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# Wave interactions in magnetohydrodynamics, and cosmic-ray-modified shocks

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Multiple-scales perturbation methods are used to study wave interactions in magnetohydrodynamics (MHD), in one Cartesian space dimension, with application to cosmic-ray-modified shocks. In particular, the problem of the propagation and interaction of short wavelength MHD waves, in a large-scale background flow, modified by cosmic rays is studied. The wave interaction equations consist of seven coupled evolution equations for the backward and forward Alfvén waves, the backward and forward fast and slow magnetoacoustic waves and the entropy wave. In the linear wave regime, the waves are coupled by wave mixing due to gradients in the background flow, cosmic-ray squeezing instability effects, and damping due to the diffusing cosmic rays. In the most general case, the evolution equations also contain nonlinear wave interaction terms due to Burgers self wave steepening for the magnetoacoustic modes, resonant three wave interactions, and mean wave field interaction terms. The form of the wave interaction equations in the ideal MHD case is also discussed. Numerical simulations of the fully nonlinear cosmic ray MHD model equations are compared with spectral code solutions of the linear wave interaction equations for the case of perpendicular, cosmic-ray-modified shocks. The solutions are used to illustrate how the different wave modes can be generated by wave mixing, and the modification of the cosmic ray squeezing instability due to wave interactions. It is shown that the Alfvén waves are coupled to the magnetoacoustic and entropy waves due to linear wave mixing, only in background flows with non-zero field aligned electric current and/or vorticity (i.e. if $\mathbf{B}\cdot\nabla\times\mathbf{B}\neq 0$ and/or $\mathbf{B}\cdot\nabla\times\mathbf{u}\neq 0$, where $\mathbf{B}$ and $\mathbf{u}$ are the magnetic field induction and fluid velocity respectively).

## 1. Introduction

Wave interactions in magnetohydrodynamics (MHD) and wave propagation in non-uniform media have wide applications in space and laboratory plasma physics. In particular, the propagation of linear waves in stratified media has an extensive literature, ranging from the propagation of radio waves in the ionosphere (see e.g. Budden 1985), to hydromagnetic wave propagation in the solar atmosphere (Ferraro and Plumpton 1958) and in the solar wind (see e.g. Heinemann and Olbert 1980; Barnes 1992). Weakly nonlinear wave interactions and resonant wave interactions in MHD have recently been discussed by Ali and Hunter (1998).

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Heinemann and Olbert (1980) obtained bidirectional evolution equations describing the propagation of toroidal Alfvén waves in the solar wind, in which the backward Alfvén wave is coupled to the forward Alfvén wave via large-scale gradients in the background flow. Zhou and Matthaeus (1990) and others subsequently developed theories for Alfvénic turbulence in the solar wind that naturally incorporated the interaction of the fluctuations with gradients in the background flow. In space plasma physics, this is commonly known as wave mixing (see e.g. Tu and Marsch 1995). Alfvén wave ponderomotive forces have been invoked as an important element in accelerating the solar wind in both WKB models (see e.g. Alazraki and Couturier 1971; Hollweg 1973, 1978; Jacques 1977; McKenzie 1994), and non-WKB models of wave-accelerated winds (see e.g. Heinemann and Olbert 1980; Barkhudarov 1991; Lou 1993; MacGregor and Charbonneau 1994; Hollweg, 1996).

Wave interactions also play an important role in cosmic ray astrophysics. Chin and Wentzel (1972) and Skilling (1975b,c) considered the role of three-wave resonant interactions and wave cascades in cosmic ray propagation problems in the galaxy. A squeezing instability for short-wavelength WKB sound waves in cosmic-ray-modified flows and shocks was investigated by Dorfi and Drury (1985), Drury and Falle (1986) and Zank and McKenzie (1987) (see also Berezhko 1986; Chalov 1988; Webb 1989; and Kang et al. 1992). More general analyses of instabilities of obliquely propagating modes in cosmic ray modified flows have been investigated by Berezhko (1986) and Zank et al. (1990). In particular, Zank et al. (1990) showed that the waves could be destabilized by squeezing and stratification effects, and by particle drifts.

Webb et al. (1997a,b,c; 1999) obtained equations describing the interaction of short-wavelength sound waves and entropy waves in two-fluid cosmic ray hydrodynamics, in a non-uniform large-scale background flow. In the high frequency limit, the equations reduce to the evolution equations for WKB sound waves obtained by Drury and Falle (1986) and Zank and McKenzie (1987). The equations also contain the effects of wave mixing, describing the interaction of the waves with each other due to gradients in the background flow, as well as nonlinear wave interaction effects. The linearized wave evolution equations were used in Webb et al. (1997a,b,c,1999) to study the effect of wave mixing on the cosmic ray squeezing instability in cosmic-ray-modified shocks and flows.

Alfvénic models of cosmic-ray-modified shocks in which the Alfvén waves that scatter the cosmic rays are generated in part by the cosmic ray streaming instability (Lerche 1967; Skilling 1975a) were developed by McKenzie and Völk (1982), Völk et al. (1984) and Medina-Tanco and Opher (1990). The cosmic-ray-generated Alfvén waves in this model were shown by McKenzie and Webb (1984), Zank (1989), Begelman and Zweibel (1994) and Ko and Jeng (1994) to drive one of the modified slow magnetoacoustic waves unstable. McKenzie and Völk (1982) and Völk et al. (1984) restricted their attention to parallel shocks, where the role of Alfvén waves on the shock structure is maximal, whereas Medina-Tanco and Opher (1990) studied the role of cosmic-ray-generated waves in general oblique MHD shocks. Ko (1992) and Ko and Jeng (1994) have considered a hydrodynamical generalization of the McKenzie and Völk (1982) model to include the effects of backward and forward Alfvén waves and second-order Fermi acceleration effects.

The main aim of this paper is to investigate the role of wave–wave interactions in magnetohydrodynamics (MHD), with application to cosmic-ray-modified shocks. We use the MHD model of oblique cosmic-ray-modified shocks of Webb (1983) and Webb et al. (1986) (see also Jun et al. 1994; and Frank et al. 1994). The model equations reduce to those of one-fluid MHD if the cosmic ray terms in the equations are dropped. Thus the wave interaction equations for standard MHD are obtained as a limiting case by dropping the cosmic ray terms in the equations. It is important to note the physical limitations of using a fluid dynamical description, rather than a collisionless, kinetic plasma description (note that this criticism also applies to Monte Carlo models). In particular, the model does not incorporate Landau damping of magnetoacoustic modes due to wave–particle interactions (see e.g. Barnes 1966, 1979). The damping rates $\gamma_L = -\Im(\omega)/\Re(\omega)$ (where $\omega$ is the wave frequency) of the magnetoacoustic waves in general increase with the plasma beta (see e.g. Figure 4 of Barnes (1979), where $\gamma_L$ is plotted as a function of the angle $\theta$ between the wave vector $\mathbf{k}$ and the background magnetic field $\mathbf{B}$ for the cases $\beta = 1$ and $\beta = 5$). The Landau damping for the fast-mode wave exhibits two peaks associated with stochastic heating of the thermal protons ($\theta \approx 10°$) and heating of the electrons ($\theta \approx 85$–$90°$) in $\beta \approx 1$ plasmas. For perpendicular propagation ($\theta = 90°$), there is no linear Landau damping of the fast mode (this corresponds to waves propagating normally to the shock in a perpendicular cosmic-ray-modified shock). In collisionless plasma theory, the entropy-wave-like modes with $\Re(\omega) = 0$ are Landau-damped (Barnes 1979). In general, without carrying out detailed calculations, it is difficult to assess whether the cosmic ray squeezing instability is sufficiently vigorous to overcome Landau damping. In a more complete theory, one should also take into account the full momentum spectrum of the cosmic rays, obtained by solving the cosmic ray transport equation (see e.g. Parker 1965) consistent with the total momentum equation for the system, in which the cosmic rays exert a force on the background flow via their pressure gradient.

The model and equations are presented in Sec. 2. Section 3 provides a discussion of the eigenvalues and eigenvectors of the MHD equations, which are central to the derivation of the wave interaction equations. The form of the eigenvectors depends on the dependent variables, or the state vector used in the analysis. Two state vectors for the MHD background fluid (omitting cosmic ray effects) are used, namely $\tilde{\mathbf{\Psi}}' = (\rho, \mathbf{u}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}, S)^{\mathrm{T}}$ and the conserved densities state vector $\tilde{\mathbf{\Psi}} = (\rho, \rho\mathbf{u}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}, \rho S)^{\mathrm{T}}$, where $\rho$, $\mathbf{u}$, $\mathbf{B}$ and $S$ denote the density, fluid velocity, magnetic induction and entropy of the MHD fluid. The relationships between the wave amplitudes $\{a_j\}$, and the right- and left-eigenvectors, the state vector perturbations, and the eigenvector symmetries are discussed. The formal derivation of the wave interaction equations is developed in Sec. 4. Section 5 discusses nonlinear and three-wave resonant interactions, and the relation of the wave equations to previous work on three-wave resonant interactions of coherent MHD waves (see e.g. Sagdeev and Galeev 1969; Chin and Wentzel 1972; Ali and Hunter 1998). Section 6 considers the wave mixing equations, describing the interaction of linear short-wavelength MHD waves in a large-scale background flow. The wave interaction coefficients describe squeezing instability effects due to the large-scale cosmic ray pressure gradient (see e.g. Drury and Falle 1986), cosmic ray damping due to diffusive cosmic ray transport (see e.g., Ptuskin 1981), and wave mixing effects due to gradients and time variations in the background flow. The wave interaction coefficients generalize the corresponding coefficients obtained by Webb et al. (1997a), describing the interaction of short-wavelength sound waves and entropy waves in cosmic-ray-modified flows. The relationship of the wave mixing equations for the Alfvén waves for planar MHD flows to the equations obtained by Heinemann and Olbert (1980) and Zhou and Matthaeus (1990) for Alfvén waves and Alfvénic turbulence in the solar wind is delineated. It turns out that for wave propagation in one Cartesian space dimension, the Alfvén waves are coupled to the magnetoacoustic and entropy waves only in flows with non-zero field-aligned electric current and/or vorticity (i.e. $\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B} \neq 0$ and/or $\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u} \neq 0$). The form of the wave mixing equations for the degenerate cases where the wave vectors $\mathbf{k} \parallel \mathbf{B}$ and $\mathbf{k} \perp \mathbf{B}$ are discussed in detail. Numerical simulations of the fully nonlinear two-fluid MHD equations are compared with solutions of the wave mixing equations for the case of perpendicular cosmic-ray-modified shocks in Sec. 7. Section 8 concludes with a summary and discussion.

## 2. Model and equations

We use the two-fluid MHD model for cosmic-ray-modified flows of Webb (1983) and Webb et al. (1986). The cosmic rays are assumed to be a hot gas with a substantial pressure $p_c$, but with negligible mass flux and momentum density compared with the thermal gas. The cosmic rays are scattered by waves or turbulence in the background flow, and the phase velocity of the waves is assumed to be negligible compared with the fluid speed (for further discussion of the two-fluid model for the non-magnetized case, see also Axford et al. 1977, 1982; Drury and Völk 1981).

For a model in which the physical variables depend only on the position coordinate $x$ of a rectangular Cartesian coordinate system $(x, y, z)$ and on the time $t$, the equations governing the system of cosmic rays, thermal gas and magnetic field $\mathbf{B}$ may be written in the form

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho u_x) + \frac{\partial}{\partial x}\left(\rho u_x^2 + p_c + p_g + \frac{B_y^2 + B_z^2}{2\mu}\right) = 0, \tag{2.2}$$

$$\frac{\partial}{\partial t}(\rho u_y) + \frac{\partial}{\partial x}\left(\rho u_x u_y - \frac{B_x B_y}{\mu}\right) = 0, \tag{2.3}$$

$$\frac{\partial}{\partial t}(\rho u_z) + \frac{\partial}{\partial x}\left(\rho u_x u_z - \frac{B_x B_z}{\mu}\right) = 0, \tag{2.4}$$

$$\frac{\partial B_y}{\partial t} + \frac{\partial}{\partial x}(u_x B_y - u_y B_x) = 0, \tag{2.5}$$

$$\frac{\partial B_z}{\partial t} + \frac{\partial}{\partial x}(u_x B_z - u_z B_x) = 0, \tag{2.6}$$

$$\frac{\partial}{\partial t}(\rho S) + \frac{\partial}{\partial x}(\rho S u_x) = 0, \tag{2.7}$$

$$\frac{\partial p_c}{\partial t} + u_x\frac{\partial p_c}{\partial x} + \gamma_c p_c \frac{\partial u_x}{\partial x} - \frac{\partial}{\partial x}\left(\kappa\frac{\partial p_c}{\partial x}\right) = 0. \tag{2.8}$$

In the above equations $\rho$, $\mathbf{u}$, $p_g$ and $S$ denote the thermal gas density, fluid velocity, pressure and entropy respectively, $p_c$, $\gamma_c$ and $\kappa$ denote the cosmic ray pressure, adiabatic index and hydrodynamical diffusion coefficient, $\mathbf{B}$ denotes the magnetic field induction, and $\mu$ is the magnetic permeability. For an ideal gas, the gas entropy $S$ has the form

$$S = C_v \ln\left(\frac{p_g}{\rho^{\gamma_g}}\right), \tag{2.9}$$

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where $C_v$ is the specific heat at constant volume and $\gamma_g = C_p/C_v$ is the ratio of specific heats for the thermal gas.

Equations (2.1)–(2.9) can be combined to yield the total energy equation for the system:

$$\frac{\partial W}{\partial t} + \frac{\partial F_x}{\partial x} = 0, \tag{2.10}$$

where

$$W = E_g + E_c + \tfrac{1}{2}\rho u^2 + \frac{B^2}{2\mu} \tag{2.11}$$

is the total energy density, and

$$\mathbf{F} = (\tfrac{1}{2}\rho u^2 + E_g + p_g)\mathbf{u} + \frac{1}{\mu}[B^2\mathbf{u} - (\mathbf{u}\cdot\mathbf{B})\mathbf{B}] + \mathbf{F}_c \tag{2.12}$$

is the total energy flux. In (2.11) and (2.12)

$$E_g = \frac{p_g}{\gamma_g - 1}, \qquad E_c = \frac{p_c}{\gamma_c - 1}, \tag{2.13}$$

$$\mathbf{F}_c = \mathbf{u}(E_c + p_c) - \mathbf{K}\cdot\frac{\partial E_c}{\partial \mathbf{x}} \tag{2.14}$$

define the internal energy densities $E_g$ and $E_c$ for the thermal and cosmic ray gases, $\mathbf{K}$ is the cosmic ray diffusion tensor (note that $\kappa = K_{xx}$), and $\mathbf{F}_c$ is the cosmic ray energy flux.

Note that (2.1)–(2.7) are in conservative form. The system of equations (2.1)–(2.8) can be written in the form

$$\frac{\partial \Psi^i}{\partial t} + \frac{\partial F^i}{\partial x} = -\frac{\partial p_c}{\partial x}\delta^i{}_2, \quad i = 1, \dots, 7, \tag{2.15}$$

$$\frac{\partial p_c}{\partial t} + \frac{M_x}{\rho}\frac{\partial p_c}{\partial x} + \gamma_c p_c \frac{\partial}{\partial x}\left(\frac{M_x}{\rho}\right) - \frac{\partial}{\partial x}\left(\kappa\frac{\partial p_c}{\partial x}\right) = 0 \tag{2.16}$$

($\delta^i{}_2$ is the Kronecker delta), where $\mathbf{M} = \rho\mathbf{u}$ is the momentum density of the thermal gas and

$$\mathbf{\Psi} = (\rho, M_x, M_y, M_z, B_y, B_z, \sigma, p_c)^{\mathrm{T}}, \qquad \sigma = \rho S, \tag{2.17}$$

defines the state vector of the system. The fluxes $\{F^i : i = 1, \dots, 7\}$ in (2.15) are

$$F^1 = M_x, \tag{2.18a}$$

$$F^2 = \frac{M_x^2}{\rho} + p_g + \tfrac{1}{2}b_0^2(B_y^2 + B_z^2), \tag{2.18b}$$

$$F^3 = \frac{M_x M_y}{\rho} - b_0^2 B_x B_y, \tag{2.18c}$$

$$F^4 = \frac{M_x M_z}{\rho} - b_0^2 B_x B_z, \tag{2.18d}$$

$$F^5 = \frac{M_x B_y - M_y B_x}{\rho}, \tag{2.18e}$$

$$F^6 = \frac{M_x B_z - M_z B_x}{\rho}, \tag{2.18f}$$

$$F^7 = \frac{M_x \sigma}{\rho}, \tag{2.18g}$$

where $b_0 = \mu^{-1/2}$ in (2.18).

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Equations (2.15)–(2.18) may also be written in terms of the dimensionless variables

$$\bar{\rho} = \frac{\rho}{\rho_0}, \qquad \bar{\mathbf{M}} = \frac{\rho \mathbf{u}}{\rho_0 V_0}, \qquad \bar{p}_g = \frac{p_g}{p_{g0}}, \qquad \bar{p}_c = \frac{p_c}{p_{g0}}, \tag{2.19a}$$

$$\bar{\mathbf{B}} = \frac{\mathbf{B}}{B_0}, \qquad \bar{\kappa} = \frac{\kappa}{V_0^2 L}, \qquad \bar{S} = \frac{S}{C_v}, \qquad \bar{\sigma} = \bar{S}\bar{\rho}, \tag{2.19b}$$

$$\bar{x} = \frac{x}{L}, \qquad \bar{t} = \frac{t}{T}. \tag{2.19c}$$

The characteristic length ($L$) and time ($T$) scales and characteristic wave speed $V_0$ are chosen so that

$$\frac{V_0 T}{L} = 1, \qquad V_0 = \left(\frac{p_{g0}}{\rho_0}\right)^{1/2}. \tag{2.20}$$

Equations (2.15)–(2.18) also have the same form in the dimensionless variables (2.19), provided that we choose

$$b_0 = \left(\frac{B_0^2}{\mu \rho_0 V_0^2}\right)^{1/2}. \tag{2.21}$$

In (2.20), $V_0$ is the isothermal sound speed.

The cosmic rays may be scattered both by resonant wave–particle interactions and by random walk of the field lines (see e.g. Jokipii 1971). The general form of the energetic particle diffusion coefficient $k_{xx}$, in the kinetic transport equation for cosmic rays (see e.g. Krymsky 1964; Parker 1965; Skilling 1975a) is of the form

$$k_{xx} = \kappa_{\|} \cos^2 \theta_{Bn} + \kappa_{\perp} \sin^2 \theta_{Bn}, \tag{2.22}$$

where $\kappa_{\|}$ and $\kappa_{\perp}$ are the particle diffusion coefficients parallel and perpendicular to the background magnetic field $B$, and $\theta_{Bn}$ is the angle between the background magnetic field and the $x$ axis, or shock normal. The parallel diffusion coefficient $\kappa_{\|}$ is determined by resonant wave–particle interactions, whereas the perpendicular diffusion coefficient $\kappa_{\perp}$ is determined both by resonant wave–particle interactions and by random walk of the field lines. For slab turbulence, in which the magnetic fluctuations are perpendicular to $\mathbf{B}$, $\kappa_{\perp}$ is determined by random walk of the field lines. In this case, one finds $\kappa_{\|} \propto 1/P_w$, where $P_w = (\delta B)^2/8\pi$ is the Alfvén wave pressure, whereas the perpendicular diffusion coefficient $\kappa_{\perp}$ is proportional to the power at zero frequency. The hydrodynamically averaged diffusion coefficient $\kappa$ is an average of $k_{xx}$ over the energetic particle momentum spectrum (see e.g. Drury and Vólk 1981). Drury and Falle (1986) note that if resonant wave–particle interactions are the main scattering mechanism then the hydrodynamically averaged diffusion coefficient $\kappa \propto 1/P_w$. From the compression of pre-existing Alfvén waves in the highly supersonic flow upstream of a cosmic-ray-modified shock, $P_w \propto \rho^{3/2}$, suggesting that $\kappa \propto \rho^{-3/2}$ for the case where $k_{xx}$ is dominated by $\kappa_{\|}$. However, Alfvén wave excitation by the resonant streaming instability will also contribute to $P_w$ (see e.g. McKenzie and Völk 1982).

On the other hand, in a quasiperpendicular shock, $k_{xx} \approx \kappa_{\perp}$ ($\theta_{Bn} \approx \frac{1}{2}\pi$ in (2.22)). If $\kappa_{\perp}$ is dominated by random walk of the field lines (Jokipii 1971, equation (65)), one can show that $\kappa_{\perp} \propto (\delta B/B)^2$. From the wave mixing equations for Alfvén

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waves (see e.g. Zhou and Matthaeus 1990), one can show that $\delta B \propto \rho$ in a quasi-perpendicular shock. Because both $\delta B \propto \rho$ and $B \propto \rho$, it follows that $\kappa \sim$ const in a quasiperpendicular shock.

These arguments suggest that

$$\kappa = \kappa(\rho) \tag{2.23}$$

is a function of the background density $\rho$. It turns out that the squeezing instability for short-wavelength sound waves in cosmic-ray-modified shocks, first investigated in detail by Drury and Falle (1986), is sensitive to the form of $\kappa(\rho)$. More specifically, the squeezing instability depends on the value of the parameter

$$\zeta = \frac{\partial \ln \kappa}{\partial \ln \rho}. \tag{2.24}$$

The instability growth rate also depends on the cosmic ray pressure gradient, and is substantially enhanced for a low-temperature background thermal gas. This is discussed in further detail in the following analysis.

In the following analysis, we take the hydrodynamical, cosmic ray diffusion coefficient $\kappa = \kappa(\rho)$, and we use the dimensionless variables (2.19), but, in an abuse of notation, we omit the overbars on the dimensionless variables.

## 3. The magnetohydrodynamic eigenequations

Before proceeding with the derivation of wave interaction equations for the two-fluid MHD cosmic ray model, it is useful to have at hand the eigenvalues and eigenvectors of the MHD equations. The MHD eigenvectors depend on the state vector (i.e. dependent variables) used in the analysis. We consider two different state vectors that are useful in deriving the wave interaction equations in Sec. 4. We consider the state vector

$$\tilde{\boldsymbol{\Psi}}' = (\rho, u_x, u_y, u_z, B_y, B_z, S)^{\mathrm{T}}, \tag{3.1}$$

and the conserved densities state vector

$$\tilde{\boldsymbol{\Psi}} = (\rho, M_x, M_y, M_z, B_y, B_z, \sigma)^{\mathrm{T}}, \tag{3.2}$$

where $\mathbf{M} = \rho\mathbf{u}$ is the mass flux and $\sigma = \rho S$ is the conserved entropy density. The eigenvalues and eigenvectors are obtained in Sec. 3.1. Section 3.2 provides formulae relating the wave amplitudes to the total perturbations of the MHD fluid. The relationships between the wave amplitudes and the individual wave mode contributions to the total perturbations are discussed. Section 3.3 discusses discrete symmetries that map eigenvectors onto eigenvectors. These symmetries imply symmetries between the wave interaction coefficients obtained in Secs 5 and 6.

### *3.1. Eigenvalues and eigenvectors*

In terms of $\tilde{\boldsymbol{\Psi}}'$, the MHD equations may be written in the form

$$\frac{\partial \tilde{\boldsymbol{\Psi}}'}{\partial t} + \mathscr{A}' \cdot \frac{\partial \tilde{\boldsymbol{\Psi}}'}{\partial x} = 0, \tag{3.3}$$

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where

$$\mathscr{A}' = \begin{pmatrix} u_x & \rho & 0 & 0 & 0 & 0 & 0 \\ a_g^2\rho^{-1} & u_x & 0 & 0 & b_0^2 B_y \rho^{-1} & b_0^2 B_z \rho^{-1} & p_g \rho^{-1} \\ 0 & 0 & u_x & 0 & -b_0^2 B_x \rho^{-1} & 0 & 0 \\ 0 & 0 & 0 & u_x & 0 & -b_0^2 B_x \rho^{-1} & 0 \\ 0 & B_y & -B_x & 0 & u_x & 0 & 0 \\ 0 & B_z & 0 & -B_x & 0 & u_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_x \end{pmatrix}. \tag{3.4}$$

The right-eigenvectors $\{\mathbf{R}'_s\}$ and left eigenvectors $\{\mathbf{L}'_s\}$ of the matrix $\mathscr{A}'$ satisfy the eigenvector equations

$$\mathbf{L}'_s \cdot (\mathscr{A}' - \lambda_s \mathbf{I}) = 0, \tag{3.5a}$$

$$(\mathscr{A}' - \lambda_s \mathbf{I}) \cdot \mathbf{R}'_s = 0, \tag{3.5b}$$

where the eigenvalues $\{\lambda_s\}$ satisfy the eigenvalue equation

$$\det(\mathscr{A}' - \lambda_s \mathbf{I}) = 0. \tag{3.6}$$

The eigenvectors are normalized so that

$$\mathbf{L}'_s \cdot \mathbf{R}'_j = \delta^s{}_j, \tag{3.7}$$

where $\delta^s{}_j$ is the Kronecker delta.

Evaluating the determinant (3.6) yields the eigenvalue equation in the form

$$\det(\mathscr{A}' - \lambda \mathbf{I}) = -\tilde{\lambda}(\tilde{\lambda}^2 - b_x^2)[\tilde{\lambda}^4 - (a_g^2 + b^2)\tilde{\lambda}^2 + a_g^2 b_x^2] = 0, \tag{3.8}$$

where $\tilde{\lambda} = \lambda - u_x$ denotes the phase speed of the waves in the fluid frame, and

$$b^2 = b_x^2 + b_\perp^2, \qquad b_x = \frac{b_0 B_x}{\rho^{1/2}}, \qquad \mathbf{b}_\perp = \frac{b_0 \mathbf{B}_\perp}{\rho^{1/2}}. \tag{3.9}$$

In (3.9), $b_x$ is the phase speed of the Alfvén wave in the fluid frame. In general, there are seven distinct solutions for $\lambda$ in (3.8), namely

$$\lambda_1 = u_x - c_f, \qquad \lambda_2 = u_x - b_x, \qquad \lambda_3 = u_x - c_s, \qquad \lambda_4 = u_x, \tag{3.10a}$$

$$\lambda_5 = u_x + c_s, \qquad \lambda_6 = u_x + b_x, \qquad \lambda_7 = u_x + c_f, \tag{3.10b}$$

where $c_f$, $c_s$ and $b_x$ denote the fast magnetoacoustic, slow magnetoacoustic, and Alfvén speeds respectively. The fast and slow phase speeds $c_f$ and $c_s$ satisfy the magnetoacoustic dispersion equation

$$c^4 - (a_g^2 + b^2)c^2 + a_g^2 b_x^2 = 0. \tag{3.11}$$

The eigenvalue $\lambda_4 = u_x$ in (3.10) corresponds to the entropy wave, or contact discontinuity eigenmode.

Following the approach of Brio and Wu (1988), Zachary and Colella (1992) and Roe and Balsara (1996), we normalize the right- and left-eigenvectors $\{\mathbf{R}'_s\}$ and $\{\mathbf{L}'_j\}$ so that a well-defined set of eigenvectors is obtained for the degenerate cases of parallel propagation for which $\mathbf{B} = (B_x, 0, 0)^{\mathrm{T}}$ and perpendicular propagation for which $\mathbf{B} = (0, B_y, B_z)^{\mathrm{T}}$. It is useful to use the notation

$$\mathbf{R}'_1 = \mathbf{R}'^-_f, \qquad \mathbf{R}'_2 = \mathbf{R}'^-_A, \qquad \mathbf{R}'_3 = \mathbf{R}'^-_s, \qquad \mathbf{R}'_4 = \mathbf{R}'_e, \tag{3.12a}$$

$$\mathbf{R}'_5 = \mathbf{R}'^+_s, \qquad \mathbf{R}'_6 = \mathbf{R}'^+_A, \qquad \mathbf{R}'_7 = \mathbf{R}'^+_f, \tag{3.12b}$$

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and a similar relabelling of the left-eigenvectors, where the subscripts $f$, $s$, $A$ and $e$ denote respectively the fast magnetoacoustic, slow magnetoacoustic, Alfvén and entropy wave eigenmodes, and the superscripts $-$ and $+$ denote the backward and forward waves.

The matrix (3.4) has non-normalized right- and left-eigenvectors for the magnetoacoustic modes of the form

$$\mathbf{r}'^{(ma)} = \left(1, \frac{c}{\rho}, -\frac{cb_x\mathbf{b}_\perp}{(c^2-b_x^2)\rho}, \frac{c^2\mathbf{b}_\perp}{(c^2-b_x^2)b_0\rho^{1/2}}, 0\right)^{\mathrm{T}}, \tag{3.13}$$

$$\mathbf{l}'^{(ma)} = \frac{\rho}{a^2}\left(\frac{a^2}{\rho}, c, -\frac{cb_x\mathbf{b}_\perp}{c^2-b_x^2}, \frac{c^2b_0\mathbf{b}_\perp}{(c^2-b_x^2)\rho^{1/2}}, \frac{a^2}{\gamma_g}\right), \tag{3.14}$$

where the superscript $(ma)$ denotes the magnetoacoustic mode, $a \equiv a_g$ is the gas sound speed, $\mathbf{b}_\perp = b_y\mathbf{e}_y + b_z\mathbf{e}_z$ is the same as in (3.9), and $c$ denotes one of the solutions of the magnetoacoustic dispersion equation (3.11). To obtain a well-defined set of normalized eigenvectors for the degenerate cases of parallel ($\mathbf{k} \parallel \mathbf{B}$) and perpendicular ($\mathbf{k} \perp \mathbf{B}$) propagation, we consider the eigenvectors

$$\mathbf{R}'^{(ma)} = k^r\mathbf{r}'^{(ma)}, \tag{3.15a}$$

$$\mathbf{L}'^{(ma)} = k^l\mathbf{l}'^{(ma)}. \tag{3.15b}$$

The eigenvectors $\{\mathbf{R}'^{(ma)}\}$ and $\{\mathbf{L}'^{(ma)}\}$ form an orthonormal set for the magnetoacoustic modes if we choose

$$k^rk^l = \frac{a^2|c^2-b_x^2|}{2c^2(c_f^2-c_s^2)}, \tag{3.16}$$

(see also Roe and Balsara 1996). The condition (3.16) for the fast and slow modes yields the equations

$$k_f^rk_f^l = \tfrac{1}{2}\alpha_f^2, \qquad k_s^rk_s^l = \tfrac{1}{2}\alpha_s^2, \tag{3.17}$$

where $\alpha_f$ and $\alpha_s$ are defined by the equations

$$\alpha_f = \left(\frac{a^2-c_s^2}{c_f^2-c_s^2}\right)^{1/2}, \qquad \alpha_s = \left(\frac{c_f^2-a^2}{c_f^2-c_s^2}\right)^{1/2}. \tag{3.18}$$

The parameters $\alpha_f$ and $\alpha_s$ are the same as those used by Roe and Balsara (1996). They satisfy the auxiliary equations

$$\alpha_f^2+\alpha_s^2 = 1, \qquad \alpha_f\alpha_s = \frac{ab_\perp}{c_f^2-c_s^2}. \tag{3.19}$$

Note also that $c_fc_s = a|b_x|$.

The choices

$$k_f^r = \alpha_f, \qquad k_f^l = \tfrac{1}{2}\alpha_f, \qquad k_s^r = \alpha_s, \qquad k_s^l = \tfrac{1}{2}\alpha_s, \tag{3.20}$$

yield well-defined eigenvectors (3.15) for the degenerate cases. Using the normalization constants (3.20), we obtain

$$\mathbf{R}'^{\pm}_f = \left(\alpha_f, \pm\frac{\alpha_fc_f}{\rho}, \mp\frac{\alpha_sc_s}{\rho}\,\mathrm{sgn}(b_x)\,\boldsymbol{\beta}_\perp, \frac{\alpha_sa\boldsymbol{\beta}_\perp}{b_0\rho^{\frac{1}{2}}}, 0\right)^{\mathrm{T}}, \tag{3.21}$$

$$\mathbf{L}'^{\pm}_f = \frac{1}{2}\left(\alpha_f, \pm\frac{\alpha_fc_f\rho}{a^2}, \mp\frac{\rho\alpha_sc_s\,\mathrm{sgn}(b_x)}{a^2}\boldsymbol{\beta}_\perp, \frac{b_0\rho^{\frac{1}{2}}\alpha_s\boldsymbol{\beta}_\perp}{a}, \frac{\alpha_f\rho}{\gamma_g}\right) \tag{3.22}$$

for the fast-mode eigenvectors, where

$$\boldsymbol{\beta}_\perp = \frac{\mathbf{B}_\perp}{B_\perp} = \beta_y \mathbf{e}_y + \beta_z \mathbf{e}_z, \tag{3.23}$$

$\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors along the $x$, $y$ and $z$ axes, and $\mathrm{sgn}(b_x)$ denotes the sign of $b_x$. Similarly, we find

$$\mathbf{R}'^{\pm}_s = \left(\alpha_s, \pm\frac{\alpha_s c_s}{\rho}, \pm\frac{\alpha_f c_f}{\rho}\,\mathrm{sgn}(b_x)\,\boldsymbol{\beta}_\perp, -\frac{\alpha_f a \boldsymbol{\beta}_\perp}{b_0 \rho^{1/2}}, 0\right)^{\mathrm{T}}, \tag{3.24}$$

$$\mathbf{L}'^{\pm}_s = \frac{1}{2}\left(\alpha_s, \pm\frac{\alpha_s c_s \rho}{a^2}, \pm\frac{\rho \alpha_f c_f\,\mathrm{sgn}(b_x)}{a^2}\boldsymbol{\beta}_\perp, -\frac{b_0 \rho^{1/2} \alpha_f \boldsymbol{\beta}_\perp}{a}, \frac{\alpha_s \rho}{\gamma_g}\right) \tag{3.25}$$

for the slow-mode eigenvectors.

A well-defined set of eigenvectors for the Alfvén modes is

$$\mathbf{R}'^{\pm}_A = \left(0, 0, \mp\frac{b_0 \beta_z}{\rho^{\frac{1}{2}}}, \pm\frac{b_0 \beta_y}{\rho^{\frac{1}{2}}}, \beta_z, -\beta_y, 0\right)^{\mathrm{T}}, \tag{3.26}$$

$$\mathbf{L}'^{\pm}_A = \frac{1}{2}\left(0, 0, \mp\frac{\rho^{\frac{1}{2}} \beta_z}{b_0}, \pm\frac{\rho^{\frac{1}{2}} \beta_y}{b_0}, \beta_z, -\beta_y, 0\right). \tag{3.27}$$

Similarly, for the entropy wave,

$$\mathbf{R}'_e = \left(1, 0, 0, 0, 0, 0, -\frac{\gamma_g}{\rho}\right)^{\mathrm{T}}, \tag{3.28}$$

$$\mathbf{L}'_e = -\frac{\rho}{\gamma_g}\,(0, 0, 0, 0, 0, 0, 1) \tag{3.29}$$

are the normalized eigenvectors.

Now consider the form of the eigenvectors for the case where the state vector is the conserved densities state vector (3.2). In this case the MHD equations may be written in the form

$$\frac{\partial \tilde{\boldsymbol{\Psi}}}{\partial t} + \mathscr{A}\cdot\frac{\partial \tilde{\boldsymbol{\Psi}}}{\partial x} = 0. \tag{3.30}$$

The matrix $\mathscr{A}$ in (3.30) is given by the formula

$$\mathscr{A}^{ij} = \frac{\partial F^i}{\partial \Psi^j}, \tag{3.31}$$

where the $\{F^i\}$ are the MHD fluxes in (2.18). Using (3.31), the matrix $\mathscr{A}$ is given by

$$\mathscr{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_g^2 - u_x^2 - \sigma p_g \rho^{-2} & 2u_x & 0 & 0 & b_0^2 B_y & b_0^2 B_z & p_g \rho^{-1} \\ -u_x u_y & u_y & u_x & 0 & -b_0^2 B_x & 0 & 0 \\ -u_x u_z & u_z & 0 & u_x & 0 & -b_0^2 B_x & 0 \\ -(u_x B_y - u_y B_x)\rho^{-1} & B_y \rho^{-1} & -B_x \rho^{-1} & 0 & u_x & 0 & 0 \\ -(u_x B_z - u_z B_x)\rho^{-1} & B_z \rho^{-1} & 0 & -B_x \rho^{-1} & 0 & u_x & 0 \\ -\sigma u_x \rho^{-1} & \sigma \rho^{-1} & 0 & 0 & 0 & 0 & u_x \end{pmatrix}. \tag{3.32}$$

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The matrix $\mathscr{A}$ in (3.32) is related to the matrix $\mathscr{A}'$ in (3.4) by the transformation

$$\mathscr{A} = \mathbf{Q}\cdot\mathscr{A}'\cdot\mathbf{P}, \qquad \mathbf{Q} = \mathbf{P}^{-1}, \tag{3.33a}$$

$$P^{\alpha\beta} = \frac{\partial\tilde{\mathbf{\Psi}}'^{\alpha}}{\partial\tilde{\mathbf{\Psi}}^{\beta}}, \qquad Q^{\alpha\beta} = \frac{\partial\tilde{\mathbf{\Psi}}^{\alpha}}{\partial\tilde{\mathbf{\Psi}}'^{\beta}}. \tag{3.33b}$$

In the perturbation analysis in Sec. 4, we use perturbations of $\tilde{\mathbf{\Psi}}$ and $\tilde{\mathbf{\Psi}}'$ of the form

$$\delta\tilde{\mathbf{\Psi}} = \epsilon\tilde{\mathbf{\Psi}}^{(1)} + O(\epsilon^2), \qquad \delta\tilde{\mathbf{\Psi}}' = \epsilon\tilde{\mathbf{\Psi}}'^{(1)} + O(\epsilon^2) \tag{3.34}$$

where

$$\tilde{\mathbf{\Psi}}^{(1)} = \sum_{s=1}^{7} a_s\mathbf{R}_s, \qquad \tilde{\mathbf{\Psi}}'^{(1)} = \sum_{s=1}^{7} a'_s\mathbf{R}'_s, \qquad a'_s = a_s, \tag{3.35}$$

and $\epsilon$ is a small parameter ordering the perturbation expansion. In (3.35), the wave amplitudes $a_s$ and $a'_s$ are chosen to be the same. For $a'_s = a_s$, the eigenvectors of the matrix $\mathscr{A}$ are related to those of the matrix $\mathscr{A}'$ by the equations

$$\mathbf{R}_s = \mathbf{Q}\cdot\mathbf{R}'_s, \tag{3.36a}$$

$$\mathbf{L}_j = \mathbf{L}'_j\cdot\mathbf{P}. \tag{3.36b}$$

Note that the $\{\mathbf{L}_j\}$ and the $\{\mathbf{R}_s\}$ satisfy the orthonormality conditions $\mathbf{L}_j\cdot\mathbf{R}_s = \delta^j{}_s$. The matrix $\mathbf{P}$ in (3.33) has the form

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_x\rho^{-1} & \rho^{-1} & 0 & 0 & 0 & 0 & 0 \\ -u_y\rho^{-1} & 0 & \rho^{-1} & 0 & 0 & 0 & 0 \\ -u_z\rho^{-1} & 0 & 0 & \rho^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \sigma\rho^{-2} & 0 & 0 & 0 & 0 & 0 & \rho^{-1} \end{pmatrix}. \tag{3.37}$$

Similarly, the matrix $\mathbf{Q} = \mathbf{P}^{-1}$ has the form

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_x & \rho & 0 & 0 & 0 & 0 & 0 \\ u_y & 0 & \rho & 0 & 0 & 0 & 0 \\ u_z & 0 & 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ S & 0 & 0 & 0 & 0 & 0 & \rho \end{pmatrix}. \tag{3.38}$$

Using the transformation matrices (3.37) and (3.38) and the eigenvectors (3.21) and (3.22) in (3.36b), we obtain the conserved densities eigenvectors for the fast mode in the form

$$\mathbf{R}_f^{\pm} = \Bigg(\alpha_f, \alpha_f(u_x \pm c_f), \alpha_f u_y \mp \alpha_s c_s\,\mathrm{sgn}(b_x)\,\beta_y, \alpha_f u_z \mp \alpha_s c_s\,\mathrm{sgn}(b_x)\,\beta_z,$$
$$\frac{\alpha_s a\beta_y}{b_0\rho^{1/2}}, \frac{\alpha_s a\beta_z}{b_0\rho^{1/2}}, \alpha_f S\Bigg)^{\mathrm{T}}, \tag{3.39}$$

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$$\mathbf{L}_f^{\pm} = \frac{1}{2}\left(\alpha_f\left(1-\frac{S}{\gamma_g}\right) \mp \frac{\alpha_f c_f u_x}{a^2} \pm \frac{\alpha_s c_s \operatorname{sgn}(b_x)}{a^2}\boldsymbol{\beta}_\perp \cdot \mathbf{u}_\perp,\right.$$
$$\pm\frac{\alpha_f c_f}{a^2}, \mp\frac{\alpha_s c_s \operatorname{sgn}(b_x)}{a^2}\beta_y, \mp\frac{\alpha_s c_s \operatorname{sgn}(b_x)}{a^2}\beta_z,$$
$$\left.\frac{\alpha_s b_0 \rho^{1/2}}{a}\beta_y, \frac{\alpha_s b_0 \rho^{1/2}}{a}\beta_z, \frac{\alpha_f}{\gamma_g}\right), \tag{3.40}$$

where $\alpha_f$, $\alpha_s$ and $\boldsymbol{\beta}_\perp$ are given by (3.18) and (3.23), and

$$\mathbf{u}_\perp = u_y \mathbf{e}_y + u_z \mathbf{e}_z. \tag{3.41}$$

Similarly, the slow-mode eigenvectors have the form

$$\mathbf{R}_s^{\pm} = \left(\alpha_s, \alpha_s(u_x \pm c_s), \alpha_s u_y \pm \alpha_f c_f \operatorname{sgn}(b_x)\,\beta_y, \alpha_s u_z \pm \alpha_f c_f \operatorname{sgn}(b_x)\,\beta_z,\right.$$
$$\left. -\frac{\alpha_f a \beta_y}{b_0 \rho^{1/2}}, -\frac{\alpha_f a \beta_z}{b_0 \rho^{1/2}}, \alpha_s S\right)^{\mathrm{T}}, \tag{3.42}$$

$$\mathbf{L}_s^{\pm} = \frac{1}{2}\left(\alpha_s\left(1-\frac{S}{\gamma_g}\right) \mp \frac{\alpha_s c_s u_x}{a^2} \mp \frac{\alpha_f c_f \operatorname{sgn}(b_x)}{a^2}\boldsymbol{\beta}_\perp \cdot \mathbf{u}_\perp,\right.$$
$$\pm\frac{\alpha_s c_s}{a^2}, \pm\frac{\alpha_f c_f \operatorname{sgn}(b_x)}{a^2}\beta_y, \pm\frac{\alpha_f c_f \operatorname{sgn}(b_x)}{a^2}\beta_z,$$
$$\left. -\frac{\alpha_f b_0 \rho^{1/2}}{a}\beta_y, -\frac{\alpha_f b_0 \rho^{1/2}}{a}\beta_z, \frac{\alpha_s}{\gamma_g}\right). \tag{3.43}$$

The Alfvén eigenvectors are

$$\mathbf{R}_A^{\pm} = (0, 0, \mp b_0 \rho^{1/2}\beta_z, \pm b_0 \rho^{1/2}\beta_y, \beta_z, -\beta_y, 0)^{\mathrm{T}}, \tag{3.44}$$

$$\mathbf{L}_A^{\pm} = \frac{1}{2}\left(\mp\frac{u_z\beta_y - u_y\beta_z}{b_0\rho^{1/2}}, 0, \mp\frac{\beta_z}{b_0\rho^{1/2}}, \pm\frac{\beta_y}{b_0\rho^{1/2}}, \beta_z, -\beta_y, 0\right). \tag{3.45}$$

The entropy wave eigenvectors are

$$\mathbf{R}_e = \left(1, u_x, u_y, u_z, 0, 0, S-\gamma_g\right)^{\mathrm{T}}, \tag{3.46}$$

$$\mathbf{L}_e = \frac{1}{\gamma_g}\,(S, 0, 0, 0, 0, 0, -1)\,. \tag{3.47}$$

The above eigenvectors are in general well defined as $\mathbf{B}_\perp \to 0$ (parallel propagation limit), provided that we specify the manner in which $B_y$ and $B_z$ tend to zero.

Both sets of eigenvectors (3.21)–(3.29) and (3.39)–(3.47) corresponding to the state vectors $\tilde{\mathbf{\Psi}}'$ and $\tilde{\mathbf{\Psi}}$ (see (3.1) and (3.2)) may be used in the perturbation analysis in Sec. 4. The wave interaction coefficients of physical interest turn out to be independent of which state vector is used in the perturbation analysis, provided that the wave amplitudes $a_s$ are chosen to be the same in both cases (see Appendix A).

### 3.2. *Wave amplitudes and eigenvector relations*

In this section, we relate the wave amplitudes $\{a_j : j = 1, \ldots, 7\}$ to the total MHD fluid perturbations, which have the form

$$\delta\tilde{\mathbf{\Psi}}' = \left(\delta\rho, \delta u_x, \delta u_y, \delta u_z, \delta B_y, \delta B_z, \delta\bar{S}\right)^{\mathrm{T}} = \epsilon(\rho^1, u_x^1, u_y^1, u_z^1, B_y^1, B_z^1, \bar{S}^1)^{\mathrm{T}}. \tag{3.48}$$

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Using the orthogonality relations $\mathbf{L}'_j \cdot \mathbf{R}'_s = \delta^j{}_s$ and (3.35) yields the formulae

$$a_j = \mathbf{L}'_j \cdot \tilde{\mathbf{\Psi}}'^{(1)} \equiv \mathbf{L}'_j \cdot (\rho^1, u^1_x, u^1_y, u^1_z, B^1_y, B^1_z, \bar{S}^1)^{\mathrm{T}} \tag{3.49}$$

for the wave amplitudes $\{a_j\}$. The total perturbations $\delta\tilde{\mathbf{\Psi}}'^{(1)}$ may in turn be split up into contributions from the different wave modes, by noting that

$$a_j \mathbf{R}'_j = \left(\rho_j, u_{jx}, u_{jy}, u_{jz}, B_{jy}, B_{jz}, \bar{S}_j\right)^{\mathrm{T}}, \quad j = 1, \ldots, 7. \tag{3.50}$$

In (3.50), the $\rho_j$, $\mathbf{u}_j$, $\mathbf{B}_j$ and $\bar{S}_j$ denote the density, velocity, magnetic induction and entropy perturbations due to the different wave modes. There are two degenerate eigenvalue cases, namely when $\mathbf{B} = (B_x, 0, 0)^{\mathrm{T}}$ and $\mathbf{B} = (0, B_y, B_z)^{\mathrm{T}}$. Particular care needs to be exercised for the case $\mathbf{B} = (B_x, 0, 0)^{\mathrm{T}}$ for which $\mathbf{k} \parallel \mathbf{B}$, since in this case one of the magnetoacoustic modes becomes incompressible, and behaves like the Alfvén mode, but with a phase shift in the eigenrelations between $\delta\mathbf{u}_\perp$ and $\delta\mathbf{B}_\perp$.

*3.2.1. Case* $\mathbf{B} \neq (B_x, 0, 0)^{\mathrm{T}}$. Use of the left-eigenvectors (3.22), (3.25), (3.27) and (3.29) in (3.49) yields the formulae

$$a_1 = a_f^-, \qquad a_7 = a_f^+, \qquad a_3 = a_s^-, \qquad a_5 = a_s^+, \tag{3.51a}$$

$$a_2 = a_A^-, \qquad a_6 = a_A^+, \qquad a_4 = a_e \tag{3.51b}$$

for the different wave modes, where

$$a_f^\pm = \frac{1}{2}\left[\alpha_f\left(\rho^1 + \frac{\rho\bar{S}^1}{\gamma_g}\right) \pm \frac{\alpha_f c_f \rho}{a^2} u^1_x + \frac{b_0 \rho^{1/2} \alpha_s}{a} \boldsymbol{\beta}_\perp \cdot \left(\mathbf{B}^1_\perp \mp \frac{c_s \rho^{1/2}}{a b_0} \mathbf{u}^1_\perp\right)\right], \tag{3.52}$$

$$a_s^\pm = \frac{1}{2}\left[\alpha_s\left(\rho^1 + \frac{\rho\bar{S}^1}{\gamma_g}\right) \pm \frac{\alpha_s c_s \rho}{a^2} u^1_x - \frac{b_0 \rho^{1/2} \alpha_f}{a} \boldsymbol{\beta}_\perp \cdot \left(\mathbf{B}^1_\perp \mp \frac{c_f \rho^{1/2}}{a b_0} \mathbf{u}^1_\perp\right)\right], \tag{3.53}$$

$$\begin{aligned} a_A^\pm &= \frac{1}{2}\left[(\beta_z B^1_y - \beta_y B^1_z) \mp \frac{\rho^{1/2}}{b_0}(\beta_z u^1_y - \beta_y u^1_z)\right] \\ &\equiv -\frac{1}{2}\mathbf{e}_x \cdot \boldsymbol{\beta}_\perp \times \left(\mathbf{B}^1_\perp \mp \frac{\rho^{1/2}}{b_0}\mathbf{u}^1_\perp\right), \end{aligned} \tag{3.54}$$

$$a_4 \equiv a_e = -\frac{\rho\bar{S}^1}{\gamma_g} \tag{3.55}$$

give the wave amplitudes for the fast, slow, Alfvén and entropy waves in terms of the total MHD fluid perturbations.

Using the right-eigenvectors $\{\mathbf{R}'_j\}$ and (3.50), we obtain the formulae

$$\rho_1 = \alpha_f a_1, \qquad \rho_3 = \alpha_s a_3, \qquad \rho_4 = a_4, \qquad \rho_5 = \alpha_s a_5, \qquad \rho_7 = \alpha_f a_7 \tag{3.56}$$

for the density perturbations of the fast and slow modes $(\rho_1, \rho_3, \rho_5, \rho_7)$ and the entropy wave $(\rho_4)$ in terms of the wave amplitudes $\{a_j\}$. Similar formulae for the fluid velocity, magnetic induction and entropy perturbations $\{\mathbf{u}_j, \mathbf{B}_j, \bar{S}_j\}$ may also be obtained from (3.50).

Similarly, for the Alfvén modes, (3.50) yields the results

$$B_{jy} = \beta_z a_j, \qquad B_{jz} = -\beta_y a_j, \tag{3.57a}$$

$$a_j = \beta_z B_{jy} - \beta_y B_{jz} \equiv \mathbf{e}_x \cdot (\boldsymbol{\beta}_\perp \times \mathbf{B}_j), \qquad j = 2, 6, \tag{3.57b}$$

relating $a_2$ and $a_6$ to the magnetic field perturbations $\mathbf{B}_2$ and $\mathbf{B}_6$.

The formulae (3.51)–(3.57) also apply for the degenerate-eigenvalue case where

$B_x = 0$ (i.e. $\mathbf{k} \perp \mathbf{B}$). It is straightforward to determine the special form of the eigenrelations (3.51)–(3.57) for the degenerate-eigenvalue cases for which $\mathbf{k} \parallel \mathbf{B}$ and $\mathbf{k} \perp \mathbf{B}$, by taking the appropriate limits in (3.51)–(3.57).

### *3.3. Eigenvector symmetries*

The eigenvectors $\{\mathbf{R}_j\}$ and $\{\mathbf{L}_j\}$ may be regarded as functions of the parameters $\mathbf{y} = (c_f, c_s, b_0, a)$. Under the transformation

$$T_a: \quad \mathbf{y} \equiv (c_f, c_s, b_0, a) \mapsto (-c_f, -c_s, -b_0, -a), \tag{3.58}$$

the forward wave eigenvectors are mapped onto the corresponding backward wave eigenvectors. In other words,

$$\mathbf{L}_j(T_a\mathbf{y}) = \mathbf{L}_{j'}(\mathbf{y}), \qquad \mathbf{R}_j(T_a\mathbf{y}) = \mathbf{R}_{j'}(\mathbf{y}), \qquad j' = 8 - j. \tag{3.59}$$

There are other maps, similar to (3.58), obtained by interchanging the parameters $\mathbf{y}$ and/or changing the sign of the parameters, that map eigenvectors onto eigenvectors. These maps imply symmetry relations between the nonlinear wave interaction coefficients (Sec. 5), and the linear wave mixing coefficients (Sec. 6). An alternative suggestive notation for $T_a$ is

$$T_a = \begin{pmatrix} c_f & c_s & b_0 & a \\ -c_f & -c_s & -b_0 & -a \end{pmatrix}, \tag{3.60}$$

where the first row in (3.60) is the domain and the second row is the range.

The map

$$T_b: \quad \mathbf{y} \equiv (c_f, c_s, b_0, a) \mapsto (c_f, c_s, -b_0, -a) \tag{3.61}$$

maps the backward Alfvén wave eigenvectors onto the forward Alfvén wave eigenvectors, but leaves the magnetoacoustic eigenvectors invariant, i.e.

$$\mathbf{R}_j(T_b\mathbf{y}) = \mathbf{R}_{j'}(\mathbf{y}), \quad j = 2, 6, \quad j' = 8 - j, \tag{3.62a}$$

$$\mathbf{R}_j(T_b\mathbf{y}) = \mathbf{R}_j(\mathbf{y}), \quad j \neq 2, 6. \tag{3.62b}$$

Similarly,

$$T_c: \quad \mathbf{y} \equiv (c_f, c_s, b_0, a) \mapsto (-c_f, -c_s, b_0, a) \tag{3.63}$$

maps the forward magnetoacoustic eigenvectors onto the backward magnetoacoustic eigenvectors, but leaves the Alfvén and entropy wave eigenvectors invariant. Thus

$$\mathbf{R}_j(T_c\mathbf{y}) = \mathbf{R}_{j'}(\mathbf{y}), \quad j \neq 2, 6, \quad j' = 8 - j, \tag{3.64a}$$

$$\mathbf{R}_j(T_c\mathbf{y}) = \mathbf{R}_j(\mathbf{y}), \quad j = 2, 4, 6. \tag{3.64b}$$

A further interesting map is

$$T_d: \quad \mathbf{y} \equiv (c_f, c_s, b_0, a) \mapsto (c_s, -c_f, -b_0, a). \tag{3.65}$$

$T_d$ maps the slow magnetoacoustic eigenvectors onto the fast magnetoacoustic eigenvectors and vice versa, in the manner

$$\mathbf{R}_1(T_d\mathbf{y}) = \mathbf{R}_3(\mathbf{y}), \qquad \mathbf{R}_3(T_d\mathbf{y}) = \mathbf{R}_7(\mathbf{y}), \qquad \mathbf{R}_7(T_d\mathbf{y}) = \mathbf{R}_5(\mathbf{y}), \qquad \mathbf{R}_5(T_d\mathbf{y}) = \mathbf{R}_1(\mathbf{y}). \tag{3.66}$$

The map $T_d$ maps the backward Alfvén wave onto the forward Alfvén wave eigenvectors, and vice versa:

$$\mathbf{R}_2(T_d\mathbf{y}) = \mathbf{R}_6(\mathbf{y}), \qquad \mathbf{R}_6(T_d\mathbf{y}) = \mathbf{R}_2(\mathbf{y}). \tag{3.67}$$

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Other maps can be constructed by composition of $T_a, T_b, T_c$ and $T_d$. For example,

$$T_d^2 = \begin{pmatrix} c_f & c_s & b_0 & a \\ -c_f & -c_s & b_0 & a \end{pmatrix} \equiv T_c, \tag{3.68a}$$

$$T_d^3 = \begin{pmatrix} c_f & c_s & b_0 & a \\ -c_s & c_f & -b_0 & a \end{pmatrix}. \tag{3.68b}$$

It is of interest to note that

$$T_a^2 = T_b^2 = T_c^2 = T_d^4 = \begin{pmatrix} c_f & c_s & b_0 & a \\ c_f & c_s & b_0 & a \end{pmatrix} \equiv \mathbf{I}, \tag{3.69}$$

where $\mathbf{I}$ is the identity map. The left-eigenvectors $\{\mathbf{L}_j\}$ transform in the same way as the right-eigenvectors in (3.62)–(3.69).

## 4. Short-wavelength waves in non-uniform flows

In this section, we use the method of multiple scales to derive equations describing weakly nonlinear wave interactions, and linear wave mixing for short-wavelength waves in a non-uniform large-scale background flow. A similar study of wave propagation for the two-fluid cosmic ray model, without MHD effects, was carried out by Webb et al. (1997a). We emphasize that wave interactions for the pure MHD case are obtained by simply dropping the cosmic ray terms in the equations. At lowest order in the perturbation analysis, one finds that cosmic ray pressure perturbations play no role in the analysis. At the next order, one obtains the standard eigenvalues and eigenvector solutions of ideal MHD. At the following order, one obtains wave evolution equations describing linear wave mixing of the different eigenmodes due to the non-uniform background flow, instability and damping terms due to the cosmic rays, and nonlinear interaction effects.

From (2.15) and (2.16), the equations governing the system can be written in the matrix form:

$$\frac{\partial \mathbf{\Psi}}{\partial t} + \mathscr{A}(\mathbf{\Psi}) \cdot \frac{\partial \mathbf{\Psi}}{\partial x} - \frac{\partial}{\partial x}\left[\hat{\mathbf{K}}(\mathbf{\Psi}) \cdot \frac{\partial \mathbf{\Psi}}{\partial x}\right] = 0, \tag{4.1}$$

where the matrix

$$\hat{K}^{ij} = \bar{\kappa}\delta^i{}_8\delta^j{}_8, \tag{4.2}$$

represents the effects of cosmic ray diffusion. As discussed in (2.23) et seq., we take $\bar{\kappa} = \bar{\kappa}(\rho)$ in the following analysis. The $8 \times 8$ matrix $\mathbf{A}$ in (4.1) has components:

$$A^{ij} = \mathscr{A}^{ij} = \frac{\partial F^i}{\partial \Psi^j}, \quad i, j = 1, \ldots, 7, \tag{4.3a}$$

$$A^{i8} = C^i = \delta^i{}_2, \quad i = 1, \ldots, 7, \tag{4.3b}$$

$$A^{8j} = D^j, \quad j = 1, \ldots, 8, \tag{4.3c}$$

where

$$\begin{aligned} \mathbf{C} &= (0, 1, 0, 0, 0, 0, 0)^{\mathrm{T}}, \\ \mathbf{D} &= (-a_c^2 u_x, a_c^2, 0, 0, 0, 0, 0, u_x), \end{aligned} \tag{4.4}$$

and

$$a_c = \left(\frac{\gamma_c p_c}{\rho}\right)^{1/2} \tag{4.5}$$

defines the ‘cosmic ray sound speed’ $a_c$.

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The Jacobian matrix $\mathscr{A}$ in (4.3) contains the MHD effects in the model, and is given by (3.32). The MHD eigenvalues and eigenvectors of the matrix $\mathscr{A}$ are given in (3.32) et seq. These play an important role in the following analysis.

### 4.1. *The perturbation expansion*

Consider the evolution of weakly nonlinear short-wavelength waves, with length scale $L_1$ and time scale $T_1$ in a large-scale background flow, with length and time scales $L$ and $T$, where $L \gg L_1$ and $T \gg T_1$. We develop a weakly nonlinear geometrical optics expansion for the waves in the perturbation parameter

$$\epsilon = \frac{L_1}{L} \equiv \frac{T_1}{T} \quad (0 < \epsilon \ll 1), \tag{4.6}$$

of the form

$$\mathbf{\Psi} = \mathbf{\Psi}^{(0)} + \epsilon \mathbf{\Psi}^{(1)} + \epsilon^2 \mathbf{\Psi}^{(2)} + \ldots, \tag{4.7}$$

where the background state $\mathbf{\Psi}^{(0)} = \mathbf{\Psi}^{(0)}(\bar{x}, \bar{t})$ depends on the slow variables $(\bar{x}, \bar{t})$ whereas the wave perturbations $\{\mathbf{\Psi}^{(i)}(\bar{x}, \bar{t}; \xi, \tau) : \ i \geqslant 1\}$ depend on both the slow variables $\bar{x}$ and $\bar{t}$ and on the fast variables:

$$\xi = \frac{\bar{x}}{\epsilon}, \qquad \tau = \frac{\bar{t}}{\epsilon}. \tag{4.8}$$

The normalized diffusion coefficient $\bar{\kappa} = O(1)$ in (2.19). Using the derivative transformations

$$\frac{\partial}{\partial \bar{t}} = \frac{\partial}{\partial \bar{t}} + \frac{1}{\epsilon}\frac{\partial}{\partial \tau}, \tag{4.9a}$$

$$\frac{\partial}{\partial \bar{x}} = \frac{\partial}{\partial \bar{x}} + \frac{1}{\epsilon}\frac{\partial}{\partial \xi}, \tag{4.9b}$$

and the perturbation expansion (4.7) in (4.1) yields the equations

$$\begin{aligned}
&\Bigg[\left(\frac{\partial}{\partial \bar{t}} + \frac{1}{\epsilon}\frac{\partial}{\partial \tau}\right) + [\mathbf{A}_0 + (\epsilon \mathbf{\Psi}^{(1)} + \epsilon^2 \mathbf{\Psi}^{(2)} + \ldots) \cdot \nabla_{\mathbf{\Psi}} \mathbf{A} \\
&\quad + \tfrac{1}{2}(\epsilon \mathbf{\Psi}^{(1)} + \epsilon^2 \mathbf{\Psi}^{(2)} + \ldots)(\epsilon \mathbf{\Psi}^{(1)} + \epsilon^2 \mathbf{\Psi}^{(2)} + \ldots) : \nabla_{\mathbf{\Psi}} \nabla_{\mathbf{\Psi}} \mathbf{A} + \ldots] \cdot \left(\frac{\partial}{\partial \bar{x}} + \frac{1}{\epsilon}\frac{\partial}{\partial \xi}\right) \\
&\quad - \left(\frac{\partial}{\partial \bar{x}} + \frac{1}{\epsilon}\frac{\partial}{\partial \xi}\right)\left\{[\hat{\mathbf{K}}_0 + (\epsilon \mathbf{\Psi}^{(1)} + \epsilon^2 \mathbf{\Psi}^{(2)} + \ldots) \cdot \nabla_{\mathbf{\Psi}} \hat{\mathbf{K}} + \ldots] \cdot \left(\frac{\partial}{\partial \bar{x}} + \frac{1}{\epsilon}\frac{\partial}{\partial \xi}\right)\right\}\Bigg] \\
&\quad (\mathbf{\Psi}^{(0)} + \epsilon \mathbf{\Psi}^{(1)} + \epsilon^2 \mathbf{\Psi}^{(2)} + \ldots) = 0.
\end{aligned} \tag{4.10}$$

As in Webb et al. (1997a), the $O(1/\epsilon^2)$ balance of terms in (4.10) is automatically satisfied, since $\mathbf{\Psi}^{(0)}$ is independent of $\xi$. The $O(1/\epsilon)$ balance in (4.10) requires that $\kappa p^1_{c,\xi\xi} = 0$, and since we require $p^1_c$ to be sublinear in $\xi$ for a uniform expansion, it follows that

$$p^1_c = 0 \tag{4.11}$$

is the appropriate solution for $p^1_c$. The balance equations at $O(1)$ may be split up into the equations

$$\frac{\partial \mathbf{\Psi}^{(0)}}{\partial \bar{t}} + \mathbf{A}_0 \cdot \frac{\partial \mathbf{\Psi}^{(0)}}{\partial \bar{x}} - \frac{\partial}{\partial \bar{x}}\left(\hat{\mathbf{K}}_0 \cdot \frac{\partial \mathbf{\Psi}^{(0)}}{\partial \bar{x}}\right) = 0 \tag{4.12}$$

for the non-uniform background state $\mathbf{\Psi}^{(0)}$, and the $O(1)$ balance for the wave perturbations

$$\frac{\partial \mathbf{\Psi}^{(1)}}{\partial \tau} + \mathbf{A}_0 \cdot \frac{\partial \mathbf{\Psi}^{(1)}}{\partial \xi} - \frac{\partial}{\partial \xi}\left[\mathbf{\Psi}^{(1)} \cdot \nabla_{\Psi} \hat{\mathbf{K}} \cdot \left(\frac{\partial \mathbf{\Psi}^{(0)}}{\partial \bar{x}} + \frac{\partial \mathbf{\Psi}^{(1)}}{\partial \xi}\right) + \hat{\mathbf{K}}_0 \cdot \left(2\frac{\partial \mathbf{\Psi}^{(1)}}{\partial \bar{x}} + \frac{\partial \mathbf{\Psi}^{(2)}}{\partial \xi}\right)\right] = 0. \quad (4.13)$$

Taking into account the form of the diffusion term in (4.1), and introducing the state vector

$$\tilde{\mathbf{\Psi}} = (\rho, M_x, M_y, M_z, B_y, B_z, \sigma)^{\mathrm{T}} \quad (4.14)$$

for the MHD background thermal fluid, the first seven equations in (4.13) reduce to:

$$\frac{\partial \tilde{\mathbf{\Psi}}^{(1)}}{\partial \tau} + \mathscr{A}_0 \cdot \frac{\partial \tilde{\mathbf{\Psi}}^{(1)}}{\partial \xi} = 0, \quad (4.15)$$

where the matrix $\mathscr{A}_0 = \mathscr{A}(\mathbf{\Psi}^{(0)})$ is given by (3.32). The eighth equation in (4.13),

$$\gamma_c p_c^0 \frac{\partial u^1}{\partial \xi} - \left(\bar{\kappa}_0 \frac{\partial^2 p_c^2}{\partial \xi^2} + \frac{\partial \bar{\kappa}}{\partial \rho}\frac{\partial \rho^1}{\partial \xi}\frac{\partial p_c^0}{\partial \bar{x}}\right) = 0, \quad (4.16)$$

couples the second-order cosmic ray perturbation $p_c^2$ to $u^1$.

Following the approach of Majda and Rosales (1984) and Webb et al. (1997a), the solutions for $\tilde{\mathbf{\Psi}}^{(1)}$ and $\tilde{\mathbf{\Psi}}^{(2)}$ are expanded in terms of the eigenvector solutions associated with the matrix $\mathscr{A}$ of the form

$$\tilde{\mathbf{\Psi}}^{(1)} = \sum_{s=1}^{7} a_s(\bar{x}, \bar{t}, \theta_s)\mathbf{R}_s, \quad (4.17)$$

$$\tilde{\mathbf{\Psi}}^{(2)} = \sum_{s=1}^{7} b_s(\bar{x}, \bar{t}, \theta_s, \tau)\mathbf{R}_s, \quad (4.18)$$

where $\{a_s\}$ and $\{b_s\}$ determine the wave amplitudes, and

$$\theta_s = k_s \xi - \omega_s \tau, \qquad \lambda_s = \frac{\omega_s}{k_s}, \quad (4.19)$$

are the phase and phase speed of the $s$th wave mode. The matrix $\mathscr{A}$ is given by (3.32). The phase velocities $\{\lambda_s : \ s = 1, \ldots, 7\}$ in (4.19) are the MHD wave eigenvelocities listed in (3.10). The right-eigenvectors $\{\mathbf{R}_s\}$ and left-eigenvectors $\{\mathbf{L}_j\}$ of the matrix $\mathscr{A}$ are given in (3.39)–(3.47).

At $O(\epsilon)$, the first seven equations in (4.10) yield the matrix equation

$$\frac{\partial \tilde{\mathbf{\Psi}}^{(2)}}{\partial \tau} + \mathscr{A}_0 \cdot \frac{\partial \tilde{\mathbf{\Psi}}^{(2)}}{\partial \xi} + \mathbf{G} = 0, \quad (4.20)$$

where

$$\mathbf{G} = \frac{\partial \tilde{\mathbf{\Psi}}^{(1)}}{\partial \bar{t}} + \mathscr{A}_0 \cdot \frac{\partial \tilde{\mathbf{\Psi}}^{(1)}}{\partial \bar{x}} + \mathbf{C}\frac{\partial p_c^2}{\partial \xi} + \tilde{\mathbf{\Psi}}^{(1)} \cdot \nabla_{\tilde{\Psi}} C \frac{\partial p_c^0}{\partial \bar{x}} + \tilde{\mathbf{\Psi}}^{(1)} \cdot \nabla_{\tilde{\Psi}} \mathscr{A} \cdot \left(\frac{\partial \tilde{\mathbf{\Psi}}^{(0)}}{\partial \bar{x}} + \frac{\partial \tilde{\mathbf{\Psi}}^{(1)}}{\partial \xi}\right). \quad (4.21)$$

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The eighth equation in (4.10) at $O(\epsilon)$ is

$$\begin{aligned}\frac{\partial p_c^2}{\partial \tau}+\mathbf{D}\cdot\left(\frac{\partial \mathbf{\Psi}^{(2)}}{\partial \xi}+\frac{\partial \mathbf{\Psi}^{(1)}}{\partial \bar{x}}\right)+\mathbf{\Psi}^{(1)}\cdot\nabla_{\mathbf{\Psi}}\mathbf{D}\cdot\left(\frac{\partial \mathbf{\Psi}^{(0)}}{\partial \bar{x}}+\frac{\partial \mathbf{\Psi}^{(1)}}{\partial \xi}\right)\\ -\frac{\partial}{\partial \bar{x}}\left(\bar{\kappa}\frac{\partial p_c^2}{\partial \xi}+\rho^1\frac{\partial \bar{\kappa}}{\partial \rho}\frac{\partial p_c^0}{\partial \bar{x}}\right)-\frac{\partial}{\partial \xi}\left[\bar{\kappa}\left(\frac{\partial p_c^2}{\partial \bar{x}}+\frac{\partial p_c^3}{\partial \xi}\right)\right.\\ \left.+\rho^1\frac{\partial \bar{\kappa}}{\partial \rho}\frac{\partial p_c^2}{\partial \xi}+\left(\rho^2\frac{\partial \bar{\kappa}}{\partial \rho}+\tfrac{1}{2}(\rho^1)^2\frac{\partial^2 \bar{\kappa}}{\partial \rho^2}\right)\frac{\partial p_c^0}{\partial \bar{x}}\right]=0.\end{aligned}\tag{4.22}$$

The vectors $\mathbf{C}$ and $\mathbf{D}$ in (4.21) and (4.22) are defined in (4.4).

Substituting the solution ansatz (4.17) and (4.18) for $\tilde{\mathbf{\Psi}}^{(1)}$ and $\tilde{\mathbf{\Psi}}^{(2)}$ in (4.20), and multiplying (4.20) on the left by the left-eigenvector $\mathbf{L}_j$, yields the system of equations

$$\frac{\partial b_j(\theta_j,\tau)}{\partial \tau_j}+\mathbf{L}_j\cdot\mathbf{G}=0,\quad j=1,\dots,7,\tag{4.23}$$

where $\partial/\partial\tau_j \equiv (\partial/\partial\tau)|_{\theta_j}$. As in Majda and Rosales (1984) and Webb et al. (1997a), we require that $b_j(\theta_j,\tau)/\tau\to 0$ as $\tau\to\infty$ in order that the perturbation expansion remain uniform for times $\tau=O(1/\epsilon)$. Using this latter condition, and integrating (4.23) with respect to $\tau$, with $\theta_j$ held fixed, yields the averaged evolution equations:

$$\langle \mathbf{L}_j\cdot\mathbf{G}\rangle_j=0,\quad j=1,\dots,7,\tag{4.24}$$

governing the short-wavelength wave modes on the long time scale. The averaging in (4.24) is defined by

$$\langle f\rangle_j=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau} f|_{\theta_j}\,d\tau',\tag{4.25}$$

(i.e. the average defined in (4.24) is carried out with $\theta_j$ held fixed). The averaged wave equations (4.24) are integro-differential evolution equations of the type obtained by Majda and Rosales (1984) describing three-wave resonant interactions for weakly nonlinear hyperbolic waves.

### 4.2. *Wave evolution equations prior to averaging*

The wave evolution equations prior to averaging may be written in the form

$$\frac{\partial b_j(\theta_j,\tau)}{\partial \tau}+T_j^{(1)}+T_j^{(2)}=0,\quad j=1,\dots,7,\tag{4.26}$$

where

$$\begin{aligned}T_j^{(1)}=\mathbf{L}_j\cdot\left(\frac{\partial\tilde{\mathbf{\Psi}}^{(1)}}{\partial \bar{t}}+\mathscr{A}_0\cdot\frac{\partial\tilde{\mathbf{\Psi}}^{(1)}}{\partial \bar{x}}+\mathbf{C}\frac{\partial p_c^2}{\partial \xi}+\tilde{\mathbf{\Psi}}^{(1)}\cdot\nabla_{\tilde{\mathbf{\Psi}}}\mathbf{C}\frac{\partial p_c^0}{\partial \bar{x}}\right.\\ \left.+\tilde{\mathbf{\Psi}}^{(1)}\cdot\nabla_{\tilde{\mathbf{\Psi}}}\mathscr{A}\cdot\frac{\partial\tilde{\mathbf{\Psi}}^{(0)}}{\partial \bar{x}}\right),\end{aligned}\tag{4.27}$$

$$T_j^{(2)}=\mathbf{L}_j\cdot\left(\tilde{\mathbf{\Psi}}^{(1)}\cdot\nabla_{\tilde{\mathbf{\Psi}}}\mathscr{A}\cdot\frac{\partial\tilde{\mathbf{\Psi}}^{(1)}}{\partial \xi}\right).\tag{4.28}$$

Next, note from (4.16) that

$$\frac{\partial p_c^{(2)}}{\partial \xi}=\frac{\rho a_c^2}{\bar{\kappa}}u_x^1-\frac{\zeta}{\rho}\frac{\partial p_c^0}{\partial \bar{x}}\rho^1\equiv\sum_{s=1}^{7}\left[\frac{\rho a_c^2}{\bar{\kappa}}(R_s^2-u_xR_s^1)-\frac{\zeta}{\rho}\frac{\partial p_c^0}{\partial \bar{x}}R_s^1\right]a_s,\tag{4.29}$$

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where $\zeta \equiv \partial \ln \kappa / \partial \ln \rho$ (see (2.24)) and $R_s^j$ denotes the $j$th component of $\mathbf{R}_s$. Using (4.29) and the eigenexpansion (4.17) for $\tilde{\mathbf{\Psi}}^{(1)}$ in (4.27), we obtain

$$T_j^{(1)} = \frac{\partial a_j}{\partial t} + \frac{\partial}{\partial \bar{x}}(\lambda_j a_j) + \sum_{s=1}^{7} \Lambda_{js} a_s \tag{4.30}$$

for the linear wave mixing term (4.27), where

$$\Lambda_{js} = -\frac{\partial \lambda_s}{\partial \bar{x}} \delta^j{}_s + \mathbf{L}_j \cdot \left[ \frac{\partial \mathbf{R}_s}{\partial \bar{t}} + \lambda_j \frac{\partial \mathbf{R}_s}{\partial \bar{x}} + \mathbf{R}_s \cdot \nabla_{\tilde{\mathbf{\Psi}}} \mathscr{A} \cdot \frac{\partial \tilde{\mathbf{\Psi}}^{(0)}}{\partial \bar{x}} \right.$$
$$\left. + \mathbf{R}_s \cdot \nabla_{\tilde{\mathbf{\Psi}}} \mathbf{C} \frac{\partial p_c^0}{\partial \bar{x}} + \mathbf{C} \left( \frac{a_c^2}{\bar{\kappa}} (R_s^2 - u_x R_s^1) - \frac{\zeta}{\rho} \frac{\partial p_c^0}{\partial \bar{x}} R_s^1 \right) \right] \tag{4.31}$$

are the wave mixing coefficients. The nonlinear wave–wave interaction term (4.28) may be written in the form

$$T_j^{(2)} = \sum_{p=1}^{7} \sum_{q=1}^{7} \Gamma_{jpq} a_q \frac{\partial a_p}{\partial \xi}, \tag{4.32}$$

where the interaction coefficients $\Gamma_{jpq}$ are given by

$$\Gamma_{jpq} = \mathbf{L}_j \cdot (\mathbf{R}_q \cdot \nabla_{\mathbf{\Psi}} \mathscr{A} \cdot \mathbf{R}_p). \tag{4.33}$$

Noting that the MHD fluxes $F^\alpha$ in (2.18) have continuous second partial derivatives, and, using $\mathscr{A}^{\alpha\beta} = \partial F^\alpha / \partial \tilde{\Psi}^\beta$ ((4.3)), one finds that the nonlinear interaction coefficients $\Gamma_{jpq}$ in (4.32) and (4.33) are symmetric in the indices $pq$. The $\Gamma_{jpq}$ are in general asymmetric if one does not use the conserved densities state vector (3.2) in the perturbation analysis. In particular, the wave interaction coefficients $\Gamma_{jpq}$ obtained in Webb et al. (1997a) are non-symmetric in the last two indices $p$ and $q$. However, after averaging (4.23) to obtain (4.24), many of the nonlinear terms have a zero average, and the remaining terms can be described by wave interaction coefficients $\hat{\Gamma}_{jpq}$, which are symmetric in the last two indices $p$ and $q$.

By differentiating the right-eigenvector equation

$$(\mathscr{A} - \lambda_p \mathbf{I}) \cdot \mathbf{R}_p = 0 \tag{4.34}$$

in the directions of the right-eigenvectors, the interaction coefficients $\Gamma_{jpq}$ in (4.33) can be written in the form

$$\Gamma_{jpq} = \delta^j{}_p \mathbf{R}_q \cdot \nabla_{\mathbf{\Psi}} \lambda_p + (\lambda_p - \lambda_j) \mathbf{L}_j \cdot (\mathbf{R}_q \cdot \nabla_{\mathbf{\Psi}} \mathbf{R}_p). \tag{4.35}$$

From (4.35), we find

$$\Gamma_{jjj} = \mathbf{R}_j \cdot \nabla_{\mathbf{\Psi}} \lambda_j, \tag{4.36a}$$

$$\Gamma_{jjq} = \mathbf{R}_q \cdot \nabla_{\mathbf{\Psi}} \lambda_j, \quad q \neq j, \tag{4.36b}$$

for the nonlinear interaction of the $j$th wave with itself and with the $q$th mode ($q \neq j$).

Using the fact that $\mathscr{A}^{ij} = \partial F^i / \partial \Psi^j$, the term involving $\nabla_{\tilde{\mathbf{\Psi}}} \mathscr{A}$ in (4.31) may be written in the form

$$\mathbf{L}_j \cdot (\mathbf{R}_s \cdot \nabla_{\tilde{\mathbf{\Psi}}} \mathscr{A}) \cdot \frac{\partial \tilde{\mathbf{\Psi}}^{(0)}}{\partial \bar{x}} = \frac{\partial \lambda_s}{\partial \bar{x}} \delta^j{}_s + (\lambda_s - \lambda_j) \mathbf{L}_j \cdot \frac{\partial \mathbf{R}_s}{\partial \bar{x}}. \tag{4.37}$$

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Using (4.37) and noting that $C^i = \delta^i{}_2$ (see (4.3)), in (4.31) results in the expressions

$$\Lambda_{js} = \mathbf{L}_j \cdot \left(\frac{\partial \mathbf{R}_s}{\partial \bar{t}} + \lambda_s \frac{\partial \mathbf{R}_s}{\partial \bar{x}}\right) + L_j^2 \left[(R_s^2 - u_x R_s^1)\frac{a_c^2}{\kappa} - \frac{\zeta}{\rho}\frac{\partial p_c}{\partial \bar{x}} R_s^1\right] \tag{4.38}$$

for the wave mixing coefficients. The relatively simple form of the wave mixing coefficients in (4.38) is a consequence of using the conserved densities state vector $\tilde{\mathbf{\Psi}}$ of (3.2). The formulae (4.38) are central results of this paper, and are used to determine the wave mixing coefficients in Sec. 6.

In the pure MHD case, when there are no cosmic ray effects (i.e. if $p_c \equiv 0$), the wave mixing coefficients have the form

$$\Lambda_{js} = \mathbf{L}_j \cdot \frac{d\mathbf{R}_s}{dt_s}, \tag{4.39}$$

where

$$\frac{d}{dt_s} = \frac{\partial}{\partial \bar{t}} + \lambda_s \frac{\partial}{\partial \bar{x}} \tag{4.40}$$

is the time derivative along the $s$th wave mode characteristic. If the background flow is an MHD simple wave associated with $\lambda = \lambda_p$ then the background state vector $\tilde{\mathbf{\Psi}}^{(0)} \equiv \tilde{\mathbf{\Psi}}^{(0)}(\varphi)$ depends only on the simple wave phase $\varphi$ satisfying the unidirectional wave equation

$$\frac{\partial \varphi}{\partial \bar{t}} + \lambda_p \frac{\partial \varphi}{\partial \bar{x}} = 0 \tag{4.41}$$

(see e.g. Boillat 1970; Webb et al. 1998). Because the eigenvectors depend only on $\varphi$, we find

$$\Lambda_{js} = \mathbf{L}_j \cdot \frac{d\mathbf{R}_s}{d\varphi}\left(\frac{\partial \varphi}{\partial \bar{t}} + \lambda_s \frac{\partial \varphi}{\partial \bar{x}}\right) \equiv (\lambda_s - \lambda_p)\mathbf{L}_j \cdot \frac{d\mathbf{R}_s}{d\varphi}\frac{\partial \varphi}{\partial \bar{x}}. \tag{4.42}$$

Thus if the background flow is a simple wave associated with the $p$th eigenmode then $\Lambda_{jp} = 0$, and the small-amplitude waves of the $p$th family do not contribute to the evolution of the other small-amplitude wave modes with $\lambda_s \neq \lambda_p$.

### 4.3. *Averaged wave evolution equations*

From Webb et al. (1997a), the wave evolution equations (4.24) after averaging may be written in the form

$$\frac{\partial a_j}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\lambda_j a_j) + \Lambda_{jj} a_j + {\sum_s}' \Lambda_{js}\langle a_s\rangle + \langle T_j^{(2)}\rangle_j = 0, \quad j = 1, \ldots, 7, \tag{4.43}$$

where ${\sum'}_s$ denotes summation over the index $s$ for all modes except $s = j$. The mean wave amplitudes $\langle a_s\rangle$ depend only on the slow variables $\bar{x}$ and $\bar{t}$. The wave interaction coefficient $\Lambda_{jj}$ used above differs from that used in Webb et al. (1997a), where the wave interaction equations equivalent to (4.43) were written in a slightly different form. The mean nonlinear interaction term $\langle T_j^{(2)}\rangle_j$ can be expressed in both a non-conservative form

$$\langle T_j^{(2)}\rangle_j = \tilde{\Gamma}_{jjj} a_j \frac{\partial a_j}{\partial \theta_j} + {\sum_q}' \tilde{\Gamma}_{jjq}\langle a_q\rangle \frac{\partial a_j}{\partial \theta_j} + {\sum_{p<q}}'{\sum_q}' \tilde{\Gamma}_{jpq}\langle a_q a_p'\rangle_j \tag{4.44}$$

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and a conservative form

$$\langle T_j^{(2)}\rangle_j = k_j \frac{\partial}{\partial\theta_j}\left(\Gamma_{jjj}\frac{a_j^2}{2} + {\sum_q}' \Gamma_{jjq}\langle a_q\rangle a_j + {\sum_{p<q}}'{\sum_q}' \hat{\Gamma}_{jpq}\langle a_q a_p\rangle_j\right), \qquad (4.45)$$

where the interaction coefficients $\tilde{\Gamma}_{jpq}$, $\bar{\Gamma}_{jpq}$ and $\hat{\Gamma}_{jpq}$ are given by

$$\tilde{\Gamma}_{jpq} = k_p\Gamma_{jpq}, \qquad (4.46a)$$

$$\bar{\Gamma}_{jpq} = \tilde{\Gamma}_{jpq} - \mu_{pjq}\tilde{\Gamma}_{jqp} \equiv k_p(\lambda_p - \lambda_j)\mathbf{L}_j\cdot[\mathbf{R}_q, \mathbf{R}_p], \qquad (4.46b)$$

$$\hat{\Gamma}_{jpq} = \frac{1}{k_j}(\mu_{jqp}\tilde{\Gamma}_{jpq} + \mu_{jpq}\tilde{\Gamma}_{jqp}) \equiv \frac{(\lambda_q - \lambda_j)(\lambda_p - \lambda_j)}{(\lambda_q - \lambda_p)}\mathbf{L}_j\cdot[\mathbf{R}_q, \mathbf{R}_p], \qquad (4.46c)$$

and

$$[\mathbf{R}_q, \mathbf{R}_p] = \mathbf{R}_q \cdot \nabla_\Psi \mathbf{R}_p - \mathbf{R}_p \cdot \nabla_\Psi \mathbf{R}_q. \qquad (4.47)$$

In (4.44) and (4.45) we use the notation $a_p' = \partial a_p(\theta_p)/\partial\theta_p$, and assume that $\lambda_p \neq \lambda_q$. Note that the coefficients $\Gamma_{jjq}$ are given by (4.36).

In (4.46), the coefficients $\{\mu_{jpq}\}$ are given by

$$\mu_{jpq} = \frac{k_j(\lambda_j - \lambda_p)}{k_q(\lambda_q - \lambda_p)} \equiv \frac{\nu_{pj}}{\nu_{pq}}, \qquad (4.48a)$$

$$\nu_{pq} = (\lambda_p - \lambda_q)k_p k_q. \qquad (4.48b)$$

For three-wave resonant interactions of the $j$th, $p$th and $q$th wave modes ($j \neq p \neq q$), the resonance conditions may be written in the form

$$\theta_j = \mu_{jpq}\theta_q + \mu_{jqp}\theta_p \qquad (4.49)$$

(see e.g. Anile et al. 1993; Webb et al. 1997a; Ali and Hunter 1998). Since $\theta_j = k_j\xi - \omega_j\tau$, (4.49) imply the equivalent equations

$$\omega_j = \mu_{jpq}\omega_q + \mu_{jqp}\omega_p, \qquad (4.50a)$$

$$k_j = \mu_{jpq}k_q + \mu_{jqp}k_p, \qquad (4.50b)$$

where

$$\omega_j = k_j\lambda_j, \qquad \omega_p = k_p\lambda_p, \qquad \omega_q = k_q\lambda_q, \qquad (4.51)$$

are the dispersion relations for the waves (see (4.19)). The condition for periodic hyperbolic waves with frequencies and wavenumbers $(\omega_p, k_p)$ and $(\omega_q, k_q)$ to resonantly interact in (4.49) is that $\mu_{jpq}$ and $\mu_{jqp}$ should be rational.

The means $\langle a_q a_p'\rangle_j$ in (4.45) are defined by

$$\langle a_q a_p'\rangle_j = \lim_{T\to\infty}\frac{1}{T}\int_0^T a_q\left(\frac{k_q\theta_j + \nu_{jq}\tau}{k_j}\right)a_p'\left(\frac{k_p\theta_j + \nu_{jp}\tau}{k_j}\right)d\tau$$
$$\equiv \lim_{|T_{jp}|\to\infty}\frac{1}{T_{jp}}\int_{\alpha_{jp}}^{\alpha_{jp}+T_{jp}} a_q\left(\frac{\nu_{qp}\theta_j + \nu_{jq}\theta_p}{\nu_{jp}}\right)a_p'(\theta_p)\,d\theta_p, \qquad (4.52)$$

where

$$T_{jp} = \frac{T\nu_{jp}}{k_j}, \qquad \alpha_{jp} = \frac{k_p\theta_j}{k_j}, \qquad \theta_j = k_j\xi - \omega_j\tau. \qquad (4.53)$$

The means $\langle a_q a_p\rangle$ are defined similarly, but with $a_p'$ replaced by $a_p$ in (4.52). The double summations over $p$ and $q$ in (4.44) and (4.45) are over all wave modes $p$, $q$ with $p < q$ and $p \neq q \neq j$. The nonlinear wave interaction terms in (4.44) and (4.45) split into three types, namely the interaction of the $j$th wave with itself (the $\Gamma_{jjj}$ term), the interaction of the $j$th wave with the mean wave field of the other waves (the $\Gamma_{jjq}$ terms), and the three-wave resonant interaction terms (the $\hat{\Gamma}_{jpq}$ terms). In (4.52), the phase $\theta_q$ of $a_q$ is given by

$$\theta_q = \mu_{qpj}\theta_j + \mu_{qjp}\theta_p, \tag{4.54}$$

which is precisely the condition for three-wave resonant interaction.

The conservative wave–wave interaction coefficients $\hat{\Gamma}_{jpq}$ in (4.46) are symmetric in the indices $p$ and $q$. If one uses the state vector $\tilde{\mathbf{\Psi}}$ of (3.2) in the perturbation analysis then the interaction coefficients $\{\Gamma_{jpq}\}$ are also symmetric in the indices $p$ and $q$. In this latter case, (4.46) yields

$$\hat{\Gamma}_{jpq} = \Gamma_{jpq}. \tag{4.55}$$

However, if one uses a different state vector, for example $\tilde{\mathbf{\Psi}}' = (\rho, \mathbf{u}^{\mathrm{T}}, B_y, B_z, S)^{\mathrm{T}}$, then the interaction coefficients $\Gamma'_{jpq}$ calculated using (4.33) are in general asymmetric in the indices $p$ and $q$ (see also Webb et al. (1997a) for the gas dynamical case). In Appendix A, it is shown that both the wave mixing coefficients $\{\Lambda_{js}\}$ and the nonlinear wave interaction coefficients $\Gamma_{jjq}$ and $\hat{\Gamma}_{jpq}$ are invariant under a change of state vector of the form $\tilde{\mathbf{\Psi}}' = \mathbf{\Phi}(\tilde{\mathbf{\Psi}})$, provided that the wave amplitudes $a_s$ are chosen to remain the same (i.e. $a'_s = a_s$).

## 5. Nonlinear and resonant wave interactions

An early discussion of three-wave resonant interactions in MHD was given by Sagdeev and Galeev (1969). More recently, Ali and Hunter (1998) have developed three-wave resonant interaction equations for magnetohydrodynamic waves in one Cartesian space dimension. In the present section, we discuss how the Ali and Hunter equations arise in the present development. We also discuss the Alfvén wave decay instability (see e.g. Sagdeev and Galeev, 1969) due to the three-wave resonant interaction between two Alfvén waves and a slow magnetoacoustic wave in a low-beta plasma.

### *5.1. Wave interaction equations*

From (4.43) and (4.45), the averaged wave evolution equations (4.43) may be written in the form

$$\begin{aligned} \frac{\partial a_j}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\lambda_j a_j) + \Lambda_{jj} a_j + {\sum_s}' \Lambda_{js} \langle a_s \rangle \\ + k_j \frac{\partial}{\partial \theta_j}\left(\Gamma_{jjj}\frac{a_j^2}{2} + {\sum_q}' \Gamma_{jjq}\langle a_q \rangle a_j + {\sum_{p<q}}' {\sum_q}' \hat{\Gamma}_{jpq}\langle a_q a_p \rangle_j\right) = 0, \end{aligned} \tag{5.1}$$

where $1 \leqslant j \leqslant 7$. The equations given in Ali and Hunter (1998) are equivalent to (5.1), when the wave mixing coefficients $\Lambda_{jj}$ and $\Lambda_{js}$ are set equal to zero, and the mean wave interaction terms (the $\Gamma_{jjq}$ terms) are set equal to zero. In this case,

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(5.1) reduces to

$$\frac{\partial a_j}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\lambda_j a_j) + k_j \frac{\partial}{\partial \theta_j}\left(\Gamma_{jjj}\frac{a_j^2}{2} + \sum_{p<q}{}^{\prime}\sum_{q}{}^{\prime}\hat{\Gamma}_{jpq}\langle a_q a_p\rangle_j\right) = 0, \quad j = 1, \ldots, 7, \quad (5.2)$$

In the present discussion, we emphasize the symmetry properties of the interaction coefficients, and consider the more general case where the waves have non-zero means. Our interaction coefficients reduce to those of Ali and Hunter when we use their wave amplitudes and eigenvectors.

In (5.1) and (5.2), the nonlinear interaction processes consist of the Burgers self-wave interactions (the $\Gamma_{jjj}$ terms), mean wave field interactions (the $\Gamma_{jjq}$ terms with $q \neq j$) and three-wave resonant interactions (the $\hat{\Gamma}_{jpq}$ terms). As noted in (4.54) et seq., the interaction coefficients may be determined by using either the conserved densities state vector $\tilde{\mathbf{\Psi}}$ in (3.2) or another appropriate state vector (e.g. $\tilde{\mathbf{\Psi}}' = (\rho, \mathbf{u}^{\mathrm{T}}, B_y, B_z, S)^{\mathrm{T}}$) in the analysis.

The Burgers interaction coefficients $\{\Gamma_{jjj}\}$ and mean wave interaction coefficients $\{\Gamma_{jjq} : \ q \neq j\}$ may be determined from (4.33) or (4.36). Using the computational algebra language Maple (see e.g. Char et al. 1992), the Burgers self wave interaction coefficients $\{\Gamma_{jjj}\}$ may be written in the form

$$\Gamma_{111} = -\frac{\alpha_f c_f}{\rho}\beta_f, \qquad \Gamma_{222} = 0, \qquad \Gamma_{333} = -\frac{\alpha_s c_s}{\rho}\beta_s, \quad (5.3a)$$

$$\Gamma_{444} = 0, \qquad \Gamma_{555} = -\Gamma_{333}, \qquad \Gamma_{666} = 0, \qquad \Gamma_{777} = -\Gamma_{111}, \quad (5.3b)$$

where

$$\beta_f = \tfrac{1}{2}[(\gamma_g + 1)\alpha_f^2 + 3\alpha_s^2], \qquad \beta_s = \tfrac{1}{2}[(\gamma_g + 1)\alpha_s^2 + 3\alpha_f^2], \quad (5.4a)$$

$$\alpha_f = \left(\frac{a_g^2 - c_s^2}{c_f^2 - c_s^2}\right)^{1/2}, \qquad \alpha_s = \left(\frac{c_f^2 - a_g^2}{c_f^2 - c_s^2}\right)^{1/2}. \quad (5.4b)$$

Note that the compressive magnetoacoustic, self wave interaction coefficients $\Gamma_{111}$, $\Gamma_{333}$, $\Gamma_{555}$ and $\Gamma_{777}$ are non-zero, whereas the Alfvén and entropy wave coefficients $\Gamma_{222}$, $\Gamma_{666}$ and $\Gamma_{444}$ are zero. One can show that the interaction coefficients $\{\Gamma_{jjp}\}$ are antisymmetric with respect to reversal of the wave speeds, namely

$$\Gamma_{jjp} = -\Gamma_{j'j'p'}, \quad j' = 8 - j, \quad p' = 8 - p, \quad (5.5)$$

where $1 \leqslant j \leqslant 7$ and $1 \leqslant p \leqslant 7$.

The non-zero mean wave field interaction coefficients $\{\Gamma_{jjp} : \ p \neq j\}$ for $j \leqslant 4$ are given by the equations

$$\Gamma_{113} = -\frac{\alpha_s}{\rho}[(\beta_f - 2)c_f + c_s], \quad \Gamma_{114} = \frac{c_f}{2\rho}, \quad (5.6a)$$

$$\Gamma_{115} = -\frac{\alpha_s}{\rho}[(\beta_f - 2)c_f - c_s], \quad \Gamma_{117} = -\frac{\alpha_f c_f(\beta_f - 2)}{\rho}, \quad (5.6b)$$

$$\Gamma_{221} = \frac{\alpha_f(|b_x| - 2c_f)}{2\rho}, \quad \Gamma_{223} = \frac{\alpha_s(|b_x| - 2c_s)}{2\rho}, \quad (5.6c)$$

$$\Gamma_{224} = \frac{|b_x|}{2\rho}, \quad \Gamma_{225} = \frac{\alpha_s(|b_x| + 2c_s)}{2\rho}, \quad \Gamma_{227} = \frac{\alpha_f(|b_x| + 2c_f)}{2\rho}, \quad (5.6d)$$

$$\Gamma_{331} = -\frac{\alpha_f}{\rho}[(\beta_s - 2)c_s + c_f], \quad \Gamma_{334} = \frac{c_s}{2\rho}, \quad (5.6e)$$

$$\Gamma_{335} = -\frac{\alpha_s c_s(\beta_s - 2)}{\rho}, \qquad \Gamma_{337} = -\frac{\alpha_f}{\rho}[(\beta_s - 2)c_s - c_f], \tag{5.6f}$$

$$\Gamma_{441} = -\frac{\alpha_f c_f}{\rho}, \qquad \Gamma_{443} = -\frac{\alpha_s c_s}{\rho}, \tag{5.6g}$$

$$\Gamma_{445} = \frac{\alpha_s c_s}{\rho}, \qquad \Gamma_{447} = \frac{\alpha_f c_f}{\rho}. \tag{5.6h}$$

For $j > 4$, the non-zero mean wave interaction coefficients may be determined by using the results (5.6) in conjunction with the symmetry relations (5.5).

The three-wave resonant interaction coefficients $\{\hat{\Gamma}_{jpq}\}$ may be determined from (4.46). An alternative expression for the $\hat{\Gamma}_{jpq}$, using (4.46), is

$$\hat{\Gamma}_{jpq} = \frac{\lambda_j - \lambda_q}{\lambda_p - \lambda_q}\Gamma_{jpq} + \frac{\lambda_p - \lambda_j}{\lambda_p - \lambda_q}\Gamma_{jqp}, \tag{5.7}$$

where $\Gamma_{jpq}$ is given by (4.33). Using the result (5.7) and (4.33), the non-zero three-wave resonant interaction coefficients for $j < 4$ are given by the equations

$$\hat{\Gamma}_{126} = -\frac{\alpha_f c_f b_0^2}{2a_g^2}, \qquad \hat{\Gamma}_{135} = -\frac{\alpha_f c_f(\beta_s - 1)}{\rho}, \qquad \hat{\Gamma}_{137} = -\frac{\alpha_s c_f(\beta_f - 1)}{\rho}, \tag{5.8a}$$

$$\hat{\Gamma}_{147} = -\frac{c_f}{2\rho}, \qquad \hat{\Gamma}_{157} = -\frac{\alpha_s c_f(\beta_f - 1)}{\rho}, \tag{5.8b}$$

$$\hat{\Gamma}_{216} = -\frac{\alpha_f |b_x|}{2\rho}, \qquad \hat{\Gamma}_{236} = -\frac{\alpha_s |b_x|}{2\rho}, \qquad \hat{\Gamma}_{246} = -\frac{|b_x|}{2\rho}, \tag{5.8c}$$

$$\hat{\Gamma}_{256} = -\frac{\alpha_s |b_x|}{2\rho}, \qquad \hat{\Gamma}_{267} = -\frac{\alpha_f |b_x|}{2\rho}, \tag{5.8d}$$

$$\hat{\Gamma}_{315} = -\frac{\alpha_f c_s(\beta_s - 1)}{\rho}, \qquad \hat{\Gamma}_{317} = -\frac{\alpha_s c_s(\beta_f - 1)}{\rho}, \qquad \hat{\Gamma}_{326} = -\frac{\alpha_s c_s b_0^2}{2a_g^2}, \tag{5.8e}$$

$$\hat{\Gamma}_{345} = -\frac{c_s}{2\rho}, \qquad \hat{\Gamma}_{357} = -\frac{\alpha_f c_s(\beta_s - 1)}{\rho}. \tag{5.8f}$$

The resonant wave interaction coefficients $\hat{\Gamma}_{4pq}$ in the entropy wave evolution equation are all zero. The non-zero resonant wave interaction coefficients $\{\hat{\Gamma}_{jpq}\}$ for $j > 4$ can be obtained by using (5.8) and the symmetry relations

$$\hat{\Gamma}_{jpq} = -\hat{\Gamma}_{j'p'q'}, \quad j' = 8 - j, \quad p' = 8 - p, \quad q' = 8 - q, \tag{5.9}$$

associated with interchanging the roles of the backward and forward waves. The non-conservative, asymmetric interaction coefficients $\bar{\Gamma}_{jpq}$ in (4.46) may be obtained by using the relations

$$\bar{\Gamma}_{jpq} = \frac{k_p(\lambda_p - \lambda_q)}{\lambda_j - \lambda_q}\hat{\Gamma}_{jpq} \tag{5.10}$$

and the results (5.8) and (5.9) for $\hat{\Gamma}_{jpq}$. The resonant wave interaction equations presented in Ali and Hunter (1998) were written in terms of the asymmetric interaction coefficients $\bar{\Gamma}_{jpq}$.

### 5.2. *Examples of three-wave resonant interactions*

There are a variety of resonant triads governed by the wave interaction equations. In order for three-wave resonant interactions to occur, two conditions must apply, namely the resonant wave interaction coefficients $\hat{\Gamma}_{jpq}$ must be non-zero

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and the resonance conditions (4.50) for the wave triad must be satisfied. Ali and Hunter (1998) point out that only four types of resonant triads are possible: slow–fast magnetoacoustic wave interaction; Alfvén–magnetoacoustic wave interaction; Alfvén–entropy wave interaction; and magnetoacoustic–entropy wave interaction. For the case of resonant periodic waves, the coefficients $\{\mu_{jpq}\}$ in the resonance conditions (4.49) and (4.50) must be rational numbers, i.e.

$$\omega_j = m\omega_q + n\omega_p, \qquad k_j = mk_q + nk_p, \qquad \mu_{jpq} = m, \qquad \mu_{jqp} = n, \tag{5.11}$$

where $m$ and $n$ are rational numbers. The simplest resonance conditions are obtained by taking $m$ and $n$ to be integers, and the wave amplitudes $\{a_s\}$ are taken to be $2\pi$-periodic functions of the wave phases $\theta_s$. We discuss two resonant triad examples below. We restrict our discussion to the pure MHD case, in which there are no cosmic ray effects, and assume a constant background state.

*5.2.1. Alfvén–magnetoacoustic wave interaction.* As an example of resonant wave interactions, consider the interaction of the backward and forward Alfvén waves with the forward slow magnetoacoustic wave in a low-beta plasma, in which $b_x > a_g$. We first consider the strictly hyperbolic case when all the wave speeds are distinct, and then consider the degenerate case when $\mathbf{B} = (B_x, 0, 0)^{\mathrm{T}}$, in which the Alfvén and fast magnetoacoustic speeds coincide.

(a) *The strictly hyperbolic case*. From (5.11), the tri-resonance condition occurs when

$$\omega_2 = m\omega_6 + n\omega_5, \qquad k_2 = mk_6 + nk_5. \tag{5.12}$$

Using (4.48b)–(4.51), the resonance conditions (5.12) are satisfied for

$$k_2 = -\frac{(b_x - c_s)m}{b_x + c_s}k_6, \qquad k_5 = -\frac{2b_x m}{(b_x + c_s)n}k_6. \tag{5.13}$$

In particular, for the case $m = -1$ and $n = 1$, we have

$$k_2 = \frac{b_x - c_s}{b_x + c_s}k_6, \qquad k_5 = \frac{2b_x}{b_x + c_s}k_6, \qquad \omega_2 = \omega_5 - \omega_6. \tag{5.14}$$

Galeev and Sagdeev (1969) have considered a similar example, in which a large-amplitude Alfvén wave (with frequency $\omega_6$), interacts resonantly with a small-amplitude sound wave (with frequency $\omega_5$), and with a small-amplitude Alfvén wave (of frequency $\omega_2$). Because $b_x > c_s$, the resonance conditions (5.14) imply that $\omega_6 > \omega_5 > 0$ and $\omega_6 > |\omega_2|$ (note that $\omega_2 < 0$). Under these circumstances, the backward Alfvén wave ($a_2$) and the forward slow magnetoacoustic wave ($a_5$) are subject to an instability (Galeev and Sagdeev 1969). For the resonant interaction case described by (5.12)–(5.14), the wave interaction equations (5.1) reduce to

$$\frac{\partial a_2}{\partial t} - b_x\frac{\partial a_2}{\partial x} + k_2\frac{\partial}{\partial \theta_2}\left(\frac{\alpha_s(b_x + 2c_s)}{2\rho}\langle a_5\rangle a_2 - \frac{\alpha_s b_x}{2\rho}\langle a_5 a_6\rangle_2\right) = 0, \tag{5.15}$$

$$\frac{\partial a_5}{\partial t} + c_s\frac{\partial a_5}{\partial x} + k_5\frac{\partial}{\partial \theta_5}\left(\frac{\alpha_s c_s \beta_s}{2\rho}a_5^2 + \frac{\alpha_s c_s b_0^2}{2a_g^2}\langle a_2 a_6\rangle_5\right) = 0, \tag{5.16}$$

$$\frac{\partial a_6}{\partial t} + b_x\frac{\partial a_6}{\partial x} + k_6\frac{\partial}{\partial \theta_6}\left[\frac{\alpha_s(2c_s - b_x)}{2\rho}\langle a_5\rangle a_6 + \frac{\alpha_s b_x}{2\rho}\langle a_5 a_2\rangle_6\right] = 0, \tag{5.17}$$

where we assume a constant background state, with $u_x = 0$. Using (3.56) and (3.57), we obtain the equations

$$B_{2y} = \beta_z a_2, \qquad B_{2z} = -\beta_y a_2, \tag{5.18a}$$

$$B_{6y} = \beta_z a_6, \qquad B_{6z} = -\beta_y a_6, \tag{5.18b}$$

$$\rho_5 = \alpha_s a_5 \tag{5.18c}$$

relating the magnetic field perturbations $\mathbf{B}_2$ and $\mathbf{B}_6$ for the backward and forward Alfvén waves and the density perturbation $\rho_5$ for the slow magnetoacoustic wave to the wave amplitudes $a_2$, $a_5$ and $a_6$.

Equations (5.15)–(5.17) show that the waves are coupled to each other, both by the mean wave field of the slow magnetoacoustic wave and also via the resonant wave interactions (the $\langle a_p a_q\rangle_j$ terms). The Burgers self wave interaction term ($a_5^2$) for the slow mode wave in (5.16) results in the generation of higher-order harmonics as the wave steepens, and considerably complicates the nature of the wave interactions. However, if one considers the initial value problem for (5.15)–(5.17) in which $a_5 = 0$, $a_2 \neq 0$ and $a_6 \neq 0$ at time $t = 0$, and assuming $\langle a_5\rangle = 0$, then at early times (5.15)–(5.17) may be approximated by the three-wave resonant interaction equations

$$\frac{\partial a_2}{\partial t} - b_x \frac{\partial a_2}{\partial x} - k_2 \frac{\partial}{\partial \theta_2}\left(\frac{\alpha_s b_x}{2\rho}\langle a_5 a_6\rangle_2\right) = 0, \tag{5.19}$$

$$\frac{\partial a_5}{\partial t} + c_s \frac{\partial a_5}{\partial x} + k_5 \frac{\partial}{\partial \theta_5}\left(\frac{\alpha_s c_s b_0^2}{2a_g^2}\langle a_2 a_6\rangle_5\right) = 0, \tag{5.20}$$

$$\frac{\partial a_6}{\partial t} + b_x \frac{\partial a_6}{\partial x} + k_6 \frac{\partial}{\partial \theta_6}\left(\frac{\alpha_s b_x}{2\rho}\langle a_5 a_2\rangle_6\right) = 0. \tag{5.21}$$

To analyse (5.19)–(5.21) further, we use the phase representation

$$a_j = A_j \exp(i\theta_j) + A_j^* \exp(-i\theta_j), \quad j = 2, 5, 6, \tag{5.22}$$

for the wave perturbations, where the complex wave amplitudes $A_j$ depend only on the slow variables $x$ and $t$. Using the representation (5.22) now allows the determination of the resonant interaction terms $\langle a_p a_q\rangle_j$ in terms of the wave amplitudes $A_s$. Thus, for example,

$$\begin{aligned}\langle a_5 a_6\rangle_2 &= \frac{1}{2\pi}\int_0^{2\pi} a_5(\theta_2+\theta_6)a_6(\theta_6)\,d\theta_6\\ &= A_5 A_6^* \exp(i\theta_2) + A_5^* A_6 \exp(-i\theta_2),\end{aligned} \tag{5.23}$$

where we have used the resonance relation $\theta_5 = \theta_2 + \theta_6$ for the wave phases. Using similar results for $\langle a_2 a_6\rangle_5$ and $\langle a_5 a_2\rangle_6$ in (5.19)–(5.21) yields the three-wave resonant interaction equations

$$\frac{\partial A_2}{\partial t} - b_x \frac{\partial A_2}{\partial x} - ik_2 \frac{\alpha_s b_x}{2\rho} A_5 A_6^* = 0, \tag{5.24}$$

$$\frac{\partial A_5}{\partial t} + c_s \frac{\partial A_5}{\partial x} + ik_5 \frac{\alpha_s c_s b_0^2}{2a_g^2} A_2 A_6 = 0, \tag{5.25}$$

$$\frac{\partial A_6}{\partial t} + b_x \frac{\partial A_6}{\partial x} + ik_6 \frac{\alpha_s b_x}{2\rho} A_5 A_2^* = 0 \tag{5.26}$$

for the complex wave amplitudes $\{A_j\}$.

The three-wave resonant interaction equations (5.24)–(5.26) (or modified versions thereof) have received considerable attention in the mathematical physics literature, since the equations comprise an integrable Hamiltonian system, in which the initial value problem may be solved by the inverse scattering method. In the present application, we are interested in the evolution of the waves for the case where $A_5 = 0$ at time $t = 0$. At early times, the resonant interaction term $A_5 A_2^*$ in (5.26) may be neglected, and (5.26) has a general solution of the form $A_6 = A_6(x + b_x t)$. Taking $A_6 = \text{const}$, (5.24) and (5.25) may be reduced to the linear wave equation

$$\left[\left(\frac{\partial}{\partial t} + c_s \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - b_x \frac{\partial}{\partial x}\right) - k_2 k_5 \frac{\alpha_s^2 c_s b_0^2 b_x}{4\rho a_g^2} |A_6|^2\right] A_2 = 0. \tag{5.27}$$

Equation (5.27) admits solutions of the form $A_2 \propto \exp[i(Kx - \nu t)]$, provided that $\nu$ satisfies the dispersion relation

$$\nu^2 + K(b_x - c_s)\nu - K^2 b_x c_s + \zeta |A_6|^2 = 0. \tag{5.28}$$

In (5.28), the parameter $\zeta$ is given by

$$\zeta = \frac{b_x(b_x - c_s)\alpha_s^2 b_x c_s}{2(b_x + c_s)^2 \rho a_g^2 \mu} k_6^2, \tag{5.29}$$

where we have used the resonance relations (5.14), and $\mu$ is the magnetic permeability. Equation (5.28) has complex solutions for $\nu$ for long-wavelength waves with wavenumbers

$$K^2 < \frac{4\zeta |A_6|^2}{(b_x + c_s)^2}. \tag{5.30}$$

Hence long-wavelength waves for $A_2$ and $A_5$ satisfying (5.30) are unstable, and experience wave growth. In the long-wavelength limit ($K = 0$), (5.28) has solutions

$$\nu = \pm i \zeta^{1/2} |A_6|, \tag{5.31}$$

showing that the instability growth rate increases with the Alfvén wave amplitude $|A_6|$. Similar results are discussed by Sagdeev and Galeev (1969), who emphasize that it is necessary that $\omega_6 > |\omega_2|, |\omega_5|$ for the instability to occur. Sagdeev and Galeev (1969) interpret the latter results quantum mechanically in terms of conservation of energy for the wave quanta involved.

(b) *The non-strictly hyperbolic case* $\mathbf{B} = (B_x, 0, 0)^{\mathrm{T}}$. Now consider the interaction of the forward slow-mode wave ($a_5$) with the backward and forward Alfvén waves ($a_2$ and $a_6$), in a low-beta plasma ($b_x > a_g$), for the degenerate case where $\mathbf{B} = (B_x, 0, 0)^{\mathrm{T}}$. As $B_\perp = (B_y^2 + B_z^2)^{1/2} \to 0$, the fast and slow magnetoacoustic speeds reduce to

$$c_f = b_x, \qquad c_s = a_g. \tag{5.32}$$

In this limit, the fast mode is incompressible and the fast-mode wave speed $c_f$ and the Alfvén speed $b_x$ coincide. Thus for $\mathbf{B} = (B_x, 0, 0)^{\mathrm{T}}$ and $b_x > a_g$, the eigenvalues are

$$\lambda_1 = \lambda_2 = u_x - b_x, \qquad \lambda_3 = u_x - a_g, \qquad \lambda_4 = u_x, \qquad \lambda_5 = u_x + a_g, \qquad \lambda_6 = \lambda_7 = u_x + b_x. \tag{5.33}$$

The parameters $\alpha_f$ and $\alpha_s$, and the parameters $\beta_f$ and $\beta_s$ (see (5.4)), describing the Burgers self wave interactions have the values

$$\alpha_f = 0, \qquad \alpha_s = 1, \quad \beta_f = \tfrac{3}{2}, \qquad \beta_s = \tfrac{1}{2}(\gamma_g + 1). \tag{5.34}$$

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Because of the degeneracy of the eigenvalues, both the Alfvén and fast-mode waves can have the same phases. It is straightforward to write down the most general system of equations (5.2) having three resonant phases $\theta_2$, $\theta_5$ and $\theta_6$ involving the wave amplitudes $\{a_1, a_2, a_5, a_6, a_7\}$. Using the complex magnetic field perturbations

$$\begin{aligned} B^- &= B_{1y} + B_{2y} + i(B_{1z} + B_{2z}) \\ &\equiv (\beta_y + i\beta_z)\left(\frac{a_g}{b_0\rho^{1/2}}a_1 - ia_2\right), \end{aligned} \tag{5.35}$$

$$\begin{aligned} B^+ &= B_{6y} + B_{7y} + i(B_{6z} + B_{7z}) \\ &\equiv (\beta_y + i\beta_z)\left(\frac{a_g}{b_0\rho^{1/2}}a_7 - ia_6\right) \end{aligned} \tag{5.36}$$

and $\rho_5 \equiv a_5$ as the wave perturbation variables, the wave interaction equations (5.2) reduce to

$$\left(\frac{\partial}{\partial t} - b_x\frac{\partial}{\partial x}\right)B^- - \frac{b_x}{2\rho}k_2\frac{\partial}{\partial\theta_2}\left(\langle\rho_5 B^+\rangle_2\right) = 0, \tag{5.37}$$

$$\left(\frac{\partial}{\partial t} + a_g\frac{\partial}{\partial x}\right)\rho_5 + k_5\frac{\partial}{\partial\theta_5}\left[\frac{(\gamma_g+1)a_g}{8\rho}\rho_5^2 + \frac{b_0^2}{4a_g}\langle B^- B^{+*} + B^+ B^{-*}\rangle_5\right] = 0, \tag{5.38}$$

$$\left(\frac{\partial}{\partial t} + b_x\frac{\partial}{\partial x}\right)B^+ + \frac{b_x}{2\rho}k_6\frac{\partial}{\partial\theta_6}\left(\langle\rho_5 B^-\rangle_6\right) = 0, \tag{5.39}$$

where the asterisk denotes the complex conjugate. In (5.37)–(5.39), the wave amplitudes are assumed to have zero means (i.e. $\langle a_s\rangle = 0$), and $u_x = 0$ is assumed. It is of interest to note that (5.37)–(5.39) allow for the possibility of circularly polarized Alfvén waves, whereas the non-degenerate equations (5.15)–(5.17) for constant $\beta_y$ and $\beta_z$ apply to plane-polarized Alfvén waves.

The above completes our discussion of resonant wave interactions in magnetohydrodynamics. A more complete classification of three-wave resonant interactions in MHD is given in Ali and Hunter (1998).

## 6. Wave mixing equations

For linear wave propagation in inhomogeneous media in which nonlinear and second-order terms are negligible, (4.26) and (4.30) yield the linear wave mixing equations

$$\frac{\partial a_j}{\partial t} + \frac{\partial}{\partial x}(\lambda_j a_j) + \sum_{s=1}^{7}\Lambda_{js}a_s = 0, \quad j = 1, \ldots, 7, \tag{6.1}$$

describing the interaction of the waves with each other due to the inhomogeneous, large-scale background flow. From (4.38), the wave mixing coefficients in (6.1) are given by

$$\Lambda_{js} = \mathbf{L}_j \cdot \frac{d\mathbf{R}_s}{dt_s} + L_j^2\left[(R_s^2 - u_x R_s^1)\frac{a_c^2}{\kappa} - \frac{\zeta}{\rho}\frac{\partial p_c}{\partial x}R_s^1\right], \tag{6.2}$$

where $d/dt_s \equiv \partial/\partial t + \lambda_s\partial/\partial x$ is the time derivative along the $s$th wave mode characteristic. Explicit formulae for the $\{\Lambda_{js}\}$ are presented in Sec. 6.1. The eigenvector symmetries discussed in Sec. 3.3 are used to obtain symmetry relations for

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the $\Lambda_{js}$. Section 6.2 discusses the form of the wave mixing equations for planar MHD flows, in which $\mathbf{B} = (B_x, B_y, 0)^{\mathrm{T}}$ and $\mathbf{u} = (u_x, u_y, 0)^{\mathrm{T}}$. In this case, the Alfvén waves are decoupled from the magnetoacoustic and entropy waves. The degenerate-eigenvalue cases where (a) $\mathbf{B} = (B_x, 0, 0)^{\mathrm{T}}$ and $\mathbf{u} = (u_x, 0, 0)^{\mathrm{T}}$, for which $\mathbf{k} \parallel \mathbf{B}$, and (b) $\mathbf{B} = (0, B_y, 0)^{\mathrm{T}}$ and $\mathbf{u} = (u_x, 0, 0)^{\mathrm{T}}$, for which $\mathbf{k} \perp \mathbf{B}$, are also discussed. Alternative forms of the wave mixing coefficients (6.2) can be obtained by using the background flow equations (2.1)–(2.8) to eliminate time derivatives.

### 6.1. *Wave mixing coefficients*

Using the formulae (3.39)–(3.47) for the right- and left-eigenvectors $\{\mathbf{R}_j\}$ and $\{\mathbf{L}_j\}$ in (6.2) yields the wave mixing coefficients for the backward fast-mode wave equation in (6.1) in the form

$$\begin{aligned}\Lambda_{11} &= \frac{1}{2}\left[-\frac{\alpha_f^2 c_f}{a^2}\frac{du_x}{dt_1} + \frac{d\ln a}{dt_1} + \alpha_s^2 \frac{d}{dt_1}\ln\left(\frac{a}{\rho^{1/2}}\right) + \frac{\alpha_f^2}{\gamma_g}\frac{d\bar{S}}{dt_1}\right.\\ &\quad \left. + \frac{\alpha_s\alpha_f c_s}{a^2}\boldsymbol{\beta}_\perp \cdot \frac{d\mathbf{u}_\perp}{dt_1} + \frac{\alpha_f^2 c_f^2}{a^2}\left(\frac{a_c^2}{\kappa} + \frac{\zeta}{\rho c_f}\frac{\partial p_c}{\partial x}\right)\right], \end{aligned} \tag{6.3a}$$

$$\begin{aligned}\Lambda_{12} &= \frac{b_0\rho^{1/2}\alpha_s(a+c_s)}{2B_\perp^2 a^2}\left(B_y\frac{dB_z}{dt_2} - B_z\frac{dB_y}{dt_2}\right)\\ &\equiv \frac{b_0\rho^{1/2}\alpha_s(a+c_s)}{2B_\perp^2 a^2}(b_x\mathbf{B}\cdot\nabla\times\mathbf{B} - B_x\mathbf{B}\cdot\nabla\times\mathbf{u}), \end{aligned} \tag{6.3b}$$

$$\begin{aligned}\Lambda_{13} &= \frac{1}{2}\left[-\frac{\alpha_s\alpha_f c_f}{a^2}\frac{du_x}{dt_3} + \frac{\alpha_s^2 c_s}{a^2}\boldsymbol{\beta}_\perp\cdot\frac{d\mathbf{u}_\perp}{dt_3} + \frac{\alpha_s\alpha_f c_f c_s}{a^2}\frac{d}{dt_3}\ln\left(\frac{c_s}{c_f}\right)\right.\\ &\quad -\alpha_s\alpha_f\frac{d}{dt_3}\ln\left(\frac{a}{\rho^{1/2}}\right) + \frac{a^2 + c_s c_f}{a^2}\alpha_s\alpha_f\frac{d}{dt_3}\ln\left(\frac{\alpha_s}{\alpha_f}\right)\\ &\quad \left. + \frac{\alpha_s\alpha_f}{\gamma_g}\frac{d\bar{S}}{dt_3} + \frac{\alpha_s\alpha_f c_f c_s}{a^2}\left(\frac{a_c^2}{\kappa} + \frac{\zeta}{\rho c_s}\frac{\partial p_c}{\partial x}\right)\right], \end{aligned} \tag{6.3c}$$

$$\Lambda_{14} = \frac{1}{2a^2}\left(-\alpha_f c_f\frac{du_x}{dt} + \alpha_s c_s\boldsymbol{\beta}_\perp\cdot\frac{d\mathbf{u}_\perp}{dt} + \alpha_f c_f\frac{\zeta}{\rho}\frac{\partial p_c}{\partial x}\right), \tag{6.3d}$$

$$\begin{aligned}\Lambda_{15} &= \frac{1}{2}\left[-\frac{\alpha_s\alpha_f c_f}{a^2}\frac{du_x}{dt_5} + \frac{\alpha_s^2 c_s}{a^2}\boldsymbol{\beta}_\perp\cdot\frac{d\mathbf{u}_\perp}{dt_5} - \frac{\alpha_s\alpha_f c_f c_s}{a^2}\frac{d}{dt_5}\ln\left(\frac{c_s}{c_f}\right)\right.\\ &\quad -\alpha_s\alpha_f\frac{d}{dt_5}\ln\left(\frac{a}{\rho^{1/2}}\right) + \frac{a^2 - c_f c_s}{a^2}\alpha_f\alpha_s\frac{d}{dt_5}\ln\left(\frac{\alpha_s}{\alpha_f}\right)\\ &\quad \left. + \frac{\alpha_s\alpha_f}{\gamma_g}\frac{d\bar{S}}{dt_5} + \frac{\alpha_s\alpha_f c_f c_s}{a^2}\left(-\frac{a_c^2}{\kappa} + \frac{\zeta}{\rho c_s}\frac{\partial p_c}{\partial x}\right)\right], \end{aligned} \tag{6.3e}$$

$$\begin{aligned}\Lambda_{16} &= \frac{b_0\rho^{1/2}\alpha_s(a-c_s)}{2B_\perp^2 a^2}\left(B_y\frac{dB_z}{dt_6} - B_z\frac{dB_y}{dt_6}\right)\\ &\equiv -\frac{b_0\rho^{1/2}\alpha_s(a-c_s)}{2B_\perp^2 a^2}(B_x\mathbf{B}\cdot\nabla\times\mathbf{u} + b_x\mathbf{B}\cdot\nabla\times\mathbf{B}), \end{aligned} \tag{6.3f}$$

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$$\Lambda_{17} = \frac{1}{2}\left[-\frac{\alpha_f^2 c_f}{a^2}\frac{du_x}{dt_7} - \frac{d\ln a}{dt_7} + \alpha_s^2\frac{d}{dt_7}\ln\left(\frac{a}{\rho^{1/2}}\right) + \frac{\alpha_s\alpha_f c_s}{a^2}\boldsymbol{\beta}_\perp\cdot\frac{d\mathbf{u}_\perp}{dt_7}\right.$$
$$\left. + \frac{\alpha_f^2}{\gamma_g}\frac{d\bar{S}}{dt_7} + \frac{\alpha_f^2 c_f^2}{a^2}\left(-\frac{a_c^2}{\kappa} + \frac{\zeta}{\rho c_f}\frac{\partial p_c}{\partial x}\right)\right]. \tag{6.3g}$$

In (6.3), we have provided two alternative forms for $\Lambda_{12}$ and $\Lambda_{16}$. The expressions for $\Lambda_{12}$ and $\Lambda_{16}$ involving $\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u}$ and $\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B}$ are obtained by using Faraday's law (2.5) and (2.6) to eliminate time derivatives.

In order to assess the role of cosmic ray squeeezing instabilities (see e.g. Dorfi and Drury 1985; Drury and Falle 1986; Zank and McKenzie 1987) it is useful to eliminate the $du_x/dt_j$ terms in the $\{\Lambda_{js}\}$ by using the normal momentum equation (2.2). Thus, for example, the expression for $\Lambda_{11}$ in (6.3) can be written in the form

$$\Lambda_{11} = \frac{1}{2}\left\{\frac{\alpha_f^2 c_f^2}{a^2}\left(\frac{a_c^2}{\kappa} + \frac{\zeta+1}{\rho c_f}\frac{\partial p_c}{\partial x}\right) + \frac{\alpha_f^2 c_f^2}{a^2}\left[\frac{1}{\rho c_f}\frac{\partial}{\partial x}\left(p_g + \frac{B_\perp^2}{2\mu}\right) + \frac{\partial u_x}{\partial x}\right]\right.$$
$$\left. + \frac{d}{dt_1}\ln a + \alpha_s^2\frac{d}{dt_1}\ln\left(\frac{a}{\rho^{1/2}}\right) + \frac{\alpha_f^2}{\gamma_g}\frac{d\bar{S}}{dt_1} + \frac{\alpha_s\alpha_f c_s}{a^2}\boldsymbol{\beta}_\perp\cdot\frac{d\mathbf{u}_\perp}{dt_1}\right\}. \tag{6.4}$$

For steady flows, we may replace $d/dt_1$ by $\lambda_1\partial/\partial x$ in (6.4). The expression (6.4) for $\Lambda_{11}$ suggests that if the cosmic ray pressure gradient $\partial p_c/\partial x$ is sufficiently large and negative, i.e.

$$\frac{a_c^2}{\kappa} + \frac{\zeta+1}{\rho c_f}\frac{\partial p_c}{\partial x} \ll 0, \tag{6.5}$$

then the backward, fast-mode wave can be driven unstable. Similar instability criteria were obtained by Dorfi and Drury (1985), Drury and Falle (1986) and Zank and McKenzie (1987), for short-wavelength sound waves in the supersonic flow upstream of a cosmic-ray-modified shock.

The wave mixing coefficients $\{\Lambda_{2s}\}$ for the backward Alfvén wave in (6.1) are given by the formulae

$$\Lambda_{21} = \frac{1}{2b_0\rho^{1/2}B_\perp^2}\left[\alpha_f B_\perp\left(B_z\frac{du_y}{dt_1} - B_y\frac{du_z}{dt_1}\right) + \alpha_s(a+c_s)\left(B_z\frac{dB_y}{dt_1} - B_y\frac{dB_z}{dt_1}\right)\right]$$
$$\equiv \frac{1}{2b_0\rho^{1/2}B_\perp^2}\{[\alpha_f b_x b_\perp - \alpha_s(a+c_s)c_f]\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B}$$
$$+[\alpha_s(a+c_s)B_x - \alpha_f B_\perp c_f]\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u}\}, \tag{6.6a}$$

$$\Lambda_{22} = \frac{1}{4}\frac{d\ln\rho}{dt_2} \equiv -\frac{1}{4}\frac{\partial}{\partial x}(u_x - 2b_x), \tag{6.6b}$$

$$\Lambda_{23} = \frac{1}{2b_0\rho^{1/2}B_\perp^2}\left[\alpha_s B_\perp\left(B_z\frac{du_y}{dt_3} - B_y\frac{du_z}{dt_3}\right) - \alpha_f(a+c_f)\left(B_z\frac{dB_y}{dt_3} - B_y\frac{dB_z}{dt_3}\right)\right]$$
$$\equiv \frac{1}{2b_0\rho^{1/2}B_\perp^2}\{[\alpha_s b_x b_\perp + \alpha_f(a+c_f)c_s]\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B}$$
$$-[\alpha_s c_s B_\perp + \alpha_f(a+c_f)B_x]\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u}\}, \tag{6.6c}$$

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$$\Lambda_{24} = \frac{1}{2b_0\rho^{1/2}B_\perp}\left(B_z\frac{du_y}{dt} - B_y\frac{du_z}{dt}\right)$$
$$\equiv \frac{b_0 B_x}{2\rho^{3/2}B_\perp}\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B}, \tag{6.6d}$$

$$\Lambda_{25} = \frac{1}{2b_0\rho^{1/2}B_\perp^2}\left[\alpha_s B_\perp\left(B_z\frac{du_y}{dt_5} - B_y\frac{du_z}{dt_5}\right) + \alpha_f(c_f - a)\left(B_z\frac{dB_y}{dt_5} - B_y\frac{dB_z}{dt_5}\right)\right]$$
$$\equiv \frac{1}{2b_0\rho^{1/2}B_\perp^2}\{[\alpha_s b_x b_\perp + \alpha_f c_s(c_f - a)]\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B}$$
$$+[\alpha_s c_s B_\perp + \alpha_f(c_f - a)B_x]\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u}\}, \tag{6.6e}$$

$$\Lambda_{26} = -\frac{1}{4}\frac{d\ln\rho}{dt_6}$$
$$\equiv \frac{1}{4}\frac{\partial}{\partial x}(u_x + 2b_x), \tag{6.6f}$$

$$\Lambda_{27} = \frac{1}{2b_0\rho^{1/2}B_\perp^2}\left[\alpha_f B_\perp\left(B_z\frac{du_y}{dt_7} - B_y\frac{du_z}{dt_7}\right) + \alpha_s(a - c_s)\left(B_z\frac{dB_y}{dt_7} - B_y\frac{dB_z}{dt_7}\right)\right]$$
$$\equiv \frac{1}{2b_0\rho^{1/2}B_\perp^2}\{[\alpha_f b_x b_\perp + \alpha_s(a - c_s)c_f]\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B}$$
$$+[\alpha_f c_f B_\perp + \alpha_s(a - c_s)B_x]\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u}\}. \tag{6.6g}$$

In (6.6), the wave mixing coefficients linking the Alfvén wave to the entropy wave and the magnetoacoustic waves are in general non-zero if the background flow has non-zero field-aligned current ($\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B} \neq 0$) and/or non-zero field aligned vorticity ($\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u} \neq 0$). For purely one-dimensional flow, dependent on $(x, t)$,

$$\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B} = B_z\frac{\partial B_y}{\partial x} - B_y\frac{\partial B_z}{\partial x}, \tag{6.7a}$$

$$\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u} = B_z\frac{\partial u_y}{\partial x} - B_y\frac{\partial u_z}{\partial x}. \tag{6.7b}$$

For planar MHD flows in which the background magnetic field $\mathbf{B} = (B_x, B_y, 0)^{\mathrm{T}}$ and fluid velocity $\mathbf{u} = (u_x, u_y, 0)^{\mathrm{T}}$ are restricted to the $(x,\, y)$ plane, $\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B} = 0$ and $\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u} = 0$. For such flows, the Alfvén wave evolution equations in (6.1) are decoupled from the magnetoacoustic and entropy waves. Webb (1983) and Webb et al. (1986) used a planar MHD model to describe cosmic-ray-modified shocks.

For pure MHD flows (with no cosmic rays), one of the simplest background flows that has non-zero field-aligned current and vorticity is an Alfvén simple wave. It is of interest to note that the magnetoacoustic simple waves are characterized by zero field-aligned current and vorticity. This suggests that a study of wave mixing phenomena in which the background flow consists of an MHD simple wave should provide further insight into the wave mixing process. It should be emphasized that there are other MHD flows and equilibria besides the Alfvén simple waves that have non-zero field-aligned current and vorticity.

The wave mixing coefficients $\{\Lambda_{3s}\}$ for the backward slow magnetoacoustic wave

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are given by the equations

$$\Lambda_{31} = \frac{1}{2}\left[-\frac{\alpha_s\alpha_f c_s}{a^2}\frac{du_x}{dt_1} - \frac{\alpha_f^2 c_f}{a^2}\boldsymbol{\beta}_\perp\cdot\frac{d\mathbf{u}_\perp}{dt_1} + \frac{\alpha_s\alpha_f c_f c_s}{a^2}\frac{d}{dt_1}\ln\left(\frac{c_f}{c_s}\right)\right.$$
$$-\alpha_s\alpha_f\frac{d}{dt_1}\ln\left(\frac{a}{\rho^{1/2}}\right) + \frac{a^2+c_f c_s}{a^2}\alpha_f\alpha_s\frac{d}{dt_1}\ln\left(\frac{\alpha_f}{\alpha_s}\right)$$
$$\left.+\frac{\alpha_s\alpha_f}{\gamma_g}\frac{d\bar{S}}{dt_1} + \frac{\alpha_s\alpha_f c_f c_s}{a^2}\left(\frac{a_c^2}{\kappa} + \frac{\zeta}{\rho c_f}\frac{\partial p_c}{\partial x}\right)\right], \tag{6.8a}$$

$$\Lambda_{32} = \frac{b_0\rho^{1/2}\alpha_f(a+c_f)}{2B_\perp^2 a^2}\left(B_z\frac{dB_y}{dt_2} - B_y\frac{dB_z}{dt_2}\right)$$
$$\equiv \frac{b_0\rho^{1/2}\alpha_f(a+c_f)}{2B_\perp^2 a^2}(B_x\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u} - b_x\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B}), \tag{6.8b}$$

$$\Lambda_{33} = \frac{1}{2}\left[-\frac{\alpha_s^2 c_s}{a^2}\frac{du_x}{dt_3} + \frac{d\ln a}{dt_3} + \alpha_f^2\frac{d}{dt_3}\ln\left(\frac{a}{\rho^{1/2}}\right) + \frac{\alpha_s^2}{\gamma_g}\frac{d\bar{S}}{dt_3}\right.$$
$$\left.-\frac{\alpha_s\alpha_f c_f}{a^2}\boldsymbol{\beta}_\perp\cdot\frac{d\mathbf{u}_\perp}{dt_3} + \frac{\alpha_s^2 c_s^2}{a^2}\left(\frac{a_c^2}{\kappa} + \frac{\zeta}{\rho c_s}\frac{\partial p_c}{\partial x}\right)\right], \tag{6.8c}$$

$$\Lambda_{34} = \frac{1}{2a^2}\left(-\alpha_s c_s\frac{du_x}{dt} - \alpha_f c_f\boldsymbol{\beta}_\perp\cdot\frac{d\mathbf{u}_\perp}{dt} + \alpha_s c_s\frac{\zeta}{\rho}\frac{\partial p_c}{\partial x}\right), \tag{6.8d}$$

$$\Lambda_{35} = \frac{1}{2}\left[-\frac{\alpha_s^2 c_s}{a^2}\frac{du_x}{dt_5} - \frac{d\ln a}{dt_5} + \alpha_f^2\frac{d}{dt_5}\ln\left(\frac{a}{\rho^{1/2}}\right) - \frac{\alpha_s\alpha_f c_f}{a^2}\boldsymbol{\beta}_\perp\cdot\frac{d\mathbf{u}_\perp}{dt_5}\right.$$
$$\left.+\frac{\alpha_s^2}{\gamma_g}\frac{d\bar{S}}{dt_5} - \frac{\alpha_s^2 c_s^2}{a^2}\left(\frac{a_c^2}{\kappa} - \frac{\zeta}{\rho c_s}\frac{\partial p_c}{\partial x}\right)\right], \tag{6.8e}$$

$$\Lambda_{36} = \frac{b_0\rho^{1/2}\alpha_f(a-c_f)}{2B_\perp^2 a^2}\left(B_z\frac{dB_y}{dt_6} - B_y\frac{dB_z}{dt_6}\right)$$
$$\equiv \frac{b_0\rho^{1/2}\alpha_f(a-c_f)}{2B_\perp^2 a^2}(b_x\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B} + B_x\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u}), \tag{6.8f}$$

$$\Lambda_{37} = \frac{1}{2}\left[-\frac{\alpha_s\alpha_f c_s}{a^2}\frac{du_x}{dt_7} - \frac{\alpha_f^2 c_f}{a^2}\boldsymbol{\beta}_\perp\cdot\frac{d\mathbf{u}_\perp}{dt_7} - \frac{\alpha_s\alpha_f c_f c_s}{a^2}\frac{d}{dt_7}\ln\left(\frac{c_f}{c_s}\right)\right.$$
$$-\alpha_s\alpha_f\frac{d}{dt_7}\ln\left(\frac{a}{\rho^{1/2}}\right) + \frac{a^2-c_s c_f}{a^2}\alpha_s\alpha_f\frac{d}{dt_7}\ln\left(\frac{\alpha_f}{\alpha_s}\right)$$
$$\left.+\frac{\alpha_s\alpha_f}{\gamma_g}\frac{d\bar{S}}{dt_7} - \frac{\alpha_s\alpha_f c_f c_s}{a^2}\left(\frac{a_c^2}{\kappa} - \frac{\zeta}{\rho c_f}\frac{\partial p_c}{\partial x}\right)\right]. \tag{6.8g}$$

The above formulae for $\{\Lambda_{3s}\}$ are similar in form to those for the backward fast magnetoacoustic wave, $\{\Lambda_{1s}\}$ in (6.3) (see also (6.9) et seq. for a discussion of a map between the $\{\Lambda_{1s}\}$ and the $\{\Lambda_{3s}\}$).

The wave mixing coefficients for the entropy wave, $\{\Lambda_{4s}\}$, are given by

$$\Lambda_{41} = -\frac{\alpha_f}{\gamma_g}\frac{d\bar{S}}{dt_1} \equiv \frac{\alpha_f c_f}{\gamma_g}\frac{\partial\bar{S}}{\partial x}, \tag{6.9a}$$

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$$\Lambda_{43} = -\frac{\alpha_s}{\gamma_g}\frac{d\bar{S}}{dt_3} \equiv \frac{\alpha_s c_s}{\gamma_g}\frac{\partial \bar{S}}{\partial x}, \tag{6.9b}$$

$$\Lambda_{45} = -\frac{\alpha_s}{\gamma_g}\frac{d\bar{S}}{dt_5} \equiv -\frac{\alpha_s c_s}{\gamma_g}\frac{\partial \bar{S}}{\partial x}, \tag{6.9c}$$

$$\Lambda_{47} = -\frac{\alpha_f}{\gamma_g}\frac{d\bar{S}}{dt_7} \equiv -\frac{\alpha_f c_f}{\gamma_g}\frac{\partial \bar{S}}{\partial x}, \tag{6.9d}$$

$$\Lambda_{42} = \Lambda_{44} = \Lambda_{46} = 0. \tag{6.9e}$$

Thus the entropy wave is modified by the magnetoacoustic waves only if $\partial\bar{S}/\partial x \neq 0$. Note that the entropy wave is not affected by the Alfvén waves, because $\Lambda_{42} = \Lambda_{46} = 0$.

*6.1.1. Mixing coefficient symmetries.* For $j > 4$, one can use the wave speed reversal symmetry $T_a$ in (3.58)–(3.59) for the eigenvectors, coupled with the formula (6.2), to obtain the results

$$\Lambda_{js}(\mathbf{y}) = \Lambda_{j's'}(-c_f, -c_s, -b_0, -a), \quad j' = 8 - j, \quad s' = 8 - s. \tag{6.10}$$

Equations (6.10) give the wave mixing coefficients $\{\Lambda_{js}\}$ for $j > 4$, where $\mathbf{y} \equiv (c_f, c_s, b_0, a)$, in terms of the mixing coefficients for $j \leqslant 4$.

From (3.61), $T_b : \mathbf{y} \mapsto (c_f, c_s, -b_0, -a)$ maps the Alfvén wave eigenvectors onto the reverse Alfvén wave eigenvectors, but leaves the entropy wave and magnetoacoustic wave eigenvectors invariant. This map, coupled with (6.2), yields the symmetry relations

$$\Lambda_{16}(\mathbf{y}) = \Lambda_{12}(T_b\mathbf{y}), \quad \Lambda_{36}(\mathbf{y}) = \Lambda_{32}(T_b\mathbf{y}), \quad \Lambda_{46}(\mathbf{y}) = \Lambda_{42}(T_b\mathbf{y}), \tag{6.11a}$$

$$\Lambda_{56}(\mathbf{y}) = \Lambda_{52}(T_b\mathbf{y}), \quad \Lambda_{76}(\mathbf{y}) = \Lambda_{72}(T_b\mathbf{y}) \tag{6.11b}$$

for the wave mixing coefficients.

From (3.63) and (3.64), $T_c : \mathbf{y} \mapsto (-c_f, -c_s, b_0, a)$ maps the magnetoacoustic eigenvectors onto the reverse magnetoacoustic eigenvectors. Using this map in (6.2) yields the symmetry relations

$$\Lambda_{25}(\mathbf{y}) = \Lambda_{23}(-c_f, -c_s, b_0, a), \quad \Lambda_{27}(\mathbf{y}) = \Lambda_{21}(-c_f, -c_s, b_0, a), \tag{6.12a}$$

$$\Lambda_{65}(\mathbf{y}) = \Lambda_{63}(-c_f, -c_s, b_0, a), \quad \Lambda_{67}(\mathbf{y}) = \Lambda_{61}(-c_f, -c_s, b_0, a), \tag{6.12b}$$

for the Alfvén wave mixing coefficients $\{\Lambda_{2s}\}$ and $\{\Lambda_{6s}\}$.

From (3.65)–(3.67), $T_d : \mathbf{y} \mapsto (c_s, -c_f, -b_0, a)$ maps the slow-mode eigenvectors onto the fast-mode eigenvectors, and vice versa, and maps the Alfvén wave eigenvectors onto the reverse Alfvén wave eigenvectors. This map applied to (6.2) yields the symmetry relations

$$\Lambda_{31}(\mathbf{y}) = \Lambda_{15}(T_d\mathbf{y}), \quad \Lambda_{32}(\mathbf{y}) = \Lambda_{16}(T_d\mathbf{y}), \quad \Lambda_{33}(\mathbf{y}) = \Lambda_{11}(T_d\mathbf{y}), \tag{6.13a}$$

$$\Lambda_{34}(\mathbf{y}) = \Lambda_{14}(T_d\mathbf{y}), \quad \Lambda_{35}(\mathbf{y}) = \Lambda_{17}(T_d\mathbf{y}), \quad \Lambda_{36}(\mathbf{y}) = \Lambda_{12}(T_d\mathbf{y}), \tag{6.13b}$$

$$\Lambda_{37}(\mathbf{y}) = \Lambda_{13}(T_d\mathbf{y}). \tag{6.13c}$$

Hence the backward slow-mode wave mixing coefficients may be obtained from the backward fast-mode coefficients by using (6.13). By using the map $T_d$, one can also derive the relations

$$\Lambda_{43}(\mathbf{y}) = \Lambda_{41}(T_d\mathbf{y}), \quad \Lambda_{46}(\mathbf{y}) = \Lambda_{42}(T_d\mathbf{y}), \quad \Lambda_{47}(\mathbf{y}) = \Lambda_{43}(T_d\mathbf{y}) \tag{6.14}$$

for the entropy wave mixing coefficients.

The symmetry relations (6.11)–(6.14) for the wave mixing coefficients $\{\Lambda_{js}\}$, associated with the maps $T_b$, $T_c$ and $T_d$, may be verified directly from (6.3), (6.6), (6.8) and (6.9). One can check the validity of the formulae (6.3)–(6.14) for the wave mixing coefficients by using the linearized conservation equations (2.1) and (2.3)–(2.7).

### 6.2. *Planar MHD flows*

In this subsection, we consider the form of the wave mixing equations for the case of planar MHD flows, in which $\mathbf{B} = (B_x, B_y, 0)^{\mathrm{T}}$ and $\mathbf{u} = (u_x, u_y, 0)^{\mathrm{T}}$. In this case, $\mathbf{B}\cdot\nabla\times\mathbf{B} = 0$, and $\mathbf{B}\cdot\nabla\times\mathbf{u} = 0$, and the Alfvén wave mixing equations in (6.1) decouple from the mixing equations for the magnetoacoustic and entropy waves. We first consider the non-degenerate case where all the eigenvalues are distinct ($B_x \neq 0$, $B_y \neq 0$), and then consider the degenerate-eigenvalue cases (a) $\mathbf{k} \parallel \mathbf{B}$, where $\mathbf{B} = (B_x, 0, 0)^{\mathrm{T}}$ and $\mathbf{u} = (u_x, 0, 0)^{\mathrm{T}}$, and (b) $\mathbf{k} \perp \mathbf{B}$ where $\mathbf{B} = (0, B_y, 0)^{\mathrm{T}}$ and $\mathbf{u} = (u_x, 0, 0)^{\mathrm{T}}$.

*6.2.1. The non-degenerate case $B_x \neq 0$ and $B_y \neq 0$.* In this case, the magnetoacoustic and entropy waves satisfy the wave mixing equations

$$\frac{\partial a_j}{\partial t} + \frac{\partial}{\partial x}(\lambda_j a_j) + \sum_{s\neq 2,6} \Lambda_{js} a_s = 0, \quad j = 1,3,4,5,7, \tag{6.15}$$

where the sum over $s$ in (6.15) is for $s = 1,3,4,5,7$ corresponding to the magnetoacoustic and entropy waves.

The Alfvén waves satisfy the separate wave mixing equations

$$\frac{\partial a_2}{\partial t} + \frac{\partial}{\partial x}(\lambda_2 a_2) + \Lambda_{22} a_2 + \Lambda_{26} a_6 = 0, \tag{6.16}$$

$$\frac{\partial a_6}{\partial t} + \frac{\partial}{\partial x}(\lambda_6 a_6) + \Lambda_{26} a_2 + \Lambda_{66} a_6 = 0, \tag{6.17}$$

where

$$\Lambda_{22} = -\tfrac{1}{4} D_x(u_x - 2b_x), \qquad \Lambda_{26} = \tfrac{1}{4} D_x(u_x + 2b_x), \tag{6.18a}$$

$$\Lambda_{62} = \tfrac{1}{4} D_x(u_x - 2b_x), \qquad \Lambda_{66} = -\tfrac{1}{4} D_x(u_x + 2b_x), \tag{6.18b}$$

and $D_x \equiv \partial/\partial x$.

Using (3.54) and the right-eigenvectors (3.26) yields the equations

$$a_2 = -\tfrac{1}{2}\,\mathrm{sgn}(B_y)\left(B_z^1 + \frac{\rho^{\frac{1}{2}}}{b_0} u_z^1\right) \equiv -\mathrm{sgn}(B_y)\, B_{2z}, \tag{6.19a}$$

$$a_6 = -\tfrac{1}{2}\,\mathrm{sgn}(B_y)\left(B_z^1 - \frac{\rho^{\frac{1}{2}}}{b_0} u_z^1\right) \equiv -\mathrm{sgn}(B_y)\, B_{6z} \tag{6.19b}$$

for the Alfvén wave amplitudes $a_2$ and $a_6$, in terms of the total magnetic and velocity fluctuations $B_z^1$ and $u_z^1$. In (6.19), $B_{2z}$ and $B_{6z}$ are related to the corresponding velocity fluctuations $u_{2z}$ and $u_{6z}$ by the eigenequations

$$u_{2z} = b_0 \rho^{-1/2} B_{2z}, \qquad u_{6z} = -b_0 \rho^{-1/2} B_{6z}. \tag{6.20}$$

Using the alternative notation

$$\delta B_z^- = B_{2z}, \qquad \delta u_z^- = u_{2z}, \qquad \delta B_z^+ = B_{6z}, \qquad \delta u_z^+ = u_{6z}, \tag{6.21}$$

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the Alfvén wave mixing equations (6.16) and (6.17) may be written in the more compact form

$$\frac{\partial \delta \mathbf{B}^{\pm}}{\partial t} + \frac{\partial}{\partial x}[(u_x \pm b_x)\delta \mathbf{B}^{\pm}] \pm \tfrac{1}{4} D_x (u_x - 2b_x)\delta \mathbf{B}^{-} \mp \tfrac{1}{4} D_x (u_x + 2b_x)\delta \mathbf{B}^{+} = 0, \quad (6.22)$$

where $\delta \mathbf{B}^{\pm} = (0, 0, \delta B_z^{\pm})^{\mathrm{T}}$, and the superscripts $-$ and $+$ refer to the backward and forward Alfvén waves respectively.

It is of interest to compare the evolution equations (6.22) with the transport equations

$$\frac{\partial \mathbf{Z}^{\pm}}{\partial t} + (\mathbf{u} \mp \mathbf{V}_A) \cdot \boldsymbol{\nabla} \mathbf{Z}^{\pm} + \tfrac{1}{4} \boldsymbol{\nabla} \cdot (\mathbf{u} \pm 2\mathbf{V}_A)\mathbf{Z}^{\pm} + \left( \boldsymbol{\nabla} \mathbf{u} \pm \frac{\boldsymbol{\nabla} \mathbf{B}}{(4\pi\rho)^{1/2}} - \tfrac{1}{4} \mathsf{I} \boldsymbol{\nabla} \cdot (\mathbf{u} \pm 2\mathbf{V}_A) \right) \cdot \mathbf{Z}^{\mp} = \mathbf{Q}^{\pm}, \quad (6.23)$$

for Alfvénic turbulence in the solar wind obtained by Zhou and Matthaeus (1990), where

$$\mathbf{Z}^{\pm} = \delta \mathbf{u} \pm \frac{b_0 \delta \mathbf{B}}{\rho^{1/2}}, \qquad b_0 = (4\pi)^{-1/2}, \quad (6.24)$$

define the Elsässer variables $\mathbf{Z}^{\pm}$; $\mathbf{Q}^{\pm}$ are nonlinear source terms for the short-scale turbulent fluctuations, and $\mathbf{V}_A = \mathbf{B}/(4\pi\rho)^{1/2}$ is the Alfvén velocity. The transport equations (6.23) for planar MHD flows, with $Q^{\pm} = 0$, reduce to

$$\frac{\partial \mathbf{Z}^{\pm}}{\partial t} + (u_x \mp b_x) \frac{\partial \mathbf{Z}^{\pm}}{\partial x} + \tfrac{1}{4} D_x (u_x \pm 2b_x) \mathbf{Z}^{\pm} - \tfrac{1}{4} D_x (u_x \pm 2b_x) \mathbf{Z}^{\mp} = 0, \quad (6.25)$$

where $\mathbf{Z}^{\pm} = (0, 0, Z_z^{\pm})^{\mathrm{T}}$ in the present case. Using the eigenrelations (6.20), it is straightforward to verify that the wave mixing equations (6.22) are equivalent to the Elsässer variable wave mixing equations (6.25).

*6.2.2. Canonical energy equation for Alfvén waves.* Consider the form of the Alfvén wave mixing equations (6.22) for the special case of a steady background flow. In this case, the mass continuity equation, and the fact that $B_x$ is constant, yield the results

$$u_x = u_{x0} \frac{\rho_0}{\rho}, \qquad b_x = b_{x0} \left( \frac{\rho_0}{\rho} \right)^{1/2}, \quad (6.26)$$

relating the $x$ component of the fluid velocity $u_x$ and the Alfvén speed $b_x$ to the density $\rho$ (note that $u_{x0}$, $\rho_0$, and $b_{x0}$ are constants). Following the approach of Heinemann and Olbert (1980), the Alfvén wave mixing equations (6.22) may be written in the form

$$\frac{\partial f}{\partial t} + (u_x - b_x) \frac{\partial f}{\partial x} = (u_x - b_x) \psi_x g, \quad (6.27)$$

$$\frac{\partial g}{\partial t} + (u_x + b_x) \frac{\partial g}{\partial x} = (u_x + b_x) \psi_x f, \quad (6.28)$$

where

$$f = \delta B_z^{-} (u_x - b_x) \rho^{1/4}, \qquad g = \delta B_z^{+} (u_x + b_x) \rho^{1/4}, \qquad \psi = \tfrac{1}{4} \ln \rho. \quad (6.29)$$

Equations (6.27)–(6.29) show that the backward and forward waves are coupled owing to the gradients in the background flow.

The wave mixing equations (6.27) and (6.28) may be combined to yield the canonical wave energy equation for both backward and forward Alfvén waves in the form:

$$\frac{\partial}{\partial t}\left(\frac{g^2}{u_x+b_x}-\frac{f^2}{u_x-b_x}\right)+\frac{\partial}{\partial x}(g^2-f^2)=0. \tag{6.30}$$

Equation (6.30) may be written in the more suggestive form

$$\frac{\partial}{\partial t}\left(\omega_-\mathscr{A}_A^-+\omega_+\mathscr{A}_A^+\right)+\frac{\partial}{\partial x}[(u_x-b_x)\omega_-\mathscr{A}_A^-+(u_x+b_x)\omega_+\mathscr{A}_A^+]=0, \tag{6.31}$$

where the equations

$$\mathscr{A}_A^\pm=\frac{E_A^\pm}{\omega'_\pm},\qquad \omega'_\pm=\pm kb_x,\qquad \omega_\pm=k(u_x\pm b_x), \tag{6.32}$$

define the Alfvén wave action densities $\mathscr{A}_A^\pm$ in terms of the physical wave energy densities

$$E_A^\pm=\frac{(\delta B_z^\pm)^2}{4\pi}, \tag{6.33}$$

and $\omega'_\pm$ and $\omega_\pm$ denote the wave frequencies measured in the fluid frame and the fixed frame respectively. In a quantum mechanical interpretation of (6.31), $\mathscr{A}_A^\pm$ correspond to the number densities of wave quasiparticles for the backward and forward Alfvén waves, and $\hbar\omega_\pm\mathscr{A}_A^\pm$ are the corresponding canonical wave energy densities, where $\hbar=h/2\pi$, and $h$ is Planck's constant (see also the discussion in Heinemann and Olbert 1980). The canonical wave energy equation or generalized wave action equation (6.31) can be generalized to the case of a time-dependent background flow, but in that case there are further terms on the right-hand side of the equation, dependent on time derivatives of the background variables.

*6.2.3. The degenerate case* $\mathbf{k}\parallel\mathbf{B}$. For the case where $\mathbf{k}\parallel\mathbf{B}$ (i.e. $\mathbf{k}=(k,0,0)^{\mathrm{T}}$ and $\mathbf{B}=(B_x,0,0)^{\mathrm{T}}$) one of the magnetoacoustic speeds coincides with the Alfvén speed $|b_x|$. In the limit as $|\mathbf{B}_\perp|\to 0$ ($\mathbf{B}_\perp=(0,B_y,B_z)^{\mathrm{T}}$), there are two possibilities to consider, namely

$$a>|b_x|:\qquad c_s\to|b_x|,\quad c_f\to a; \tag{6.34a}$$

$$a<|b_x|:\qquad c_s\to a,\quad c_f\to|b_x|. \tag{6.34b}$$

Note that there is a triple degeneracy in the eigenspeeds when $a=|b_x|$, which has interesting implications for nonlinear wave interactions between the Alfvén and magnetoacoustic waves at this point (see e.g. Brio 1989; Hada 1993; Webb et al. 1995). The form of the wave mixing coefficients depends on the manner in which $|B_\perp|\to 0$. For the sake of definiteness, we consider the limiting case where

$$\mathbf{B}=(B_x,B_y,0)^{\mathrm{T}},\qquad a>|b_x|,\qquad c_s\to|b_x|,\qquad c_f\to a, \tag{6.35}$$

and $B_y\to 0$. The background flow velocity is assumed to be directed along the $x$ axis (i.e. $\mathbf{u}=(u_x,0,0)^{\mathrm{T}}$). In the above limit, the eigenvalues are

$$\lambda_1=u_x-a,\qquad \lambda_2=\lambda_3=u_x-b_x,\qquad \lambda_4=u_x, \tag{6.36a}$$

$$\lambda_5=\lambda_6=u_x+b_x,\qquad \lambda_7=u_x+a, \tag{6.36b}$$

where we assume $b_x>0$. Thus the slow magnetoacoustic and Alfvén speeds coincide, and the fast magnetoacoustic speed equals the gas sound speed $a$.

The wave mixing equations (6.1) in the above case split into three separate, non-interacting subsystems in the limit as $B_y \to 0$. The first subsystem describes the interaction of the backward and forward sound waves ($\rho_1$ and $\rho_7$) and the entropy wave ($\rho_4$); the second system describes the backward and forward Alfvén waves, and the third describes the backward and forward slow magnetoacoustic waves. The sound waves and entropy wave satisfy the wave mixing equations

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}(\lambda_1 \rho_1) + \Lambda_{11}\rho_1 + \Lambda_{14}\rho_4 + \Lambda_{17}\rho_7 = 0, \tag{6.37a}$$

$$\frac{\partial \rho_4}{\partial t} + \frac{\partial}{\partial x}(\lambda_4 \rho_4) + \Lambda_{41}\rho_1 + \Lambda_{44}\rho_4 + \Lambda_{47}\rho_7 = 0, \tag{6.37b}$$

$$\frac{\partial \rho_7}{\partial t} + \frac{\partial}{\partial x}(\lambda_7 \rho_7) + \Lambda_{71}\rho_1 + \Lambda_{74}\rho_4 + \Lambda_{77}\rho_7 = 0, \tag{6.37c}$$

where $\rho_1$, $\rho_4$ and $\rho_7$ are the wave density perturbations (note that $\alpha_f = 1$ and $\alpha_s = 0$ in the present case). The wave interaction coefficients in (6.37) may be written in the form

$$\Lambda_{11} = \frac{1}{2}\left(\frac{3-\gamma_g}{2}R_x^+ - \frac{a_g \bar{S}_x}{\gamma_g - 1} + \frac{a_c^2}{\kappa} + \frac{\zeta+1}{\rho a_g}\frac{\partial p_c}{\partial x}\right), \tag{6.38a}$$

$$\begin{aligned}\Lambda_{14} &= \frac{1}{2}\left[\frac{R_x^+ - R_x^-}{2} - \frac{a_g \bar{S}_x}{\gamma_g(\gamma_g - 1)} + \frac{\zeta+1}{\rho a_g}\frac{\partial p_c}{\partial x}\right] \\ &\equiv -\frac{1}{2a_g}\frac{du_x}{dt} + \frac{\zeta}{2\rho a_g}\frac{\partial p_c}{\partial x},\end{aligned} \tag{6.38b}$$

$$\Lambda_{17} = \frac{1}{2}\left[\frac{\gamma_g - 3}{2}R_x^- + \frac{(\gamma_g - 2)a_g \bar{S}_x}{\gamma_g(\gamma_g - 1)} + \frac{\zeta+1}{\rho a_g}\frac{\partial p_c}{\partial x} - \frac{a_c^2}{\kappa}\right], \tag{6.38c}$$

$$\Lambda_{41} = -\Lambda_{47} = \frac{a_g \bar{S}_x}{\gamma_g}, \tag{6.38d}$$

$$\Lambda_{44} = 0, \tag{6.38e}$$

$$\Lambda_{71} = \frac{1}{2}\left[\frac{\gamma_g - 3}{2}R_x^+ - \frac{(\gamma_g - 2)a_g \bar{S}_x}{\gamma_g(\gamma_g - 1)} - \frac{\zeta+1}{\rho a_g}\frac{\partial p_c}{\partial x} - \frac{a_c^2}{\kappa}\right], \tag{6.38f}$$

$$\Lambda_{74} = -\Lambda_{14}, \tag{6.38g}$$

$$\Lambda_{77} = \frac{1}{2}\left(\frac{3-\gamma_g}{2}R_x^- + \frac{a_g \bar{S}_x}{\gamma_g - 1} + \frac{a_c^2}{\kappa} - \frac{\zeta+1}{\rho a_g}\frac{\partial p_c}{\partial x}\right), \tag{6.38h}$$

where $\zeta = \partial \ln \kappa / \partial \ln \rho$ (see (2.24)), $du_x/dt = \partial u_x/\partial t + u_x \partial u_x/\partial x$ is the $x$ component of the fluid acceleration vector, and

$$R^{\pm} = u_x \pm \frac{2a_g}{\gamma_g - 1} \tag{6.39}$$

are the Riemann invariants of isentropic gas dynamics. The wave mixing equations (6.37) and wave mixing coefficients (6.38) are the same as those in Webb et al. (1997a) for two-fluid cosmic-ray-modified flows in which the magnetic field plays no dynamical role.

It turns out that both the slow magnetoacoustic and Alfvén wave mixing equations in the limit as $B_y \to 0$ (assuming $a > b_x$) can be written in the form (6.22), where $\delta\mathbf{B}^{\pm} = (0, 0, \delta B_z^{\pm})^{\mathrm{T}}$ for the Alfvén modes, but $\delta\mathbf{B}^{\pm} = (0, \delta B_y^{\pm}, 0)^{\mathrm{T}}$ for the slow-mode waves. Note that the equations for $\delta B_z^{\pm}$ (Alfvén modes) are decoupled from the equations for $\delta B_y^{\pm}$ (slow mode waves). Similarly, one can show that the Elsässer variable wave mixing equations (6.25) also apply to both the Alfvén ($\mathbf{Z}^{\pm} = (0, 0, Z_z^{\pm})^{\mathrm{T}}$) and the slow-mode ($\mathbf{Z}^{\pm} = (0, Z_y^{\pm}, 0)^{\mathrm{T}}$) waves.

*6.2.4. The degenerate case* $\mathbf{k} \perp \mathbf{B}$. For the case where $\mathbf{B} = (0, B_y, 0)^{\mathrm{T}}$, $\mathbf{u} = (u_x, 0, 0)^{\mathrm{T}}$ and $\mathbf{k} = (k, 0, 0)^{\mathrm{T}}$, with $\mathbf{k} \perp \mathbf{B}$, the eigenvalues are

$$\lambda_1 = u_x - c_f, \qquad \lambda_7 = u_x + c_f, \qquad \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = u_x. \tag{6.40}$$

The slow-, intermediate- and fast-mode wave speeds are given by the equations

$$c_s = 0, \qquad b_x = 0, \qquad c_f = (a^2 + b_\perp^2)^{1/2}, \tag{6.41}$$

Because $b_x = c_s = 0$, the slow magnetoacoustic, Alfvén and entropy wave have the same phase speed $\lambda = u_x$, but have distinct eigenvectors. The above flow configuration corresponds to a cosmic-ray-modified perpendicular shock (see e.g. Webb 1983; Webb et al. 1986). The parameters $\alpha_f$ and $\alpha_s$ in (3.18) reduce to

$$\alpha_f = \frac{a}{c_f}, \qquad \alpha_s = \frac{b_\perp}{c_f}, \tag{6.42}$$

where $c_f$ is given by (6.41).

From (6.3), the non-zero wave mixing coefficients for the backward fast-mode wave reduce to

$$\begin{aligned}\Lambda_{11} = \frac{1}{2}\Bigg[&-\frac{\alpha_f}{a}\frac{du_x}{dt_1} + \frac{d\ln a}{dt_1} + \alpha_s^2\frac{d}{dt_1}\ln\left(\frac{a}{\rho^{1/2}}\right) \\ &+\frac{\alpha_f^2}{\gamma_g}\frac{d\bar{S}}{dt_1} + \left(\frac{a_c^2}{\kappa} + \frac{\zeta}{\rho c_f}\frac{\partial p_c}{\partial x}\right)\Bigg],\end{aligned} \tag{6.43a}$$

$$\Lambda_{13} = \frac{\alpha_s}{2a}\left[-\frac{du_x}{dt} + \alpha_f a\frac{d}{dt}\ln\left(\frac{B_y}{a^2}\right) + \frac{\zeta}{\rho}\frac{\partial p_c}{\partial x}\right], \tag{6.43b}$$

$$\Lambda_{14} = \frac{1}{2a}\left(-\frac{du_x}{dt} + \frac{\zeta}{\rho}\frac{\partial p_c}{\partial x}\right), \tag{6.43c}$$

$$\begin{aligned}\Lambda_{15} &= \frac{\alpha_s}{2a}\left[-\frac{du_x}{dt} + \alpha_f a\frac{d}{dt}\ln\left(\frac{B_y}{a^2}\right) + \frac{\zeta}{\rho}\frac{\partial p_c}{\partial x}\right] \\ &\equiv \Lambda_{13},\end{aligned} \tag{6.43d}$$

$$\begin{aligned}\Lambda_{17} = \frac{1}{2}\Bigg[&-\frac{\alpha_f}{a}\frac{du_x}{dt_7} - \frac{d}{dt_7}\ln a + \alpha_s^2\frac{d}{dt_7}\ln\left(\frac{a}{\rho^{1/2}}\right) \\ &+\frac{\alpha_f^2}{\gamma_g}\frac{d\bar{S}}{dt_7} - \left(\frac{a_c^2}{\kappa} - \frac{\zeta}{\rho c_f}\frac{\partial p_c}{\partial x}\right)\Bigg].\end{aligned} \tag{6.43e}$$

From (6.8), the corresponding non-zero interaction coefficients for the backward slow-mode wave simplify to

$$\Lambda_{31} = \frac{\alpha_s\alpha_f}{2}\left[\frac{d}{dt_1}\ln\left(\frac{\rho}{B_y}\right) + \frac{1}{\gamma_g}\frac{d\bar{S}}{dt_1}\right], \tag{6.44a}$$

$$\Lambda_{33} = \frac{1}{2}\left[\frac{d\ln a}{dt} + \alpha_f^2\frac{d}{dt}\ln\left(\frac{a}{\rho^{1/2}}\right)\right], \tag{6.44b}$$

$$\Lambda_{35} = \frac{1}{2}\left[-\frac{d\ln a}{dt} + \alpha_f^2 \frac{d}{dt}\ln\left(\frac{a}{\rho^{1/2}}\right)\right], \tag{6.44c}$$

$$\Lambda_{37} = \frac{\alpha_f \alpha_s}{2}\left[\frac{d}{dt_7}\ln\left(\frac{\rho}{B_y}\right) + \frac{1}{\gamma_g}\frac{d\bar{S}}{dt_7}\right]. \tag{6.44d}$$

From (6.9), the non-zero interaction coefficients for the entropy wave are

$$\Lambda_{41} = -\frac{\alpha_f}{\gamma_g}\frac{d\bar{S}}{dt_1} \equiv \frac{a}{\gamma_g}\frac{\partial \bar{S}}{\partial x}, \tag{6.45a}$$

$$\Lambda_{47} = -\frac{\alpha_f}{\gamma_g}\frac{d\bar{S}}{dt_7} \equiv -\frac{a}{\gamma_g}\frac{\partial \bar{S}}{\partial x}. \tag{6.45b}$$

The non-zero wave mixing coefficients $\{\Lambda_{js}\}$ for $j > 4$ may be obtained by use of the wave speed reversal symmetry (6.10), used in conjunction with the mixing coefficients (6.43)–(6.45).

The wave mixing equations for the Alfvén waves reduce to (6.22), but with $b_x = 0$, i.e.

$$\frac{\partial \delta\mathbf{B}^{\pm}}{\partial t} + \frac{\partial}{\partial x}\left(u_x \delta\mathbf{B}^{\pm}\right) \pm \tfrac{1}{4}D_x(u_x)\delta\mathbf{B}^{-} \mp \tfrac{1}{4}D_x(u_x)\delta\mathbf{B}^{+} = 0, \tag{6.46}$$

where $\delta\mathbf{B}^{\pm} = (0, 0, \delta B_z^{\pm})^{\mathrm{T}}$. The Elsässer variable form of the wave mixing equations (6.25) and the canonical wave energy equation for Alfvén waves in steady flows also apply, but with $b_x = 0$.

It is of interest to note that if the background flow is a steady cosmic-ray-modified perpendicular shock, in which $\partial\bar{S}/\partial x = 0$ at a generic point in the flow (i.e. $S = \text{const}$), then $\Lambda_{31} = \Lambda_{37} = 0$, because both $\partial\bar{S}/\partial x = 0$ and $B_y \propto \rho$ in a steady perpendicular shock (see e.g. Webb 1983; Webb et al. 1986). Similarly, because $\partial\bar{S}/\partial x = 0$, the entropy wave interaction coefficients $\Lambda_{41} = \Lambda_{47} = 0$. This implies that in a steady perpendicular shock, the slow magnetoacoustic waves are unaffected by the fast-mode waves and the entropy wave. Even although the slow magnetoacoustic waves are unaffected by the other waves, they act as source terms in the fast magnetoacoustic wave equations. Similarly, the entropy waves are unaffected by the other waves, but contribute as a source term in the fast magnetoacoustic equations. Wave interactions in a perpendicular cosmic-ray-modified shock are explored numerically in the next section.

## 7. Numerical examples

In this section, we present examples of wave interactions in a perpendicular, cosmic-ray-modified shock. A typical steady, smooth transition, perpendicular cosmic-ray-modified shock is depicted in Fig. 1, which shows the variation of $(\rho, \mathbf{u}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}, p_g, p_c)^{\mathrm{T}}$ in the shock frame, for a shock transition in which the long-wavelength Mach number $M_{l0} = u_{x0}/(a_{g0}^2 + a_{c0}^2 + V_{A0}^2)^{1/2} = 10$, $p_{c0} = p_{g0} = \rho_0 = 1$ far upstream ($x \to \infty$ far upstream). The adiabatic indices of the cosmic ray and thermal gases $\gamma_c = 1.5$ and $\gamma_g = \frac{5}{3}$, and the hydrodynamically averaged diffusion coefficient $\kappa = 1$ was assumed. Note that a constant diffusion coefficient is appropriate for a perpendicular shock configuration, when the particle diffusive transport across the magnetic field is due to random walk of the field lines (see (2.22) et seq.). The shock transition in Fig. 1 was obtained by solving the steady-state shock structure equation for cosmic-ray-modified shocks (see e.g. Webb 1983; Webb et al. 1986).

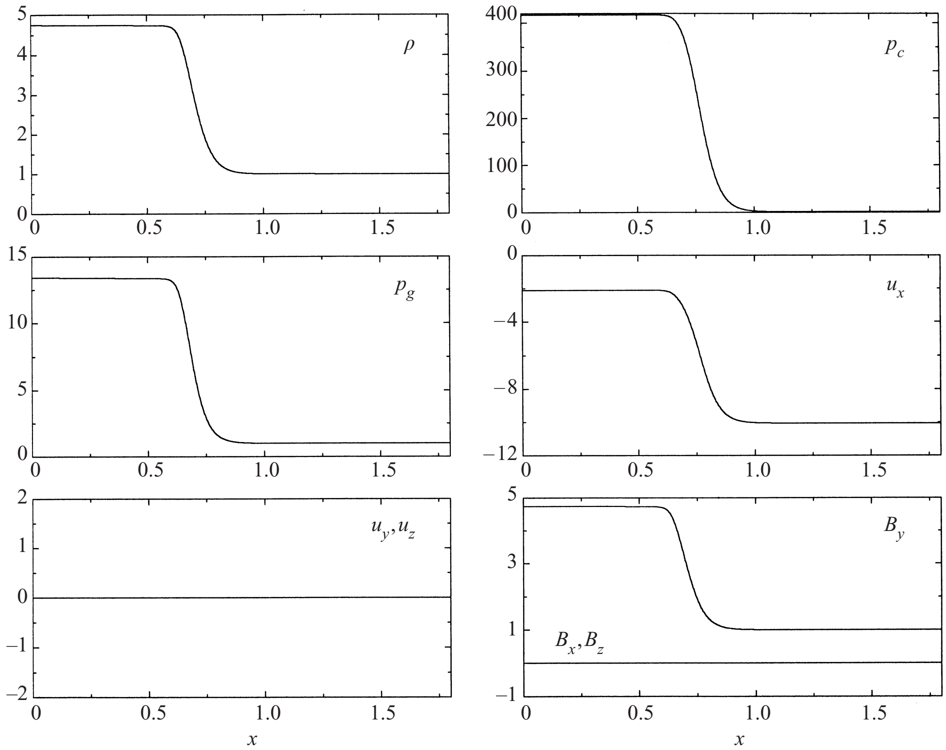


**Figure 1.** A steady-state, smooth-transition, perpendicular cosmic ray modified shock in which the fluid velocity $\mathbf{u} = (u_x, 0, 0)^{\mathrm{T}}$ and magnetic field induction $\mathbf{B} = (0, B_y, 0)^{\mathrm{T}}$ are perpendicular to each other throughout the shock transition. The figure shows the profiles of $(\rho, \mathbf{u}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}, p_g, p_c)^{\mathrm{T}}$ throughout the shock. The diffusion coefficient $\kappa = 1$, $\gamma_g = \frac{5}{3}$ and $\gamma_c = 1.5$. Far upstream, the long-wavelength Mach number $M_{l0} = 10$, and $p_{c0} = p_{g0} = 1$ are the values of $p_c$ and $p_g$ far upstream as $x \to \infty$.

Note that $u_y = u_z = 0$ and $B_x = B_z = 0$ throughout the shock transition. We study how a single wave mode, initially present in the upstream medium, generates and interacts with the other wave modes. The waves are assumed to have their wave vectors $\mathbf{k} = (k_x, 0, 0)^{\mathrm{T}}$ along the $x$ axis, perpendicular to the background magnetic field $\mathbf{B} = (0, B_y, 0)^{\mathrm{T}}$. For this configuration, the slow magnetoacoustic phase speed $c_s = 0$ and the Alfvén phase speed $b_x = 0$, but the corresponding group velocities $\mathbf{V}_{gs} = \pm ab/(a^2 + b^2)^{1/2}\mathbf{e}_b$ for the slow modes and $\mathbf{V}_{gA} = \pm \mathbf{b}$ for the Alfvén modes are non-zero, where $\mathbf{e}_b$ is the unit vector along the magnetic field.

Both spectral code solutions of the wave mixing equations (6.1) and analytical solution results for the Alfvén and slow-mode waves are compared with fully non-linear numerical solutions of the two-fluid MHD cosmic ray model equations (2.1)–(2.8). The basic strategy in solving the initial value problem for the time-dependent model equations (2.1)–(2.8) consists of two steps:

($a$) solve the cosmic ray energy equation (2.8) for $p_c$ for a given flow velocity profile (for example by using a Crank–Nicholson scheme or by using an explicit scheme with subcycling);

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(*b*) solve the MHD equations (2.1)–(2.7) using the Zeus-2D MHD code (Stone and Norman 1992a,b), suitably modified by the inclusion of the cosmic ray pressure gradient.

For the smooth transition shock in Fig. 1, the gas entropy $S$ is constant throughout the structure. In this case, the wave mixing equations (6.1) split up into four subsystems, namely the Alfvén wave interaction equations

$$\frac{\partial a_2}{\partial t} + \frac{\partial}{\partial x}(u_x a_2) + \frac{1}{4}\frac{\partial u_x}{\partial x}(a_6 - a_2) = 0, \tag{7.1a}$$

$$\frac{\partial a_6}{\partial t} + \frac{\partial}{\partial x}(u_x a_6) + \frac{1}{4}\frac{\partial u_x}{\partial x}(a_2 - a_6) = 0, \tag{7.1b}$$

the slow magnetoacoustic subsystem

$$\frac{\partial a_3}{\partial t} + \frac{\partial}{\partial x}(u_x a_3) + \Lambda_{33} a_3 + \Lambda_{35} a_5 = 0, \tag{7.2a}$$

$$\frac{\partial a_5}{\partial t} + \frac{\partial}{\partial x}(u_x a_5) + \Lambda_{53} a_3 + \Lambda_{55} a_5 = 0, \tag{7.2b}$$

the entropy wave equation

$$\frac{\partial a_4}{\partial t} + \frac{\partial}{\partial x}(u_x a_4) = 0, \tag{7.3}$$

and the fast-mode equations:

$$\frac{\partial a_1}{\partial t} + \frac{\partial}{\partial x}(\lambda_1 a_1) + \Lambda_{11} a_1 + \Lambda_{13} a_3 + \Lambda_{14} a_4 + \Lambda_{15} a_5 + \Lambda_{17} a_7 = 0, \tag{7.4a}$$

$$\frac{\partial a_7}{\partial t} + \frac{\partial}{\partial x}(\lambda_7 a_7) + \Lambda_{71} a_1 + \Lambda_{73} a_3 + \Lambda_{74} a_4 + \Lambda_{75} a_5 + \Lambda_{77} a_7 = 0. \tag{7.4b}$$

The detailed forms of the wave interaction coefficients in (7.2)–(7.4) are given by (6.43)–(6.45) and by the wave reversal symmetry relations (6.10). From (7.1)–(7.4), the Alfvén waves, the slow magnetoacoustic waves and the entropy wave are unaffected by the fast-mode waves. The slow magnetoacoustic and entropy waves act as source terms in the fast-mode equations (7.4), and hence the fast-mode waves can be generated from the slow-mode waves and the entropy wave by wave mixing. However, the Alfvén waves cannot generate fast-mode waves by linear wave mixing, since there are no Alfvén wave source terms in the fast-mode equations (7.4).

For the above configuration, the solutions for the Alfvén waves and slow-mode waves can be obtained in closed form. Noting that $b_x = 0$ in a perpendicular shock, the Alfvén wave mixing equations (6.27)–(6.29) reduce to

$$\frac{\partial f}{\partial t} + u_x \frac{\partial f}{\partial x} = u_x \frac{\partial \psi}{\partial x} g, \tag{7.5a}$$

$$\frac{\partial g}{\partial t} + u_x \frac{\partial g}{\partial x} = u_x \frac{\partial \psi}{\partial x} f, \tag{7.5b}$$

where $\psi = \frac{1}{4}\ln\rho$ and

$$f = \delta B_z^- u_x \rho^{1/4}, \tag{7.6a}$$

$$g = \delta B_z^+ u_x \rho^{1/4}. \tag{7.6b}$$

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The general solutions of (7.5) are

$$f = A(\theta)\rho^{1/4} + B(\theta)\rho^{-1/4}, \tag{7.7}$$

$$g = A(\theta)\rho^{1/4} - B(\theta)\rho^{-1/4}, \tag{7.8}$$

where $A(\theta)$ and $B(\theta)$ are arbitrary functions of the phase variable

$$\theta = \omega\left(t - \int_{x_0}^{x} \frac{dx}{u_x}\right) \tag{7.9}$$

(see Appendix B), and $\omega$ is a constant frequency. It is of interest to note from (7.7) and (7.8) that

$$f^2 - g^2 = 2AB, \tag{7.10}$$

and hence $f^2 - g^2$ is constant if $B(\theta) \propto 1/A(\theta)$. Equation (7.10) is clearly related to the wave action equation (6.30), where $g^2 - f^2$ is the wave action flux. From (7.6), the magnetic field perturbations $\delta B_z^{\pm}$ for the backward and forward Alfvén waves are of the form

$$\delta B_z^{\pm} = \hat{A}(\theta)\bar{\rho} \mp \hat{B}(\theta)\bar{\rho}^{1/2}, \tag{7.11}$$

where

$$\bar{\rho} = \frac{\rho}{\rho_0}, \qquad u_x = \frac{u_{x0}}{\bar{\rho}}, \tag{7.12a}$$

$$\hat{A}(\theta) = \frac{A(\theta)}{u_{x0}}, \qquad \hat{B}(\theta) = \frac{B(\theta)}{u_{x0}\rho_0^{1/2}}, \tag{7.12b}$$

and $x = x_0$ is a fixed point in the flow. In particular, the choice $\hat{A} = \hat{B} = \frac{1}{2}\delta B_{z0}^{-} \sin\theta$ in (7.11) and (7.12) yields the solutions:

$$\delta B_z^{-} = \tfrac{1}{2}\delta B_{z0}^{-} \sin\theta(\bar{\rho} + \bar{\rho}^{1/2}), \tag{7.13a}$$

$$\delta B_z^{+} = \tfrac{1}{2}\delta B_{z0}^{-} \sin\theta(\bar{\rho} - \bar{\rho}^{1/2}), \tag{7.13b}$$

for $\delta B_z^{-}$ and $\delta B_z^{+}$.

Figure 2 shows an example of a spectral code solution of the Alfvén wave mixing equations (7.1), for the case where $\delta B_z^{-} \equiv B_{2z}$ is initially specified at time $t = 0$ as a sine wave profile far upstream of the shock of Fig. 1 (top panel). The solutions for $B_{2z}$ and $B_{6z}$ are shown at two later times $t_2 = 0.03 \equiv 17.24t_d$ and $t_3 = 0.09 \equiv 51.71t_d$, where $t_d = \kappa/u_{\rm sh}^2$ is the convection diffusion time scale and $u_{\rm sh}$ is the shock speed, or upstream flow speed relative to the shock frame (lower two panels). The lower panels show the interaction of the forward Alfvén wave $B_{6z}$ with the backward wave $B_{2z}$ due to wave mixing. The amplitudes of both waves increase and the wavelengths decrease as the waves pass through the shock transition into the downstream region. Far downstream (bottom panel), both backward and forward waves have constant wave amplitude and wavelength, and are advected downstream with the flow.

From (7.13), the ratio of the wave amplitudes far downstream of the shock is given by

$$\frac{\delta B_{zd}^{-}}{\delta B_{zd}^{+}} = \frac{r_c^{1/2} + 1}{r_c^{1/2} - 1}, \tag{7.14}$$

where $r_c = \rho_d/\rho_u$ is the shock compression ratio, and the subscripts $u$ and $d$ refer to the upstream and downstream states. For the shock in Figs 1 and 2, $\delta B_{zd} = 0.0336$,

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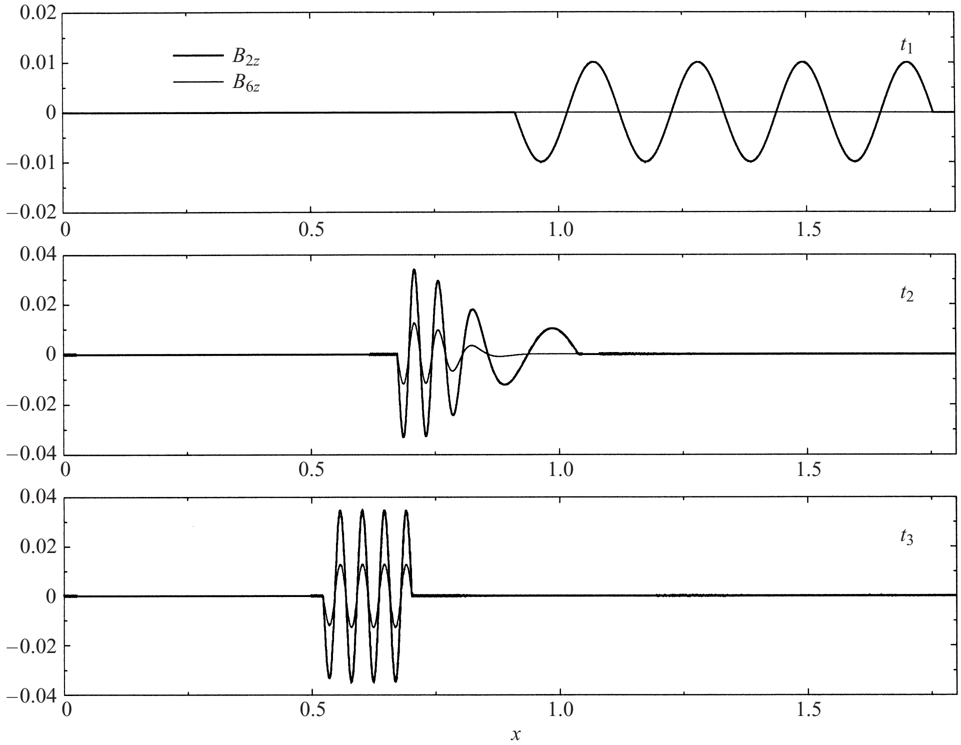


**Figure 2.** Spectral code solutions of the Alfvén wave mixing equations (7.1) for the case where the backward Alfvén wave $B_{2z} \equiv \delta B_z^-$ is initially specified (at time $t = t_1 = 0$) far upstream of the shock of Fig. 1 (top panels). The magnetic field perturbations $B_{2z}$ and $B_{6z}$ for the backward and forward Alfvén waves are shown at times $t_2 = 0.03$ and $t_3 = 0.09$. Numerical solutions of the two-fluid equations (2.1)–(2.8) with the same initial conditions yield solutions for $B_{2z}$ and $B_{6z}$ that are indistinguishable from the spectral code solutions.

$\delta B_{zu} = 0.0124$ and the ratio $\delta B_{zd}^-/\delta B_{zd}^+ = 2.71$. The shock compression ratio $r_c = 4.74$. Hence the spectral code solutions yield a value for $\delta B_{zd}^-/\delta B_{zd}^+$ that agrees with the analytical solution result (7.14). The local length scale of the sinusoidally varying profile in (7.13) is given by $l = \theta_x^{-1} = u_x/\omega$, and hence

$$l_d = \frac{l_u}{r_c} \tag{7.15}$$

gives the downstream scale length $l_d$ in terms of the upstream scale length $l_u$ and the shock compression ratio $r_c$. The ratio $l_d/l_u = 0.21$ from (7.15), is in reasonable agreement with the spectral code solution results ($l_d = 0.0215$ and $l_u = 0.105$). The decrease in the wavelength as the fluid compresses is apparent in the spectral code solutions in Fig. 2.

The spectral code solutions of the wave mixing equations (7.1) were also compared with numerical solutions of the two-fluid MHD cosmic ray model equations (2.1)–(2.8). The numerical solutions are also plotted in Fig. 2, but the differences between the two solutions are very small, and are not apparent in the figure.

Figure 3 shows spectral code solutions of the wave mixing equations (7.1)–(7.4) for the case where the initial data consists of a backward slow-mode wave train far upstream of the shock in Fig. 1. The left panels show the density perturbations

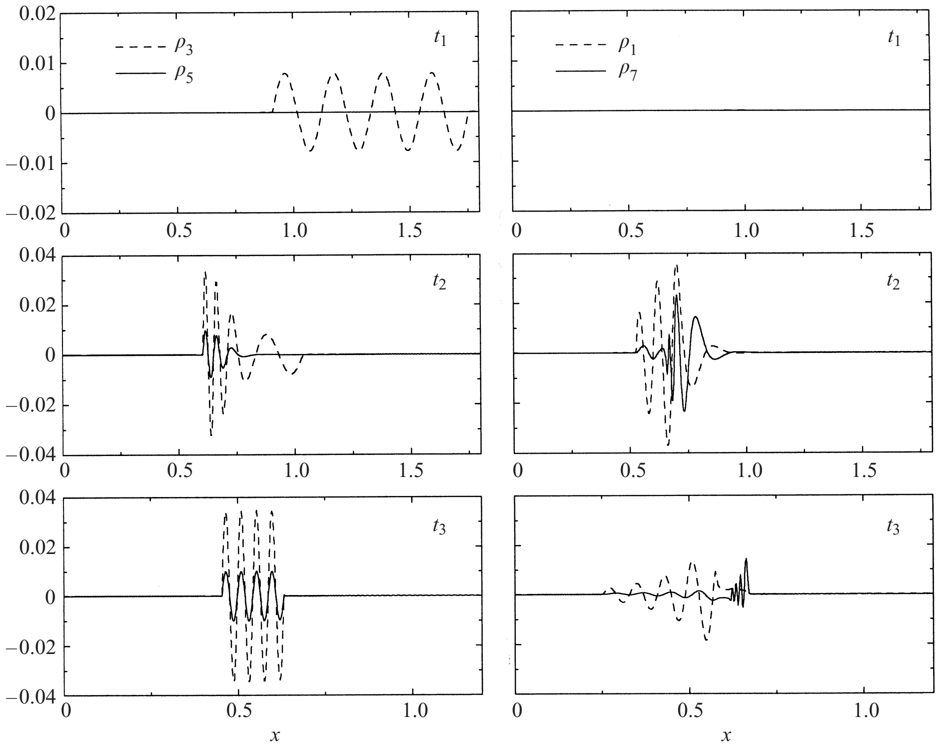


**Figure 3.** Spectral code solutions of the wave mixing equations (7.1)–(7.4), for the case where a backward slow-mode wave ($\rho_3$) is specified far upstream, at time $t = t_1 = 0$, of the cosmic-ray-modified shock in Fig. 1. The left panels show the density perturbations $\rho_3$ and $\rho_5$ for the backward and forward slow-mode waves at times $t_2 = 0.03$ (middle panel) and $t_3 = 0.06$ (bottom panel). The density perturbations $\rho_1$ and $\rho_7$ for the backward and forward fast-mode waves are shown in the right panels.

$\rho_3$ and $\rho_5$ for the backward and forward slow-mode waves at times $t_2 = 0.03$ and $t_3 = 0.06$, whereas the right panels show the fast-mode wave density perturbations $\rho_1$ and $\rho_7$ that have been generated by wave mixing. The wave coupling coefficients (6.43) for the backward fast-mode wave ($\rho_1$) and the corresponding coefficients for the forward fast-mode wave ($\rho_7$) indicate that fast-mode waves generated from the slow-mode waves are subject to the cosmic ray squeezing instability, and to wave damping due to the diffusing cosmic rays, and are modified by MHD wave mixing effects. The backward fast-mode wave is amplified owing to the cosmic ray squeezing instability, which is similar to the results obtained by Webb et al. (1999) for the case of backward-propagating sound waves upstream of a cosmic-ray-modified shock, in which the magnetic field plays no dynamical role, and for which $\kappa = \text{const}$. Note that the forward fast-mode solution $\rho_7$ is approximately $\frac{1}{2}\pi$ out of phase with the backward fast mode solution $\rho_1$.

The slow-mode solutions $\rho_3$ and $\rho_5$ in Fig. 3 are similar to the Alfvén wave solutions depicted in Fig. 2. One can show that the solutions of the slow-mode wave mixing equations (7.2) for the wave density perturbations $\rho_3$ and $\rho_5$ have the form

$$\rho_3 = \frac{\alpha_s^{3/2}}{u_x a^{1/2}}[A_s(\theta)\rho^{1/4} + B_s(\theta)\rho^{-1/4}], \tag{7.16a}$$

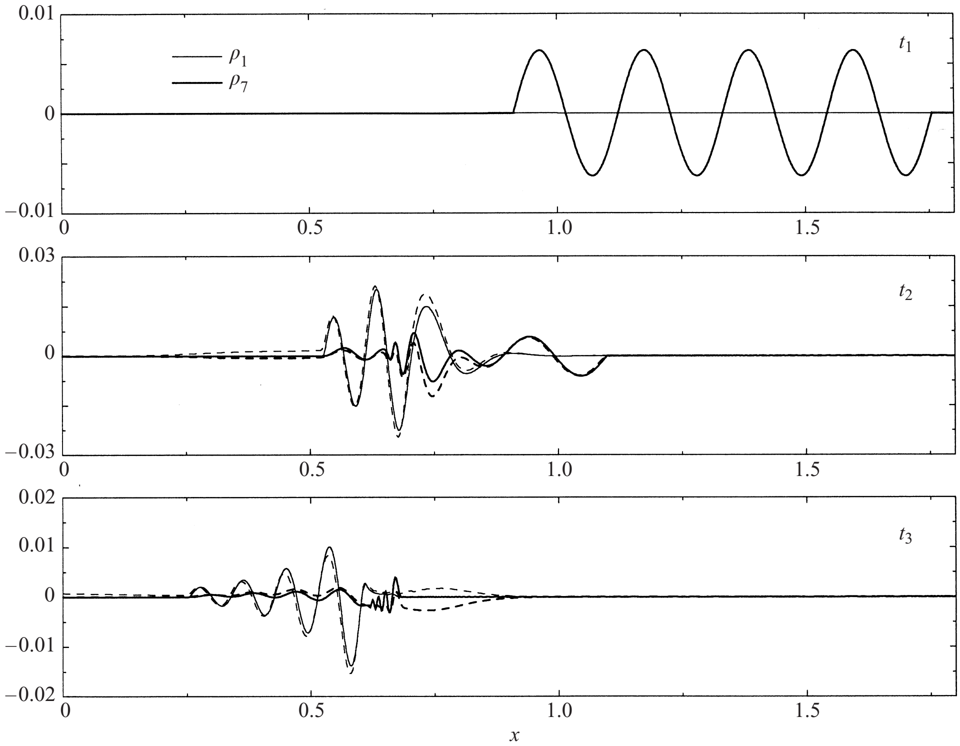


**Figure 4.** Spectral code solutions of the wave mixing equations (7.4) for the backward ($\rho_1$) and forward ($\rho_7$) fast-mode wave density perturbations are compared with numerical solutions of (2.1)–(2.8) for the shock of Fig. 1, in which a forward fast-mode wave ($\rho_7$) is initially specified far upstream at time $t = t_1 = 0$. The two lower panels show the spectral code solutions (solid lines) and numerical solutions (dashed curves) at times $t_2 = 0.03$ and $t_3 = 0.06$.

$$\rho_5 = \frac{\alpha_s^{3/2}}{u_x a^{1/2}}[A_s(\theta)\rho^{1/4} - B_s(\theta)\rho^{-1/4}], \tag{7.16b}$$

where $A_s(\theta)$ and $B_s(\theta)$ are arbitrary functions of $\theta$. Thus the slow-mode solutions (7.16) and the solutions for $\rho_3$ and $\rho_5$ in Fig. 3 have similar wave forms to the Alfvén wave solutions in Fig. 2.

Figure 4 shows spectral code solutions of the wave mixing equations (7.4) for the case where a forward fast-mode wave ($\rho_7$) is initially specified upstream of the shock (top panel). The two lower panels show the spectral code solutions (solid lines) at two later time instants $t_2 = 0.03$ and $t_3 = 0.06$. For comparison, the dashed curves show solutions of the fully nonlinear two-fluid equations (2.1)–(2.8) with the same initial conditions. The fully nonlinear solutions follow the wave mixing solutions fairly well, except at late times ($t = t_3$), where there is some discrepancy between the solutions in the region $x > 0.7$. There is a significant wave growth due to the steep cosmic ray pressure gradient in the middle of the shock transition. At late times $t = t_3$, the generated backward wave is more dominant than the forward wave. Far downstream, both waves are strongly damped owing to the $a_c^2/\kappa$ terms in the wave mixing coefficients.

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It is clear from (7.1)–(7.4), that fast-mode waves can be generated from entropy waves initially present upstream of the shock.

## 8. Summary and discussion

The main aim of this paper has been a study of wave interactions in magnetohydrodynamics, with application to cosmic-ray-modified shocks.

The method of multiple scales has been used to obtain equations describing the interaction of weakly nonlinear short-wavelength MHD waves in a non-uniform large-scale background flow. The linear terms in the equations describe the interaction of the waves due to gradients and time dependence of the background flow (wave mixing), wave damping due to the diffusing cosmic rays (Ptuskin 1981), and squeezing instability terms associated with the large-scale cosmic ray pressure gradient (see e.g. Dorfi and Drury 1985; Drury and Falle 1986; Zank and McKenzie 1987). The averaged wave evolution equations (4.43) also contain weakly nonlinear interaction terms describing (a) Burgers self wave steepening for the magnetoacoustic modes; (b) three-wave resonant interactions; and (c) mean wave interaction effects representing the interaction of a given wave mode with the mean wave field of the other waves.

For the case of a uniform background flow, in which there are no cosmic ray and mean wave field effects, the equations reduce to MHD wave interaction equations obtained by Ali and Hunter (1998) (Sec. 5). The wave interaction equations of Ali and Hunter (1998), consist of coupled integro-differential Burgers equations, with integral terms describing the resonant interaction of two waves to generate a third wave (or the decay of one of the waves into two lower-frequency waves), if the resonance conditions are satisfied. Resonant interactions are more liable to be significant for long nearly periodic wave trains than for short wave trains or pulses, since the wave interactions are strengthened the longer the the waves interact. For wave propagation in non-uniform media, the waves in resonance in a localized region of $(x, t)$ space will in general pass out of resonance, since the frequencies and wavenumbers of the waves will detune in a non-uniform medium. Similar equations were obtained by Majda and Rosales (1984) describing the resonant interaction of sound waves and entropy waves in one Cartesian space dimension (see also Hunter et al. (1986) for similar equations describing the resonant interaction of sound waves, entropy waves and vortex eigenmodes in two or more space dimensions). Galeev and Oraevski (1963) (see also Sagdeev and Galeev 1969) derived related equations describing the resonant decay of a high-frequency Alfvén wave into a lower-frequency Alfvén wave and a lower-frequency sound wave. The equations are complementary to weak turbulence equations involving resonant wave interactions (see e.g. Galeev and Karpman 1963; Zakharov et al. 1992).

For linear wave propagation in inhomogeneous media, one obtains wave mixing equations of the form (6.1), namely

$$\frac{\partial a_j}{\partial t} + \frac{\partial}{\partial x}(\lambda_j a_j) + \sum_{s=1}^{7} \Lambda_{js} a_s = 0, \quad j = 1, \ldots, 7, \tag{8.1}$$

where the $\{a_j\}$ denote the wave amplitudes and $\mathbf{V}_{pj} = \lambda_j \mathbf{e}_x$ is the characteristic eigen-velocity of the $j$th wave mode along the $x$ axis. The wave mixing coefficients $\{\Lambda_{js}\}$ in (8.1) depend on the gradients in the background flow, cosmic ray coupling effects due to the large scale cosmic-ray pressure gradient, and damping of the waves due to the diffusing cosmic rays (Ptuskin 1981). The character of the cosmic ray squeezing instability depends on the large-scale cosmic ray pressure gradient and also on the parameter $\zeta = \partial \ln \kappa / \partial \ln \rho$, where $\kappa = \kappa(\rho)$ is the effective hydrodynamical cosmic ray diffusion coefficient and $\rho$ is the density of the thermal gas. In the case of MHD wave propagation in the absence of cosmic rays, the wave mixing coefficients $\{\Lambda_{js}\}$ coupling the different wave modes have the simple form

$$\Lambda_{js} = \mathbf{L}_j \cdot \frac{d\mathbf{R}_s}{dt_s}, \quad 1 \leqslant j \leqslant 7, \quad 1 \leqslant s \leqslant 7. \tag{8.2}$$

In (8.2), the $\{\mathbf{L}_j\}$ and $\{\mathbf{R}_s\}$ are the left- and right-eigenvectors for the MHD equations, for the case where the conserved densities $\tilde{\boldsymbol{\Psi}} = (\rho, \rho\mathbf{u}^{\mathrm{T}}, B_y, B_z, \rho S)^{\mathrm{T}}$ are used as the state vector, and $d/dt_s = \partial/\partial t + \lambda_s \partial/\partial x$ is the time derivative along the $s$th wave mode characteristic (the more general form of the $\Lambda_{js}$ when cosmic ray effects are included is given by (6.2)). For MHD wave propagation in one Cartesian space dimension, there are 49 wave interaction coefficients in (8.2). The detailed expressions for the $\{\Lambda_{js}\}$ in (6.3)–(6.10) reveal that the Alfvén waves are decoupled from the magnetoacoustic and entropy waves for flows in which $\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u} = \mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B} = 0$. In particular, for planar MHD flows in which $\mathbf{B} = (B_x, B_y, 0)^{\mathrm{T}}$ and $\mathbf{u} = (u_x, u_y, 0)^{\mathrm{T}}$, the Alfvén wave equations are a special case of the wave mixing equations for Alfvén waves, and Alfvénic turbulence in the solar wind (see e.g. Heinemann and Olbert 1980; Zhou and Matthaeus 1990).

The formula (8.2) also reveals the special role of simple wave background flows. For a simple wave flow of the $n$th wave mode, $d\mathbf{R}_n/dt_n = 0$, and hence $\Lambda_{jn} = 0$. In this case, the $n$th wave mode does not affect the other modes with $j \neq n$. In general, $\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u} \neq 0$, and $\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B} \neq 0$ for Alfvén simple waves. In this case, the Alfvén waves are modified by their interaction with the magnetoacoustic and entropy waves, but the magnetoacoustic waves and entropy waves, in turn, are not affected by the Alfvén waves. On the other hand, magnetoacoustic simple waves are characterized by zero field aligned current and vorticity (i.e. $\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{u} = \mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{B} = 0$). In the latter case, the Alfvén waves do not interact with the magnetoacoustic and entropy waves, since the Alfvén wave interaction coefficients $\Lambda_{2j}$, $\Lambda_{j2}$, $\Lambda_{6j}$ and $\Lambda_{j6}$, $j \neq 2, 6$, are all zero.

Numerical simulations of the fully nonlinear two-fluid MHD equations (2.1)–(2.8) have been compared with spectral code solutions of the wave mixing equations (6.1) for the case of a steady-state perpendicular cosmic-ray-modified shock (Sec. 7). These calculations complement a similar study of the wave mixing of sound waves and entropy waves in cosmic-ray-modified shocks by Webb et al. (1999) for the case where the magnetic field plays no dynamical role. Further simulations are needed to more fully understand the role of the magnetic field in the wave mixing of the MHD waves in oblique MHD cosmic-ray-modified shocks.

For wave propagation perpendicular to the magnetic field, the Alfvén waves, the entropy wave and the slow magnetoacoustic phase speeds are all zero in the reference frame of the background fluid. For this configuration, the wave mixing equations split into four separate subsystems, namely (a) the Alfvén wave mixing equations (7.1); (b) the slow-mode equations (7.2); (c) the entropy wave equation (7.3); and (d) the fast-mode equations (7.4). The systems (a), (b) and (c) for the Alfvén, slow and entropy waves are independent of each other, but the fast-mode equations (7.4) contain source terms due to the slow-mode and entropy waves.

It is of interest to note that, for wave propagation exactly perpendicular to **B**, the fast-mode waves are not Landau-damped, but the slow-mode waves are damped (Barnes 1966, 1979). For wave propagation oblique to the magnetic field, both slow-mode and fast-mode waves are Landau-damped. Landau damping of the magnetoacoustic waves in general increases with increasing the plasma beta. It is in general difficult to assess the role of Landau damping of the waves, without detailed calculations. In a more complete model, one should also take into account the momentum spectrum of the cosmic rays, but these issues are beyond the scope of the present paper.

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## Appendix A

In this appendix, we show that the wave mixing coefficients $\{\Lambda_{js}\}$ of (4.31) and (4.38) are invariant under a transformation of the form:

$$\tilde{\boldsymbol{\Psi}}' = \boldsymbol{\Phi}(\tilde{\boldsymbol{\Psi}}), \tag{A 1}$$

where $\det(\partial\tilde{\Psi}'^{\alpha}/\partial\tilde{\Psi}^{\beta}) \neq 0$, provided that the wave amplitudes $\{a'_s\}$ are chosen to be invariant under the transformation (i.e. $a'_s = a_s$). We also show that the nonlinear wave interaction coefficients $\Gamma_{jjq}$ (4.36), and the symmetric interaction coefficients $\hat{\Gamma}_{jpq}$, (4.46), are also invariant under the transformation, but the coefficients $\Gamma_{jpq}$ in (4.33) are in general not invariant under the transformation.

To prove the invariance of the linear wave mixing coefficients ($\Lambda'_{js} = \Lambda_{js}$), first note from (4.31) that

$$\begin{aligned}\Lambda'_{js} = -\frac{\partial\lambda_s}{\partial\bar{x}}\delta^j{}_s + \mathbf{L}'_j\cdot\left[\frac{\partial\mathbf{R}'_s}{\partial\bar{t}} + \lambda_j\frac{\partial\mathbf{R}'_s}{\partial\bar{x}} + \mathbf{R}'_s\cdot\boldsymbol{\nabla}_{\tilde{\boldsymbol{\Psi}}'}\mathscr{A}'\cdot\frac{\partial\tilde{\boldsymbol{\Psi}}'^{(0)}}{\partial\bar{x}}\right.\\ \left. +\mathbf{R}'_s\cdot\boldsymbol{\nabla}_{\tilde{\boldsymbol{\Psi}}'}\mathbf{C}'\frac{\partial p_c^0}{\partial\bar{x}} + \mathbf{C}'\left(\frac{a_c^2}{\bar{\kappa}}(R_s^2 - u_xR_s^1) - \frac{\zeta}{\rho}\frac{\partial p_c^0}{\partial\bar{x}}R_s^1\right)\right]\end{aligned} \tag{A 2}$$

is the general form of the wave mixing coefficients when $\tilde{\boldsymbol{\Psi}}'$ is used as the state vector in the perturbation analysis. Using the results (3.33)–(3.36b), we have the transformations:

$$\mathscr{A}' = \mathsf{P}\cdot\mathscr{A}\cdot\mathsf{Q},\qquad \mathbf{C}' = \mathsf{P}\cdot\mathbf{C},\qquad \mathbf{R}'_s = \mathsf{P}\cdot\mathbf{R}_s,\qquad \mathbf{L}'_j = \mathbf{L}_j\cdot\mathsf{Q}, \tag{A 3}$$

relating $\mathscr{A}'$, $\mathbf{C}'$, $\mathbf{R}'_s$ and $\mathbf{L}'_j$ to their corresponding forms when $\tilde{\boldsymbol{\Psi}}$ is used as the state vector. Note $\mathbf{C}$ is defined in (4.3), and $\mathbf{C}'$ is the corresponding form for $\mathbf{C}$ when $\tilde{\boldsymbol{\Psi}}'$ is used as the state vector.

Using the results (A 3) in (A 2) we find

$$\mathbf{L}'_j\cdot\frac{\partial\mathbf{R}'_s}{\partial\bar{t}} = \mathbf{L}_j\cdot\frac{\partial\mathbf{R}_s}{\partial\bar{t}} - \mathbf{L}_j\cdot\mathsf{Q}\cdot(\mathbf{R}_s\cdot\boldsymbol{\nabla}_{\tilde{\boldsymbol{\Psi}}}\mathsf{P})\cdot\left(\mathscr{A}\cdot\frac{\partial\tilde{\boldsymbol{\Psi}}^{(0)}}{\partial\bar{x}} + \mathbf{C}\frac{\partial p_c^0}{\partial\bar{x}}\right), \tag{A 4a}$$

$$\lambda_j \mathbf{L}'_j \cdot \frac{\partial \mathbf{R}'_s}{\partial \bar{x}} = \lambda_j \mathbf{L}_j \cdot \left( \frac{\partial \mathbf{R}_s}{\partial \bar{x}} + \mathsf{Q} \cdot (\mathbf{R}_s \cdot \nabla_{\tilde{\Psi}} \mathsf{P}) \cdot \frac{\partial \tilde{\Psi}^{(0)}}{\partial \bar{x}} \right), \tag{A 4b}$$

$$\mathbf{L}'_j \cdot (\mathbf{R}'_s \cdot \nabla_{\tilde{\Psi}'} \mathscr{A}') \cdot \frac{\partial \tilde{\Psi}'^{(0)}}{\partial \bar{x}} = \mathbf{L}_j \cdot \left[ \mathbf{R}_s \cdot \nabla_{\tilde{\Psi}} \mathscr{A} \cdot \frac{\partial \tilde{\Psi}^{(0)}}{\partial \bar{x}} + \mathsf{Q} \cdot (\mathbf{R}_s \cdot \nabla_{\tilde{\Psi}} \mathsf{P}) \cdot (\mathscr{A} - \lambda_j \mathsf{I}) \cdot \frac{\partial \tilde{\Psi}^{(0)}}{\partial \bar{x}} \right], \tag{A 4c}$$

$$\mathbf{L}'_j \cdot (\mathbf{R}'_s \cdot \nabla_{\tilde{\Psi}'} \mathbb{C}') \frac{\partial p_c^0}{\partial \bar{x}} = \mathbf{L}_j \cdot \mathsf{Q} \cdot (\mathbf{R}_s \cdot \nabla_{\tilde{\Psi}} \mathsf{P}) \cdot \mathbb{C} \frac{\partial p_c^0}{\partial \bar{x}}, \tag{A 4d}$$

$$\mathbf{L}'_j \cdot \mathbb{C}' = \mathbf{L}_j \cdot \mathbb{C}. \tag{A 4e}$$

Using the results (A 4) in (A 2) we find

$$\Lambda'_{js} = \Lambda_{js}, \tag{A 5}$$

where $\Lambda_{js}$ is given by (4.31). If the state vector $\tilde{\Psi}$ is the conserved densities state vector (3.2) then $\Lambda_{js}$ may be reduced to the simpler form (4.38).

To prove that the nonlinear wave interaction coefficients $\Gamma_{jjq}$ are invariant under (A 1), we note that

$$\Gamma'_{jjq} = \mathbf{R}'_q \cdot \nabla_{\tilde{\Psi}'} \lambda_j = (\mathsf{P} \cdot \mathbf{R}_q) \cdot (\nabla_{\tilde{\Psi}} \lambda_j \cdot \mathsf{Q}) \equiv \mathbf{R}_q \cdot \nabla_{\tilde{\Psi}} \lambda_j = \Gamma_{jjq}, \tag{A 6}$$

where we have used the fact that $\mathsf{Q} \cdot \mathsf{P} = \mathsf{I}$. To prove that the symmetric nonlinear wave interaction coefficient $\hat{\Gamma}_{jpq}$ is invariant under the transformation, we first note from (4.46) that

$$\hat{\Gamma}'_{jpq} = \frac{k_j(\lambda_q - \lambda_j)(\lambda_p - \lambda_j)}{\lambda_q - \lambda_p} \mathbf{L}'_j \cdot [\mathbf{R}'_q, \mathbf{R}'_p]. \tag{A 7}$$

Using the transformation (A 3), we find

$$\mathbf{L}'_j \cdot [\mathbf{R}'_q, \mathbf{R}'_p] = \mathbf{L}_j \cdot [\mathbf{R}_q, \mathbf{R}_p]. \tag{A 8}$$

Using the result (A 8) in (A 7) implies that $\hat{\Gamma}'_{jpq} = \hat{\Gamma}_{jpq}$. However, the non-symmetric wave interaction coefficients

$$\Gamma'_{jpq} = \mathbf{L}'_j \cdot (\mathbf{R}'_q \cdot \nabla_{\tilde{\Psi}'} \mathscr{A}' \cdot \mathbf{R}'_p), \tag{A 9}$$

in (4.33) are in general not invariant under the transformation (A 1). From (4.35),

$$\Gamma'_{jpq} = (\lambda_p - \lambda_j) \mathbf{L}'_j \cdot (\mathbf{R}'_q \cdot \nabla_{\tilde{\Psi}'} \mathbf{R}'_p) \tag{A 10}$$

for $p \neq j$. Using the transformation (3.36b) in (A 10), we find

$$\Gamma'_{jpq} = \Gamma_{jpq} + (\lambda_p - \lambda_j) \mathbf{L}_j \cdot \mathsf{Q} \cdot (\mathbf{R}_q \cdot \nabla_{\tilde{\Psi}} \mathsf{P}) \cdot \mathbf{R}_p, \tag{A 11}$$

and hence, in general, $\Gamma'_{jpq} \neq \Gamma_{jpq}$.

## Appendix B

In this appendix, we discuss solutions of the Alfvén wave mixing equations (6.27) and (6.28). For the case of a perpendicular cosmic-ray-modified shock, we derive the exact wave mixing equation solutions (7.7) and (7.8).

For a steady background flow, (6.27) and (6.28) for $f$ and $g$ have solutions of the form

$$f = F \exp(i\omega t) + \text{c.c.}, \tag{B 1a}$$

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$$g = G\exp(i\omega t) + \text{c.c.}, \tag{B 1b}$$

where $F$ and $G$ satisfy the ordinary differential equations

$$\frac{dF}{dx} + \frac{i\omega}{u_x - b_x}G = \psi_x G, \tag{B 2a}$$

$$\frac{dG}{dx} + \frac{i\omega}{u_x + b_x}G = \psi_x F. \tag{B 2b}$$

Equations (B 2) can also be written in the form

$$\frac{d\hat{F}}{dx} = \psi_x \hat{G}\exp(i\omega\phi), \tag{B 3a}$$

$$\frac{d\hat{G}}{dx} = \psi_x \hat{F}\exp(-i\omega\phi), \tag{B 3b}$$

where

$$\hat{F} = F\exp(i\omega\phi_-), \tag{B 4a}$$

$$\hat{G} = \exp(i\omega\phi_+), \tag{B 4b}$$

$$\phi_\pm = \int_{x_0}^{x} \frac{dx}{u_x \pm b_x}, \qquad \phi = \phi_- - \phi_+. \tag{B 5}$$

Equations (B 3) may be combined to yield the integral

$$|\hat{F}|^2 - |\hat{G}|^2 \equiv |F|^2 - |G|^2 = \text{const}, \tag{B 6}$$

which is the Fourier-space wave action integral (for the case of Alfvén waves in the solar wind, see e.g. Heinemann and Olbert 1980).

Because $\psi = \frac{1}{4}\ln\rho$ is a monotonic function of $x$ in the smoothed upstream foreshock of a cosmic-ray-modified shock, (B 3) may be written in the form

$$\frac{d\hat{F}}{d\psi} = \hat{G}\exp(i\omega\phi), \tag{B 7a}$$

$$\frac{d\hat{G}}{d\psi} = \hat{F}\exp(-i\omega\phi), \tag{B 7b}$$

where $\psi$ is the new independent variable. From (B 7), we find that $\hat{F}$ and $\hat{G}$ satisfy the ordinary differential equations:

$$\frac{d^2\hat{F}}{d\psi^2} - i\omega\frac{d\phi}{d\psi}\frac{d\hat{F}}{d\psi} - \hat{F} = 0, \tag{B 8a}$$

$$\frac{d^2\hat{G}}{d\psi^2} + i\omega\frac{d\phi}{d\psi}\frac{d\hat{G}}{d\psi} - \hat{G} = 0, \tag{B 8b}$$

For the case of a perpendicular cosmic-ray-modified shock, the Alfvén phase speed $b_x = 0$, and hence, from (B 5), $\phi_+ = \phi_-$ and $\phi = 0$. In this case, (B 8) reduce to

$$\frac{d^2\hat{F}}{d\psi^2} - \hat{F} = 0, \tag{B 9a}$$

$$\frac{d^2\hat{G}}{d\psi^2} - \hat{G} = 0. \tag{B 9b}$$

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From (B 7) and (B 9), the general solutions for $F$ and $G$ are

$$\hat{F} = \tilde{A}\exp(\psi) + \tilde{B}\exp(-\psi), \tag{B 10a}$$

$$\hat{G} = \tilde{A}\exp(\psi) - \tilde{B}\exp(-\psi), \tag{B 10b}$$

where $\tilde{A}(\omega)$ and $\tilde{B}(\omega)$ are arbitrary functions of $\omega$. The corresponding solutions (B 1) for $f$ and $g$ have the form

$$f_\omega = \exp(i\omega\theta)[\tilde{A}(\omega)\rho^{1/4} + \tilde{B}(\omega)\rho^{-1/4}] + \text{c.c.}, \tag{B 11a}$$

$$g_\omega = \exp(i\omega\theta)[\tilde{A}(\omega)\rho^{1/4} - \tilde{B}(\omega)\rho^{-1/4}] + \text{c.c.} \tag{B 11b}$$

Linear superposition of the Fourier mode solutions (B 11), and use of Fourier's theorem, now yields the solutions (7.7) and (7.8) for $f$ and $g$.

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