H_a is trivial on M_a^1. This map H_a can be extended as a function over $\mathrm{O}_{m+2a}(\mathbb{A})$, via the Iwasawa decomposition or the Langlands decomposition above. By direct computation, we know that

$$\mathrm{H}_a(m(x,h))(s) = |\det x|^s, \qquad \mathrm{H}_a(m(x,h))(\rho_{Q_a}) = |\det x|^{(m+a-1)/2} \tag{2.9}$$

where $s \in \mathbb{C}$ and $m = m(x,h) \in M_a(\mathbb{A})$, with $x \in \mathrm{GL}_a(\mathbb{A})$ and $h \in \mathrm{O}_m(\mathbb{A})$.

We denote by $\mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ the set of irreducible, automorphic, cuspidal representations of $\mathrm{O}_m(\mathbb{A})$, occurring as subspaces in the space of cuspidal automorphic forms on $\mathrm{O}_m(\mathbb{A})$. For $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ and a quadratic character χ, of $\mathbb{A}^\times/k^\times$, we denote by $\mathcal{E}_{a;\sigma,\chi}$ the space of smooth $\mathbb{C}$-valued functions $\phi_{a;\sigma,\chi}$ on $N_a(\mathbb{A})M_a(k)\backslash \mathrm{O}(X_a)(\mathbb{A})$ satisfying the following properties:

- $\phi_{a;\sigma,\chi}$ is right K_a-finite;
- for any $x \in \mathrm{GL}_a(\mathbb{A})$ and $g \in \mathrm{O}(X_a)(\mathbb{A})$,

  $$\phi_{a;\sigma,\chi}(m(x,\mathrm{I}_X)g) = \chi(\det x)\cdot|\det x|^{(m+a-1)/2}\cdot\phi_{a;\sigma,\chi}(g);$$

  and
- for any fixed $t \in K_a$, and for any $h \in \mathrm{O}_m(\mathbb{A})$, the function

  $$h \mapsto \phi_{a;\sigma,\chi}(m(\mathrm{I}_a,h)t)$$

  is a smooth and right $K_a \cap M_a^1$-finite vector, in the space V_σ of the irreducible, automorphic, cuspidal representation σ, realized in the cuspidal spectrum on $\mathrm{O}_m(\mathbb{A})$ (in the σ-isotypic component of the cuspidal spectrum if there is multiplicity).

For any $s \in \mathbb{C}$ and $\phi_{a;\sigma,\chi} \in \mathcal{E}_{a;\sigma,\chi}$, define

$$\varPhi_s(g;\phi_{a;\sigma,\chi}) := \mathrm{H}_a(g)(s)\phi_{a;\sigma,\chi}(g). \tag{2.10}$$

It is clear that for $g = nm(x,h)t \in Q_a(\mathbb{A})\cdot K_a$, we have

$$\varPhi_s(g;\phi_{a;\sigma,\chi}) = \chi(\det(x))|\det(x)|^{s+((m+a-1)/2)}\phi_{a;\sigma,\chi}(m(1_a,h)t). \tag{2.11}$$

Equivalently, one may identify $\varPhi_s(g;\phi_{a;\sigma,\chi})$ as a smooth section of the normalized induced representation

$$\mathrm{I}(s;\sigma) := \mathrm{Ind}_{Q_a(\mathbb{A})}^{\mathrm{O}_{m+2a}(\mathbb{A})}(\chi\cdot|\det|^s\otimes\sigma). \tag{2.12}$$

Attached to such a function $\varPhi_s(g;\phi_{a;\sigma,\chi})$, we define an Eisenstein series, on $\mathrm{O}_{m+2a}(\mathbb{A})$, by

$$E^{Q_a}(g;\phi_{a;\sigma,\chi},s) := \sum_{\gamma\in Q_a(k)\backslash \mathrm{O}_{m+2a}(k)} \varPhi_s(\gamma g;\phi_{a;\sigma,\chi}). \tag{2.13}$$

By the Langlands theory of Eisenstein series [**22**], this Eisenstein series converges absolutely, for $\mathrm{Re}(s) > \frac{1}{2}(m+a-1)$, and has a meromorphic continuation to the whole

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s-plane, with finitely many possible poles for $\mathrm{Re}(s) > 0$, which in this case, turn out to be real.

We denote by $\mathcal{P}(\sigma \otimes \chi, Q_a)$, the set of positive poles of the Eisenstein series $E^{Q_a}(g; \phi_{a;\sigma,\chi}, s)$, when $\phi_{a;\sigma,\chi}$ runs in $\mathcal{E}_{a;\sigma,\chi}$. Hence, if $s_0 > 0$ is a number in the set $\mathcal{P}(\sigma \otimes \chi, Q_a)$, there exists a $\phi_{a;\sigma,\chi} \in \mathcal{E}_{a;\sigma,\chi}$ such that $E^{Q_a}(g; \phi_{a;\sigma,\chi}, s)$ has a pole at s_0; and if $s_0 > 0$ is the maximal number in the set $\mathcal{P}(\sigma \otimes \chi, Q_a)$, then for all $\phi_{a;\sigma,\chi} \in \mathcal{E}_{a;\sigma,\chi}$, the Eisenstein series $E^{Q_a}(g; \phi_{a;\sigma,\chi}, s)$ is holomorphic for $\mathrm{Re}(s) > s_0$. As in [**20**] and [**21**], we have the following proposition.

Proposition 2.1. *Assume that $\mathcal{P}(\sigma \otimes \chi, Q_1)$ is non-empty, and let s_0 be its maximum. Then, for all integers $a \geqslant 1$, $s_a := s_0 + \frac{1}{2}(a-1)$ lies in $\mathcal{P}(\sigma \otimes \chi, Q_a)$.*

The proof is the same as that in § 1.1 of [**20**]. We omit the details here.

Proposition 2.2. *Assume that the partial L-function $L^S(s, \sigma \otimes \chi)$ has a pole at $s = s_0 > 0$, and is holomorphic for $\mathrm{Re}(s) > s_0$. Then, for all integers $a \geqslant 1$, the Eisenstein series $E^{Q_a}(g; \phi_{a;\sigma,\chi}, s)$ has a pole at $s_a := s_0 + \frac{1}{2}(a-1)$.*

Proof. The proof is the same as in § 1.2 of [**20**]. We sketch some details here for completeness. By the Langlands theory of the constant terms of Eisenstein series, one has to calculate the constant terms of $E^{Q_a}(g; \phi_{a;\sigma,\chi}, s)$ along various parabolic subgroups of O_{m+2a}. As in § 1.2 of [**20**], since $s = s_0$ is the maximal pole of the partial L-function $L^S(s, \sigma \otimes \chi)$, it is enough to consider the poles of the intertwining operator attached to the longest Weyl group element w_a in $Q_a \backslash \mathrm{O}_{m+2a} / Q_a$. By the Gindikin–Karpelevich formula at unramified finite places, when m is even, the product of partial L-functions reads

$$\prod_{i=1}^{a} \frac{L^S(s - \frac{1}{2}(a-1) + i - 1, \sigma \otimes \chi)}{L^S(s - \frac{1}{2}(a-1) + i, \sigma \otimes \chi)} \prod_{1 \leqslant i < j \leqslant a} \frac{\zeta^S(2s - a + i + j - 1)}{\zeta^S(2s - a + i + j)}; \tag{2.14}$$

and when m is odd, it reads

$$\prod_{i=1}^{a} \frac{L^S(s - \frac{1}{2}(a-1) + i - 1, \sigma \otimes \chi)}{L^S(s - \frac{1}{2}(a-1) + i, \sigma \otimes \chi)} \prod_{1 \leqslant i \leqslant j \leqslant a} \frac{\zeta^S(2s - a + i + j - 1)}{\zeta^S(2s - a + i + j)}. \tag{2.15}$$

Here, $\zeta^S(s)$ is the partial Dedekind zeta function of k. After cancellation, when m is even, it reduces to

$$\frac{L^S(s - \frac{1}{2}(a-1), \sigma \otimes \chi)}{L^S(s + \frac{1}{2}(a+1), \sigma \otimes \chi)} \frac{\prod_{i=1}^{[a/2]} \zeta^S(2s - a + 2i)}{\prod_{i=[(a+1)/2]}^{a-1} \zeta^S(2s - a + 2i + 1)}; \tag{2.16}$$

and when m is odd, it reduces to

$$\frac{L^S(s - \frac{1}{2}(a-1), \sigma \otimes \chi)}{L^S(s + \frac{1}{2}(a+1), \sigma \otimes \chi)} \frac{\prod_{i=1}^{[(a+1)/2]} \zeta^S(2s - a + 2i - 1)}{\prod_{i=[a/2]+1}^{a} \zeta^S(2s - a + 2i)}. \tag{2.17}$$

Now we evaluate the products of partial L-functions at $s = s_a = s_0 + \frac{1}{2}(a-1)$ and obtain the following. When m is even, it reads

$$\frac{\mathrm{res}_{s=s_0} L^S(s, \sigma \otimes \chi)}{L^S(s_0 + a, \sigma \otimes \chi)} \frac{\prod_{i=1}^{[a/2]} \zeta^S(2s_0 - 1 + 2i)}{\prod_{i=[(a+1)/2]}^{a-1} \zeta^S(2s_0 + 2i)}; \tag{2.18}$$

and when m is odd, it reads

$$\frac{\mathrm{res}_{s=s_0} L^S(s, \sigma \otimes \chi)}{L^S(s_0 + a, \sigma \otimes \chi)} \frac{\prod_{i=1}^{[(a+1)/2]} \zeta^S(2s_0 + 2i - 2)}{\prod_{i=[a/2]+1}^{a} \zeta^S(2s_0 + 2i - 1)}. \tag{2.19}$$

In both cases, the residue $\mathrm{res}_{s=s_0} L^S(s, \sigma \otimes \chi)$ cannot be cancelled by the denominator. Hence the product of local intertwining operators at finite unramified places has a pole at $s = s_a = s_0 + \frac{1}{2}(a-1)$. We remark that since s_0 is the maximal pole, it cannot be cancelled by the possible poles from the intertwining operators attached to other Weyl group elements. Therefore, the constant term of $E^{Q_a}(g; \phi_{a;\sigma,\chi}, s)$ has a pole at $s = s_a = s_0 + \frac{1}{2}(a-1)$. $\square$

Note that by [**17**], the possible positive poles of the partial L-functions $L^S(s, \sigma \otimes \chi)$ are at $s = \frac{1}{2}m - j > 0$. We want to prove the following proposition.

Proposition 2.3. *Let s_0 be the maximum of $\mathcal{P}(\sigma \otimes \chi, Q_1)$. Then s_0 must be of form $\frac{1}{2}m - j$, with $j = 1, 2, \dots, [\frac{1}{2}(m-1)]$, when $\mathrm{O}(X)$ is not k-anisotropic. When $\mathrm{O}(X)$ is k-anisotropic, the Eisenstein series $E^{Q_1}(g; \phi_{1;\sigma,\chi}, s)$ has a pole at $s = \frac{1}{2}m$, if and only if σ is the composition of χ with the spinor norm of $\mathrm{O}(X)$.*

This is a consequence of Theorem 3.1 in § 3, which will be proved by the generalized doubling method.

3. Generalized doubling method

We consider the generalized doubling method, for general orthogonal groups O_m, following the ideas and arguments of Moeglin in § 2.1 of [**20**], where the case of even orthogonal groups is studied. See also § 3.1 of [**15**] for the treatment for the case of $\widetilde{\mathrm{Sp}}_{2j}$. As a consequence, we prove a stronger result than Proposition 2.3.

Theorem 3.1. *Let O_m be the orthogonal group attached to a quadratic k-vector space of dimension m, and let χ be a quadratic character of $k^\times \backslash \mathbb{A}^\times$, and $\sigma \in \mathcal{A}^c(\mathrm{O}_m/k)$. If the Eisenstein series $E^{Q_1}(g; \phi_{1;\sigma,\chi}, s)$ has a pole at $s = s_0 > 0$ and is holomorphic for $\mathrm{Re}(s) > s_0$, i.e. s_0 is the maximum of $\mathcal{P}(\sigma \otimes \chi, Q_1)$, then $s_0 = \frac{1}{2}m - j$, for some integer j, and there is an automorphic sign character ϵ of $\mathrm{O}_m(\mathbb{A})$, such that the ψ-theta lifting $\theta^{2j}_{\psi,m}((\sigma \otimes \chi) \otimes \epsilon)$ of $(\sigma \otimes \chi) \otimes \epsilon$ to $\mathrm{Mp}_{2j}(\mathbb{A})$ is non-zero, i.e. the lowest occurrence*

$$\mathrm{LO}_\psi(\sigma \otimes \chi) \leqslant \mathrm{FO}_\psi((\sigma \otimes \chi) \otimes \epsilon) \leqslant 2j.$$

Moreover, we also have $2j \geqslant r$, where r is the Witt index of the quadratic form b, which defines O_m. In particular, if O_m is k-split, then $2j \geqslant [\frac{1}{2}m]$.

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This theorem was proved by Moeglin in [**20**], for even orthogonal groups. The proof for m odd is similar. For completeness, we review the proof, in general, following Moeglin's argument. To prove Theorem 3.1, we recall the generalized doubling method. We keep using the notation introduced in § 2.

Let $(X', b') := (X, -b)$ be the quadratic k-vector space obtained by taking the same vector space as X, but with the non-degenerate symmetric bilinear form $b' = -b$. Take $W = X \perp X'$, the 'doubled' k-vector space, and define, for any non-negative integer a,

$$W_a = X_a \perp X' = W \perp (\ell_a^+ \oplus \ell_a^-) = W \perp \mathcal{H}_a, \tag{3.1}$$

where X_a, $\ell_a^{\pm}$ and $\mathcal{H}_a$ are given in (2.1) and (2.2). One may call W_a the generalized doubled k-vector space. Define

$$X^{\Delta,\pm} := \{(x, \pm x) \in W \mid x \in X\} \quad \text{and} \quad X_a^{\Delta,\pm} = X^{\Delta,\pm} \oplus \ell_a^{\pm}. \tag{3.2}$$

Then we have

$$W = X^{\Delta,+} \oplus X^{\Delta,-} \quad \text{and} \quad W_a = X_a^{\Delta,+} \oplus X_a^{\Delta,-}. \tag{3.3}$$

These are the polarizations of W and W_a, with respect to the maximal totally isotropic subspaces $X^{\Delta,\pm}$ and $X_a^{\Delta,\pm}$, respectively. In particular, both W and W_a are k-split quadratic k-vector spaces.

Let P_a be the maximal parabolic subgroup of $\mathrm{O}(W_a)$, stabilizing the maximal totally isotropic subspace $X_a^{\Delta,+}$ of W_a. Then we write $P_a = M_{P_a} U_{P_a}$ as the Levi decomposition of P_a, with the Levi part $M_{P_a} \cong \mathrm{GL}_{m+a}$. It is clear that the unipotent radical U_{P_a} is abelian. There is a maximal compact subgroup K'_a of $\mathrm{O}(W_a)(\mathbb{A})$ such that

$$K'_a \cap \mathrm{O}(X_a)(\mathbb{A}) = K_a \quad \text{and} \quad \mathrm{O}(W_a)(\mathbb{A}) = P_a(\mathbb{A}) K'_a, \tag{3.4}$$

where K_a is the maximal compact subgroup of $\mathrm{O}(X_a)(\mathbb{A})$, as defined in (2.4). Let

$$\iota = \iota_1 \times \iota_2 : \mathrm{O}(X_a) \times \mathrm{O}(X') \to \mathrm{O}(W_a) \tag{3.5}$$

be the natural embedding according to (3.1).

Replacing σ by $\sigma \otimes \chi$, we may assume that $\chi = 1$.

As in [**20**], we denote by $\mathbb{E}'_{a,s}$ the space of automorphic forms f on

$$U_{P_a}(\mathbb{A}) M_{P_a}(k) \backslash \mathrm{O}(W_a)(\mathbb{A}),$$

such that

$$f(m(x)g) = |\det(x)|_{\mathbb{A}}^s \delta_{P_a}^{1/2}(m(x)) f(g),$$

where

$$m(x) = \begin{pmatrix} x & 0 \\ 0 & x^* \end{pmatrix} \in M_{P_a}(\mathbb{A}),$$

with $x \in \mathrm{GL}_{m+a}(\mathbb{A})$, $g \in \mathrm{O}(W_a)(\mathbb{A})$. For $f \in \mathbb{E}'_{a,0}$ (i.e. $s = 0$), define

$$f_s(g) := [h_{P_a}(g)]^s \cdot f(g), \tag{3.6}$$

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where h_{P_a} is the same as the function H_a, defined in (2.8), replacing the parabolic subgroup Q_a there by the parabolic subgroup P_a here. It is clear that f_s defines a section in the normalized induced representation $\mathrm{Ind}_{P_a(\mathbb{A})}^{\mathrm{O}(W_a)(\mathbb{A})}(|\det|^s)$. We define the Eisenstein series attached to f_s by

$$E^{P_a}(g; f_s) := \sum_{\gamma \in P_a(k)\backslash \mathrm{O}(W_a)(k)} f_s(\gamma g). \tag{3.7}$$

In [**17**], the possible positive poles of the Eisenstein series $E^{P_a}(g; f_s)$ are proved to be in the following set

$$\{\tfrac{1}{2}(m+a-1) - j \mid j = 0, 1, 2, \dots, [\tfrac{1}{2}(m+a)] - 1\}. \tag{3.8}$$

For any $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}(X)/k)$, we define, for any $\phi_\sigma \in V_\sigma$, $f \in \mathbb{E}'_{a,0}$ and $g_a \in \mathrm{O}(X_a)(\mathbb{A})$, the following function

$$f_{\phi_\sigma,s}(g_a) := \int_{\mathrm{O}(X)(\mathbb{A})} f_s(\iota_1(g^{-1}g_a))\phi_\sigma(g)\,\mathrm{d}g, \tag{3.9}$$

where f_s is given as in (3.6). We have the following results, which extend § 2.1 of [**20**] to cover both even and odd orthogonal groups. For $\widetilde{\mathrm{Sp}}_{2j}$, it is Proposition 3.3 of [**15**].

Proposition 3.2. *With the notation above, the following hold.*

(1) *The integral defining $f_{\phi_\sigma,s}(g_a)$ converges absolutely for* $\mathrm{Re}(s) > \frac{1}{2}(m+a-1)$.

(2) *The integral defining $f_{\phi_\sigma,s}(g_a)$ has meromorphic continuation to the whole complex plane $\mathbb{C}$, with possible poles contained in the set of poles of the Eisenstein series $E^{P_0}(g; (f_s)|_{\mathrm{O}(W)(\mathbb{A})})$, which is*

$$\{\pm(\tfrac{1}{2}(m-1) - j) \neq 0 \mid j = 0, 1, 2, \dots, \tfrac{1}{2}m - 1\},$$

where $(f_s)|_{\mathrm{O}(W)(\mathbb{A})}$ denotes the restriction of f_s to the subgroup $\mathrm{O}(W)(\mathbb{A})$.

(3) *The function $f_{\phi_\sigma,s}(g_a)$ is a K_a-finite section in the normalized induced representation* $\mathrm{I}(s;\sigma) = \mathrm{Ind}_{Q_a(\mathbb{A})}^{\mathrm{O}(X_a)(\mathbb{A})}(|\det|^s \otimes \sigma)$ *of* $\mathrm{O}(X_a)(\mathbb{A})$ *as defined in (2.12).*

The proof of Proposition 3.2 is exactly the same as in § 2.1 of [**20**], which is for even orthogonal groups. The same result for $\widetilde{\mathrm{Sp}}_{2j}$ is in Proposition 3.3 of [**15**]. We omit the details here. By using the section $f_{\phi_\sigma,s}(g_a)$, we define the Eisenstein series $E^{Q_a}(g_a; f_{\phi_\sigma,s})$ as in (2.13). As in Proposition 2.1 of [**20**] and in Theorem 3.5 in [**15**], we consider the maximal parabolic subgroup $P_{a,\Delta}$ of $\mathrm{O}(W_a)$, which stabilizes the maximal totally isotropic subspace $X_a^{\Delta,-}$ (see (3.3)). Let δ_0 be an element in $\mathrm{O}(W_a)$ such that

$$\delta_0 \cdot P_{a,\Delta} \cdot \delta_0^{-1} = P_a.$$

Then as in (3.6) of [**15**] or in § 2.1 of [**20**] we define

$$f'_s(g) := f_s(\delta_0 \cdot g).$$

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Then f'_s is a section in $\mathrm{Ind}_{P_{a,\Delta}}^{\mathrm{O}(W_a)(\mathbb{A})}(\mathbb{A})(|\det_{X_a^{\Delta,-}}|^s)$, and we have

$$E^{P_{a,\Delta}}(g; f'_s) = E^{P_a}(\delta_0 \cdot g; f_s). \tag{3.10}$$

We have the following result, the proof of which is the same as in § 2 of [**20**] and § 3 of [**15**], and we will omit the details here.

Proposition 3.3. *For any section f'_s in $\mathrm{Ind}_{P_{a,\Delta}(\mathbb{A})}^{\mathrm{O}(W_a)(\mathbb{A})}(|\det_{X_a^{\Delta,-}}|^s)$, and ϕ_σ in the space of $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}(X_a)/k)$, the following identity holds*

$$\int_{\mathrm{O}(X)(k)\backslash \mathrm{O}(X)(\mathbb{A})} E^{P_{a,\Delta}}(\iota(g_a, g); f'_s)\phi_\sigma(g)\,\mathrm{d}g = E^{Q_a}(g_a; f_{\phi_\sigma,s}).$$

The only thing we want to discuss here is that, for a large enough, the section $f_{\phi_\sigma,s}$ is general enough to detect the poles of the Eisenstein series corresponding to $\mathrm{I}(s;\sigma) = \mathrm{Ind}_{Q_a(\mathbb{A})}^{\mathrm{O}(X_a)(\mathbb{A})}(|\det|^s \otimes \sigma)$. This is important for the determination of the possible poles of the Eisenstein series $E^{Q_a}(g_a; \phi_{a;\sigma}, s)$ in terms of the possible poles of the degenerate Eisenstein series $E^{P_{a,\Delta}}(g; f'_s)$ or $E^{P_a}(g; f_s)$. We do this as in § 2 of [**20**].

By (3.9), the integral which defines the section $f_{\phi_\sigma,s}$ can be viewed as an $\mathrm{O}(X_a)(\mathbb{A})$-intertwining mapping from

$$\mathrm{Ind}_{P_a(\mathbb{A})}^{\mathrm{O}(W_a)(\mathbb{A})}(|\det|^s) \otimes V_\sigma$$

to $\mathrm{I}(s;\sigma)$, as defined in (2.12). We want to show that this intertwining mapping produces arbitrary sections in $\mathrm{I}(s;\sigma)$. It is clear that one has to prove this for every local place v of k.

Let $P_a^0 = M_{P_a}^0 \cdot U_{P_a}$ be the subgroup of the Siegel parabolic subgroup $P_a = M_{P_a} \cdot U_{P_a}$, with $M_{P_a}^0 = \mathrm{SL}_{m+a}$. Let $Q_a^0 = M_a^0 \cdot N_a$ be the subgroup of the maximal parabolic subgroup $Q_a = M_a \cdot N_a$, defined in (2.3), with $M_a^0 = \mathrm{SL}(\ell_a^+) \times \mathrm{I}_X$, and $\hat{t} = \mathrm{diag}(t, \mathrm{I}_{a-1}) \in \mathrm{GL}(\ell_a^+)$ for $t \in k_v^\times$. We have the following natural projection,

$$\varphi_v \mapsto \int_{k_v^\times \times \mathrm{O}(X)(k_v)} |t|^{-s}\delta_{Q_a}(\hat{t})^{-1/2}\sigma_v(g^{-1})\varphi_v(\hat{t}gg_a)\,\mathrm{d}^\times t\,\mathrm{d}g, \tag{3.11}$$

from the space, $C_c^\infty(Q_a^0(k_v)\backslash \mathrm{O}(X_a)(k_v), V_{\sigma_v})$, consisting of all V_{σ_v}-valued, smooth, compactly supported functions, to the induced representation $\mathrm{I}(s;\sigma_v)$. It is clear that this projection is an $\mathrm{O}(X_a)(k_v)$-intertwining map. As in § 2 of [**20**], after the normalization by a product of local L-functions defined by σ_v, (3.11) produces arbitrary holomorphic $K_{a,v}$-finite sections in $\mathrm{I}(s;\sigma_v)$, if a is sufficiently large (depending on σ only). The condition on a is needed only when σ_v has ramification.

By applying the argument used in the original doubling method, we know, from (3.5), that the natural embedding

$$\iota_1 : (k_v^\times \times Q_a^0(k_v))\backslash \mathrm{O}(X_a)(k_v) \to (k_v^\times \times P_a^0(k_v))\backslash \mathrm{O}(W_a)(k_v) \tag{3.12}$$

gives the Zariski open orbit of the action of $\mathrm{O}(X_a)(k_v) \times \mathrm{O}(X)(k_v)$ on the generalized flag variety $P_a(k_v)\backslash \mathrm{O}(W_a)(k_v)$. Hence functions in $C_c^\infty(Q_a^0(k_v)\backslash \mathrm{O}(X_a)(k_v))$ can be naturally extended to functions in $C_c^\infty(P_a^0(k_v)\backslash \mathrm{O}(W_a)(k_v))$. Moreover, it is easy to check

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that $K_{a,v}$-finite functions in $C_c^\infty(Q_a^0(k_v)\backslash \mathrm{O}(X_a)(k_v))$ extend to $K'_{a,v}$-finite functions in $C_c^\infty(P_a^0(k_v)\backslash \mathrm{O}(W_a)(k_v))$. Via the natural isomorphism, we have

$$C_c^\infty(Q_a^0(k_v)\backslash \mathrm{O}(X_a)(k_v)) \hat{\otimes} V_{\sigma_v} \cong C_c^\infty(Q_a^0(k_v)\backslash \mathrm{O}(X_a)(k_v), V_{\sigma_v}).$$

One can also check that this isomorphism carries

$$C_c^\infty(Q_a^0(k_v)\backslash \mathrm{O}(X_a)(k_v), V_{\sigma_v})_{K_{a,v}\text{-finite}}$$

into

$$C_c^\infty(Q_a^0(k_v)\backslash \mathrm{O}(X_a)(k_v))_{K_{a,v}\text{-finite}} \hat{\otimes} V_{\sigma_v}.$$

This is clear when v is a finite place of k, since the smooth functions are locally constant. At an archimedean place v of k, for any φ in $C_c^\infty(Q_a^0(k_v)\backslash \mathrm{O}(X_a)(k_v), V_{\sigma_v})$, we write

$$\varphi(g_a) = \sum_{n=1}^{\infty} \varphi_n(g_a) \cdot \nu_n$$

where $\{\nu_n \mid n = 1, 2, \dots\}$ forms a countable orthonormal basis in the unitary completion of the representation σ_v. If φ is $K_{a,v}$-finite, with $K_{a,v}$-type ρ_v and orthonormal basis $\varphi_1, \dots, \varphi_f$ in ρ_v, then for $k \in K_{a,v}$, we have

$$\begin{aligned}
\varphi(g_a k) &= \sum_{i=1}^{f} \langle \rho_v(k)\varphi, \varphi_i\rangle \cdot \varphi_i(g_a) \\
&= \sum_{i=1}^{f} \langle \rho_v(k)\varphi, \varphi_i\rangle \sum_{n=1}^{\infty} \varphi_{i,n}(g_a) \cdot \nu_n \\
&= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{f} \langle \rho_v(k)\varphi, \varphi_i\rangle \varphi_{i,v}(g_a) \right) \nu_n.
\end{aligned}$$

It follows that

$$\varphi_n(g_a k) = \sum_{i=1}^{f} \langle \rho_v(k)\varphi, \varphi_i\rangle \varphi_{i,v}(g_a).$$

Hence the $K_{a,v}$-finite φ can be approximated by elements in

$$C_c^\infty(Q_a^0(k_v)\backslash \mathrm{O}(X_a)(k_v))_{K_{a,v}\text{-finite}} \otimes V_{\sigma_v}.$$

This space is embedded, by the explanation right after (3.12), inside the space $C_c^\infty(P_a^0(k_v)\backslash \mathrm{O}(W_a)(k_v)) \otimes V_{\sigma_v}$. As in (3.11), there is a natural $\mathrm{O}(W_a)(k_v)$-intertwining projection from $C_c^\infty(P_a^0(k_v)\backslash \mathrm{O}(W_a)(k_v))$ onto

$$\mathrm{Ind}_{P_a(k_v)}^{\mathrm{O}(W_a)(k_v)}(|\det|^s).$$

Therefore, the local version of (3.9) produces arbitrary holomorphic $K_{a,v}$-finite sections in $\mathrm{I}(s;\sigma_v)$ for each place v, as long as a is sufficiently large (depending on the local ramification of σ_v), and so we obtain that for a sufficiently large, the integral (3.9) produces arbitrary global holomorphic K_a-finite sections in $\mathrm{I}(s;\sigma)$. This proves the following proposition.

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Proposition 3.4. *For a sufficiently large, the family of Eisenstein series formed by the integrals (3.9) has the same set of poles as the whole family of Eisenstein series corresponding to* $\mathrm{I}(s;\sigma)$.

Now we can prove Proposition 2.3, which is the first part of Theorem 3.1. (We keep the assumption $\chi = 1$.)

Let $s_0 > 0$ be the maximal number in $\mathcal{P}(\sigma, Q_1)$, i.e. the Eisenstein series $E^{Q_1}(g; \phi_{1;\sigma}, s)$ is holomorphic for $\mathrm{Re}(s) > s_0$, and has a pole at s_0. Then, by Proposition 2.1, for all integers $a \geqslant 1$, $s_a = s_0 + \frac{1}{2}(a-1)$ is in $\mathcal{P}(\sigma, Q_a)$, i.e. the Eisenstein series $E^{Q_a}(g; \phi_{a;\sigma}, s)$ has a pole at s_a. By Propositions 3.4 and 3.3, for a sufficiently large, there is a holomorphic K'_a-finite section f_s in $\mathbb{E}'_{a,s}$ such that the Eisenstein series $E^{P_a}(g; f_s)$ has a pole at

$$s = s_a = s_0 + \tfrac{1}{2}(a-1).$$

On the other hand, by [**17**, (3.8)], there exists an integer $j \geqslant 0$ such that $s_a = \frac{1}{2}(m + a - 1) - j$. Hence, we must have

$$s_0 = \tfrac{1}{2}m - j.$$

When $j = 0$, the residue at $s = \frac{1}{2}(m+a-1)$ of the Eisenstein series $E^{P_a}(g; f_s)$ generates the trivial one-dimensional automorphic representation of $\mathrm{O}(W_a)(\mathbb{A})$. In order to get a non-zero residue at $s = \frac{1}{2}(m + a - 1)$, via the identity in Proposition 3.3, σ must be the trivial representation. Thus, the Eisenstein series $E^{Q_a}(g; \phi_{a;\sigma}, s)$ is holomorphic at $s = \frac{1}{2}(m + a - 1)$ when $\mathrm{O}(X)$ is not k-anisotropic. In this case, by Proposition 2.1, the Eisenstein series $E^{Q_1}(g; \phi_{1;\sigma}, s)$ must be holomorphic at $s = \frac{1}{2}m$. When $\mathrm{O}(X)$ is k-anisotropic, the assertion of Proposition 2.3 follows also from Proposition 3.3. This proves Proposition 2.3.

We continue to complete the proof of Theorem 3.1 below, by using a certain version of the regularized Siegel–Weil formula for the case under consideration.

Let $s_0 = \frac{1}{2}m - j$ be the maximum of $\mathcal{P}(\sigma, Q_1)$. Then by Proposition 2.1, for all integers $a \geqslant 1$, $s_a = \frac{1}{2}(m + a - 1) - j$ is a pole of the Eisenstein series $E^{Q_a}(g; \phi_{a;\sigma}, s)$. By Propositions 3.3 and 3.4, for a sufficiently large, the following integral (period)

$$\int_{\mathrm{O}(X)(k)\backslash \mathrm{O}(X)(\mathbb{A})} \mathrm{res}_{s=s_a} E^{P_a}(\iota(g_a, g), f_s)\phi_\sigma(g)\,\mathrm{d}g \tag{3.13}$$

is non-zero for some choice of data. To understand $\mathrm{res}_{s=s_a} E^{P_a}(\iota(g_a, g), f_s)$, we use the idea of the regularized Siegel–Weil formula of Kudla and Rallis [**18**]. In this particular case, we follow § 3 of [**20**] and § 2 of [**15**]. By § 3.1 of [**20**], the space generated by the square-integrable automorphic forms $\mathrm{res}_{s=s_a} E^{P_a}(\iota(g_a, g), f_s)$ is contained in the following direct sum

$$\bigoplus_{\epsilon} \Pi_{1,\epsilon},$$

where the representation $\Pi_{1,\epsilon}$ is as defined in § 3.1 of [**20**], with $\eta = 1$. Now by § 3.2 of [**20**] and § 2 of [**15**], for a given f, there are automorphic sign characters ϵ and φ_ϵ in $\mathcal{S}(W_a(\mathbb{A})^j)$ such that

$$\mathrm{res}_{s=s_a} E^{P_a}(h, f_s) = \sum_{\epsilon} c_\epsilon \cdot \epsilon(\det h) \int_{\mathrm{Sp}_{2j}(k)\backslash \mathrm{Sp}_{2j}(\mathbb{A})} \theta_\psi^{\omega_{\psi_{v_0}}(1,\alpha_{v_0})\circ\varphi_\epsilon}(h, x)\,\mathrm{d}x, \tag{3.14}$$

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where c_ϵ is a non-zero constant as given in the regularized Siegel–Weil formula. In the integral, ψ is the fixed character which defines the Weil representation ω_ψ on $\mathcal{S}(W_a(\mathbb{A})^j)$, and α_{v_0} is a Hecke algebra element at a finite place v_0, depending on the section f_s. This Hecke algebra element α_{v_0} regularizes the theta function $\theta_\psi^\varphi(g,x)$, so that it decays rapidly on the variable x and so that the integral converges absolutely. Formula (3.14) is the version of the regularized Siegel–Weil formula for the non-connected group O_m and will be used below in the proof of Theorem 3.1.

By (3.14), the non-vanishing of (3.13) implies that there is a sign character ϵ, as above, such that the following integral does not vanish identically:

$$\int_{\mathrm{O}(X)(k)\backslash\,\mathrm{O}(X)(\mathbb{A})}\int_{\mathrm{Sp}_{2j}(k)\backslash\,\mathrm{Sp}_{2j}(\mathbb{A})}\theta_\psi^{\omega_{\psi_{v_0}}(1,\alpha_{v_0})\circ\varphi}(\iota(g_a,g),x)\epsilon(\det g)\phi_\sigma(g)\,\mathrm{d}x\,\mathrm{d}g. \tag{3.15}$$

Consider the separation of variables in the Weil representation, and obtain

$$\omega_\psi^{\mathrm{O}(W_a)\times\mathrm{Sp}_{2j}}|_{[\mathrm{O}(X_a)(\mathbb{A})\times\mathrm{O}(X)(\mathbb{A})]\times\mathrm{Mp}_{2j}(\mathbb{A})}\cong\omega_\psi^{\mathrm{O}(X_a)\times\mathrm{Mp}_{2j}}\otimes\omega_\psi^{\mathrm{O}(X)\times\mathrm{Mp}_{2j}}. \tag{3.16}$$

One may choose $\varphi=\varphi_{X_a}\otimes\varphi_X$ so that one gets

$$\theta_\psi^\varphi((g_a,g),x)=\theta_\psi^{\varphi_{X_a}}(g_a,x)\theta_\psi^{\varphi_X}(g,x). \tag{3.17}$$

Hence the integral (3.15) can be written as

$$\int_x\theta_\psi^{\omega_{\psi_{v_0}}(1,\alpha_{v_0})\circ\varphi_{X_a}}(g_a,x)\int_g\theta_\psi^{\omega_{\psi_{v_0}}(1,\alpha_{v_0})\circ\varphi_X}(g,x)\epsilon(\det g)\phi_\sigma(g)\,\mathrm{d}g\,\mathrm{d}x, \tag{3.18}$$

where the $\mathrm{d}x$-integration is over $\mathrm{Sp}_{2j}(k)\backslash\,\mathrm{Sp}_{2j}(\mathbb{A})$, and the $\mathrm{d}g$-integration is over $\mathrm{O}(X)(k)\backslash\,\mathrm{O}(X)(\mathbb{A})$. Now the non-vanishing of (3.15) implies the non-vanishing of (3.18), and hence implies the non-vanishing of the inner integration

$$\int_{\mathrm{O}(X)(k)\backslash\,\mathrm{O}(X)(\mathbb{A})}\theta_\psi^{\omega_{\psi_{v_0}}(1,\alpha_{v_0})\circ\varphi_X}(g,x)\epsilon(\det g)\phi_\sigma(g)\,\mathrm{d}g. \tag{3.19}$$

Since σ is cuspidal, integral (3.19) converges absolutely for all theta functions $\theta_\psi^{\varphi_X}(g,x)$ with all Schwartz–Bruhat functions φ_X in $\mathcal{S}(X(\mathbb{A})^j)$. Hence the following integral

$$\int_{\mathrm{O}(X)(k)\backslash\,\mathrm{O}(X)(\mathbb{A})}\theta_\psi^{\varphi_X}(g,x)\epsilon(\det g)\phi_\sigma(g)\,\mathrm{d}g$$

does not vanish for some choices of data. This is equivalent to saying that the ψ-theta lifting of $\sigma\otimes\epsilon$ to $\mathrm{Mp}_{2j}(\mathbb{A})$ is non-zero. Finally, let $\mathrm{FO}_\psi(\sigma\otimes\epsilon)=2j'$. Then $2j'\leqslant 2j$. Denote $\pi=\theta_{\psi,m}^{2j'}(\sigma\otimes\epsilon)$. This is a cuspidal representation of $\mathrm{Mp}_{2j'}(\mathbb{A})$. By the theorem in § 2 of [**21**] and by Theorem 1.2 in [**15**], π is irreducible and $\sigma\otimes\epsilon=\theta_{\psi^{-1},2j'}^m(\pi)$. Write $m=m_0+2r$, where r is the Witt index of the quadratic form b. Then it is known that always, $\theta_{\psi^{-1},2j'}^{m_0+4j'}(\pi)\neq 0$. Since $\mathrm{FO}_{\psi^{-1}}(\pi)=m$, we conclude that $m\leqslant m_0+4j'$, and hence $r\leqslant 2j'\leqslant 2j$. The proof of Theorem 3.1 is now complete.

4. Proof of Theorem 1.1

The proof of Theorem 1.1 can be carried out exactly as in [**17**], using the doubling method directly for O_m, where the role of the regularized Siegel–Weil formula is played by (3.14). This will prove the theorem except one case, namely when $O_m = O_{2n+1}$, $L^S(\sigma\otimes\chi, \frac{1}{2}) \neq 0$ and $L^S(\sigma\otimes\chi, s)$ is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}$. So, as in Theorem 3.1, we consider again the maximal positive pole s_0 (assuming it exists) of the Eisenstein series $E^{Q_1}(g;\phi_{1;\sigma,\chi}, s)$. As in the previous section, we may assume that $\chi = 1$, since we can replace σ by $\sigma\otimes\chi$. Since we only consider case $a = 1$ in the rest of this section, we redenote $E^{Q_1}(g;\phi_{1;\sigma}, s)$ by $E(g;\phi_{1;\sigma}, s)$. We now prove Theorem 1.1.

Assume first, that the partial L-function $L^S(s,\sigma)$ has a pole at $s = \frac{1}{2}m - j > 0$. Consider its maximal pole, say $s_0' = \frac{1}{2}m - j_0'$, $j_0' \leqslant j$. Then, by Proposition 2.2, the Eisenstein series $E(g;\phi_{1;\sigma}, s)$ has a pole at $s_0' = \frac{1}{2}m - j_0'$. Hence there is an $s_0 \geqslant \frac{1}{2}m - j_0' > 0$ such that $E(g;\phi_{1;\sigma}, s)$ has a pole at s_0 and is holomorphic for $\mathrm{Re}(s) > s_0$. By Proposition 2.3, there is $j_0 \leqslant j_0' \leqslant j$, such that $s_0 = \frac{1}{2}m - j_0$. By Theorem 3.1, there is an automorphic sign character ϵ, such that the ψ-theta lift $\theta_{\psi,m}^{2j_0}(\sigma\otimes\epsilon)$ of $\sigma\otimes\epsilon$ to $\mathrm{Mp}_{2j_0}(\mathbb{A})$ is non-zero. By the Rallis tower property [**24**], the ψ-theta lift $\theta_{\psi,m}^{2j}(\sigma\otimes\epsilon)$ to $\mathrm{Mp}_{2j}(\mathbb{A})$ is non-zero, since $j_0 \leqslant j$. Assume next, that $m = 2n+1$, and $L^S(\sigma, \frac{1}{2}) \neq 0$. If $L^S(\sigma, s)$ has a pole at $\mathrm{Re}(s) > \frac{1}{2}$, then by what we just proved, there is an automorphic sign character ϵ, such that $\theta_{\psi,m}^{2j}(\sigma\otimes\epsilon) \neq 0$, for some $j < n$, and hence, by Rallis tower property, $\theta_{\psi,m}^{2n}(\sigma\otimes\epsilon) \neq 0$. If $L^S(\sigma, s)$ is holomorphic at $\mathrm{Re}(s) > \frac{1}{2}$, then we repeat the calculation (2.19), with $a = 1$. In the calculation of the constant term of $E(g,\phi_{1;\sigma}, s)$ along the unipotent radical of Q_1, the associated intertwining operator is applied to a decomposable section and then evaluated at the identity, and we get

$$\frac{L^S(s,\sigma)\zeta^S(2s)}{L^S(s+1,\sigma)\zeta^S(2s+1)},$$

up to a function that is holomorphic and non-zero at $s_0 = \frac{1}{2}$. Since at $s_0 = \frac{1}{2}$,

$$\frac{L^S(s,\sigma)\zeta^S(2s)}{L^S(s+1,\sigma)\zeta^S(2s+1)}$$

has a pole, we obtain that $E(g,\phi_{1;\sigma}, s)$ has a pole at $s = \frac{1}{2}$. Then, by Theorem 3.1, there is an automorphic sign character ϵ, such that $\theta_{\psi,m}^{2n}(\sigma\otimes\epsilon) \neq 0$. This proves part (1) of the theorem.

Part (2) follows from part (1). Assume that the lowest occurrence $\mathrm{LO}_\psi(\sigma)$ is $2j_0 < m$. If the partial L-function $L^S(s,\sigma)$ is not holomorphic for $\mathrm{Re}(s) > s_0 = \frac{1}{2}m - j_0$, then there is an integer $j < j_0$ such that the partial L-function $L^S(s,\sigma)$ has a pole at $s = \frac{1}{2}m - j$. Then by part (1), we know that there is an automorphic sign character ϵ_0, such that the ψ-theta lift $\theta_{\psi,m}^{2j}(\sigma\otimes\epsilon_0)$ of $\sigma\otimes\epsilon_0$ to $\mathrm{Mp}_{2j}(\mathbb{A})$ is non-zero. Since $j < j_0$, we must obtain that the lowest occurrence $\mathrm{LO}_\psi(\sigma) = 2j_0 > 2j$. This contradiction proves part (2).

Finally, assume that the lowest occurrence $\mathrm{LO}_\psi(\sigma)$ is $2j_0 \geqslant m$. If the partial L-function $L^S(s,\sigma)$ has a pole at $s = \frac{1}{2}m - j \geqslant \frac{1}{2}$, then by part (1), there is an automorphic sign character ϵ_0, such that the ψ-theta lift $\theta^{2j}_{\psi,m}(\sigma \otimes \epsilon_0)$ to $\mathrm{Mp}_{2j}(\mathbb{A})$ is non-zero. Hence we have

$$m \leqslant 2j_0 \leqslant \mathrm{FO}_\psi(\sigma \otimes \epsilon_0) \leqslant 2j < m.$$

This is impossible. This completes the proof of Theorem 1.1.

Remark 4.1. Theorem 1.1 is the orthogonal group version of Theorem 7.2.5 in [**18**], which explains the poles of the partial L-function $L^S(s,\sigma)$ for Sp_{2n} in terms of theta liftings to orthogonal groups.

5. Proof of Theorem 1.3

We first state the following theorem, which is important to the proof of Theorem 1.3.

Theorem 5.1. *Let O_m be the orthogonal group attached to a quadratic k-vector space of dimension m. For any $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$, if the first occurrence of σ is $\mathrm{FO}_\psi(\sigma) = 2j_0 \neq m$, with the property that either $2j_0 < m$ or $\frac{1}{2}m < j_0 < m-2$ with $6 < m$, then the Eisenstein series $E(g;\phi_{1;\sigma},s)$ has a pole at $s = \frac{1}{2}m - j_0$.*

Note that in this theorem, we consider the specific first occurrence $\mathrm{FO}_\psi(\sigma)$ of σ, but not the lowest occurrence $\mathrm{LO}_\psi(\sigma)$ of the family of all twists by sign characters $\sigma \otimes \epsilon$ of σ. In the proof, we have to study a certain period integral of the Arthur truncation of the Eisenstein series. The condition that $\frac{1}{2}m < j_0 < m-2$ with $6 < m$ is imposed to avoid the technical complication of proving absolute convergence of certain integrals in this case. See its proof in § 6 for details. As result, when $2j_0 < m$, we are going to prove a result (Theorem 5.3) which is stronger than Theorem 5.1 in this case.

5.1. Proof of Theorem 1.3

Let us show how Theorem 3.1 and Theorem 5.1 imply Theorem 1.3.

We may assume, as before, that $\chi = 1$. Assume that the Eisenstein series $E(g;\phi_{1;\sigma},s)$ has a pole at $s_0 = \frac{1}{2}m - j_0 > 0$ and is holomorphic for $\mathrm{Re}(s) > s_0$. By Theorem 3.1, there is an automorphic sign character ϵ_0, such that the ψ-theta lifting of $\sigma \otimes \epsilon_0$ to $\mathrm{Mp}_{2j_0}(\mathbb{A})$, $\theta^{2j_0}_{\psi,m}(\sigma \otimes \epsilon_0)$ does not vanish. If the lowest occurrence $\mathrm{LO}_\psi(\sigma) < 2j_0$, then there is an automorphic sign character ϵ (could be ϵ_0), such that the first occurrence $\mathrm{FO}_\psi(\sigma \otimes \epsilon) = 2j_1 < 2j_0$. Then by Theorem 5.1, the Eisenstein series $E(g;\phi_{1;\sigma\otimes\epsilon},s)$ must have a pole at $s = \frac{1}{2}m - j_1 > s_0$. Clearly, we have

$$E(g;\phi_{1;\sigma\otimes\epsilon},s) = \epsilon(\det g)E(g;\phi_{1;\sigma},s).$$

Hence $E(g;\phi_{1;\sigma},s)$ has a pole at $s = \frac{1}{2}m - j_1 > s_0$. This contradicts the assumption. Hence we must have that $\mathrm{LO}_\psi(\sigma) = 2j_0$. This proves one direction of Theorem 1.3.

Conversely, assume that the lowest occurrence $\mathrm{LO}_\psi(\sigma) = 2j_0 < m$. Let ϵ_0 be an automorphic sign character of $\mathrm{O}_m(\mathbb{A})$, such that the lowest occurrence $\mathrm{LO}_\psi(\sigma)$ is achieved by the first occurrence of $\sigma \otimes \epsilon_0$, i.e.

$$\mathrm{FO}_\psi(\sigma \otimes \epsilon_0) = 2j_0 = \mathrm{LO}_\psi(\sigma).$$

Then by Theorem 5.1, the Eisenstein series $E(g;\phi_{1;\sigma\otimes\epsilon_0},s)$, and hence $E(g;\phi_{1;\sigma},s)$, has a pole at $s_0=\frac{1}{2}m-j_0>0$. If $E(g;\phi_{1;\sigma},s)$ is not holomorphic for $\mathrm{Re}(s)>s_0$, then there is an $s_1>s_0$, such that $E(g;\phi_{1;\sigma},s)$ has a pole at s_1, and is holomorphic for $\mathrm{Re}(s)>s_1$. By Proposition 2.3, there is an integer $j_1<j_0$ such that $s_1=\frac{1}{2}m-j_1$. Now, by Theorem 3.1, there is an automorphic sign character ϵ_1, such that the ψ-theta lifting of $\sigma\otimes\epsilon_1$ to $\mathrm{Mp}_{2j_1}(\mathbb{A})$, $\theta^{2j_1}_{\psi,m}(\sigma\otimes\epsilon_1)$ is non-zero. Hence we must have

$$\mathrm{FO}_\psi(\sigma\otimes\epsilon_1)\leqslant 2j_1<2j_0=\mathrm{LO}_\psi(\sigma).$$

This is a contradiction to the definition of the lowest occurrence of σ (see (1.2)).

Therefore, Theorem 3.1 and Theorem 5.1 imply Theorem 1.3.

5.2. Periods of certain residues

In order to state Theorem 5.3, we have to recall some basics in the theory of theta correspondence and define the periods of Eisenstein series and its residue.

We write the elements of the symplectic group Sp_{2j} with respect to the skew-symmetric matrix J^-_{2j}, which is defined inductively as follows:

$$J^-_{2j}:=\begin{pmatrix}0 & & 1\\ & J^-_{2j-2} & \\ -1 & & 0\end{pmatrix}.$$

Consider the unipotent radical U of the Siegel parabolic subgroup $Q=LU$ of Sp_{2j}. We write the elements of U as

$$u(S)=\begin{pmatrix}I_j & S\\ 0 & I_j\end{pmatrix}\in U\subset \mathrm{Sp}_{2j}\,. \tag{5.1}$$

Let $\ell:=(l_1,l_2,\dots,l_j)\in(k^\times)^j$ be any j-tuple. For a non-trivial character ψ of $\mathbb{A}/k$, define a character ψ_ℓ of $U(\mathbb{A})$ by

$$\psi_\ell(u(S)):=\psi(l_1s_{1,j}+l_2s_{2,j-1}+\cdots+l_js_{j,1}). \tag{5.2}$$

This is trivial on $U(k)$. Let $\tilde{\pi}$ be an irreducible, automorphic, cuspidal representation of $\mathrm{Mp}_{2j}(\mathbb{A})$. By a theorem of Li [**19**], there exists an $\ell=(l_1,l_2,\dots,l_j)$ as above, such that the following ψ_ℓ-Fourier coefficient

$$\int_{U(k)\backslash U(\mathbb{A})}\phi_{\tilde{\pi}}(ug)\psi_\ell^{-1}(u)\,\mathrm{d}u \tag{5.3}$$

does not vanish, for some $\phi_{\tilde{\pi}}$ in the space of $\tilde{\pi}$.

In the following, for any $\sigma\in\mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$, we take $\tilde{\pi}=\theta^{2j}_{\psi,m}(\sigma)$ and assume that $\tilde{\pi}$ is an irreducible, automorphic, cuspidal representation of $\mathrm{Mp}_{2j}(\mathbb{A})$, i.e. the first occurrence of σ is $\mathrm{FO}_\psi(\sigma)=2j$, with $1\leqslant j\leqslant m$. The calculation of the ψ_ℓ-Fourier coefficient of automorphic forms $\phi_{\tilde{\pi}}$ in the space of $\tilde{\pi}$ is standard [**10**, **24**].

Let $v_1,\dots,v_j$ be vectors in (X,b) (as introduced in § 2), with the property that $b(v_s,v_t)=0$ if $s\neq t$ for $s,t=1,2,\dots,j$; and $b(v_t,v_t)=l_t$ for $t=1,2,\dots,j$. Then the pointwise stabilizer H_ℓ, in O_m, of the subspace generated by $v_1,v_2,\dots,v_j$, is a k-rational form of O_{m-j}. Note that if $j=m$, then H_ℓ is the identity group. In this case, the first statement of the next proposition is meaningless, and the meaning of the proposition is really its second part. Note also, that if $j=m-1$, then H_ℓ is isomorphic to the group of two elements.

Proposition 5.2. *Let $\sigma\in\mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$. Assume that the first occurrence of σ is $\mathrm{FO}_\psi(\sigma)=2j$, for some positive integer $j\leqslant m$.*

(1) *There exists an $\ell=(l_1,l_2,\dots,l_j)\in(k^\times)^j$, such that the following period*

$$\int_{H_\ell(k)\backslash H_\ell(\mathbb{A})}\phi_\sigma(h)\,\mathrm{d}h \tag{5.4}$$

does not vanish, for some ϕ_σ in the space of σ.

(2) *For any orthogonal subgroup $H_{\ell+}$ of O_m, with a conjugate $gH_{\ell+}g^{-1}$, $g\in\mathrm{O}_m$, containing H_ℓ as a proper subgroup, the following period*

$$\int_{H_{\ell+}(k)\backslash H_{\ell+}(\mathbb{A})}\phi_\sigma(h)\,\mathrm{d}h \tag{5.5}$$

vanishes for all choices of the data.

(3) *The period integrals in (5.4) and (5.5) converge absolutely.*

Proof. It is not hard to show that the period integrals in (5.4) and (5.5) are absolutely convergent. In fact, the cuspidal automorphic form ϕ_σ is rapidly decreasing over the Siegel domain of $\mathrm{O}_m(k)\backslash\mathrm{O}_m(\mathbb{A})$. In particular, it is bounded over the Siegel domain of O_m and hence over the group $\mathrm{O}_m(\mathbb{A})$. When H_ℓ or $H_{\ell+}(k)$ is not the split orthogonal group in two variables, both $H_\ell(k)\backslash H_\ell(\mathbb{A})$ and $H_{\ell+}(k)\backslash H_{\ell+}(\mathbb{A})$ are of finite volume with respect to their canonical Haar measures. This proves that both period integrals in (5.4) and (5.5) are absolutely convergent.

In case of the split orthogonal group in two variables, we can use a suitable basis to get that the connected part becomes the diagonal group

$$t(x)=\begin{pmatrix}x&&\\&I_{m-2}&\\&&x^{-1}\end{pmatrix},$$

and $\phi_\sigma(t(x))$ is rapidly decreasing at infinity. It is also rapidly decreasing near zero, since if w_0 is a Weyl element of O_m which takes $t(x)$ to $t(x^{-1})$, then $\phi_\sigma(t(x))=\phi_\sigma(t(x^{-1})w_0)$. Thus,

$$\int_{k^*\backslash\mathbb{A}^*}|\phi_\sigma(t(x))|\,\mathrm{d}^*x<\infty,$$

and clearly this convergence implies the absolute convergence of the period integral along the split orthogonal group in two variables. This proves part (3) of the theorem.

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The standard calculation of the ψ_ℓ-Fourier coefficient of automorphic forms $\phi_{\tilde{\pi}}$ in the space of $\tilde{\pi}$ [**10, 24**] expresses the period (5.4) as an inner integration of the ψ_ℓ-Fourier coefficient (5.3). This implies part (1).

The same calculation shows that any period as in (5.5) will occur as an inner integral of a certain Fourier coefficient of the ψ-theta lifting of σ to $\mathrm{Mp}_{2j_0}(\mathbb{A})$ with $j_0 < j$. Since the first occurrence of σ is $2j$, the period as in (5.5) must vanish identically.

More detailed arguments and calculations will be given in § 6.1 in the proof of Lemma 6.2. □

As in (2.2), we consider $X_1 = X \perp (\ell_1^+ \oplus \ell_1^-)$. We may write

$$X_1 = \ell_1^+ \oplus X \oplus \ell_1^-. \tag{5.6}$$

Denote the corresponding bilinear form on X_1 by b_1. We choose a basis in X_1 of form

$$e_0^+, x_1, \ldots, x_m, e_0^-, \tag{5.7}$$

where $x_1, \ldots, x_m$ is a basis for X, and $\ell_1^\pm = k \cdot e_0^\pm$ with $b_1(e_0^+, e_0^-) = 1$.

As in (2.3), for $a = 1$, we take $Q_1 = M_1 N_1$ to be the maximal parabolic subgroup of $\mathrm{O}(X_1)$, which stabilizes the isotropic line $k \cdot e_0^+ = \ell_1^+$. Hence, the Levi subgroup M_1 is isomorphic to $\mathrm{GL}_1 \times \mathrm{O}(X) = \mathrm{GL}_1 \times \mathrm{O}_m$. For any $\ell = (l_1, l_2, \ldots, l_j) \in (k^\times)^j$, we choose, as before, vectors $v_1, \ldots, v_j$ in (X, b), with the property that $b(v_s, v_t) = 0$, if $s \neq t$, for $s, t = 1, 2, \ldots, j$; and $b(v_t, v_t) = l_t$, for $t = 1, 2, \ldots, j$. We denote by Y_j the subspace of X generated by $v_1, \ldots, v_j$. Then Y_j is a non-degenerate subspace of X, and we have

$$X = Y_j \perp Z_{m-j}. \tag{5.8}$$

As in Proposition 5.2, we have $H_\ell = \mathrm{O}(Z_{m-j})$. It follows that

$$X_1 = \ell_1^+ \oplus X \oplus \ell_1^- = Y_j \perp Z_{m-j+2} = Y_j \perp (\ell_1^+ \oplus Z_{m-j} \oplus \ell_1^-). \tag{5.9}$$

We set $G_\ell = \mathrm{O}(Z_{m-j+2})$. For simplicity, we set here $\mathrm{O}_m = \mathrm{O}(X)$ and $\mathrm{O}_{m+2} = \mathrm{O}(X_1)$.

Now we are ready to state a refinement of Theorem 5.1 for the poles at $s = \frac{1}{2}m - j > 0$ of the Eisenstein series $E(g; \phi_{1;\sigma}, s)$. By a theorem of Langlands [**22**, § IV.1.11], these poles have no multiplicity.

Theorem 5.3. *Let $E(g; \phi_{1;\sigma}, s)$ be the Eisenstein series on $\mathrm{O}_{m+2}(\mathbb{A})$, attached to the cuspidal datum $(Q_1, 1 \otimes \sigma)$, as defined in § 2. Assume that $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ has the first occurrence $\mathrm{FO}_\psi(\sigma) = 2j$ for some positive integer $j < \frac{1}{2}m$. Then*

(1) *the Eisenstein series $E(g; \phi_{1;\sigma}, s)$ has a simple pole at $s = \frac{1}{2}m - j > 0$;*

(2) *there is an $\ell = (l_1, l_2, \ldots, l_j) \in (k^\times)^j$, such that the residue at $s_0 = \frac{1}{2}m - j$ of $E(g; \phi_{1;\sigma}, s)$, denoted by $\mathcal{E}_{m/2-j}(g; \phi_{1;\sigma})$, is G_ℓ-distinguished, i.e. the following period*

$$\int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \mathcal{E}_{m/2-j}(h; \phi_{1;\sigma}) \, \mathrm{d}h \tag{5.10}$$

is absolutely convergent for all choices of data, and is non-zero for some choices of data; and

(3) *all residues at other positive (simple) poles* $s_0 \neq \frac{1}{2}m - j$ *of* $E(g; \phi_{1;\sigma}, s)$ *cannot be* G_ℓ*-distinguished.*

It is clear that part (1) of Theorem 5.3 is contained in Theorem 5.1 for the positive poles. In the next section we will prove Theorem 5.1 by studying the period of type (5.10) for the Arthur truncation of the Eisenstein series. Based on this, we continue to study period of type (5.10) for Arthur truncation of the residue of the Eisenstein series, which proves Theorem 5.3.

6. Proof of Theorems 5.1 and 5.3

We start with proof of Theorem 5.1. For any $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$, we assume that the first occurrence of σ is $\mathrm{FO}_\psi(\sigma) = 2j_0 \neq m$, with the property that either $2j_0 < m$ or $\frac{1}{2}m < j_0 < m-2$ with $6 < m$. This is the assumption of Theorem 5.1.

We consider the period of the Eisenstein series

$$\int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} E(h; \phi_{1;\sigma}, s)\,\mathrm{d}h. \tag{6.1}$$

This integral may diverge. We first regularize this integral by Arthur's truncation. Then we show that the period over G_ℓ of the truncated Eisenstein series has a pole at $s = \frac{1}{2}m - j \neq 0$, and hence the Eisenstein series $E(g; \phi_{1;\sigma}, s)$ has a pole at $s = \frac{1}{2}m - j$. This will prove Theorem 5.1. When $2j_0 > m$, we introduce the restriction on j_0 and m to avoid the technical complication of proving absolute convergence of certain integrals which are introduced in the regularization process via the Arthur truncation method. Then, when the pole above is positive (i.e. $2j_0 < m$), by applying the Arthur truncation to the residue $\mathcal{E}_{m/2-j}(g; \phi_{1;\sigma})$, we express the period of $\mathcal{E}_{m/2-j}(g; \phi_{1;\sigma})$ in terms of the period of the truncated Eisenstein series and deduce (5.10). This will prove Theorem 5.3.

6.1. The Arthur truncation method

We recall from [**1**] and [**2**] Arthur's truncation formula, in our special case. Since Q_1 is maximal, $\mathfrak{a}_{Q_1}$ is one dimensional. We identity $\mathfrak{a}_{Q_1}$ with $\mathbb{R}$, as in § 2. Let c be a real number $c \in \mathbb{R}_{>1} \subset \mathfrak{a}_{Q_1}$, where $\mathbb{R}_{>1}$ is the set of real numbers, which are greater than one. Let τ^{c} ($c \in \mathbb{R}_{>1}$) be the characteristic function over $\mathbb{R}_{>0}$ of the subset $\mathbb{R}_{\geqslant c}$ and $\tau_c = 1_{\mathbb{R}_{>0}} - \tau^{\mathrm{c}}$. By § 2.13 of [**22**], the truncation of the Eisenstein series is defined as follows:

$$\Lambda^{\mathrm{c}} E(g; \phi_{1;\sigma}, s) = E(g; \phi_{1;\sigma}, s) - \sum_{\gamma \in Q_1 \backslash \mathrm{O}_{m+2}} E_{Q_1}(\gamma g; \phi_{1;\sigma}, s)\tau^{\mathrm{c}}(H(\gamma g)), \tag{6.2}$$

where $E_{Q_1}(g; \phi_{1;\sigma}, s)$ is the constant term of $E(g; \phi_{1;\sigma}, s)$ along the maximal parabolic subgroup Q_1. We remark that we are free to take c as large as we may need. Below we will need that $c > c_0$, for a certain constant c_0, which will be specified later. For $\mathrm{Re}(s) > \frac{1}{2}m$, we have the identity

$$E_{Q_1}(g; \phi_{1;\sigma}, s) = \Phi_s(g; \phi_{1;\sigma}) + \mathcal{M}(s, \sigma, w_1)(\Phi_s)(g; \phi_{1;\sigma}),$$

which holds for all s as meromorphic functions. Here $\mathcal{M}(s,\sigma,w_1)$ is the standard (global) intertwining operator attached to the maximal parabolic subgroup Q_1 and the Weyl element w_1, which has the property that $w_1 M_1 w_1^{-1} = M_1$ and $w_1 N_1 w_1^{-1} = N_1^-$ (the opposite of N_1).

We remark that the summation in (6.2) has only finitely many terms (depending on g) and converges. Further, for $\mathrm{Re}(s) > \frac{1}{2}m$, the truncated Eisenstein series may be written as follows:

$$\begin{aligned}\Lambda^{\mathrm{c}} E(g;\phi_{1;\sigma},s) &= \sum_{\gamma\in Q_1\backslash \mathrm{O}_{m+2}} \Phi_s(\gamma g;\phi_{1;\sigma})\tau_c(H(\gamma g)) \\ &\qquad - \sum_{\gamma\in Q_1\backslash \mathrm{O}_{m+2}} \mathcal{M}(s,\sigma,w_1)(\Phi_s)(\gamma g;\phi_{1;\sigma})\tau^{\mathrm{c}}(H(\gamma g)) \\ &= \theta_1^{\mathrm{c}}(g) - \theta_2^{\mathrm{c}}(g), \end{aligned} \tag{6.3}$$

where

$$\begin{aligned}\theta_1^{\mathrm{c}}(g) &:= \sum_{\gamma\in Q_1\backslash \mathrm{O}_{m+2}} \Phi_s(\gamma g;\phi_{1;\sigma})\tau_c(H(\gamma g)), \\ \theta_2^{\mathrm{c}}(g) &:= \sum_{\gamma\in Q_1\backslash \mathrm{O}_{m+2}} \mathcal{M}(s,\sigma,w_1)(\Phi_s)(\gamma g;\phi_{1;\sigma})\tau^{\mathrm{c}}(H(\gamma g)).\end{aligned}$$

Note that $\theta_1^{\mathrm{c}}(g)$ converges absolutely for $\mathrm{Re}(s) > \frac{1}{2}m$, while $\theta_2^{\mathrm{c}}(g)$ has only finitely many terms (depending on g). Both have meromorphic continuation to the whole complex plane.

We will calculate first the period of the truncated Eisenstein series

$$\int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \Lambda^{\mathrm{c}} E(g;\phi_{1;\sigma},s)\,\mathrm{d}g = \int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \theta_1^{\mathrm{c}}(g)\,\mathrm{d}g - \int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \theta_2^{\mathrm{c}}(g)\,\mathrm{d}g. \tag{6.4}$$

The truncated Eisenstein series $\Lambda^{\mathrm{c}} E(g;\phi_{1;\sigma},s)$ is rapidly decaying over a Siegel domain of $\mathrm{O}_{m+2}(k)\backslash \mathrm{O}_{m+2}(\mathbb{A})$, and hence, by the proof of part (3) of Proposition 5.2, the period integral on the left-hand side of (6.4) is absolutely convergent. The right-hand side of (6.4) is a difference of two integrals. We will show that the term with $i=1$ converges absolutely for $\mathrm{Re}(s)$ sufficiently large and has a meromorphic continuation to the whole plane, and we will determine the location of its poles precisely. We will also show that the term with $i=2$ converges absolutely for all s, where the intertwining operator is defined, provided we take the truncation parameter $c > c_0$ for some c_0, which will be specified later. Put

$$I_i^{\mathrm{c}} = \int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \theta_i^{\mathrm{c}}(g)\,\mathrm{d}g. \tag{6.5}$$

For the calculation below, write, for $i=1,2$,

$$\theta_i^{\mathrm{c}}(g) = \sum_{\gamma\in Q_1\backslash \mathrm{O}_{m+2}} \xi_{i,s}^{\mathrm{c}}(\gamma g), \tag{6.6}$$

where we have

$$\xi^{\mathrm{c}}_{i,s}(g) = \begin{cases} \Phi_s(g;\phi_{1;\sigma})\tau_c(H(g)) & \text{if } i=1, \\ \mathcal{M}(s,\sigma,w_1)(\Phi_s)(g;\phi_{1;\sigma})\tau^{\mathrm{c}}(H(g)) & \text{if } i=2. \end{cases} \tag{6.7}$$

We may unfold formally the integral (6.5) as follows:

$$\begin{aligned} I^{\mathrm{c}}_i &= \int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \sum_{\gamma\in Q_1\backslash \mathrm{O}_{m+2}} \xi^{\mathrm{c}}_{i,s}(\gamma g)\,\mathrm{d}g \\ &= \sum_{\gamma\in Q_1\backslash \mathrm{O}_{m+2}/G_\ell} \int_{G^\gamma_\ell\backslash G_\ell(\mathbb{A})} \xi^{\mathrm{c}}_{i,s}(\gamma g)\,\mathrm{d}g, \end{aligned} \tag{6.8}$$

where $G^\gamma_\ell = \gamma^{-1}Q_1\gamma \cap G_\ell$. We will show that each integral in the summation (6.8) converges absolutely. Since there are infinitely many G_ℓ-orbits in the generalized flag variety $Q_1\backslash \mathrm{O}_{m+2}$, we have to show that the series is absolutely convergent. More precisely, we will show that

$$\sum_{\gamma\in Q_1\backslash \mathrm{O}_{m+2}/G_\ell} \int_{G^\gamma_\ell\backslash G_\ell(\mathbb{A})} |\xi^{\mathrm{c}}_{i,s}(\gamma g)|\,\mathrm{d}g < \infty, \tag{6.9}$$

for $\mathrm{Re}(s)$ sufficiently large, when $i=1$, and for all s, where the intertwining operator is defined, when $i=2$, provided $c>c_0$. We will show that the integrals in (6.8) are equal to zero, for all G_ℓ-orbits $Q_1\gamma G_\ell$, except the G_ℓ-orbit with representative $\gamma=1$. Hence the series over $\gamma\in Q_1\backslash \mathrm{O}_{m+2}/G_\ell$ has at most one non-zero term. We will compute this term explicitly.

We first classify the G_ℓ-orbits in the generalized flag variety $Q_1\backslash \mathrm{O}_{m+2}$. It is clear that the double coset decomposition $Q_1\backslash \mathrm{O}_{m+2}/G_\ell$ is the same as the orbit decomposition of the right action of G_ℓ on the generalized flag variety $Q_1\backslash \mathrm{O}_{m+2}$, which is isomorphic to the projective variety $\mathcal{I}(X_1)$, which is the isotropic cone in X_1 consisting of all isotropic lines in the quadratic vector space X_1.

Let x and x' be two non-zero isotropic vectors in X_1, which belong to the same G_ℓ-orbit. Then by (5.7), we have $b_1(x,x)=b_1(x',x')=0$ and there is an $h\in G_\ell$ and $\alpha\in k^\times$ such that $\alpha x' = h^{-1}\cdot x$. Following the decomposition (5.9), we write $x=y+z$ and $x'=y'+z'$. It follows that $h^{-1}\cdot x = y + h^{-1}\cdot z = \alpha y' + \alpha z' = \alpha x'$. Hence we obtain that if $x=y+z$ and $x'=y'+z'$ are two non-zero isotropic vectors in X_1, which belong to the same G_ℓ-orbit, then

$$y'=\beta y \quad\text{and}\quad z'=\beta\cdot h^{-1}\cdot z$$

for some $\beta\in k^\times$ and $h\in G_\ell$. For a double coset $Q_1\gamma G_\ell$, in the double coset decomposition $Q_1\backslash \mathrm{O}_{m+2}/G_\ell$, we denote by $k\cdot x_\gamma$ the corresponding isotropic line in X_1, and similarly, for an isotropic line $k\cdot x$ in X_1, we denote by $Q_1\gamma_x G_\ell$ the corresponding double coset in the double coset decomposition $Q_1\backslash \mathrm{O}_{m+2}/G_\ell$.

We decompose the isotropic cone $\mathcal{I}(X_1)$ into a disjoint union as follows:

$$Q_1\backslash \mathrm{O}_{m+2} = \mathcal{I}(X_1) = \Omega_{1,0}\cup\Omega_{0,1}\cup\Omega_{1,1}\cup\Omega_{2,2}. \tag{6.10}$$

Here $\Omega_{1,0}$ consists of all $x = y + z \in \mathcal{I}(X_1)$ such that $z = 0$; $\Omega_{0,1}$ consists of all $x = y + z \in \mathcal{I}(X_1)$ such that $y = 0$; $\Omega_{1,1}$ consists of all $x = y + z \in \mathcal{I}(X_1)$ such that both y and z are non-zero isotropic vectors; and $\Omega_{2,2}$ consists of all $x = y + z \in \mathcal{I}(X_1)$ such that both y and z are anisotropic vectors. It is not hard to check that $\Omega_{1,0}$, $\Omega_{0,1}$, $\Omega_{1,1}$ and $\Omega_{2,2}$ are all G_ℓ-stable. Hence we have

$$Q_1 \backslash \mathrm{O}_{m+2} / G_\ell = [\Omega_{1,0}/G_\ell] \cup [\Omega_{0,1}/G_\ell] \cup [\Omega_{1,1}/G_\ell] \cup [\Omega_{2,2}/G_\ell].$$

It follows that the series in (6.8) can be written as

$$\begin{aligned}\sum_{\gamma \in Q_1 \backslash \mathrm{O}_{m+2} / G_\ell} \int_{G_\ell^\gamma \backslash G_\ell(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma g)\,\mathrm{d}g &= \sum_{x \in \Omega_{1,0}/G_\ell} \int_{G_\ell^{\gamma_x} \backslash G_\ell(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma_x g)\,\mathrm{d}g \\ &= \sum_{x \in \Omega_{0,1}/G_\ell} \int_{G_\ell^{\gamma_x} \backslash G_\ell(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma_x g)\,\mathrm{d}g \\ &= \sum_{x \in \Omega_{1,1}/G_\ell} \int_{G_\ell^{\gamma_x} \backslash G_\ell(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma_x g)\,\mathrm{d}g \\ &= \sum_{x \in \Omega_{2,2}/G_\ell} \int_{G_\ell^{\gamma_x} \backslash G_\ell(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma_x g)\,\mathrm{d}g. \end{aligned} \tag{6.11}$$

It is enough to prove (6.9) for each of these series.

6.2. The series over $\Omega_{1,0}/G_\ell$

For $x \in \Omega_{1,0}$, we have $x = y \in Y_j$ with $z = 0$. Then $k \cdot y$ is a G_ℓ-orbit. Since $b_1(y, y) = b_1(x, x) = 0$, there is a $\gamma \in \mathrm{O}_{m+2}(k)$ such that $\gamma^{-1} \cdot e_0^+ = y$. This implies that $x = x_\gamma$, and

$$\begin{aligned} G_\ell^\gamma &= \gamma^{-1} \cdot Q_1 \cdot \gamma \cap G_\ell \\ &= \{g \in G_\ell \mid g^{-1}(k \cdot y) = k \cdot y\} \\ &= G_\ell. \end{aligned}$$

It follows that $\gamma \cdot G_\ell \cdot \gamma^{-1} \subset Q_1$. We choose a representative γ for the double coset $Q_1 \gamma G_\ell$ as follows. Let $y' \in Y_j$ be an isotropic vector, which is dual to y, i.e. $b_1(y, y') = 1$. Then we choose $\gamma \in \mathrm{O}_{m+2}(k)$ such that $\gamma^{-1} \cdot e_0^+ = y$, $\gamma^{-1} \cdot y = e_0^+$, $\gamma^{-1} \cdot e_0^- = y'$, $\gamma^{-1} \cdot y' = e_0^-$, and the restriction of γ^{-1} to the subspace $\mathrm{Span}(y, y')^\perp \cap Y_j \oplus Z_{m-j}$ is the identity. It follows that

$$\gamma \cdot G_\ell \cdot \gamma^{-1} \subset \begin{pmatrix} 1 & & 0 \\ & \mathrm{O}_m & \\ 0 & & 1 \end{pmatrix} = M_1^0. \tag{6.12}$$

More precisely, we have

$$\gamma \cdot G_\ell \cdot \gamma^{-1} = \mathrm{O}((k \cdot y \oplus k \cdot y') \oplus Z_{m-j}) \subset \mathrm{O}(X) = \mathrm{O}_m.$$

For such a γ or a double coset $Q_1\gamma G_\ell$, the integral in (6.8) can be written as

$$\int_{G_\ell^\gamma\backslash G_\ell(\mathbb{A})} \xi^{\mathrm{c}}_{i,s}(\gamma g)\,\mathrm{d}g = \int_{G_\ell^\gamma\backslash G_\ell(\mathbb{A})} \xi^{\mathrm{c}}_{i,s}(\gamma g\gamma^{-1}\cdot\gamma)\,\mathrm{d}g$$
$$= \int_{\mathrm{O}((k\cdot y\oplus k\cdot y')\oplus Z_{m-j})\backslash\,\mathrm{O}((k\cdot y\oplus k\cdot y')\oplus Z_{m-j})(\mathbb{A})} \xi^{\mathrm{c}}_{i,s}(h\gamma)\,\mathrm{d}h.$$

By the definition of $\xi^{\mathrm{c}}_{i,s}(g)$ (see (6.7) and (2.10), (2.11)), we have

$$\xi^{\mathrm{c}}_{i,s}(h\gamma) = \begin{cases} \phi_{1;\sigma}(h\gamma)\tau_c(H(\gamma))H(\gamma)(s) & \text{if } i=1,\\ \mathcal{M}(s,\sigma,w_1)(\phi_{1;\sigma})(h\gamma)\tau^{\mathrm{c}}(H(\gamma))H(\gamma)(-s) & \text{if } i=2.\end{cases}$$

Hence the dh-integration defines a period of the cuspidal automorphic form ϕ_σ in the space of σ over the reductive subgroup

$$\mathrm{O}((k\cdot y\oplus k\cdot y')\oplus Z_{m-j})$$

of O_{m+2}. By Proposition 5.2, it is absolutely convergent. Note that the dh-integration of $H(\gamma)(-\frac{1}{2}m)|\phi_{1;\sigma}(h\gamma)|$ is bounded, as γ varies as above. Note also that, for all y, y' as above, $\mathrm{O}((k\cdot y\oplus k\cdot y')\oplus Z_{m-j})$ is k-isomorphic to G_ℓ. We use the fact that $H(\gamma)(-\frac{1}{2}m)|\phi_{1;\sigma}(h\gamma)|$ is uniformly bounded, for h, γ as above, and the fact that $G_\ell(k)\backslash G_\ell(\mathbb{A})$ has finite measure (since $m-j+2\neq 2$). Thus, it is enough to consider the convergence of the series

$$\sum_{x\in\Omega_{1,0}/G_\ell} \tau_c(H(\gamma_x))H(\gamma_x)(\mathrm{Re}(s)+\tfrac{1}{2}m)$$

if $i=1$; and

$$\sum_{x\in\Omega_{1,0}/G_\ell} \tau^{\mathrm{c}}(H(\gamma_x))H(\gamma_x)(-\mathrm{Re}(s)+\tfrac{1}{2}m)$$

if $i=2$.

When $i=1$, the series is majorized by the Eisenstein series

$$\sum_{\gamma\in Q_1\backslash\,\mathrm{O}_{m+2}} H(\gamma)(\mathrm{Re}(s)+\tfrac{1}{2}m),$$

which converges absolutely, for $\mathrm{Re}(s)$ large enough. When $i=2$, the series above is a finite sum, due to the presence of $\tau^{\mathrm{c}}(H(\gamma))$.

On the other hand, by Proposition 5.2, the dh-integrations above must vanish, for all choices of data because of the assumption of the first occurrence of σ. We summarize this as the following proposition.

Proposition 6.1. *Let $\sigma\in\mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ have the first occurrence $\mathrm{FO}_\psi(\sigma)=2j$, such that j and m satisfy the condition in Theorem 5.1. Then the series*

$$\sum_{x\in\Omega_{1,0}/G_\ell} \int_{G_\ell^{\gamma_x}\backslash G_\ell(\mathbb{A})} \xi^{\mathrm{c}}_{i,s}(\gamma_x g)\,\mathrm{d}g$$

converges absolutely, i.e.

$$\sum_{x\in\Omega_{1,0}/G_\ell}\int_{G_\ell^{\gamma_x}\backslash G_\ell(\mathbb{A})}|\xi^{\mathrm{c}}_{i,s}(\gamma_x g)|\,\mathrm{d}g<\infty,$$

for $\mathrm{Re}(s)$ *sufficiently large, when* $i=1$*, and for all* s*, where the intertwining operator is defined, when* $i=2$*. Moreover, the integral in the summation above attached to* γ_x*,*

$$\int_{G_\ell^{\gamma_x}\backslash G_\ell(\mathbb{A})}\xi^{\mathrm{c}}_{i,s}(\gamma_x g)\,\mathrm{d}g,$$

is absolutely convergent for all s *(where the intertwining operator is defined, when* $i=2$*) and is identically zero, for all choices of data.*

In Proposition 6.1, the only restriction we needed on j was that $j<m$.

6.3. The series over $\Omega_{1,1}/G_\ell$

For $x=y+z$ with $z\neq 0$, we have three cases for y:

(1) $y=0$, which means that $x=y+z=z$ is isotropic, and hence $x\in\Omega_{0,1}/G_\ell$;

(2) $y\neq 0$ is isotropic, which implies that z is isotropic, and hence $x\in\Omega_{1,1}/G_\ell$ (of course, if b is k-anisotropic, this is impossible and hence $\Omega_{1,1}$ is empty); and

(3) $y\neq 0$ is anisotropic, which means that z must be anisotropic, and hence $x\in\Omega_{2,2}/G_\ell$.

We treat the case (2) in this subsection and cases (1) and (3) in the following subsections.

For $x=y+z\in\Omega_{1,1}/G_\ell$, y is an non-zero isotropic vector in Y_j. This means that b is assumed to be k-isotropic. In this case, z is isotropic in Z_{m-j+2}. It is easy to see that, for each y, with the above property, the set of isotropic lines of the following type

$$\{k\cdot(y+z)\mid 0\neq z\in Z_{m-j+2},\ b_1(z,z)=0\}\tag{6.13}$$

is a G_ℓ-orbit. Note that this also holds for $x\in\Omega_{0,1}$, i.e. $y=0$ case.

When $y\neq 0$ ($x=y+z\in\Omega_{1,1}/G_\ell$), we may choose the isotropic line $k\cdot(y+e_0^+)$, as the representative of the G_ℓ-orbit, and γ is the corresponding representative for $Q_1\gamma G_\ell$. For $h\in G_\ell$, which stabilizes the isotropic line $k\cdot(y+e_0^+)$, we have $h\cdot(y+e_0^+)=\alpha(y+e_0^+)$. We must have $\alpha=1$ and $h\cdot e_0^+=e_0^+$. Hence, the stabilizer of the isotropic line $k\cdot(y+e_0^+)$ in G_ℓ, G_ℓ^γ is contained in Q_1^0, where $Q_1^0=M_1^0N_1\subset Q_1$, with M_1^0 being defined in (6.12). It is easy to check that

$$G_\ell^\gamma=G_\ell\cap Q_1^0.$$

Next, note that $k\cdot(y_1+e_0^+)$ and $k\cdot(y_2+e_0^+)$ (y_1, y_2 non-zero isotropic vectors in Y_j) lie in the same G_ℓ orbit, if and only if y_1,y_2 are proportional by an element of k^*. Thus, for this family of orbits, y varies in the set of non-zero isotropic vectors of Y_j, modulo k^*. Now, we may modify the representative γ, so that γ has the following additional properties. Let y' be an isotropic vector in Y_j, which is dual to y. It is easy to check that we can choose a γ within the double coset $Q_1\gamma G_\ell$, with the following properties:

$$\gamma^{-1}\cdot e_0^+ = y + e_0^+, \qquad \gamma^{-1}\cdot e_0^- = \tfrac{1}{2}(y' + e_0^-),$$
$$\gamma^{-1}\cdot y = -y + e_0^+, \qquad \gamma^{-1}\cdot y' = \tfrac{1}{2}(-y' + e_0^-),$$

and the restriction of γ^{-1} to the subspace $\mathrm{Span}(y,y')^\perp \cap Y_j \oplus Z_{m-j}$ is the identity.

For $h \in G_\ell \cap Q_1^0 = G_\ell^\gamma$, we want to know the action of $\gamma h\gamma^{-1}$ according to the decomposition of X_1:

$$X_1 = k\cdot e_0^+ \oplus k\cdot y \oplus Z_{m-j} \oplus Y_j \cap \mathrm{Span}(y,y')^\perp \oplus k\cdot y' \oplus k\cdot e_0^-.$$

Note that $\gamma h\gamma^{-1} \in Q_1$. Further we have

$$\begin{aligned}
\gamma h\gamma^{-1}\cdot e_0^+ &= e_0^+,\\
\gamma h\gamma^{-1}\cdot y &= y,\\
\gamma h\gamma^{-1}\cdot z' &= \tfrac{1}{2}\beta e_0^+ + \tfrac{1}{2}\beta y + z'(h) \quad (z' \in Z_{m-j},\ \beta \in k),\\
\gamma h\gamma^{-1}\cdot y_0 &= y_0 \quad (y_0 \in Y_j \cap \mathrm{Span}(y,y')^\perp),\\
\gamma h\gamma^{-1}\cdot y' &= \tfrac{1}{4}\alpha e_0^+ + \tfrac{1}{4}\alpha y + \tfrac{1}{2}z + y',\\
\gamma h\gamma^{-1}\cdot e_0^- &= \tfrac{1}{4}\alpha e_0^+ + \tfrac{1}{4}\alpha y + \tfrac{1}{2}z + e_0^- \quad (\alpha = \alpha(h) \in k,\ z = z(h) \in Z_{m-j}),
\end{aligned}$$

and $\gamma h\gamma^{-1}\cdot(-y' + e_0^-) = (-y' + e_0^-)$. Hence the element $\gamma h\gamma^{-1}$ can be expressed as the following matrix:

$$\begin{pmatrix}
1 & 0 & \tfrac{1}{2}\beta & 0 & \tfrac{1}{4}\alpha & \tfrac{1}{4}\alpha\\
0 & 1 & \tfrac{1}{2}\beta & 0 & \tfrac{1}{4}\alpha & \tfrac{1}{4}\alpha\\
0 & 0 & h' & 0 & z & z\\
0 & 0 & 0 & I_{j-2} & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \tag{6.14}$$

where $h' \in \mathrm{O}(Z_{m-j}) = H_\ell$, β is a row vector and z is an appropriate column vector. Note that in (6.14) we have

$$g(h) := \begin{pmatrix}
1 & \tfrac{1}{2}\beta & 0 & \tfrac{1}{4}\alpha\\
0 & h' & 0 & z\\
0 & 0 & I_{j-2} & 0\\
0 & 0 & 0 & 1
\end{pmatrix} \in \mathrm{O}_m = \mathrm{O}(X),$$

according the following decomposition of the quadratic vector space X:

$$X = k\cdot y \oplus Z_{m-j} \oplus Y_j \cap \mathrm{Span}(y,y')^\perp \oplus k\cdot y'.$$

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It is easy to check that the mapping $\gamma h\gamma^{-1} \mapsto g(h)$ is a group homomorphism from $\gamma \cdot G_\ell^\gamma \cdot \gamma^{-1}$ onto the subgroup $H_{y,j}$ of $\mathrm{O}_m = \mathrm{O}(X)$, where $H_{y,j}$ is defined to be

$$\left\{ g = \begin{pmatrix} 1 & u & 0 & v \\ 0 & h' & 0 & u' \\ 0 & 0 & I_{j-2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{O}(X) \;\middle|\; h' \in \mathrm{O}(Z_{m-j}),\ u \in k^{m-j},\ v \in k \right\}. \tag{6.15}$$

In other words, for any g in (6.15), we have

$$\gamma^{-1} g\gamma \in G_\ell \cap Q_1^0 = G_\ell^\gamma.$$

Lemma 6.2. *For $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$, if the first occurrence $\mathrm{FO}_\psi(\sigma) = 2j$, then the following period*

$$\int_{H_{y,j}\backslash H_{y,j}(\mathbb{A})} \phi_\sigma(h)\,\mathrm{d}h$$

is absolutely convergent, and vanishes identically for all automorphic forms ϕ_σ in the space of σ.

This lemma will be proven later in § 6.8. The absolute convergence in the lemma is clear, using Proposition 5.2. Note that $H_{y,j}$ is a semi-direct product of $\mathrm{O}(Z_{m-j})$ and a unipotent group.

The formal calculation of the integral in (6.8), attached to the representative γ above, is as follows:

$$\begin{aligned} \int_{G_\ell^\gamma\backslash G_\ell(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma g)\,\mathrm{d}g &= \int_{G_\ell^\gamma(\mathbb{A})\backslash G_\ell(\mathbb{A})} \int_{G_\ell^\gamma\backslash G_\ell^\gamma(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma h g)\,\mathrm{d}h\,\mathrm{d}g \\ &= \int_{G_\ell^\gamma(\mathbb{A})\backslash G_\ell(\mathbb{A})} \int_{G_\ell^\gamma\backslash G_\ell^\gamma(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma h\gamma^{-1}\cdot\gamma g)\,\mathrm{d}h\,\mathrm{d}g \\ &= \int_{G_\ell^\gamma(\mathbb{A})\backslash G_\ell(\mathbb{A})} \int_{H_{y,j}\backslash H_{y,j}(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(h\cdot\gamma g)\,\mathrm{d}h\,\mathrm{d}g. \end{aligned} \tag{6.16}$$

The last equality holds because $\xi_{i,s}^{\mathrm{c}}(ng) = \xi_{i,s}^{\mathrm{c}}(g)$ for any $n \in N_1(\mathbb{A})$. By the definition of $\xi_{i,s}^{\mathrm{c}}(g)$ (see (6.7) and (2.10), (2.11)), we have

$$\xi_{i,s}^{\mathrm{c}}(h\cdot\gamma g) = \begin{cases} \phi_{1;\sigma}(h\gamma g)\tau_c(H(\gamma g))H(\gamma g)(s) & \text{if } i=1, \\ \mathcal{M}(s,\sigma,w_1)(\phi_{1;\sigma})(h\gamma g)\tau^{\mathrm{c}}(H(\gamma g))H(\gamma g)(-s) & \text{if } i=2. \end{cases}$$

Hence the inner integration in (6.16)

$$\int_{H_{y,j}\backslash H_{y,j}(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(h\cdot\gamma g)\,\mathrm{d}h$$

defines a period as in Lemma 6.2. Hence it is identically zero for all choices of data since the first occurrence $\mathrm{FO}_\psi(\sigma)$ is $2j$.

Now we prove that the series

$$\sum_{x\in\Omega_{1,1}/G_\ell}\int_{G_\ell^{\gamma_x}(\mathbb{A})\backslash G_\ell(\mathbb{A})}\int_{H_{y,j}\backslash H_{y,j}(\mathbb{A})}\xi^{\mathrm{c}}_{i,s}(h\cdot\gamma_x g)\,\mathrm{d}h\,\mathrm{d}g \tag{6.17}$$

converges absolutely, i.e.

$$\sum_{x\in\Omega_{1,1}/G_\ell}\int_{G_\ell^{\gamma_x}(\mathbb{A})\backslash G_\ell(\mathbb{A})}\int_{H_{y,j}\backslash H_{y,j}(\mathbb{A})}|\xi^{\mathrm{c}}_{i,s}(h\cdot\gamma_x g)|\,\mathrm{d}h\,\mathrm{d}g<\infty,$$

for $\mathrm{Re}(s)$ sufficiently large, when $i=1$, and for all s, where the intertwining operator is defined, when $i=2$, provided the truncation parameter $c>c_0$, which is a constant to be specified soon. Since each G_ℓ-orbit has representative $x=y+e_0^+$, we may abuse the notation by setting $\gamma_x=\gamma_y$ for $y\in Y_j$. In this case, y must be non-zero and isotropic.

Let us start with $i=1$. Put $G_\ell\cap Q_1=Q_{\ell,1}$. This is the parabolic subgroup of G_ℓ, which preserves the line through e_0^+. Clearly, the elements of $Q_{\ell,1}$ can be written uniquely in the form $gh_0(t)$, where $g\in G_\ell^\gamma$ and, for $t\in k^*$, $h_0(t)e_0^+=te_0^+$, $h_0(t)e_0^-=t^{-1}e_0^-$, and the restriction of $h_0(t)$ to Z_{m-j} is the identity. By the Iwasawa decomposition in $G_\ell(\mathbb{A})$, it is enough to consider

$$\sum_y\int_{\mathbb{A}^*}|t|^{j-m}\int_{H_{y,j}\backslash H_{y,j}(\mathbb{A})}H(\gamma_y h_0(t))(-\tfrac{1}{2}m)|\phi_{1;\sigma}(h\gamma_y h_0(t))|$$
$$\times\,\tau_c(H(\gamma_y h_0(t)))H(\gamma_y h_0(t))(\mathrm{Re}(s)+\tfrac{1}{2}m)\,\mathrm{d}h\,\mathrm{d}t,$$

where the summation is over the non-zero isotropic vectors $y\in Y_j$, modulo k^*. Note that the matrix form of $H_{y,j}$ is independent of y. Since σ is cuspidal and $H_{y,j}\backslash H_{y,j}(\mathbb{A})$ has finite measure (due to our assumption that $m-j\neq 2$, if $j\geqslant 2$) we may bound $H(\gamma_y h_0(t))(-\frac{1}{2}m)|\phi_{1;\sigma}(h\gamma_y h_0(t))|$ by a constant (depending on the section). Thus, it is enough to consider

$$\sum_y\int_{\mathbb{A}^*}|t|^{j-m}\tau_c(H(\gamma_y h_0(t)))H(\gamma_y h_0(t))(\mathrm{Re}(s)+\tfrac{1}{2}m)\,\mathrm{d}^*t, \tag{6.18}$$

where the summation over y is as before. Let us use the k-basis for $Y'=ke_0^+\oplus Y_j\oplus ke_0^-$, $\{e_0^+,v_1,\dots,v_j,e_0^-\}$ in order to give coordinates on $Y'(k_v)$, for all places v, and identify $Y'(k_v)=k_v^{j+2}$, using this basis (see the paragraph after (5.7)). Consider for each place v the local norm on $Y'(k_v)=k_v^{j+2}$, which is the maximum norm of the coordinates, when v is finite, the usual Euclidean norm, when v is real, and the square of the usual Euclidean norm, when v is complex. Define now, for $w\in Y'(\mathbb{A})$,

$$\|w\|=\prod_v\|w_v\|_v.$$

Note that for any non-zero vector $w\in Y'$ (i.e. with rational coordinates), we have $\|w\|\geqslant 1$. Now, in the last integral, by our choice of the representatives $\gamma=\gamma_y$, we

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may think of γ_y and of $h_0(t)$ as elements of the adelic points of the orthogonal group $\mathrm{O}(Y')$ corresponding to Y', because they act as the identity on Z_{m-j}. Denote by Q'_1 the parabolic subgroup of $\mathrm{O}(Y')$, which preserves the line through e_0^+. The Levi part of Q'_1 is naturally isomorphic to $\mathrm{GL}_1 \times \mathrm{O}(Y_j)$, where the elements of GL_1 are realized as the $h_0(t)$ above. Denote the unipotent radical of Q'_1 by U' and put $(Q'_1)^0 = \mathrm{O}(Y_j)U'$. Choose a good maximal subgroup K'_v of $\mathrm{O}(Y')_v$, with respect to Q'_1 and let $K'_{\mathbb{A}} = \prod_v K'_v$. Let $g \in \mathrm{O}(Y')_{\mathbb{A}}$ and write its Iwasawa decomposition

$$g = q^0(g)h_0(t(g))k'(g),$$

where $q^0(g) \in (Q'_1)^0(\mathbb{A})$, $t(g) \in \mathbb{A}^*$, $k'(g) \in K'_{\mathbb{A}}$. We have

$$\|g^{-1}e_0^+\| = |t(g)|^{-1}\|k'(g)^{-1}e_0^+\| = H(g)(-1)\|k'(g)^{-1}e_0^+\|,$$

and hence

$$H(g)(1) = \|g^{-1}e_0^+\|^{-1}\|k'(g)^{-1}e_0^+\|.$$

Clearly, there are positive constants, d_1, d_2, such that

$$d_1 \leqslant \|k'e_0^+\| \leqslant d_2,$$

for all $k' \in K'_{\mathbb{A}}$, hence $H(g)(1)$ is comparable to $\|g^{-1}e_0^+\|^{-1}$, and so in the integral (6.18) we may replace $H(\gamma_y h_0(t))(\mathrm{Re}(s) + \frac{1}{2}m)$ by

$$\|(\gamma_y h_0(t))^{-1}e_0^+\|^{-\mathrm{Re}(s)-m/2} = \|t^{-1}e_0^+ + y\|^{-\mathrm{Re}(s)-m/2}.$$

Thus, it is enough to consider the convergence of

$$\sum_y \int_{\mathbb{A}^*} |t|^{m-j}\|te_0^+ + y\|^{-\mathrm{Re}(s)-m/2}\,\mathrm{d}^*t,$$

where the sum is taken over all non-zero isotropic vectors $y \in Y_j$, modulo k^*. Note that $\|te_0^+ + y\| \geqslant \|y\| \geqslant 1$. For a finite place v, we have

$$\int_{k_v^*} |t|_v^{m-j}\|te_0^+ + y\|_v^{-s-m/2}\,\mathrm{d}t$$
$$= \|y\|_v^{-s-m/2}\int_{|t|_v \leqslant \|y\|_v} |t|_v^{m-j}\,\mathrm{d}^*t + \int_{|t|_v > \|y\|_v} |t|_v^{-s+m/2-j}\,\mathrm{d}^*t.$$

Now, a straightforward calculation gives

$$\int_{k_v^*} |t|_v^{m-j}\|te_0^+ + y\|_v^{-s-m/2}\,\mathrm{d}^*t = \|y\|_v^{-s+m/2-j}\frac{\zeta_v(m-j)\zeta_v(s-\frac{1}{2}m+j)}{\zeta_v(s+\frac{1}{2}m)},$$

provided $\mathrm{Re}(s) > \frac{1}{2}m - j$. (Note that our assumptions on j imply, in particular, that $m - j \geqslant 1$.) Here, ζ_v denotes the local zeta function. We get the same condition, when v is archimedean. For example, if $k_v = \mathbb{R}$, then

$$\int_{k_v^*} |t|_v^{m-j}\|te_0^+ + y\|_v^{-s-m/2}\,\mathrm{d}^*t = \int_{\mathbb{R}^*} (|t|^2 + \|y\|^2)^{-s/2-m/4}|t|^{m-j}\,\mathrm{d}^*t$$
$$= \|y\|^{-s+m/2-j}\int_{\mathbb{R}^*} (|t|^2 + 1)^{-s/2-m/4}|t|^{m-j}\,\mathrm{d}^*t.$$

Now, it is clear that this integral converges absolutely when $\mathrm{Re}(s) > \frac{1}{2}m - j$ (and $m - j \geqslant 1$). Thus, we get, for $s \in \mathbb{R}$,

$$\sum_y \int_{\mathbb{A}^*} |t|^{m-j} \|te_0^+ + y\|^{-s-m/2} \, \mathrm{d}^* t$$
$$= \prod_v \frac{\zeta_v(m-j)\zeta_v(s - \frac{1}{2}m + j)}{\zeta_v(s + \frac{1}{2}m)} I_\infty(s) \sum_y \|y\|^{-s+m/2-j},$$

where $I_\infty(s)$ is the product over the archimedean places of the integrals

$$\int_{k_v^*} |t|_v^{m-j} \|te_0^+ + y\|_v^{-s-m/2} \, \mathrm{d}^* t.$$

Thus, for convergence, we must require that $s > \frac{1}{2}m - j + 1$ (by our assumptions on j, we have $m - j \geqslant 2$) and that

$$\sum_y \|y\|^{-s+m/2-j} < \infty.$$

For this, fix a non-zero isotropic vector $e_1 \in Y_j$. Let R be the parabolic subgroup of $\mathrm{O}(Y_j)$ which preserves the line through e_1. Recall that in the last sum y varies over the non-zero isotropic vectors of Y_j, modulo k^*, thus we may write $y = \eta^{-1} e_1$, where $\eta \in R \backslash \mathrm{O}(Y_j)$, and hence we consider

$$\sum_{\eta \in R \backslash \mathrm{O}(Y_j)} \|\eta^{-1} e_1\|^{-s+m/2-j}.$$

This is an Eisenstein series on $\mathrm{O}(Y_j)$, corresponding to the representation induced from a character of the form $\delta_R^{s'}$, where

$$s' = \frac{s - \frac{1}{2}(m - j - 2)}{j - 2},$$

and δ_R is the modular character of $R(\mathbb{A})$. Thus, this series converges when s is sufficiently large (s is real now). This completes the proof of absolute convergence of the series (6.17) which is a subseries in (6.8) when $i = 1$.

The case $i = 2$ is much simpler. Indeed, as in case $i = 1$, we have to examine the convergence of the following series, which is analogous to (6.18),

$$\sum_y \int_{\mathbb{A}^*} |t|^{j-m} \tau^{\mathrm{c}}(H(\gamma_y h_0(t))) H(\gamma_y h_0(t))(-s + \tfrac{1}{2}m) \, \mathrm{d}^* t.$$

Again, we take s real. In the last integral, if $\tau^{\mathrm{c}}(H(\gamma_y h_0(t))) \neq 0$, we must have $c \leqslant H(\gamma h_0(t))$. Since we have $H(g)(1) \leqslant d_2 \|g^{-1} e_0^+\|^{-1}$, for all $g \in \mathrm{O}(Y')_{\mathbb{A}}$, we get that (recall that $\gamma = \gamma_y$)

$$c \leqslant d_2 \|(\gamma_y h_0(t))^{-1} e_0^+\|^{-1} = d_2 \|t^{-1} e_0^+ + y\|^{-1},$$

and hence $\|t^{-1}e_0^+ + y\| \leqslant d_2 c^{-1}$. Since we always have

$$\|y\| \leqslant \|t^{-1}e_0^+ + y\|,$$

and also $1 \leqslant \|y\|$, for all non-zero y in Y_j, we get that $1 \leqslant d_2 c^{-1}$. Hence, when $c > d_2$, we always have that

$$\tau^{\mathrm{c}}(H(\gamma_y h_0(t))) = 0.$$

Recall that $c > 1$ is the truncation parameter. We may take it as large as we want. Note that d_2 is of course fixed and depends on j and our choice the basis of Y_j. Hence, when $i = 2$, every term in the sum over y above is zero when c is sufficiently large. Now we define the constant c_0 by

$$c_0 = \max\{1, d_2\}.$$

Proposition 6.3. *Let $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ have the first occurrence $\mathrm{FO}_\psi(\sigma) = 2j$, such that j and m satisfy the condition in Theorem 5.1. Then the integral attached to $\gamma_x = \gamma_y$,*

$$\int_{G_\ell^{\gamma_x}\backslash G_\ell(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma_x g)\,\mathrm{d}g,$$

is absolutely convergent and identically zero for all choices of data. Moreover, when $i = 1$, the series

$$\sum_{x \in \Omega_{1,1}/G_\ell} \int_{G_\ell^{\gamma_x}\backslash G_\ell(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma_x g)\,\mathrm{d}g$$

converges absolutely, i.e.

$$\sum_{x \in \Omega_{1,1}/G_\ell} \int_{G_\ell^{\gamma_x}\backslash G_\ell(\mathbb{A})} |\xi_{i,s}^{\mathrm{c}}(\gamma_x g)|\,\mathrm{d}g < \infty,$$

for $\mathrm{Re}(s)$ sufficiently large; while, when $i = 2$, there is a constant c_0 such that for all $c > c_0$, the following holds:

$$\sum_{x \in \Omega_{1,1}/G_\ell} \int_{G_\ell^{\gamma_x}\backslash G_\ell(\mathbb{A})} |\xi_{2,s}^{\mathrm{c}}(\gamma_x g)|\,\mathrm{d}g = 0$$

for all s where the intertwining operator is defined.

In Proposition 6.3, we need the restrictions that $j < m-1$ and $j \neq m-2$, when $j \geqslant 2$.

6.4. The series over $\Omega_{0,1}/G_\ell$

For $x = y + z \in \Omega_{0,1}/G_\ell$ with $y = 0$ and $z \neq 0$. By Witt's theorem, $\Omega_{0,1}/G_\ell$ is a single G_ℓ-orbit. We may choose e_0^+ as a representative in the G_ℓ-orbit $\Omega_{0,1}/G_\ell$. Correspondingly, the double coset is $Q_1 \cdot G_\ell$, i.e. $\gamma = 1$. Hence we have $G_\ell^\gamma = G_\ell \cap Q_1$. Recall that

$$G_\ell = \mathrm{O}(Z_{m-j+2}) = \mathrm{O}(k \cdot e_0^+ \oplus Z_{m-j} \oplus k \cdot e_0^-).$$

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Recall that $G_\ell \cap Q_1 = Q_{\ell,1}$ is the maximal parabolic subgroup of G_ℓ, which stabilizes the isotropic line $k \cdot e_0^+$ in Z_{m-j+2}. It has Levi decomposition $Q_{\ell,1} = M_{\ell,1}N_{\ell,1}$, the Levi part of which is $M_{\ell,1} = \mathrm{GL}_1 \times H_\ell = \mathrm{GL}_1 \times \mathrm{O}_{m-j}$. The integral in (6.8) attached to $\gamma = 1$ is

$$\int_{Q_{\ell,1}(k)\backslash G_\ell(\mathbb{A})} \xi^{\mathrm{c}}_{i,s}(g)\,\mathrm{d}g. \tag{6.19}$$

By the Iwasawa decomposition of $G_\ell(\mathbb{A})$, $G_\ell(\mathbb{A}) = Q_{\ell,1}(\mathbb{A}) \cdot K_{\ell,1}$ ($K_{\ell,1}$ is a product of local good maximal compact subgroups, with respect to $Q_{\ell,1}$) we can write $g = nm(t,h)r$. Integral (6.19) equals

$$\int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \int_{k^\times\backslash\mathbb{A}^\times} \xi^{\mathrm{c}}_{i,s}(m(t,h)r)\delta^{-1}_{Q_{\ell,1}}(m(t,1))\,\mathrm{d}^\times t\,\mathrm{d}h\,\mathrm{d}r, \tag{6.20}$$

because the quotient $N_{\ell,1}(k)\backslash N_{\ell,1}(\mathbb{A})$ has volume one.

When $i = 1$, integral (6.20) equals

$$\int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \phi_{1;\sigma}(hr)\,\mathrm{d}h\,\mathrm{d}r \cdot \int_{k^\times\backslash\mathbb{A}^\times,\ |t|_{\mathbb{A}}\leqslant c} |t|_{\mathbb{A}}^{s-m/2+j}\,\mathrm{d}^\times t, \tag{6.21}$$

and this converges absolutely, when $\mathrm{Re}(s) > \frac{1}{2}m - j$. It is easy to see that

$$\int_{k^\times\backslash\mathbb{A}^\times,\ |t|_{\mathbb{A}}\leqslant c} |t|_{\mathbb{A}}^{s-m/2+j}\,\mathrm{d}^\times t = \mathrm{vol}(k^\times\backslash\mathbb{A}^1)\frac{c^{s-m/2+j}}{s-\frac{1}{2}m+j}. \tag{6.22}$$

Also the integral

$$\int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \phi_{1;\sigma}(hr)\,\mathrm{d}h\,\mathrm{d}r$$

converges absolutely, by Proposition 5.2. Hence integral (6.21) and therefore, integral (6.19) with $\gamma = 1$ and $i = 1$ converges absolutely for $\mathrm{Re}(s) > \frac{1}{2}m - j$, has meromorphic continuation to the whole complex plane $\mathbb{C}$ and has at most a simple pole at $s = \frac{1}{2}m - j$ with residue

$$\mathrm{vol}(k^\times\backslash\mathbb{A}^1)\int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \phi_{1;\sigma}(hr)\,\mathrm{d}h\,\mathrm{d}r. \tag{6.23}$$

This residue is indeed not identically zero since by part (1) of Proposition 5.2, there is $\phi_{1;\sigma}$ such that the period

$$\int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \phi_{1;\sigma}(hg)\,\mathrm{d}h$$

is non-zero. By the argument in [**11**] (see also the same argument in [**6**], [**12**] or [**7**]), one can extend the non-zero period of $\phi_{1;\sigma}$ over H_ℓ through the compact integration over $K_{\ell,1}$ and obtain a non-zero integral

$$\int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \phi_{1;\sigma}(hr)\,\mathrm{d}h\,\mathrm{d}r.$$

When $i = 2$, integral (6.20) equals

$$\int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \mathcal{M}(s,\sigma,w_1)(\phi_{1;\sigma})(hr)\,\mathrm{d}h\,\mathrm{d}r \int_{k^\times\backslash\mathbb{A}^\times,\,|t|_\mathbb{A}\geqslant c} |t|_\mathbb{A}^{-s-m/2+j}\,\mathrm{d}^\times t. \qquad (6.24)$$

Hence this integral converges absolutely for $\mathrm{Re}(s) > j - \frac{1}{2}m$, has meromorphic continuation to $\mathbb{C}$, and is equal to

$$\frac{\mathrm{vol}(k^\times\backslash\mathbb{A}^1)c^{-s-m/2+j}}{s+\frac{1}{2}m-j} \int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \mathcal{M}(s,\sigma,w_1)(\phi_{1;\sigma})(hr)\,\mathrm{d}h\,\mathrm{d}r, \qquad (6.25)$$

as meromorphic functions.

We summarize the above calculation as the following proposition.

Proposition 6.4. *Let $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ have the first occurrence $\mathrm{FO}_\psi(\sigma) = 2j$, such that j and m satisfy the condition in Theorem 5.1. Then the integral attached to $\gamma = 1$ converges absolutely for $\mathrm{Re}(s)$ sufficiently large and can be expressed as follows: when $i = 1$,*

$$\int_{G_\ell^\gamma\backslash G_\ell(\mathbb{A})} \xi_{1,s}^{\mathrm{c}}(\gamma g)\,\mathrm{d}g = c_k \frac{c^{s-m/2+j}}{s-\frac{1}{2}m+j} \int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \phi_{1;\sigma}(hr)\,\mathrm{d}h\,\mathrm{d}r,$$

where $c_k := \mathrm{vol}(k^\times\backslash\mathbb{A}^1)$; and when $i = 2$,

$$\int_{G_\ell^\gamma\backslash G_\ell(\mathbb{A})} \xi_{2,s}^{\mathrm{c}}(\gamma g)\,\mathrm{d}g = c_k \frac{c^{-s-m/2+j}}{s+\frac{1}{2}m-j} \int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \mathcal{M}(s,\sigma,w_1)(\phi_{1;\sigma})(hr)\,\mathrm{d}h\,\mathrm{d}r.$$

These identities provide the meromorphic continuation of the integrals on the left-hand side to the whole plane. Finally, the integral

$$\int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \phi_{1;\sigma}(hr)\,\mathrm{d}h\,\mathrm{d}r$$

is not identically zero.

For Proposition 6.4, we did not use any of our restrictions on j.

6.5. The series over $\Omega_{2,2}/G_\ell$

Finally, we consider $x = y + z \in \Omega_{2,2}/G_\ell$, which is isotropic lines $k \cdot x$ in X_1 with $y \in Y_j$ and $z \in Z_{m-j+2}$ not isotropic. In this case, we have

$$b_1(y,y) = -b_1(z,z) \neq 0.$$

For a fixed $y \in Y_j$, the following set of isotropic lines in X_1

$$\{k(y+z) \mid b_1(y,y) = -b_1(z,z) \neq 0\}$$

is one G_ℓ-orbit. We may take $k(y + z_y)$ as a representative of this G_ℓ-orbit, where

$$z_y = e_0^+ - \tfrac{1}{2}b_1(y,y)e_0^-.$$

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If $h \in G_\ell$ stabilizes the isotropic line $k(y+z_y)$, then we have $h \cdot y = y$ and $h \cdot z_y = z_y$. It follows that $\mathrm{Stab}_{G_\ell}(k(y+z_y)) = \mathrm{Stab}_{G_\ell}(z_y)$. Take $z'_y = e_0^+ + \frac{1}{2}b_1(y,y)e_0^-$. It is clear that $\mathrm{Span}(e_0^+, e_0^-) = \mathrm{Span}(z_y, z'_y)$ is the hyperbolic plane. Hence we have

$$\begin{aligned}\mathrm{Stab}_{G_\ell}(k(y+z_y)) &= \mathrm{Stab}_{G_\ell}(z_y)\\ &= \mathrm{O}(Z_{m-j} \oplus k \cdot z'_y)\\ &= \mathrm{O}(z_y^\perp \cap Z_{m-j+2}).\end{aligned} \tag{6.26}$$

The representative γ for the corresponding double coset $Q_1 \cdot \gamma \cdot G_\ell$ can be chosen such that

$$\begin{aligned}\gamma^{-1} \cdot e_0^+ &= y + z_y,\\ \gamma^{-1} \cdot e_0^- &= \frac{1}{2b_1(y,y)}(y - z_y),\\ \gamma^{-1} \cdot y &= z'_y,\end{aligned}$$

and the restriction of γ^{-1} to the subspace $y^\perp \cap X$ is the identity. Then we have

$$G_\ell^\gamma = \gamma^{-1} \cdot Q_1 \cdot \gamma \cap G_\ell = \mathrm{Stab}_{G_\ell}(k(y+z_y)) = \mathrm{O}(Z_{m-j} \oplus k \cdot z'_y),$$

which is the same as (6.26). We want to write down the explicit embedding of $\gamma \cdot G_\ell^\gamma \cdot \gamma^{-1}$ into $\mathrm{O}(X)$.

For $h \in G_\ell^\gamma = \mathrm{O}(Z_{m-j} \oplus k \cdot z'_y)$, it is easy to check that the following hold

$$\begin{aligned}\gamma h \gamma^{-1} \cdot e_0^+ &= e_0^+,\\ \gamma h \gamma^{-1} \cdot e_0^- &= e_0^-,\\ \gamma h \gamma^{-1} \cdot y &= \gamma h \cdot z'_y = \gamma \cdot (z_0 + \alpha z'_y) = z_0 + \alpha y,\end{aligned}$$

where $z_0 = z_0(h) \in Z_{m-j}$ and $\alpha \in k$; for any $z \in Z_{m-j}$, we have $\gamma^{-1} \cdot z = z$ since $Z_{m-j} \subset y^\perp \cap X$, and

$$\gamma h \gamma^{-1} \cdot z = \gamma h \cdot z = \gamma \cdot (z^* + \beta z'_y) = z^* + \beta z'_y,$$

where $z^* = z^*(h)$ and $\beta = \beta(h) \in k$; and finally, the restriction of $\gamma h \gamma^{-1}$ to the subspace $y^\perp \cap Y_j$ is the identity. According to the following decomposition of X_1

$$X_1 = k \cdot e_0^+ \oplus Z_{m-j} \oplus k \cdot y \oplus y^\perp \cap Y_j \oplus k \cdot e_0^-,$$

we can write $\gamma h \gamma^{-1}$ in matrix form:

$$\begin{pmatrix} 1 &&&& \\ & h^* & * & 0 & \\ & * & * & 0 & \\ & 0 & 0 & I_{j-1} & \\ &&&& 1 \end{pmatrix}, \tag{6.27}$$

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where h^* is the composition of h with the projection from $h \cdot z = z^* + \beta y$ to z^*. It is easy to check that the following element from (6.27)

$$g(h) = \begin{pmatrix} h^* & * & 0 \\ * & * & 0 \\ 0 & 0 & I_{j-1} \end{pmatrix}$$

belongs to $\mathrm{O}(X)$ and the set $\{g(h) \mid h \in G_\ell^\gamma\}$ is a subgroup of $\mathrm{O}(X)$ which is isomorphic to $\mathrm{O}(Z_{m-j} \oplus k \cdot y)$.

For such a $\gamma = \gamma_x$, or a double coset $Q_1 \gamma G_\ell$, the integral attached to γ_x can be written as

$$\begin{aligned} \int_{G_\ell^{\gamma_x} \backslash G_\ell(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma_x g) \, \mathrm{d}g &= \int_{G_\ell^{\gamma_x} \backslash G_\ell(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma_x g \gamma_x^{-1} \cdot \gamma_x) \, \mathrm{d}g \\ &= \int_{\mathrm{O}(k \cdot y \oplus Z_{m-j}) \backslash \mathrm{O}(k \cdot y \oplus Z_{m-j})(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(h\gamma) \, \mathrm{d}h. \end{aligned} \tag{6.28}$$

By the definition of $\xi_{i,s}^{\mathrm{c}}(g)$ (see (6.7) and (2.10), (2.11)), we have

$$\xi_{i,s}^{\mathrm{c}}(h\gamma) = \begin{cases} \phi_{1;\sigma}(h) \tau_c(H(\gamma)) H(\gamma)(s) & \text{if } i = 1, \\ \mathcal{M}(s, \sigma, w_1)(\phi_{1;\sigma})(h) \tau^{\mathrm{c}}(H(\gamma)) H(\gamma)(-s) & \text{if } i = 2. \end{cases}$$

Hence the $\mathrm{d}h$-integration in (6.28) defines a period of cuspidal automorphic functions in σ over the reductive subgroup $\mathrm{O}(k \cdot y \oplus Z_{m-j})$ of O_m, which is absolutely convergent (as in the proof of part (3) of Proposition 5.2). Again, by Proposition 5.2, the integral must vanish identically for all choices of data because of the assumption of the first occurrence of σ. Finally, exactly as in the proof of Proposition 6.1, we get that for $\mathrm{Re}(s)$ sufficiently large

$$\sum_{x = y + z_y \in \Omega_{2,2}/G_\ell} \int_{\mathrm{O}(k \cdot y \oplus Z_{m-j}) \backslash \mathrm{O}(k \cdot y \oplus Z_{m-j})(\mathbb{A})} |\xi_{1,s}^{\mathrm{c}}(h\gamma_x)| \, \mathrm{d}h < \infty,$$

where the summation is over the representatives γ as above. Similarly, for all s where the intertwining operator is defined

$$\sum_{x = y + z_y \in \Omega_{2,2}/G_\ell} \int_{\mathrm{O}(k \cdot y \oplus Z_{m-j}) \backslash \mathrm{O}(k \cdot y \oplus Z_{m-j})(\mathbb{A})} |\xi_{2,s}^{\mathrm{c}}(h\gamma_x)| \, \mathrm{d}h < \infty.$$

Indeed the last series has finitely many non-zero terms. We state this result as the following proposition.

Proposition 6.5. *Let $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ have the first occurrence $\mathrm{FO}_\psi(\sigma) = 2j$, such that j and m satisfy the condition in Theorem 5.1. For $x = y + z_y \in \Omega_{2,2}/G_\ell$, the integral in the summation (6.8) attached to γ_x,*

$$\int_{G_\ell^{\gamma_x} \backslash G_\ell(\mathbb{A})} \xi_{i,s}^{\mathrm{c}}(\gamma_x g) \, \mathrm{d}g,$$

is absolutely convergent and identically zero, for all choices of data. Moreover, the following absolute convergence

$$\sum_{x=y+z_y\in\Omega_{2,2}/G_\ell}\int_{\mathrm{O}(k\cdot y\oplus Z_{m-j})\backslash\,\mathrm{O}(k\cdot y\oplus Z_{m-j})(\mathbb{A})}|\xi^{\mathrm{c}}_{i,s}(h\gamma_x)|\,\mathrm{d}h<\infty,$$

holds for $\mathrm{Re}(s)$ *sufficiently large, when* $i=1$*, and for all* s *where the intertwining operator is defined, when* $i=2$*.*

In Proposition 6.5, we used the restriction $j<m-1$.

6.6. Proof of Theorem 5.1

We complete the proof of Theorem 5.1 in this subsection.

By Propositions 6.1, 6.3, 6.4 and 6.5, we have proved that (6.9) holds, that is, each integral in the summation in (6.8) converges absolutely, and the series in (6.8) converges absolutely for $\mathrm{Re}(s)$ sufficiently large, when $i=1$, and for all s where the intertwining operator is defined, when $i=2$ and the truncation parameter $c>c_0$ (which is defined before Proposition 6.3). Further we prove that all the summands in the expression (6.8) are identically zero except the summand attached to the representative $\gamma=1$. In this case, the summand is given by Proposition 6.4.

Combining this with formulae (6.4), (6.6), (6.7) and (6.8), we obtain, for $\mathrm{Re}(s)$ large, $c>c_0$ and j satisfying the condition of Theorem 5.1, a formula for the period of the truncated Eisenstein series

$$\int_{G_\ell(k)\backslash G_\ell(\mathbb{A})}\Lambda^{\mathrm{c}}E(g;\phi_{1;\sigma},s)\,\mathrm{d}g=I^{\mathrm{c}}_1-I^{\mathrm{c}}_2,\tag{6.29}$$

where

$$I^{\mathrm{c}}_1=c_k\frac{c^{s-m/2+j}}{s-\frac{1}{2}m+j}\int_{K_{\ell,1}}\int_{H_\ell(k)\backslash H_\ell(\mathbb{A})}\phi_{1;\sigma}(hr)\,\mathrm{d}h\,\mathrm{d}r\tag{6.30}$$

and

$$I^{\mathrm{c}}_2=c_k\frac{c^{-s-m/2+j}}{s+\frac{1}{2}m-j}\int_{K_{\ell,1}}\int_{H_\ell(k)\backslash H_\ell(\mathbb{A})}\mathcal{M}(s,\sigma,w_1)(\phi_{1;\sigma})(hr)\,\mathrm{d}h\,\mathrm{d}r,\tag{6.31}$$

where the constant c_k is as in Proposition 6.4. Note that we proved the identity (6.29) for $\mathrm{Re}(s)$ sufficiently large, and since the left-hand side is meromorphic in the whole plane, then, by (6.30), (6.31) the identity (6.29) holds as meromorphic functions in $\mathbb{C}$. It is clear that I^{c}_1 has only one pole, and it is the simple pole at $s=\frac{1}{2}m-j$. (See also the proof of Theorem 6.6.)

Assume now, that $2j\neq m$ (j as above). We claim that the intertwining operator $\mathcal{M}(s,\sigma,w_1)(\phi_{1;\sigma})$ has a pole at $s=\frac{1}{2}m-j$, for some $\phi_{1;\sigma}$. Otherwise, I^{c}_2 is holomorphic at $s=\frac{1}{2}m-j$. Then by (6.29), the period of the truncated Eisenstein series has a pole at $s=\frac{1}{2}m-j$. In particular, the Eisenstein series $E(g;\phi_{1;\sigma},s)$ has a pole at $s=\frac{1}{2}m-j$, and hence the intertwining operator $\mathcal{M}(s,\sigma,w_1)(\phi_{1;\sigma})$ must have a pole at $s=\frac{1}{2}m-j$ for some $\phi_{1;\sigma}$. This is a contradiction. This proves that the Eisenstein series $E(g;\phi_{1;\sigma},s)$

must have a pole at $s = \frac{1}{2}m - j$ under the assumption that the first occurrence of σ is $\mathrm{FO}_\psi(\sigma) = 2j$, when $2j \neq m$, $j < m - 1$ and if $j \geqslant 2$, then $j \neq m - 2$. This proves Theorem 5.1.

6.7. Proof of Theorem 5.3

In order to finish the proof of Theorem 5.3, we assume that $2j < m$, and hence the pole at $s = \frac{1}{2}m - j > 0$ is simple (by the general theory of Eisenstein series due to Langlands [**22**]). We will consider the period (and its convergence) of the residue at $s = \frac{1}{2}m - j$ of the Eisenstein series $E(g;\phi_{1;\sigma}, s)$, i.e. the period in (5.10):

$$\int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \mathcal{E}_{m/2-j}(h;\phi_{1;\sigma})\,\mathrm{d}h. \tag{6.32}$$

Let $s_0 > 0$ be a positive pole of the Eisenstein series $E(g;\phi_{1;\sigma}, s)$, and denote the residue by $\mathcal{E}_{s_0}(g;\phi_{1;\sigma})$. It is clear that the constant term of the residue is

$$\mathcal{E}_{s_0,P_1}(g,\phi_{1;\sigma}) = \mathrm{res}_{s=s_0}\,\mathcal{M}(s,\sigma,w_1)(\phi_{1;\sigma})(g) := \mathcal{M}_{s_0}(\sigma,w_1)(\phi_{1;\sigma})(g).$$

Then the truncation of the residue is

$$\begin{aligned} \Lambda^{\mathrm{c}}\mathcal{E}_{s_0}(g,\phi_{1;\sigma}) &= \mathcal{E}_{s_0}(g,\phi_{1;\sigma}) - \sum_{\gamma\in P_1\backslash G} \mathcal{M}_{s_0}(\sigma,w_1)(\phi_{1;\sigma})(\gamma g)\tau^{\mathrm{c}}(H(\gamma g)) \\ &:= \mathcal{E}_{s_0}(g,\phi_{1;\sigma}) - \theta_3^{\mathrm{c}}(g). \end{aligned} \tag{6.33}$$

Note that $\Lambda^{\mathrm{c}}\mathcal{E}_{s_0}(g,\phi_{1;\sigma}) = \mathrm{res}_{s=s_0}\,\Lambda^{\mathrm{c}}E(g,\phi_{1;\sigma})$. We can repeat the proofs of Propositions 6.1–6.5, for $i = 2$, and get, in particular, that $|\theta_3^{\mathrm{c}}|$ is integrable over $G_\ell(k)\backslash G_\ell(\mathbb{A})$. Note that in Proposition 6.4 $s_0 + \frac{1}{2}m - j > s_0 > 0$, and so (6.24) and (6.25) are valid for θ_3^{c} as well. We conclude from (6.33) that $|\mathcal{E}_{s_0}(g;\phi_{1;\sigma})|$ is integrable over $G_\ell(k)\backslash G_\ell(\mathbb{A})$.

Now we can express the period of the residue $\mathcal{E}_{s_0}(g;\phi_{1;\sigma})$ over the subgroup G_ℓ as follows:

$$\begin{aligned} \int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \mathcal{E}_{s_0}(h,\phi_{1;\sigma})\,\mathrm{d}h &= \int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} [\theta_3^{\mathrm{c}}(h) + \Lambda^{\mathrm{c}}\mathcal{E}_{s_0}(h;\phi_{1;\sigma})]\,\mathrm{d}h \\ &= \int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \theta_3^{\mathrm{c}}(h)\,\mathrm{d}h + \mathrm{res}_{s=s_0}(I_1^{\mathrm{c}} - I_2^{\mathrm{c}}). \end{aligned} \tag{6.34}$$

We obtain from the proof of the case when $i = 2$ in Proposition 6.4 that

$$\mathrm{res}_{s=s_0}\, I_2^{\mathrm{c}} = \int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \theta_3^{\mathrm{c}}(h)\,\mathrm{d}h, \tag{6.35}$$

since for $s_0 > 0$, the pole of I_2^{c} is also simple. By (6.34), we obtain

$$\int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \mathcal{E}_{s_0}(h,\phi_{1;\sigma})\,\mathrm{d}h = \mathrm{res}_{s=s_0}\, I_1^{\mathrm{c}}.$$

Thus, we see that if $s_0 \neq \frac{1}{2}m - j$, then since $\operatorname{res}_{s=s_0} I_1^{\mathrm{c}} = 0$, we have that

$$\int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \mathcal{E}_{s_0}(h, \phi_{1;\sigma})\,\mathrm{d}h = 0.$$

This proves part (3) of Theorem 5.3.

When $s_0 = \frac{1}{2}m - j$, we have

$$\int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \mathcal{E}_{m/2-j}(h, \phi_{1;\sigma})\,\mathrm{d}h = c_k \int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \phi_{1;\sigma}(hr)\,\mathrm{d}h\,\mathrm{d}r, \tag{6.36}$$

where the constant c is as Proposition 6.4. Finally, the proof of part (2) of Theorem 5.3 follows easily from part (1) of Proposition 5.2 and the identity (6.36). In fact, when a $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ has the first occurrence $\mathrm{FO}_\psi(\sigma) = 2j$ for some integer $j \in \{1, 2, \dots, [\frac{1}{2}(m-1)]\}$, the period

$$\int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \phi_{1;\sigma}(hr)\,\mathrm{d}h$$

is non-zero for some choice of data, by part (1) of Proposition 5.2. Now it is a standard argument to extend this non-vanishing property through the compact integration in (6.36) [**7, 11, 12**], so that the right-hand side of the identity (6.36) is non-zero for some choice of data. This proves part (2) of Theorem 5.3, and hence completes the proof of Theorem 5.3.

We summarize the above discussion and Proposition 6.4 as the following theorem.

Theorem 6.6. *Let $E(g; \phi_{1;\sigma}, s)$ be the Eisenstein series on $\mathrm{O}_{m+2}(\mathbb{A})$ as in Theorem 5.3. Assume that $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ has the first occurrence $\mathrm{FO}_\psi(\sigma) = 2j$ for some integer $j \in \{1, 2, \dots, [\frac{1}{2}(m-1)]\}$. Let $s_0 > 0$ be a pole of the Eisenstein series. Then the residue at s_0, $\mathcal{E}_{s_0}(g; \phi_{1;\sigma})$ of $E(g; \phi_{1;\sigma}, s)$ is not G_ℓ-distinguished for all $\ell = (l_1, \dots, l_j) \in (k^\times)^j$, except for $s_0 = \frac{1}{2}m - j$. At $s_0 = \frac{1}{2}m - j$ the Eisenstein series has a simple pole, and there is an $\ell = (l_1, \dots, l_j) \in (k^\times)^j$ such that the period of the residue $\mathcal{E}_{m/2-j}(g; \phi_\sigma)$ is non-trivial. This period is expressed by the following formula*

$$\int_{G_\ell(k)\backslash G_\ell(\mathbb{A})} \mathcal{E}_{m/2-j}(h, \phi_{1;\sigma})\,\mathrm{d}h = c_k \cdot \int_{K_{\ell,1}} \int_{H_\ell(k)\backslash H_\ell(\mathbb{A})} \phi_{1;\sigma}(hr)\,\mathrm{d}h\,\mathrm{d}r,$$

where the constant $c_k = \operatorname{vol}(k^\times\backslash\mathbb{A}^1)$.

6.8. Proof of Lemma 6.2

The nature of the proof of Lemma 6.2 is essentially the same as that of Proposition 5.2, where we sketched the main ideas. For completeness, we give here a more detailed proof of Lemma 6.2.

For a Schwartz–Bruhat function φ in $\mathcal{S}(X(\mathbb{A})^{j-1})$ and for ϕ_σ in the space V_σ, the ψ-theta lifting of ϕ_σ to $\mathrm{Mp}_{2j-2}(\mathbb{A})$ is given by the following integral (as in (1.1)):

$$\theta_{\psi,m}^{2j-2}(g; \phi_\sigma, \varphi) = \int_{\mathrm{O}_m(k)\backslash \mathrm{O}_m(\mathbb{A})} \theta_{\psi,\varphi}(h, g)\phi_\sigma(h)\,\mathrm{d}h, \tag{6.37}$$

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where $g \in \mathrm{Mp}_{2j-2}(\mathbb{A})$. The ψ-theta lifting of σ, which is denoted by $\theta^{2j-2}_{\psi,m}(\sigma)$ as before, consists of all automorphic functions, $\theta^{2j-2}_{\psi,m}(g;\phi_\sigma,\varphi)$ with φ running in $\mathcal{S}(X(\mathbb{A})^{j-1})$ and ϕ_σ running in V_σ.

To prove Lemma 6.2, we show that if $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ has the first occurrence $\mathrm{FO}_\psi(\sigma) = 2j$ with j and m satisfying the condition of Theorem 5.1, then the period

$$\int_{H_{y,j}\backslash H_{y,j}(\mathbb{A})} \phi_\sigma(h)\,\mathrm{d}h \tag{6.38}$$

is identically zero for all choices of ϕ_σ in V_σ (see (6.15) for the definition of the subgroup $H_{y,j}$). (Recall that we already explained why this integral converges absolutely, right after the statement of this lemma in the previous subsection.)

In order to obtain the vanishing statement above, we calculate certain Fourier coefficients of $\theta^{2j-2}_{\psi,m}(g;\phi_\sigma,\varphi)$. More precisely, as in § 5, we take the Siegel parabolic subgroup $Q = LU$ of Sp_{2j-2} and take $\ell = (l_1,\dots,l_{j-2},0)$ with $l_i \in k^\times$. Then we calculate the ψ_ℓ-Fourier coefficient of $\theta^{2j-2}_{\psi,m}(g;\phi_\sigma,\varphi)$, which is given by

$$\mathcal{F}^{\psi_\ell}(\theta^{2j-2}_{\psi,m}(\cdot)) := \int_{U(k)\backslash U(\mathbb{A})} \theta^{2j-2}_{\psi,m}(u;\phi_\sigma,\varphi)\psi_\ell^{-1}(u)\,\mathrm{d}u. \tag{6.39}$$

By (6.37), the ψ_ℓ-Fourier coefficient (6.39) can be written as

$$\begin{aligned}\mathcal{F}^{\psi_\ell}(\theta^{2j-2}_{\psi,m}(\cdot)) &= \int_{U(k)\backslash U(\mathbb{A})}\int_{\mathrm{O}_m(k)\backslash \mathrm{O}_m(\mathbb{A})} \theta_{\psi,\varphi}(h,u)\phi_\sigma(h)\,\mathrm{d}h\psi_\ell^{-1}(u)\,\mathrm{d}u \\ &= \int_h \phi_\sigma(h)\int_{U(k)\backslash U(\mathbb{A})} \theta_{\psi,\varphi}(h,u)\psi_\ell^{-1}(u)\,\mathrm{d}u\,\mathrm{d}h, \end{aligned} \tag{6.40}$$

where the $\mathrm{d}h$-integration is over $\mathrm{O}_m(k)\backslash \mathrm{O}_m(\mathbb{A})$. We consider first the inner integration

$$\int_{U(k)\backslash U(\mathbb{A})} \theta_{\psi,\varphi}(h,u)\psi_\ell^{-1}(u)\,\mathrm{d}u. \tag{6.41}$$

To calculate this Fourier coefficient explicitly, we write

$$\xi = (\xi_1,\xi_2,\dots,\xi_{j-1}) \in X^{j-1},$$

and $u = u(S)$ as in (5.1). By definition, we have

$$\begin{aligned}\theta_{\psi,\varphi}(h,u) &= \sum_{\xi\in X^{j-1}(k)} \omega_\psi(u(S))\omega_\psi(h)\varphi(\xi) \\ &= \sum_{\xi\in X^{j-1}(k)} \omega_\psi(h)\cdot\varphi(\xi)\psi(\tfrac{1}{2}\,\mathrm{tr}\,\mathrm{Gr}(\xi)\cdot S\cdot J_{j-1}),\end{aligned} \tag{6.42}$$

where the matrix J_i is defined by

$$J_i = \begin{pmatrix} 0 & 1 \\ J_{i-1} & 0 \end{pmatrix}$$

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inductively, and $\mathrm{Gr}(\xi)$ is defined as the $(j-1)\times(j-1)$-matrix $(b(\xi_t,\xi_s))$. Hence the Fourier coefficient (6.41) can be written as

$$\sum_{\xi\in X^{j-1}(k)} \omega_\psi(h)\cdot\varphi(\xi)\int_{U(k)\backslash U(\mathbb{A})} \psi(\tfrac{1}{2}\operatorname{tr}\mathrm{Gr}(\xi)\cdot S\cdot J_{j-1})\psi_\ell^{-1}(u)\,\mathrm{d}u. \tag{6.43}$$

By the orthogonality relations of characters, we know that the summands parametrized by $\xi\in X^{j-1}(k)$ may be non-zero only if $\xi=(\xi_1,\xi_2,\dots,\xi_{j-1})\in X^{j-1}$ satisfy the following geometric conditions: $b(\xi_t,\xi_s)=0$ if $t\neq s$ for $t,s=1,2,\dots,j-1$; $b(\xi_t,\xi_t)=l_t$ for $t=1,2,\dots,j-2$; and $b(\xi_{j-1},\xi_{j-1})=0$. We denote by X_ℓ^{j-1} the subset of all ξ satisfying the above geometric conditions. It is clear from Witt's theorem that the action of $\mathrm{O}_m(k)=\mathrm{O}(X)(k)$ on X_ℓ^{j-1} decomposes X_ℓ^{j-1} into two orbits: one is with $\xi_{j-1}=0$ and the second is with $\xi_{j-1}\neq 0$ (but isotropic).

We fix a representative $\xi^0:=(v_1,\dots,v_{j-2},0)$ for the first orbit, and a representative $\xi^y:=(v_1,\dots,v_{j-2},y)$ for the second orbit, where v_t for $t=1,2,\dots,j-2$ are given as in § 5. In particular we have that $b(v_t,v_t)=l_t$. One can check that the stabilizer of the representative ξ^y in $\mathrm{O}(X)$ is $H_{y,j}$ as in Lemma 6.2. The stabilizer of the representative ξ^0 in $\mathrm{O}(X)$ is denoted by H_{m-j+2} as in § 5.2. It is a k-rational form of O_{m-j+2}. Hence we may write (6.43) and hence (6.41) as a sum of two terms:

$$\sum_{g\in H_{y,j}(k)\backslash \mathrm{O}_m(k)} \varphi(g^{-1}\xi^y)+\sum_{g\in H_{m-j+2}(k)\backslash \mathrm{O}_m(k)} \varphi(g^{-1}\xi^0). \tag{6.44}$$

We go back to calculate the ψ_ℓ-Fourier coefficient $\mathcal{F}^{\psi_\ell}(\theta_{\psi,m}^{2j-2}(\cdot))$ via the second integral in (6.40). We have

$$\begin{aligned}\mathcal{F}^{\psi_\ell}(\theta_{\psi,m}^{2j-2}(\cdot)) &= \int_{H_{y,j}(\mathbb{A})\backslash \mathrm{O}_m(\mathbb{A})} \varphi(g^{-1}\xi^y)\int_{H_{y,j}(k)\backslash H_{y,j}(\mathbb{A})}\phi_\sigma(hg)\,\mathrm{d}h\,\mathrm{d}g\\ &\quad+\int_g \varphi(g^{-1}\xi^0)\int_{H_{m-j+2}(k)\backslash H_{m-j+2}(\mathbb{A})}\phi_\sigma(hg)\,\mathrm{d}h\,\mathrm{d}g,\end{aligned} \tag{6.45}$$

where the dg-integration in the second summand in (6.45) is over $H_{m-j+2}(\mathbb{A})\backslash \mathrm{O}_m(\mathbb{A})$. Note that each summand in (6.45) converges absolutely as a double integral. This is due to the fact that φ is a Schwartz function, ϕ_σ is bounded over the Siegel domain of O_m, and both $H_{y,j}(k)\backslash H_{y,j}(\mathbb{A})$ and $H_{m-j+2}(k)\backslash H_{m-j+2}(\mathbb{A})$ have finite measures. Since the first occurrence of σ, $\mathrm{FO}_\psi(\sigma)=2j$, by part (2) of Proposition 5.2, the period

$$\int_{H_{m-j+2}(k)\backslash H_{m-j+2}(\mathbb{A})}\phi_\sigma(hg)\,\mathrm{d}h$$

must be zero identically in V_σ. (The proof for this is a repetition of the calculation (6.43), where we compute $\mathcal{F}^{\psi_{\ell'}}(\theta_{\psi,m}^{2j-4}(\cdot))$, for $\ell'=(\ell_1,\dots,\ell_{j-2})$.) Hence by (6.45), if the first occurrence of σ, $\mathrm{FO}_\psi(\sigma)=2j$, we have

$$\mathcal{F}^{\psi_\ell}(\theta_{\psi,m}^{2j-2}(\cdot))=\int_{H_{y,j}(\mathbb{A})\backslash \mathrm{O}_m(\mathbb{A})} \varphi(g^{-1}\xi^y)\int_{H_{y,j}(k)\backslash H_{y,j}(\mathbb{A})}\phi_\sigma(hg)\,\mathrm{d}h\,\mathrm{d}g. \tag{6.46}$$

If the period (the inner integral)

$$\int_{H_{y,j}(k)\backslash H_{y,j}(\mathbb{A})} \phi_\sigma(hg)\,\mathrm{d}h$$

is non-zero for some choice of $\phi_\sigma \in V_\sigma$, then it is easy to show that there exists a choice of φ in $\mathcal{S}(X^{j-1}(\mathbb{A}))$ such that the whole integral in (6.46) is non-zero. This proves that the ψ-theta lifting of σ to $\mathrm{Mp}_{2j-2}(\mathbb{A})$, $\theta^{2j-2}_{\psi,m}(\sigma)$ has a non-zero ψ_ℓ-Fourier coefficient, and hence $\theta^{2j-2}_{\psi,m}(\sigma)$ is non-zero. But this contradicts the assumption that $\mathrm{FO}_\psi(\sigma) = 2j$. This completes the proof of Lemma 6.2.

7. Proofs of Theorems 1.5, 1.6 and 1.7

In this section, we first prove Theorem 7.1 below. Combining this with Theorem 1.3, and the results in [**13**], [**14**] and [**15**], we prove Theorems 1.5, 1.6 and 1.7. To state Theorem 7.1 below, we have to introduce some notation from the theory of local theta correspondence.

Let v be a finite local place of the number field k. We recall briefly from [**23**] the local theta correspondence over the local field k_v. For a non-trivial character ψ_v of k_v, let ω_{ψ_v} be the Weil representation of the reductive dual pair $\mathrm{O}_m(k_v) \times \mathrm{Mp}_{2j}(k_v)$ acting on the local Schrödinger model $\mathcal{S}(X^j(k_v))$. The detailed discussion of the splitting of the double cover and the related the cocycles can be found in [**15**]. See [**16**] for general reductive dual pairs.

Let (σ_v, V_{σ_v}) (and $(\tilde{\pi}_v, V_{\tilde{\pi}_v})$, respectively) be an irreducible admissible representation of $\mathrm{O}_m(k_v)$ (and $\mathrm{Mp}_{2j}(k_v)$, respectively). If the following space

$$\mathrm{Hom}_{\mathrm{O}_m(k_v)\times \mathrm{Mp}_{2j}(k_v)}(\omega_{\psi_v}, V_{\sigma_v} \otimes V_{\tilde{\pi}_v}) \neq 0, \tag{7.1}$$

then we say that $\tilde{\pi}_v$ is a local ψ_v-theta lift of σ_v, and σ_v is a local ψ_v-theta lift of $\tilde{\pi}_v$. We do not assume that the local Howe duality conjecture holds for the case we are discussing here. The local Howe duality conjecture was proved by Waldspurger [**27**], when the residual characteristic of k is odd. In such a circumstance, the local ψ_v-theta lift is the same as the local ψ_v-Howe lift. We refer to [**23**] for more detailed discussions.

Our definition of the first occurrence for the local ψ_v-theta liftings is based on (7.1). More precisely, we say that the first occurrence of σ_v is $\mathrm{FO}_{\psi_v}(\sigma_v) = 2j_0$ if the following space

$$\mathrm{Hom}_{\mathrm{O}_m(k_v)\times \mathrm{Mp}_{2j_1}(k_v)}(\omega_{\psi_v}, V_{\sigma_v} \otimes V_{\tilde{\pi}_{v,j_1}})$$

is zero for all $j_1 < j_0$ and for all irreducible admissible representations $\tilde{\pi}_{v,j_1}$ of $\mathrm{Mp}_{2j_1}(k_v)$, but there exists at least one irreducible admissible representation $\tilde{\pi}_{v,j_0}$ of $\mathrm{Mp}_{2j_0}(k_v)$ such that

$$\mathrm{Hom}_{\mathrm{O}_m(k_v)\times \mathrm{Mp}_{2j_0}(k_v)}(\omega_{\psi_v}, V_{\sigma_v} \otimes V_{\tilde{\pi}_{v,j_0}}) \neq 0.$$

We note that the above definition of the first occurrence for the local ψ_v-theta liftings can be extended to the archimedean local places. We will not repeat it here.

In analogy to (1.2), where we defined the (global) lowest occurrence of σ, we define now the lowest occurrence at the v-component of $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$ (at a local place v of k):

$$\mathrm{LO}_{\psi_v}(\sigma) := \min\{\mathrm{FO}_{\psi_v}(\sigma_v), \mathrm{FO}_{\psi}(\sigma_v \otimes \det)\}. \tag{7.2}$$

Theorem 7.1. *Let $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{O}_m/k)$. Assume that there is a local place v of k, such that $\mathrm{LO}_{\psi_v}(\sigma) = 2j_0$. If $2j_0 < m$, then the partial L-function $L^S(s, \sigma)$ is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}m - j_0$. If $2j_0 \geqslant m$, then $L^S(s, \sigma)$ is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}$.*

Proof. Let the (global) lowest occurrence $\mathrm{LO}_{\psi}(\sigma) = 2j_1$. Assume that $\mathrm{FO}_{\psi}(\sigma \otimes \epsilon_0) = 2j_1$ for some sign character ϵ_0 of $\mathrm{O}_m(\mathbb{A})$. Then the ψ-theta lifting $\theta_{\psi,m}^{2j_1}(\sigma \otimes \epsilon_0)$ is an irreducible cuspidal automorphic representation of $\mathrm{Mp}_{2j_1}(\mathbb{A})$ [**15**, **21**]. In particular, the local ψ_v-theta lift of $\sigma_v \otimes \epsilon_{0,v}$ to $\mathrm{Mp}_{2j_1}(k_v)$ is non-zero. This implies that $2j_0 \leqslant 2j_1$, i.e. $j_0 \leqslant j_1$.

Assume that $2j_0 < m$. If the partial L-function $L^S(s, \sigma)$ has a pole at $\mathrm{Re}(s) > \frac{1}{2}m - j_0$, then this pole must be at $s = \frac{1}{2}m - j_1'$ with $j_1' < j_0$. By Theorem 1.1, there is an automorphic sign character ϵ, such that the global ψ-theta lift, $\theta_{\psi,m}^{2j_1'}(\sigma \otimes \epsilon)$ is non-zero. This implies that

$$j_1 \leqslant j_1' < j_0 \leqslant j_1,$$

which is a contradiction. Finally, assume that $2j_0 \geqslant m$. Then $2j_1 \geqslant m$, and by Theorem 1.1 $L^S(s, \sigma)$ is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}$. This proves the theorem. □

Remark 7.2. By Theorem 1.3, we can conclude a stronger result that under the assumption of Theorem 7.1, the Eisenstein series $E^{Q_1}(g, \phi_{1;\sigma}, s)$ is holomorphic at $\mathrm{Re}(s) > \frac{1}{2}m - j_0$. We omit the details here.

Remark 7.3. We expect that the lowest occurrence at the v-component, $\mathrm{LO}_{\psi_v}(\sigma)$ should be characterized in terms of the generalized Gelfand–Graev models for irreducible admissible representations of $\mathrm{O}_m(k_v)$. To illustrate this key point, we prove Theorems 1.5 and 1.7 by Theorem 7.1. The general cases will be treated in our forthcoming work.

7.1. Proof of Theorem 1.5

Consider $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{SO}_m/k)$, i.e. σ is an irreducible automorphic cuspidal representation of $\mathrm{SO}_m(\mathbb{A})$, where $\mathrm{SO}_m = \mathrm{SO}(X)$ is as before. Assume that there is a place v_0, such that $\mathrm{SO}_m(k_{v_0})$ is k_{v_0}-quasisplit and the local v_0-component, σ_{v_0} of σ has a non-zero local Whittaker model, i.e. σ is locally generic at v_0. Then, for any quadratic character χ, the representation $\sigma \otimes \chi$ is also locally generic at v_0. It is enough to prove Theorem 1.5 for σ.

We 'extend' σ to an irreducible, automorphic, cuspidal representation σ' of $\mathrm{O}_m(\mathbb{A})$ as follows. When m is odd, then $\mathrm{O}_m = \mathrm{SO}_m \times \mathrm{Z}_2$, where $\mathrm{Z}_2 = \{\pm I_m\}$. We choose a character μ of $\mathrm{Z}_2(k) \backslash \mathrm{Z}_2(\mathbb{A})$, and extend σ to $\sigma' = \sigma_\mu$, on $\mathrm{O}_m(\mathbb{A})$, by letting the central subgroup $\mathrm{Z}_2(\mathbb{A})$ act by μ. Similarly, we extend the cusp forms in the space of σ to $\mathrm{O}_m(\mathbb{A})$. It is clear that $\sigma'|_{\mathrm{SO}_m(\mathbb{A})} = \sigma$. When m is even, we fix $\alpha \in \mathrm{O}_m(k)$, with $\det(\alpha) = -1$. In this case, we define $\mathrm{Z}_2 = \{I_m, \alpha\}$. Then $\mathrm{O}_m = \mathrm{SO}_m \rtimes \mathrm{Z}_2$. Consider the map T, on $\mathrm{Ind}_{\mathrm{SO}_m(\mathbb{A})}^{\mathrm{O}_m(\mathbb{A})} \sigma$, defined by

$$Tf(h) = f(h, 1) + f(\alpha h, 1),$$

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where f is a function in the last induced representation, viewed as a complex function on $\mathrm{O}_m(\mathbb{A}) \times \mathrm{SO}_m(\mathbb{A})$, such that $g \mapsto f(h,g)$ is a cusp form in the space of σ, for each h. Clearly, the image of T, $\mathrm{Im}(T)$ is an automorphic, cuspidal representation of $\mathrm{O}_m(\mathbb{A})$. Fix an irreducible summand σ' of $\mathrm{Im}(T)$. Note that, for $g \in \mathrm{SO}_m(\mathbb{A})$,

$$Tf(g) = f(1,g) + f(\alpha, g^{\alpha}),$$

where $g^{\alpha} = \alpha g \alpha$. Thus, $Tf|_{\mathrm{SO}_m(\mathbb{A})} \in \sigma + \sigma^{\alpha}$.

Assume that the (global) lowest occurrence

$$\mathrm{LO}_{\psi}(\sigma) = \mathrm{FO}_{\psi}(\sigma' \otimes \epsilon_0) = 2j_0.$$

Then the ψ-theta lifting $\tilde{\pi} := \theta^{2j_0}_{\psi,m}(\sigma' \otimes \epsilon_0)$ is an irreducible, automorphic, cuspidal representation of $\mathrm{Mp}_{2j_0}(\mathbb{A})$ (see [**21**] for m even and [**15**] for m odd). Then the local ψ_{v_0}-theta lift of $\sigma'_{v_0} \otimes \epsilon_{0,v_0}$ to $\mathrm{Mp}_{2j_0}(k_{v_0})$ is non-trivial. Hence $\sigma'_{v_0}|_{\mathrm{SO}_m(k_{v_0})}$ has a local non-trivial ψ_{v_0}-theta lift to $\mathrm{Mp}_{2j_0}(k_{v_0})$. In case m is odd, this means that $\theta^{2j_0}_{\psi_{v_0},m}(\sigma_{v_0}) \neq 0$. In case m is even, it is clear that $\sigma'_{v_0}|_{\mathrm{SO}_m(k_{v_0})}$ is either σ_{v_0}, $\sigma^{\alpha}_{v_0}$, or the direct sum of these two representations. Since σ_{v_0} is locally generic, we know that the local first occurrence of σ_{v_0} (or, in case m is even, consider $\sigma^{\alpha}_{v_0}$ as well; it is also generic) is greater than or equal to $2[\frac{1}{2}(m-1)]$. When v_0 is finite, this is [**13**, Proposition 2.1] for m odd, and by the local analogue of the global results in [**6**] for m even. When v_0 is archimedean, the same result can be shown by adapting in a simple way these proofs to the archimedean setting. We omit the details. Hence we have

$$2j_0 \geqslant 2[\tfrac{1}{2}(m-1)].$$

Hence,

$$\tfrac{1}{2}m - j_0 \leqslant \tfrac{1}{2}m - [\tfrac{1}{2}(m-1)] = \begin{cases} 1 & \text{if } m \text{ is even,} \\ \frac{1}{2} & \text{if } m \text{ is odd.} \end{cases}$$

If $\frac{1}{2}m - j_0 > 0$, then, by Theorem 7.1, the partial L-function $L^S(s,\sigma)$ is holomorphic at $\mathrm{Re}(s) > \frac{1}{2}m - [\frac{1}{2}(m-1)]$. Similarly, if $\frac{1}{2}m - j_0 \leqslant 0$, then, by Theorem 7.1, $L^S(s,\sigma)$ is holomorphic at $\mathrm{Re}(s) > \frac{1}{2}$. This proves Theorem 1.5.

Remark 7.4. As in Remark 7.2, by Theorem 1.3, we conclude a stronger result, that under the assumption of Theorem 1.5, the Eisenstein series $E(g, \phi_{1;\sigma}, s)$ is holomorphic at $\mathrm{Re}(s) > \frac{1}{2}$ if m is odd and at $\mathrm{Re}(s) > 1$ if m is even.

Here, the Eisenstein series $E(g, \phi_{1;\sigma}, s)$ is defined analogously on $\mathrm{SO}_{m+2}(\mathbb{A})$. It is easy to see that this family of Eisenstein series, on $\mathrm{SO}_{m+2}(\mathbb{A})$, has exactly the same set of poles as that of the family of Eisenstein series $E(g, \phi'_{1;\sigma'}, s)$, on $\mathrm{O}_{m+2}(\mathbb{A})$.

7.2. Proof of Theorem 1.6

In this section, we assume that $m = 2n+1$ is odd and SO_{2n+1} is the k-split odd orthogonal group.

Recall briefly, from § 2 of [**14**], the definition of Bessel model of special type for cuspidal automorphic form ϕ on $\mathrm{SO}_{2n+1}(\mathbb{A})$. Define

$$J_{2n+1} = \begin{pmatrix} 0 & & 1 \\ & J_{2n-1} & \\ 1 & & 0 \end{pmatrix},$$

inductively, so that the quadratic vector space (X, b), and SO_{2n+1} are defined with respect to J_{2n+1}. As in § 2.2 of [**14**], we consider the unipotent radical N_n^{n-1}, consisting of the unipotent matrices of following type

$$n(u, x, z) = \begin{pmatrix} u & x & z \\ & I_3 & x^* \\ & & u^* \end{pmatrix} \in \mathrm{SO}_{2n+1}, \tag{7.3}$$

where $u \in U_{n-1}$, the maximal upper triangular unipotent subgroup of GL_{n-1}. Note that the Levi subgroup, which normalizes N_n^{n-1} is $L_n^{n-1} = \mathrm{GL}_1^{n-1} \times \mathrm{SO}_3$. As in (2.9) of [**14**], we define

$$\psi_{n,n-1;\lambda}(n(u, x, z)) := \psi(u_{1,2} + \cdots + u_{n-2,n-1})\psi(b(x \cdot u_\lambda, e_{n-1})) \tag{7.4}$$

for each $\lambda \in k^\times \mod (k^\times)^2$. As in (2.10) of [**14**], the $\psi_{n,n-1;\lambda}$-Fourier coefficient of cuspidal automorphic form ϕ is defined by

$$\mathcal{F}^{\psi_{n,n-1;\lambda}}(g; \phi) := \int_{N_n^{n-1}(k)\backslash N_n^{n-1}(\mathbb{A})} \phi(ng)\psi^{-1}_{n,n-1;\lambda}(n)\,\mathrm{d}n. \tag{7.5}$$

This is a Fourier coefficient attached to the subregular nilpotent orbit in the sense of § 2 of [**14**], which is also-called a *Bessel–Fourier coefficient* of ϕ attached to the subregular nilpotent orbit. Note that the Whittaker–Fourier coefficient is the one attached to the regular nilpotent orbit. The connected component of the stabilizer of $\psi_{n,n-1;\lambda}$ in L_n^{n-1} is isomorphic to a k-rational form of SO(2), which is denoted by D_λ as in § 2 of [**14**]. We say that a cuspidal automorphic form ϕ has a Bessel model of special type if the following integral

$$\mathcal{B}^{\psi_{n,n-1;\lambda}}(\phi) := \int_{D_\lambda(k)\backslash D_\lambda(\mathbb{A})} \mathcal{F}^{\psi_{n,n-1;\lambda}}(h; \phi)\,\mathrm{d}h \neq 0, \tag{7.6}$$

as in (2.11) of [**14**].

Let $\sigma = \otimes_v \sigma_v \in \mathcal{A}^{\mathrm{c}}(\mathrm{SO}_{2n+1}/k)$ have a non-zero Bessel model of special type. By [**15**, Theorem 1.5], σ is either nearly equivalent to an irreducible generic (globally) cuspidal automorphic representation σ_0 of $\mathrm{SO}_{2n+1}(\mathbb{A})$, i.e. the local components σ_v are isomorphic to the local component $\sigma_{0,v}$ at almost all finite places v of k, or σ is a CAP automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$. In the second case, by [**15**, Theorem 1.5] again, the partial L-function $L^S(s, \sigma_{\alpha_0} \otimes \chi)$ has a pole at $s = \frac{3}{2}$ for some quadratic character χ.

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Assume that there exists a place v_0, such that σ_{v_0} is generic. By Theorem 1.5, just proved above, the partial L-function $L^S(s, \sigma \otimes \chi)$ is holomorphic at $\mathrm{Re}(s) > \frac{1}{2}$, and, in particular, it is holomorphic at $s = \frac{3}{2}$, for all quadratic characters χ. This implies that σ cannot be a CAP automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$. Therefore, by [**15**, Theorem 1.5], σ must be nearly equivalent to an irreducible generic (globally) cuspidal automorphic representation σ_0 of $\mathrm{SO}_{2n+1}(\mathbb{A})$. This proves Theorem 1.6.

7.3. A sketch of the proof of Theorem 1.7

We sketch the proof of Theorem 1.7 in order to illustrate the idea that the Gelfand–Graev models for irreducible cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ determine the structure of irreducible cuspidal automorphic representations. This idea should work for other classical groups. The general discussion will be included in our forthcoming work.

Assume that $\sigma \in \mathcal{A}^{\mathrm{c}}(\mathrm{SO}_{2n+1}/k)$ has a non-zero Bessel–Fourier coefficient as defined in (7.4). First, we can prove that $\mathrm{FO}_\psi(\sigma) \geqslant 2n-2$. The proof uses just one place v, and the fact that σ_v has a non-trivial Jacquet module, with respect to the character at v, which defines the Bessel–Fourier coefficient. We show that, for $2j < 2n-2$, a local theta lift from $\mathrm{Mp}_{2j}(k_v)$ to SO_{2n+1} cannot have a non-trivial such Jacquet module. Following the same argument as in the proof of Theorem 1.5 above, we conclude that the partial L-function $L^S(s, \sigma \otimes \chi)$ is holomorphic for $\mathrm{Re}(s) > \frac{3}{2}$, for all quadratic characters χ of $k^\times \backslash \mathbb{A}^\times$. If the partial L-function $L^S(s, \sigma \otimes \chi)$ has a pole at $s = \frac{3}{2}$, for some quadratic character χ, then the first occurrence $\mathrm{FO}_\psi(\sigma)$ must be $2n-2$. Thus, the ψ-theta lift to $\mathrm{Mp}_{2n-2}(\mathbb{A})$, $\theta^{2n-2}_{\psi,2n+1}(\sigma)$ is a (non-trivial) automorphic, cuspidal representation. We show that this representation is also generic, with respect to a character, determined by the Bessel–Fourier coefficient above. Let $\tilde{\pi}$ be an irreducible summand of $\theta^{2n-2}_{\psi,2n+1}(\sigma)$. Then $\sigma = \theta^{2n+1}_{\psi,2n-2}(\tilde{\pi})$. We calculate directly the Bessel–Fourier coefficient above, for σ, in terms of Whittaker coefficients of $\tilde{\pi}$, and we get that this Bessel–Fourier coefficient already provides a Bessel model of special type as in (7.5). By [**15**, Theorem 1.5], σ must be a CAP automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$, since $L^S(s, \sigma \otimes \chi)$ has a pole at $s = \frac{3}{2}$, for some quadratic character χ. Now the conclusion of Theorem 1.7 follows from Theorem 1.5 of [**15**]. We omit the details here.

It is clear that Theorem 1.7 restricts the global Arthur parameters for irreducible cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ with non-zero Fourier coefficient attached to the subregular nilpotent orbit. We expect that this is a general phenomenon. We will get back to this topic in our future work.

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References

1. J. Arthur, A trace formula for reductive groups, I, Terms associated to classes in $G(Q)$, *Duke Math. J.* **45**(4) (1978), 911–952.
2. J. Arthur, A trace formula for reductive groups, II, Applications of a truncation operator, *Compositio Math.* **40**(1) (1980), 87–121.
3. J. Arthur, An introduction to the trace formula, in *Harmonic analysis, the trace formula, and Shimura varieties*, Clay Mathematical Proceedings, Volume 4, pp. 1–263 (American Mathematical Society, Providence, RI, 2005).
4. J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro and F. Shahidi, On lifting from classical groups to GL_N, *Publ. Math. IHES* **93** (2001), 5–30.
5. S. Gelbart, I. Piatetski-Shapiro and S. Rallis, *Explicit constructions of automorphic L-functions*, Lecture Notes in Mathematics, Volume 1254 (Springer, 1987).
6. D. Ginzburg, S. Rallis and D. Soudry, Periods, poles of L-functions and symplectic-orthogonal theta lifts, *J. Reine Angew. Math.* **487** (1997), 85–114.
7. D. Ginzburg, D. Jiang and S. Rallis, Non-vanishing of the central value of the Rankin–Selberg L-functions, I, *J. Am. Math. Soc.* **17**(3) (2004), 679–722.
8. D. Ginzburg, D. Jiang and D. Soudry, Poles of L-functions and theta liftings for orthogonal groups, II, in *Shahidi's 60th Birthday Conference Volume*, in press.
9. R. Howe, θ-series and invariant theory, in *Automorphic forms, representations and L-functions, Oregon State University, Corvallis, OR, 1977*, Part 1, pp. 275–285, Proceedings of Symposia in Pure Mathematics, Volume 33 (American Mathematical Society, Providence, RI, 1979).
10. R. Howe and I. Piatetski-Shapiro, Some examples of automorphic forms on Sp_4, *Duke Math. J.* **50**(1) (1983), 55–106.
11. H. Jacquet and S. Rallis, Symplectic periods, *J. Reine Angew. Math.* **423** (1992), 175–197.
12. D. Jiang, G_2-periods and residual representations, *J. Reine Angew. Math.* **497** (1998), 17–46.
13. D. Jiang and D. Soudry, The local converse theorem for $\mathrm{SO}(2n+1)$ and applications, *Annals Math. (2)* **157**(3) (2003), 743–806.
14. D. Jiang and D. Soudry, On the genericity of cuspidal automorphic forms of SO_{2n+1}, *J. Reine Angew. Math.* **604** (2007), 187–209.
15. D. Jiang and D. Soudry, On the genericity of cuspidal automorphic forms of SO_{2n+1}, II, *Compositio Math.* **143**(3) (2007), 721–748.
16. S. S. Kudla, Splitting metaplectic covers of dual reductive pairs, *Israel J. Math.* **87** (1994), 361–401.
17. S. S. Kudla and S. Rallis, Poles of Eisenstein series and L-functions, in *Festschrift in Honor of I. I. Piatetski-Shapiro on the Occasion of His Sixtieth Birthday, Part II, Ramat Aviv, 1989*, Israel Mathematical Conference Proceedings, Volume 3, pp. 81–110 (Weizmann, Jerusalem, 1990).
18. S. S. Kudla and S. Rallis, A regularized Siegel–Weil formula: the first term identity, *Annals Math. (2)* **140**(1) (1994), 1–80.
19. J.-S. Li, Nonexistence of singular cusp forms, *Compositio Math.* **83**(1) (1992), 43–51.
20. C. Moeglin, Non nullité de certains relêvements par séries théta, *J. Lie Theory* **7** (1997), 201–229.
21. C. Moeglin, Quelques propriétés de base des séries théta, *J. Lie Theory* **7** (1997), 231–238.
22. C. Moeglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge Tracts in Mathematics, Volume 113 (Cambridge University Press, 1995).
23. C. Moeglin, M.-F. Vignéras and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique* ('Howe correspondences over a p-adic field'), Lecture Notes in Mathematics, Volume 1291 (Springer, 1987; in French)

24. S. Rallis, On the Howe duality conjecture, *Compositio Math.* **51**(3) (1984), 333–399.
25. S. Rallis, Poles of standard L functions, in *Proc. Int. Congress of Mathematicians, Kyoto, 1990*, Volumes I and II, pp. 833–845 (Mathematical Society of Japan, Tokyo, 1991).
26. F. Shahidi, On the Ramanujan conjecture and the finiteness of poles of certain L-functions, *Annals Math. (2)* **127** (1988), 547–584.
27. J.-L. Waldspurger, Démonstration d'une conjecture de dualité de Howe dans le cas p-adique, $p \neq 2$ ('Proof of a duality conjecture of Howe in the p-adic case, $p \neq 2$'), in *Festschrift in Honor of I. I. Piatetski-Shapiro on the Occasion of His Sixtieth Birthday, Part II, Ramat Aviv, 1989*, Israel Mathematical Conference Proceedings, Volume 2, pp. 267–324 (Weizmann, Jerusalem, 1990; in French).