Adv. Appl. Prob. **49,** 1144–1169 (2017)
*doi:*10.1017/apr.2017.37

EXPONENTIAL ERGODICITY OF AN AFFINE TWO-FACTOR MODEL BASED ON THE α-ROOT PROCESS

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Abstract

We study an affine two-factor model introduced by Barczy *et al.* (2014). One component of this two-dimensional model is the so-called α-root process, which generalizes the well-known Cox–Ingersoll–Ross process. In the $\alpha = 2$ case, this two-factor model was used by Chen and Joslin (2012) to price defaultable bonds with stochastic recovery rates. In this paper we prove exponential ergodicity of this two-factor model when $\alpha \in (1, 2)$. As a possible application, our result can be used to study the parameter estimation problem of the model.

Keywords: Affine process; exponential ergodicity; α-root process; transition density; Foster–Lyapunov function

2010 Mathematics Subject Classification: Primary 60J25; 37A25
Secondary 60J35; 60J75

1. Introduction

In this paper we study a two-dimensional affine process $(Y, X) := (Y_t, X_t)_{t\geq 0}$ determined by the following stochastic differential equation (SDE):

$$\begin{aligned} \mathrm{d}Y_t &= (a - bY_t)\,\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\,\mathrm{d}L_t, \qquad t \geq 0, \qquad Y_0 \geq 0 \quad \text{almost surely}, \\ \mathrm{d}X_t &= (m - \theta X_t)\,\mathrm{d}t + \sqrt{Y_t}\,\mathrm{d}B_t, \qquad t \geq 0, \end{aligned} \tag{1.1}$$

where $a > 0, b > 0, \theta, m \in \mathbb{R}$, $\alpha \in (1, 2)$, $(L_t)_{t\geq 0}$ is a spectrally positive α-stable Lévy process with the Lévy measure $C_\alpha z^{-1-\alpha}\,\mathbf{1}_{\{z>0\}}\,\mathrm{d}z$ with $C_\alpha := (\alpha\Gamma(-\alpha))^{-1}$, and $(B_t)_{t\geq 0}$ is an independent standard Brownian motion. Note that if (Y_0, X_0) is independent of $(L_t, B_t)_{t\geq 0}$ then the existence and uniqueness of a strong solution to SDE (1.1) follow from [4, Theorem 2.1]. We remark that the two-factor model (1.1) is actually well-defined for an arbitrary real constant b (see [4]), while in this paper we restrict ourselves to the so-called subcritical case $(b > 0)$.

The process $(Y_t, X_t)_{t\geq 0}$ given by (1.1) was introduced by Barczy *et al.* [4]. There, it was proved that $(Y_t, X_t)_{t\geq 0}$ belongs to the class of regular affine processes (with state space $\mathbb{R}_{\geq 0}\times\mathbb{R}$). The process Y is the so-called α-root process (sometimes referred as the stable Cox–Ingersoll–Ross (CIR) process, shortened to SCIR; see [22]) and is also an affine process (with state

Received 25 July 2016; revision received 12 June 2017.
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space $\mathbb{R}_{\geq 0}$). It can be considered as an extension of the CIR process. The general theory of affine processes on the canonical state space $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ was initiated by Duffie *et al.* [8] and further developed in [7]. An affine process on $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ is a continuous-time Markov process taking values in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, whose log-characteristic function depends in an affine way on the initial state vector of the process, i.e. the log-characteristic function is linear with respect to the initial state vector. Affine processes are particularly important in financial mathematics because of their computational tractability. For example, the models of Cox *et al.* [6], Heston [13], and Vasicek [32] are all based on affine processes.

An important issue for the application of affine processes is the calibration of their parameters. This has been investigated for some well-known affine models; see, e.g. [1]–[3], [28], and [29]. To study the asymptotic properties of estimators of the parameters, a comprehension of the long-time behavior of the underlying affine processes is very often required. This is one of the reasons why the stationary, ergodic, and recurrent properties of affine processes have recently attracted many investigations; see, e.g. [4], [9], [15]–[17], and [20]–[22].

Concerning the two-factor model defined in (1.1), it was shown in [4] that $(Y_t, X_t)_{t\geq 0}$ has a stationary distribution. Using the same argument as in [19, p. 80], it can easily be seen that the stationary distribution for $(Y_t, X_t)_{t\geq 0}$ is actually unique. If one allows $\alpha = 2$ and replaces $(L_t)_{t\geq 0}$ in (1.1) by a standard Brownian motion $(W_t)_{t\geq 0}$ (independent of $(B_t)_{t\geq 0}$), then the process Y becomes the CIR process; in this case, the ergodicity of $(Y_t, X_t)_{t\geq 0}$ was proved in [4]. However, the ergodicity of $(Y_t, X_t)_{t\geq 0}$ in the $1 < \alpha < 2$ case is still not known.

In this work we study the ergodicity problem for the two-factor model in (1.1) when $1 < \alpha < 2$. As our main result (see Theorem 6.1 below), we show that $(Y_t, X_t)_{t\geq 0}$ in (1.1) is exponentially ergodic if $\alpha \in (1, 2)$, complementing the results of [4]. Our approach is very close to that of [15]. The first step is to show the existence of positive transition densities of the α-root process Y_t. To achieve this, we calculate explicitly the Laplace transform of Y_t. Through a careful analysis of the decay rate of the Laplace transform of Y_t at ∞, we are able to show the positivity of the density function of Y_t using the inverse Fourier transform. The positivity of the density function of Y_t plays an essential role in the proof of the exponential ergodicity for $(Y_t, X_t)_{t\geq 0}$, since it enables us to show that the Lebesgue measure is an irreducibility measure for the skeleton chains of $(Y_t, X_t)_{t\geq 0}$. We would like to remark that our approach of proving the existence of a positive density function for Y_t is purely analytic. In the literature there are some probabilistic methods to study the positivity of densities for jump-diffusions; see, e.g. [10]. It is an interesting question if the method of [10] can be applied in our case. In the second step, we construct a Foster–Lyapunov function for the process $(Y_t, X_t)_{t\geq 0}$. Using the general theory in [24]–[26] on the ergodicity of Markov processes, we are then able to obtain the exponential ergodicity of the process $(Y_t, X_t)_{t\geq 0}$ in (1.1).

Finally, we remark that the exponential ergodicity for a large class of affine processes on $\mathbb{R}_{\geq 0}$, including the α-root process $(Y_t)_{t\geq 0}$, was derived in [22] by a coupling method. We do not know if a similar coupling argument would work for the two-dimensional affine process $(Y_t, X_t)_{t\geq 0}$ in (1.1).

The rest of the paper is organized as follows. In Section 2 we recall some basic facts on the process $(Y_t, X_t)_{t\geq 0}$. In Section 3 we derive the Laplace transform of the α-root process Y. In Section 4 we prove that the α-root process Y possesses positive transition densities. In Section 5 we construct a Foster–Lyapunov function for the process $(Y_t, X_t)_{t\geq 0}$. In Section 6 we show that the process $(Y_t, X_t)_{t\geq 0}$ is exponentially ergodic.

2. Preliminaries

In this section we recall some key facts on the affine process $(Y, X) := (Y_t, X_t)_{t\geq 0}$ defined by (1.1), mainly due to [4].

Let $\mathbb{N}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$ denote the sets of positive integers, nonnegative integers, real numbers, nonnegative real numbers, and strictly positive real numbers, respectively. Let $\mathbb{C}$ be the set of complex numbers. For $z \in \mathbb{C} \setminus \{0\}$, we denote by $\arg(z)$ the principal value of its argument and by $\bar{z}$ its conjugate. We define the following subsets of $\mathbb{C}$:

$$\mathcal{U}_- := \{u \in \mathbb{C}\colon \operatorname{Re} u \leq 0\}, \qquad \mathcal{U}_+ := \{u \in \mathbb{C}\colon \operatorname{Re} u \geq 0\},$$
$$\mathcal{U}_-^{\mathrm{o}} := \{u \in \mathbb{C}\colon \operatorname{Re} u < 0\}, \qquad \mathcal{U}_+^{\mathrm{o}} := \{u \in \mathbb{C}\colon \operatorname{Re} u > 0\},$$

and

$$\mathcal{O} := \mathbb{C} \setminus \{-x : x \in \mathbb{R}_{\geq 0}\}.$$

For $z \in \mathbb{C} \setminus \{0\}$, let $\log(z)$ be the principal value of the complex logarithm of z, i.e. $\log(z) = \ln(|z|) + \mathrm{i}\arg(z)$. In this paper we define $\arg(x) := \pi$ for $x \in (-\infty, 0)$. For $\beta \in \mathbb{R}$, define the complex power function z^β as

$$z^\beta := \exp\{\beta \log z\}, \qquad z \in \mathbb{C} \setminus \{0\}. \tag{2.1}$$

By $C^2(S, \mathbb{R})$, $C_c^2(S, \mathbb{R})$, $C_b^2(S, \mathbb{R})$, and $C^\infty(S, \mathbb{C})$, we denote the sets of $\mathbb{R}$-valued or $\mathbb{C}$-valued functions on S that are twice continuously differentiable, that are twice continuously differentiable with compact support, that are bounded continuous with bounded continuous first- and second-order partial derivatives, and that are smooth, respectively, where the space S can be $\mathbb{R}$, $\mathbb{R}_{\geq 0} \times \mathbb{R}$, or $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}$ in this paper.

We assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$.

Let $(B_t)_{t\geq 0}$ be a standard $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion and $(L_t)_{t\geq 0}$ be a spectrally positive $(\mathcal{F}_t)_{t\geq 0}$-Lévy process with the Lévy measure $C_\alpha z^{-1-\alpha}\mathbf{1}_{\{z>0\}}\,\mathrm{d}z$, where $1 < \alpha < 2$. Assume that $(B_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ are independent. Note that the characteristic function of L_1 is given by

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}uL_1}] = \exp\biggl\{\int_0^\infty (\mathrm{e}^{\mathrm{i}uz} - 1 - \mathrm{i}uz) C_\alpha z^{-1-\alpha}\,\mathrm{d}z\biggr\}, \qquad u \in \mathbb{R}.$$

Let $N(\mathrm{d}s, \mathrm{d}z)$ be a Poisson random measure on $\mathbb{R}_{>0}^2$ with intensity measure given by $C_\alpha z^{-1-\alpha}\mathbf{1}_{\{z>0\}}\,\mathrm{d}s\,\mathrm{d}z$ and $\hat{N}(\mathrm{d}s, \mathrm{d}z)$ its compensator. Then the Lévy–Itô representation of L takes the form

$$L_t = \gamma t + \int_0^t \int_{\{|z|<1\}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{\{|z|\geq 1\}} zN(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0, \tag{2.2}$$

where

$$\gamma := -\mathbb{E}\biggl[\int_0^1 \int_{\{|z|\geq 1\}} zN(\mathrm{d}s, \mathrm{d}z)\biggr]$$

and $\tilde{N}(\mathrm{d}s, \mathrm{d}z) := N(\mathrm{d}s, \mathrm{d}z) - \hat{N}(\mathrm{d}s, \mathrm{d}z)$ with $\hat{N}(\mathrm{d}s, \mathrm{d}z) = C_\alpha z^{-1-\alpha}\mathbf{1}_{\{z>0\}}\,\mathrm{d}s\,\mathrm{d}z$, is the compensated Poisson random measure on $\mathbb{R}_{>0}^2$ that corresponds to $N(\mathrm{d}s, \mathrm{d}z)$. We remark that $\gamma t = -\int_0^t \int_{\{|z|\geq 1\}} z\hat{N}(\mathrm{d}s, \mathrm{d}z)$ and

$$\int_0^t \int_{\{|z|\geq 1\}} zN(\mathrm{d}s, \mathrm{d}z) + \gamma t, \qquad t \geq 0,$$

is thus a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. It follows from [4, Theorem 2.1] that if (Y_0, X_0) is independent of $(L_t, B_t)_{t\geq 0}$ and $\mathbb{P}(Y_0 \geq 0) = 1$, then there is a unique strong solution $(Y_t, X_t)_{t\geq 0}$ of SDE (1.1) with

$$Y_t = \mathrm{e}^{-bt}\bigg(Y_0 + a\int_0^t \mathrm{e}^{bs}\,\mathrm{d}s + \int_0^t \mathrm{e}^{bs}\sqrt[\alpha]{Y_{s-}}\,\mathrm{d}L_s\bigg),$$

and

$$X_t = \mathrm{e}^{-\theta t}\bigg(X_0 + m\int_0^t \mathrm{e}^{\theta s}\,\mathrm{d}s + \int_0^t \mathrm{e}^{\theta s}\sqrt{Y_s}\,\mathrm{d}B_s\bigg)$$

for all $t \geq 0$. Moreover, $(Y_t, X_t)_{t\geq 0}$ is a regular affine process, and the infinitesimal generator $\mathcal{A}$ of (Y, X) is given by

$$\begin{aligned}(\mathcal{A}f)(y, x) = {}& (a - by)\frac{\partial}{\partial y}f(y, x) + (m - \theta x)\frac{\partial}{\partial x}f(y, x) + \frac{1}{2}y\frac{\partial^2}{\partial x^2}f(y, x)\\ &+ y\int_0^\infty\bigg(f(y + z, x) - f(y, x) - z\frac{\partial}{\partial y}f(y, x)\bigg)C_\alpha z^{-1-\alpha}\,\mathrm{d}z,\end{aligned}\tag{2.3}$$

where $(y, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ and $f \in C_c^2(\mathbb{R}_{\geq 0} \times \mathbb{R}, \mathbb{R})$.

3. Laplace transform of the α-root process Y

In this section we study the α-root process $(Y_t)_{t\geq 0}$ defined by

$$\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\,\mathrm{d}L_t, \qquad t \geq 0, \qquad Y_0 \geq 0 \quad \text{almost surely,} \tag{3.1}$$

where $a \geq 0$, $b > 0$, $\alpha \in (1, 2)$, and $(L_t)_{t\geq 0}$ is a spectrally positive α-stable Lévy process with the Lévy measure $C_\alpha z^{-1-\alpha}\,\mathbf{1}_{\{z>0\}}\,\mathrm{d}z$. Without any further specification, we always assume that Y_0 is independent of $(L_t)_{t\geq 0}$.

We remark that we have allowed $a = 0$ in (3.1), which is different as in (1.1). In this case, SDE (3.1) becomes

$$\mathrm{d}Y_t = -bY_t\,\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\,\mathrm{d}L_t, \qquad t \geq 0, \qquad Y_0 \geq 0 \quad \text{almost surely,} \tag{3.2}$$

and, by [12, Theorem 6.2 and Corollary 6.3], a unique strong solution of (3.2) also exists. The α-root process Y is thus well-defined for all $a \geq 0$. From now on and till the end of this section, we assume temporally that $a \geq 0$.

The solution of SDE (3.1) depends obviously on its initial value Y_0. From now on, we denote by $(Y_t^y)_{t\geq 0}$ the α-root process starting from a constant initial value $y \in \mathbb{R}_{\geq 0}$, i.e. $(Y_t^y)_{t\geq 0}$ satisfies

$$\mathrm{d}Y_t^y = (a - bY_t^y)\,\mathrm{d}t + \sqrt[\alpha]{Y_{t-}^y}\,\mathrm{d}L_t, \qquad t \geq 0, \qquad Y_0^y = y. \tag{3.3}$$

Since the α-root process is an affine process, the corresponding characteristic functions of $(Y_t^y)_{t\geq 0}$ are of affine form, namely,

$$\mathbb{E}[\mathrm{e}^{uY_t^y}] = \mathrm{e}^{\phi(t,u)+y\psi(t,u)}, \qquad u \in \mathcal{U}_-. \tag{3.4}$$

The functions ϕ and ψ, in turn, are given as solutions of the generalized Riccati equations

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t,u) &= F(\psi(t,u)), \qquad \phi(0,u)=0, \\ \frac{\partial}{\partial t}\psi(t,u) &= R(\psi(t,u)), \qquad \psi(0,u)=u\in\mathcal{U}_-, \end{aligned} \tag{3.5}$$

with

$$F(u)=au \quad \text{and} \quad R(u)=-bu+\frac{(-u)^\alpha}{\alpha};$$

see [4, Theorem 3.1]. An equivalent equation for ψ (see (3.6) below) was studied in [4, Theorem 3.1]. In particular, it follows from [4, Theorem 3.1] that (3.6) below has a unique solution. However, the explicit form of the solution to (3.6) was not derived in [4]. In order to study the transition densities of the α-root process, we will find the explicit form of the solution to (3.6) in the following theorem.

Proposition 3.1. *Let $a\ge 0$, $b>0$. Define $v_t(\lambda):=-\psi(t,-\lambda)$, $\lambda\in\mathbb{R}_{>0}$. Then $v_t(\lambda)$ solves the differential equation*

$$\frac{\partial}{\partial t}v_t(\lambda)=-bv_t(\lambda)-\frac{1}{\alpha}(v_t(\lambda))^\alpha, \qquad t\ge 0, \qquad v_0(\lambda)=\lambda, \tag{3.6}$$

where $\lambda\in\mathbb{R}_{>0}$. The unique solution to (3.6) is given by

$$v_t(\lambda)=\bigg(\bigg(\frac{1}{\alpha b}+\lambda^{1-\alpha}\bigg)\mathrm{e}^{b(\alpha-1)t}-\frac{1}{\alpha b}\bigg)^{1/(1-\alpha)}, \qquad t\ge 0. \tag{3.7}$$

Moreover, the Laplace transform of Y_t^y is given by

$$\mathbb{E}[\mathrm{e}^{-\lambda Y_t^y}]=\exp\bigg\{-a\int_0^t v_s(\lambda)\,\mathrm{d}s-yv_t(\lambda)\bigg\} \quad \textit{for all } t\ge 0 \textit{ and } \lambda\in\mathbb{R}_{>0}. \tag{3.8}$$

Proof. Equation (3.6) is a Bernoulli differential equation which can be transformed into a linear differential equation through a change of variables. More precisely, if we write $u_t(\lambda):=(v_t(\lambda))^{1-\alpha}$ then

$$\frac{\partial}{\partial t}u_t(\lambda)=b(\alpha-1)u_t(\lambda)+(1-\alpha^{-1}), \qquad t\ge 0, \qquad u_0(\lambda)=\lambda^{1-\alpha}. \tag{3.9}$$

By solving (3.9), we obtain, for $t\ge 0$ and $\lambda\in\mathbb{R}_{>0}$,

$$v_t(\lambda)=(u_t(\lambda))^{1/(1-\alpha)}=\bigg(\bigg(\frac{1}{\alpha b}+\lambda^{1-\alpha}\bigg)\mathrm{e}^{b(\alpha-1)t}-\frac{1}{\alpha b}\bigg)^{1/(1-\alpha)}.$$

Moreover, by (3.4) and (3.5) and noting that $v_t(\lambda)=-\psi(t,-\lambda)$, we obtain (3.8). □

We remark that we have assumed $\lambda\in\mathbb{R}_{>0}$ in Proposition 3.1. However, (3.8) holds for the trivial $\lambda=0$ case as well, which can be seen by taking the limit $\lambda\downarrow 0$.

Let

$$\begin{aligned} \varphi_1(t,\lambda,y) &:= \exp\bigg\{-y\bigg(\bigg(\frac{1}{\alpha b}+\lambda^{1-\alpha}\bigg)\mathrm{e}^{b(\alpha-1)t}-\frac{1}{\alpha b}\bigg)^{1/(1-\alpha)}\bigg\}, \\ \varphi_2(t,\lambda) &:= \exp\bigg\{-a\int_0^t\bigg(\bigg(\frac{1}{\alpha b}+\lambda^{1-\alpha}\bigg)\mathrm{e}^{b(\alpha-1)s}-\frac{1}{\alpha b}\bigg)^{1/(1-\alpha)}\,\mathrm{d}s\bigg\}. \end{aligned}$$

Then $\mathbb{E}[\exp\{-\lambda Y_t^y\}]=\varphi_1(t,\lambda,y)\varphi_2(t,\lambda)$. Keeping this decomposition of the Laplace transform of Y_t^y in mind, we take a closer look at the following two special cases.

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3.1. The special case $a = 0$

To avoid abuse of notation, we use $(Z_t^y)_{t\geq 0}$ to denote the strong solution of the SDE

$$\mathrm{d}Z_t^y = -bZ_t^y\,\mathrm{d}t + \sqrt[\alpha]{Z_{t-}^y}\,\mathrm{d}L_t, \qquad t \geq 0, \qquad Z_0^y = y \geq 0. \tag{3.10}$$

According to (3.8), the corresponding Laplace transform of Z_t^y coincides with $\varphi_1(t, \lambda, y)$. Noting that $b > 0$, we obtain

$$\lim_{\lambda\to\infty} v_t(\lambda) = \left(\frac{1}{\alpha b}(\mathrm{e}^{b(\alpha-1)t} - 1)\right)^{1/(1-\alpha)} =: d(t), \tag{3.11}$$

where $d(t) \in (0, \infty)$ for all $t > 0$. Furthermore, by the dominated convergence theorem, we have

$$\begin{aligned} \mathrm{e}^{-yd(t)} &= \lim_{\lambda\to\infty} \mathrm{e}^{-yv_t(\lambda)} &\text{(3.12)}\\ &= \lim_{\lambda\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda Z_t^y}] \\ &= \lim_{\lambda\to\infty} (\mathbb{E}[\mathrm{e}^{-\lambda Z_t^y}\,\mathbf{1}_{\{Z_t^y=0\}}] + \mathbb{E}[\mathrm{e}^{-\lambda Z_t^y}\,\mathbf{1}_{\{Z_t^y>0\}}]) \\ &= \mathbb{P}(Z_t^y = 0) \\ &> 0 \quad \text{for all } t > 0 \text{ and } y \geq 0. &\text{(3.13)} \end{aligned}$$

3.2. The special case $y = 0$

Consider $(Y_t^0)_{t\geq 0}$ that satisfies

$$\mathrm{d}Y_t^0 = (a - bY_t^0)\,\mathrm{d}t + \sqrt[\alpha]{Y_{t-}^0}\,\mathrm{d}L_t, \qquad t \geq 0, \qquad Y_0^0 = 0. \tag{3.14}$$

In view of (3.8), we can easily see that the Laplace transform of Y_t^0 is equal to $\varphi_2(t, \lambda)$.

Summarizing the results in Subsections 3.1 and 3.2, we have the following proposition.

Proposition 3.2. *Let $a \geq 0$ and $b > 0$. Consider the processes $(Y_t^y)_{t\geq 0}$ and $(Z_t^y)_{t\geq 0}$ defined as the unique strong solutions of SDEs (3.3) and (3.10), respectively. Let $\mu_{Y_t^y}$ and $\mu_{Z_t^y}$ be the probability laws of Y_t^y and Z_t^y induced on $(\mathbb{R}_{\geq 0}, \mathcal{B}(\mathbb{R}_{\geq 0}))$, respectively. Then $\mu_{Y_t^y} = \mu_{Y_t^0} * \mu_{Z_t^y}$, where '$*$' denotes the convolution of measures.*

4. Transition densities of the α-root process Y

In this section we show that the α-root process Y has positive and continuous transition densities. Our approach is essentially based on the inverse Fourier transform.

Recall that the function $v_t(\cdot)$ given by (3.7) is defined on $\mathbb{R}_{>0}$. By considering the complex power functions, the domain of definition for $v_t(\cdot)$ can be extended to $\mathbb{C} \setminus \{0\}$. Indeed, the function

$$v_t(z) = \left(\left(\frac{1}{\alpha b} + z^{1-\alpha}\right)\mathrm{e}^{b(\alpha-1)t} - \frac{1}{\alpha b}\right)^{1/(1-\alpha)}, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{4.1}$$

is well defined, where the complex power function is given by (2.1).

We next establish two estimates on $\int_0^t v_s(z)\,\mathrm{d}s$. Since the proofs are of pure analytic nature, we postpone them to Appendix A.

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Lemma 4.1. *Let $T > 1$. Then there exists a sufficiently small constant $\varepsilon_0 > 0$ such that*

$$\mathrm{Re}\left(\int_0^t v_s(z)\,\mathrm{d}s\right) \geq -C_1 + C_2|z|^{2-\alpha} \tag{4.2}$$

when $|\arg(z)| \in [\pi/2 - \varepsilon_0, \pi/2 + \varepsilon_0]$ and $T^{-1} \leq t \leq T$, where $C_1, C_2 > 0$ are constants depending only on a, b, α, ε_0, and T.

Proof. See Appendix A. □

Lemma 4.2. *Let ε_0 be as in the previous lemma. Then, for each $t \geq 0$, we can find constants $C_3, C_4 > 0$, which depend only on a, b, α, ε_0, and t, such that*

$$\left|\int_0^t v_s(z)\,\mathrm{d}s\right| \leq C_3 + C_4|z|^{2-\alpha}$$

when $\arg(z) \in [\pi/2 + \varepsilon_0, \pi]$ *and* $|z| > 0$.

Proof. See Appendix A. □

Now consider the process $(Y_t^0)_{t\geq 0}$ given by (3.14). As the Laplace transform of Y_t^0, the function $u \mapsto \mathbb{E}[\exp\{-uY_t^0\}]$ is continuous on $\mathcal{U}_+$ and holomorphic on $\mathcal{U}_+^{\mathrm{o}}$. On the other hand, the function $z \mapsto v_t(z)$ given in (4.1) is continuous on $\mathcal{U}_+$ and holomorphic on $\mathcal{U}_+^{\mathrm{o}}$ for each $t \geq 0$ as well. Therefore, we have

$$\mathbb{E}[\mathrm{e}^{-uY_t^0}] = \exp\left\{-a\int_0^t v_s(u)\,\mathrm{d}s\right\}, \qquad u \in \mathcal{U}_+. \tag{4.3}$$

Indeed, (4.3) holds at least for $u \in \mathbb{R}_{>0}$ by (3.8). This and the identity theorem for holomorphic functions (see, e.g. [11, Theorem III.3.2]) imply (4.3) for all $u \in \mathcal{U}_+$, since both sides of (4.3) are functions that are continuous on $\mathcal{U}_+$ and holomorphic on $\mathcal{U}_+^{\mathrm{o}}$. In particular, the characteristic function of Y_t^0 with $t > 0$ is given by

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Y_t^0}] = \exp\left\{-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s\right\}, \qquad \xi \in \mathbb{R}.$$

Remark 4.1. Recall that the process $(Z_t^y)_{t\geq 0}$ is given in (3.10). By repeating the same arguments as above for Z_t^y, we see that its characteristic function is given by

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Z_t^y}] = \exp\{-yv_t(-\mathrm{i}\xi)\}, \qquad \xi \in \mathbb{R}.$$

In the next lemma we obtain the existence of a density function for Y_t^0 when $t > 0$. Note that by [4, Theorem 2.1], we have $Y_t^0 \geq 0$ almost surely for each $t \geq 0$.

Lemma 4.3. *Assume that $a > 0$ and $b > 0$. Then, for each $t > 0$, Y_t^0 possesses a density function $f_{Y_t^0}$ given by*

$$f_{Y_t^0}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\xi}\exp\left\{-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s\right\}\mathrm{d}\xi, \qquad x \geq 0. \tag{4.4}$$

Moreover, the function $f_{Y_t^0}(x)$ is jointly continuous in $(t, x) \in (0, \infty) \times \mathbb{R}_{\geq 0}$, and $f_{Y_t^0}(\cdot) \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{C})$ for each $t > 0$.

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Proof. Let $T > 1$ be fixed. By Lemma 4.1, there exist constants $c_1, c_2 > 0$ such that

$$\left|\exp\left\{-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s\right\}\right| = \exp\left\{\operatorname{Re}\left(-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s\right)\right\} \le c_1\exp\{-c_2|\xi|^{2-\alpha}\} \quad (4.5)$$

for all $\xi \in \mathbb{R}$ and $t \in [1/T, T]$, which implies that $\xi \mapsto \exp\{-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s\}$ is integrable on $\mathbb{R}$. Therefore, by the Fourier inversion formula, Y_t^0 has a density $f_{Y_t^0}$ given by (4.4). The joint continuity of the density $f_{Y_t^0}(x)$ in (t, x) follows from (4.4), (4.5), and the dominated convergence theorem. The smoothness property of $f_{Y_t^0}(\cdot)$ is a consequence of (4.5) and [30, Proposition 28.1]. □

We remark that for each $t > 0$, the function $f_{Y_t^0}(x)$ given in (4.4) is actually well defined also for $x < 0$, although $f_{Y_t^0}(x) \equiv 0$ for $x \le 0$, which is due to the fact that $Y_t^0 \ge 0$ almost surely. Next, we would like to know if $f_{Y_t^0}(x) > 0$ when $x > 0$. The next lemma partly answers this question.

Lemma 4.4. *Assume that $a > 0$ and $b > 0$. For each $t > 0$, the density function $f_{Y_t^0}(\cdot)$ of Y_t^0 is almost everywhere positive on $\mathbb{R}_{\ge 0}$.*

Proof. Basically, the idea of the proof is as follows. We will show the following.

Claim 4.1. *The function $x \mapsto f_{Y_t^0}(x)$, $x \in \mathbb{R}_{>0}$, can be extended to a holomorphic function on $\mathcal{U}_+^{\mathrm{o}}$.*

If this claim is true then the set $A_n := \{x > 1/n\colon f_{Y_t^0}(x) = 0\}$ with $n \in \mathbb{N}$ must be discrete, i.e. for each $x \in A_n$, one can find a neighborhood of x whose intersection with A_n is equal to x; otherwise, the identity theorem for holomorphic functions (see, e.g. [11, Proposition III.3.1]) implies that $f_{Y_t^0}(x) \equiv 0$ for $x > 0$. As a consequence, A_n is countable, which implies that $A := \bigcup_{n\in\mathbb{N}} A_n$ is also countable and thus has Lebesgue measure 0.

Let $x > 0$ be fixed. We will complete the proof of the above claim in five steps.

Step 1. We derive a simpler representation for $f_{Y_t^0}(x)$. We have

$$\begin{aligned} f_{Y_t^0}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\xi}\exp\left\{-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s\right\}\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}x\xi}\exp\left\{-a\int_0^t v_s(\mathrm{i}\xi)\,\mathrm{d}s\right\}\mathrm{d}\xi \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}x\xi}\exp\left\{-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s\right\}\mathrm{d}\xi. \end{aligned} \quad (4.6)$$

For $\xi < 0$, we have $\overline{v_s(-\mathrm{i}\xi)} = v_s(\mathrm{i}\xi)$, which implies that

$$\overline{\mathrm{e}^{-\mathrm{i}x\xi}\exp\left\{-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s\right\}} = \mathrm{e}^{\mathrm{i}x\xi}\exp\left\{-a\int_0^t v_s(\mathrm{i}\xi)\,\mathrm{d}s\right\}. \quad (4.7)$$

By (4.6) and (4.7), we obtain

$$f_{Y_t^0}(x) = \operatorname{Re}\left(\frac{1}{\pi}\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}x\xi}\exp\left\{-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s\right\}\mathrm{d}\xi\right). \quad (4.8)$$

For simplicity, let

$$I := \frac{1}{\pi}\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}x\xi}\exp\left\{-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s\right\}\mathrm{d}\xi. \quad (4.9)$$

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Step 2. We calculate I by contour integration. By a change of variables $z := -\mathrm{i}\xi$, we obtain

$$
\begin{aligned}
I &= \frac{-\mathrm{i}}{\pi}\int_0^{\mathrm{i}\infty} \mathrm{e}^{xz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\}\,\mathrm{d}z \\
&= \lim_{K\to\infty}\frac{-\mathrm{i}}{\pi}\int_{\mathrm{i}K^{-1}}^{\mathrm{i}K} \mathrm{e}^{xz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\}\,\mathrm{d}z. \qquad (4.10)
\end{aligned}
$$

Define two paths $\Gamma_{1,K}$ and $\Gamma_{2,K}$ by

$$
\Gamma_{1,K}(\vartheta) := K\mathrm{e}^{\mathrm{i}\vartheta},\quad \vartheta\in\Big[\frac{\pi}{2},\pi\Big],\qquad \Gamma_{2,K}(\vartheta) := K^{-1}\mathrm{e}^{\mathrm{i}\vartheta},\quad \vartheta\in\Big[\frac{\pi}{2},\pi\Big].
$$

According to (4.1), we see that the function

$$
z\mapsto \mathrm{e}^{yz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\},\qquad z\in\mathcal{O}_1 := \Big\{\rho\mathrm{e}^{\mathrm{i}\vartheta}:\rho>0,\ \vartheta\in\Big[\frac{\pi}{2},\pi\Big]\Big\},
$$

can be extended to a holomorphic function on $\mathcal{O}_2 := \{\rho\mathrm{e}^{\mathrm{i}\vartheta}:\rho>0,\ \vartheta\in(0,3\pi/2)\}$. Therefore, we have

$$
\begin{aligned}
&\int_{\mathrm{i}K^{-1}}^{\mathrm{i}K} \mathrm{e}^{xz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\}\,\mathrm{d}z \\
&\quad= \int_{-K^{-1}}^{-K} \mathrm{e}^{xz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\}\,\mathrm{d}z - \int_{\Gamma_{1,K}} \mathrm{e}^{xz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\}\,\mathrm{d}z \\
&\qquad+ \int_{\Gamma_{2,K}} \mathrm{e}^{xz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\}\,\mathrm{d}z. \qquad (4.11)
\end{aligned}
$$

Since $\lim_{z\to 0}\mathrm{e}^{xz}\exp\{-a\int_0^t v_s(z)\,\mathrm{d}s\} = 1$, it follows that

$$
\lim_{K\to\infty}\int_{\Gamma_{2,K}} \mathrm{e}^{xz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\}\,\mathrm{d}z = 0. \qquad (4.12)
$$

To estimate the second term on the right-hand side of (4.11), we divide the path $\Gamma_{1,K}$ into two parts, namely,

$$
\Gamma_{11,K}(\vartheta) := K\mathrm{e}^{\mathrm{i}\vartheta},\quad \vartheta\in\Big[\frac{\pi}{2},\frac{\pi}{2}+\varepsilon_0\Big],\qquad \Gamma_{12,K}(\vartheta) =: K\mathrm{e}^{\mathrm{i}\vartheta},\quad \vartheta\in\Big[\frac{\pi}{2}+\varepsilon_0,\pi\Big],
$$

with $\varepsilon_0 > 0$ being the constant appearing in Lemmas 4.1 and 4.2. Then

$$
\begin{aligned}
&\int_{\Gamma_{1,K}} \mathrm{e}^{xz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\}\,\mathrm{d}z \\
&\quad= \int_{\Gamma_{11,K}} \mathrm{e}^{xz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\}\,\mathrm{d}z + \int_{\Gamma_{12,K}} \mathrm{e}^{xz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\}\,\mathrm{d}z \\
&\quad:= \mathrm{I}_1(K) + \mathrm{I}_2(K).
\end{aligned}
$$

If we can show that $\lim_{K\to\infty}\mathrm{I}_1(K) = 0$ and $\lim_{K\to\infty}\mathrm{I}_2(K) = 0$, then it follows from (4.10)–(4.12) that

$$
I = \frac{-\mathrm{i}}{\pi}\int_0^{-\infty} \mathrm{e}^{xz}\exp\Big\{-a\int_0^t v_s(z)\,\mathrm{d}s\Big\}\,\mathrm{d}z. \qquad (4.13)
$$

Step 3. We show that $\lim_{K\to\infty} \mathrm{I}_1(K) = 0$. If $\vartheta \in [\pi/2, \pi/2 + \varepsilon_0]$ then

$$|\exp\{xK\mathrm{e}^{\mathrm{i}\vartheta}\}| = \exp\{\mathrm{Re}(xK\mathrm{e}^{\mathrm{i}\vartheta})\} = \exp\{xK\cos(\vartheta)\} \le 1.$$

By Lemma 4.1, we obtain

$$\begin{aligned}\Big|\mathrm{I}_1(K)\Big| &= \bigg|\int_{\pi/2}^{\pi/2+\varepsilon_0} \mathrm{i}K\exp\{\mathrm{i}\vartheta\}\exp\{xK\mathrm{e}^{\mathrm{i}\vartheta}\}\exp\bigg\{-a\int_0^t v_s(K\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}s\bigg\}\,\mathrm{d}\vartheta\bigg| \\ &\le K\int_{\pi/2}^{\pi/2+\varepsilon_0}\bigg|\exp\bigg\{-a\int_0^t v_s(K\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}s\bigg\}\bigg|\,\mathrm{d}\vartheta \\ &\le K\varepsilon_0\exp\{aC_1 - aC_2K^{2-\alpha}\},\end{aligned}$$

which implies that $\lim_{K\to\infty}|\mathrm{I}_1(K)| = 0$.

Step 4. We show that $\lim_{K\to\infty} \mathrm{I}_2(K) = 0$. In the $\vartheta \in [\pi/2 + \varepsilon_0, \pi]$ case, we have

$$\begin{aligned}|\exp\{xK\mathrm{e}^{\mathrm{i}\vartheta}\}| &= \exp\{\mathrm{Re}(xK\mathrm{e}^{\mathrm{i}\vartheta})\} \\ &= \exp\{xK\cos(\vartheta)\} \\ &\le \exp\bigg\{xK\cos\bigg(\frac{\pi}{2}+\varepsilon_0\bigg)\bigg\} \\ &= \exp\{-xK\sin(\varepsilon_0)\}.\end{aligned}$$

So

$$\begin{aligned}|\mathrm{I}_2(K)| &= \bigg|\int_{\pi/2+\varepsilon_0}^{\pi} \mathrm{i}K\mathrm{e}^{\mathrm{i}\vartheta}\exp\{xK\mathrm{e}^{\mathrm{i}\vartheta}\}\exp\bigg\{-a\int_0^t v_s(K\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}s\bigg\}\,\mathrm{d}\vartheta\bigg| \\ &\le K\exp\{-xK\sin(\varepsilon_0)\}\int_{\pi/2+\varepsilon_0}^{\pi}\exp\bigg\{a\bigg|\int_0^t v_s(K\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}s\bigg|\bigg\}\,\mathrm{d}\vartheta.\end{aligned}$$

By Lemma 4.2, we obtain

$$\lim_{K\to\infty}|\mathrm{I}_2(K)| \le \lim_{K\to\infty} K\bigg(\frac{\pi}{2}-\varepsilon_0\bigg)\exp\{-xK\sin(\varepsilon_0)\}\exp\{aC_3\}\exp\{aC_4K^{2-\alpha}\} = 0.$$

Step 5. Finally, we prove that $x \mapsto f_{Y_t^0}(x)$ is holomorphic on $\mathcal{U}_+^{\mathrm{o}}$. By (4.8), (4.9), and (4.13), we have

$$\begin{aligned}f_{Y_t^0}(x) &= \mathrm{Re}\bigg(\frac{-\mathrm{i}}{\pi}\int_0^{-\infty}\mathrm{e}^{xz}\exp\bigg\{-a\int_0^t v_s(z)\,\mathrm{d}s\bigg\}\,\mathrm{d}z\bigg) \\ &= \frac{1}{\pi}\int_0^{\infty}\mathrm{e}^{-xz}\bigg\{-\mathrm{Im}\bigg(\exp\bigg\{-a\int_0^t v_s(-z)\,\mathrm{d}s\bigg\}\bigg)\bigg\}\,\mathrm{d}z.\end{aligned}$$

Let $x_0 > 0$ be fixed. Consider $x \in \mathbb{C}$ with $\mathrm{Re}(x) > x_0$ and $(x_n) \subset \mathbb{C}$ such that $\mathrm{Re}(x_n) > x_0$ and $x_n \to x$ as $n \to \infty$. For $z \in \mathbb{R}_{\ge 0}$, we can use Lemma 4.2 to obtain

$$\begin{aligned}z\mathrm{e}^{-x_0z}\bigg|\mathrm{Im}\bigg(\exp\bigg\{-a\int_0^t v_s(-z)\,\mathrm{d}s\bigg\}\bigg)\bigg| &\le z\mathrm{e}^{-x_0z}\bigg|\exp\bigg\{-a\int_0^t v_s(-z)\,\mathrm{d}s\bigg\}\bigg| \\ &\le z\mathrm{e}^{-x_0z}\exp\{aC_3 + aC_4|z|^{2-\alpha}\}, \qquad (4.14)\end{aligned}$$

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where the right-hand side of (4.14) is an integrable function (with the variable z) on $\mathbb{R}_{\geq 0}$. Note that $|\exp\{-x_n z\}-\exp\{-xz\}| \leq z\exp\{-x_0 z\}|x_n - x|$ for $z \geq 0$. By the dominated convergence theorem, we see that the function

$$x \mapsto \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-xz}\bigg\{-\operatorname{Im}\bigg(\exp\bigg\{-a\int_0^t v_s(-z)\,\mathrm{d}s\bigg\}\bigg)\bigg\}\,\mathrm{d}z, \qquad x \in \mathcal{U}_+^{\mathrm{o}},$$

is holomorphic, which means that $x \mapsto f_{Y_t^0}(x)$ has a holomorphic extension on $\mathcal{U}_+^{\mathrm{o}}$. □

With the help of the previous lemma, we are now able to prove the main result of this section. Recall that the process $(Y_t^y)_{t\geq 0}$ is given by (3.3).

Proposition 4.1. *Assume that $a > 0$ and $b > 0$. Then, for each $y \geq 0$ and $t > 0$, Y_t^y possesses a density function $f_{Y_t^y}$ given by*

$$f_{Y_t^y}(x) := \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}x\xi}\exp\bigg\{-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s - yv_t(-\mathrm{i}\xi)\bigg\}\,\mathrm{d}\xi, \qquad x \geq 0, \tag{4.15}$$

where $f_{Y_t^y}(\cdot) \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{C})$ and $f_{Y_t^y}(x) > 0$ for all $x > 0$. Moreover, the function $f_{Y_t^y}(x)$ is jointly continuous in $(t, y, x) \in (0,\infty)\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}$.

Proof. In view of Proposition 3.2, we have

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Y_t^y}] = \mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Y_t^0}]\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Z_t^y}] = \exp\bigg\{-a\int_0^t v_s(-\mathrm{i}\xi)\,\mathrm{d}s - yv_t(-\mathrm{i}\xi)\bigg\}, \tag{4.16}$$

where $\xi \in \mathbb{R}$. It follows from (4.5) that

$$|\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Y_t^y}]| \leq |\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Y_t^0}]| \leq c_1\exp\{-c_2|\xi|^{2-\alpha}\} \quad \text{for all } \xi \in \mathbb{R} \text{ and } t \in [1/T, T],$$

where $T > 1$ and $c_1, c_2 > 0$ are constants depending on T. It follows that, for $t > 0$, Y_t^y has a density $f_{Y_t^y}$ given by (4.15). Proceeding in the same way as in Lemma 4.3, we obtain the desired continuity and smoothness properties of $f_{Y_t^y}$.

We next show that if $t > 0$ then $f_{Y_t^y}(x) > 0$ for all $x > 0$. According to (4.16), we see that the law of Y_t^y, denoted by $\mu_{Y_t^y}$, is the convolution of the laws of Z_t^y and Y_t^0, which we denote by $\mu_{Z_t^y}$ and $\mu_{Y_t^0}$, respectively. So $\mu_{Y_t^y} = \mu_{Z_t^y} * \mu_{Y_t^0}$. From this we deduce that, for all $x > 0$,

$$f_{Y_t^y}(x) = \int_{(0,\infty)} f_{Y_t^0}(x-z)\mu_{Z_t^y}(\mathrm{d}z) + f_{Y_t^0}(x)\mu_{Z_t^y}(\{0\}). \tag{4.17}$$

By Lemma 4.4, the density function $f_{Y_t^0}(x)$ of Y_t^0 is strictly positive for almost all $x > 0$. In the following, we consider a fixed $x > 0$ and distinguish between two cases.

Case 1: $f_{Y_t^0}(x) > 0$. It follows from (4.17) that $f_{Y_t^y}(x) \geq f_{Y_t^0}(x)\mu_{Z_t^y}(\{0\}) > 0$, since $\mu_{Z_t^y}(\{0\}) = \mathbb{P}(Z_t^y = 0) > 0$, as shown in (3.13).

Case 2: $f_{Y_t^0}(x) = 0$. Then $x \in A_n$ for a large enough n, where the set A_n is the same as in the proof of Lemma 4.4. Since A_n is discrete, we can find a small enough $\delta > 0$ such that

$$f_{Y_t^0}(x-z) > 0 \quad \text{for all } z \in (0,\delta]. \tag{4.18}$$

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We next show that $\mu_{Z_t^y}((0,\delta]) > 0$. By (3.11), (3.13), and L'Hospital's rule, we obtain

$$\begin{aligned}
&\lim_{\lambda\to\infty}(\mathbb{E}[\exp\{-\lambda(Z_t^y-\delta)\}] - \mathbb{E}[\exp\{-\lambda(Z_t^y-\delta)\}\mathbf{1}_{\{Z_t^y=0\}}]) \\
&\quad = \lim_{\lambda\to\infty} \mathrm{e}^{\lambda\delta}(\mathbb{E}[\exp\{-\lambda Z_t^y\}] - \mathbb{P}(Z_t^y=0)) \\
&\quad = \lim_{\lambda\to\infty} \mathrm{e}^{\lambda\delta}(\exp\{-yv_t(\lambda)\} - \mathrm{e}^{-yd}) \\
&\quad = \lim_{\lambda\to\infty} \delta^{-1}\mathrm{e}^{\lambda\delta} y\exp\{-yv_t(\lambda)\}(v_t(\lambda))^{\alpha}\exp\{b(\alpha-1)t\}\lambda^{-\alpha} \\
&\quad = \infty. \qquad (4.19)
\end{aligned}$$

Suppose that $\mathbb{P}(Z_t^y \in (0,\delta]) = 0$. Then we can use the dominated convergence theorem to obtain

$$\begin{aligned}
&\lim_{\lambda\to\infty}(\mathbb{E}[\exp\{-\lambda(Z_t^y-\delta)\}] - \mathbb{E}[\exp\{-\lambda(Z_t^y-\delta)\}\mathbf{1}_{\{Z_t^y=0\}}]) \\
&\quad = \lim_{\lambda\to\infty}(\mathbb{E}[\exp\{-\lambda(Z_t^y-\delta)\}\mathbf{1}_{\{0<Z_t^y\le\delta\}}] + \mathbb{E}[\exp\{-\lambda(Z_t^y-\delta)\}\mathbf{1}_{\{Z_t^y>\delta\}}]) \\
&\quad = 0,
\end{aligned}$$

which contradicts (4.19). Consequently, the assumption that $\mathbb{P}(Z_t^y \in (0,\delta]) = 0$ does not hold and we thus obtain $\mathbb{P}(Z_t^y \in (0,\delta]) > 0$. Now, by (4.17) and (4.18), we have

$$f_{Y_t^y}(x) \ge \int_{(0,\delta]} f_{Y_t^0}(x-z)\mu_{Z_t^y}(\mathrm{d}z) > 0.$$

Summarizing the above two cases, we have $f_{Y_t^y}(x) > 0$ for all $x > 0$. □

5. A Foster–Lyapunov function for (Y, X)

We now turn back to the two-dimensional affine process $(Y, X) = (Y_t, X_t)_{t\ge0}$ defined in (1.1). Our aim of this section is to construct a Foster–Lyapunov function for (Y, X).

For a functional $\Phi(Y, X)$ based on the process (Y, X), we use $\mathbb{E}_{(y,x)}[\Phi(Y, X)]$ to indicate that the process (Y, X) considered under the expectation is with the initial condition $(Y_0, X_0) = (y, x)$, where $(y, x) \in \mathbb{R}_{\ge0}\times\mathbb{R}$ is constant. The notation $\mathbb{P}_{(y,x)}(\Phi(Y, X) \in \cdot)$ is similarly defined.

Lemma 5.1. *Let $h \in C^{\infty}(\mathbb{R},\mathbb{R})$ be such that $h(x) \ge 1$ for all $x \in \mathbb{R}$ and $h(x) = |x|$ whenever $|x| \ge 2$. Define*

$$V(y, x) := \beta y + h(x), \qquad y \ge 0,\ x \in \mathbb{R},$$

where $\beta > 0$ is a constant. If β is sufficiently large then V is a Foster–Lyapunov function for (Y, X), i.e. there exist constants $c, M > 0$ such that

$$\mathbb{E}_{(y,x)}[V(Y_t, X_t)] \le \mathrm{e}^{-ct}V(y, x) + \frac{M}{c} \qquad (5.1)$$

for all $(y, x) \in \mathbb{R}_{\ge0}\times\mathbb{R}$ and $t \ge 0$.

Proof. We start by checking that $\mathbb{R}_{\ge0}^2\times\Omega \ni (s, z, \omega) \mapsto z\mathbf{1}_{\{|z|<1\}}\sqrt[\alpha]{Y_{s-}} \in F_p^{2,\mathrm{loc}}$ and $\mathbb{R}_{\ge0}^2\times\Omega \ni (s, z, \omega) \mapsto z\mathbf{1}_{\{|z|\ge1\}}\sqrt[\alpha]{Y_{s-}} \in F_p^1$, where the definition of the classes $F_p^{2,\mathrm{loc}}$ and F_p^1

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can be found in [14, pp. 61, 62]. Let $\tau_n := \inf\{t \in \mathbb{R}_{>0} : Y_t > n\}$, $n \in \mathbb{N}$. Noting that

$$\begin{aligned}\mathbb{E}\bigg[\int_0^{t\wedge\tau_n}\int_{\{|z|<1\}}(z\sqrt[\alpha]{Y_{s-}})^2 C_\alpha z^{-1-\alpha}\,\mathrm{d}s\,\mathrm{d}z\bigg] &\le C_\alpha\int_0^1 z^{1-\alpha}\,\mathrm{d}z\,\mathbb{E}\bigg[\int_0^{t\wedge\tau_n} n^{2/\alpha}\,\mathrm{d}s\bigg]\\ &= \frac{C_\alpha}{2-\alpha}tn^{2/\alpha}\\ &< \infty, \end{aligned}\tag{5.2}$$

it follows that $\mathbb{R}_{\ge 0}^2 \times \Omega \ni (s, z, \omega) \mapsto \mathbf{1}_{\{|z|<1\}}\, z\sqrt[\alpha]{Y_{s-}} \in F_p^{2,\mathrm{loc}}$. Similarly, since, for any $0 < \varepsilon < \alpha$, we have

$$\mathbb{E}_y[Y_t^\varepsilon] \le c_1\bigg(1 + y^\varepsilon \exp\bigg\{-\frac{\varepsilon bt}{\alpha}\bigg\}\bigg) \quad \text{for } t \ge 0, \tag{5.3}$$

by [22, Proposition 2.8], where $c_1 > 0$ is some constant, we obtain

$$\begin{aligned}\mathbb{E}\bigg[\int_0^t\int_{\{|z|\ge1\}}|z\sqrt[\alpha]{Y_{s-}}|C_\alpha z^{-1-\alpha}\,\mathrm{d}s\,\mathrm{d}z\bigg] &= C_\alpha\int_1^\infty z^{-\alpha}\,\mathrm{d}z\int_0^t \mathbb{E}[\sqrt[\alpha]{Y_{s-}}]\,\mathrm{d}s\\ &\le c_1\frac{C_\alpha}{\alpha-1}\int_0^t\bigg(1 + {Y_0}^{1/\alpha}\exp\bigg\{\frac{-bs}{\alpha^2}\bigg\}\bigg)\,\mathrm{d}s\\ &< \infty, \end{aligned}\tag{5.4}$$

which verifies that $\mathbb{R}_{\ge 0}^2 \times \Omega \ni (s, z, \omega) \mapsto \mathbf{1}_{\{|z|\ge1\}}\, z\sqrt[\alpha]{Y_{s-}} \in F_p^1$.

Define $g(t, y, x) := \exp\{ct\}V(y, x)$, where $c > 0$ is a constant to be determined later. It is easy to see that $g \in C^2(\mathbb{R}_{\ge0} \times \mathbb{R}_{\ge0} \times \mathbb{R}, \mathbb{R})$. We define the functions g_1', g_2', g_3', and $g_{3,3}''$ by

$$g_1'(t, y, x) := \frac{\partial}{\partial t}g(t, y, x) = c\mathrm{e}^{ct}V(y, x), \qquad g_2'(t, y, x) := \frac{\partial}{\partial y}g(t, y, x) = \beta\mathrm{e}^{ct},$$

$$g_3'(t, y, x) := \frac{\partial}{\partial x}g(t, y, x) = \mathrm{e}^{ct}\frac{\partial}{\partial x}h(x), \qquad g_{3,3}''(t, y, x) := \frac{\partial^2}{\partial x^2}g(t, y, x) = \mathrm{e}^{ct}\frac{\partial^2}{\partial x^2}h(x).$$

If the process $(Y_t, X_t)_{t\ge0}$ starts from (y, x), i.e. $(Y_0, X_0) = (y, x)$, then we can use the Lévy–Itô decomposition of $(L_t)_{t\ge0}$ in (2.2) to obtain, for each $t \ge 0$,

$$\begin{aligned}Y_t &= y + \int_0^t \gamma\sqrt[\alpha]{Y_s}\,\mathrm{d}s + \int_0^t (a - bY_s)\,\mathrm{d}s + \int_0^t\int_{\{|z|<1\}} z\sqrt[\alpha]{Y_{s-}}\tilde{N}(\mathrm{d}s,\ \mathrm{d}z)\\ &\quad + \int_0^t\int_{\{|z|\ge1\}} z\sqrt[\alpha]{Y_{s-}}N(\mathrm{d}s,\ \mathrm{d}z),\\ X_t &= x + \int_0^t (m - \theta X_s)\,\mathrm{d}s + \int_0^t \sqrt{Y_s}\,\mathrm{d}B_s, \end{aligned}\tag{5.5}$$

where γ, and $N(\mathrm{d}s,\ \mathrm{d}z)$ and $\tilde{N}(\mathrm{d}s,\ \mathrm{d}z)$ are as in (2.2). By (5.5) and applying Itô's formula for g (see [31, Theorem 94]), we obtain, for each $t \ge 0$,

$$g(t, Y_t, X_t) - g(0, Y_0, X_0) = \int_0^t (\mathcal{L}g)(s, Y_s, X_s)\,\mathrm{d}s + \int_0^t g_1'(s, Y_s, X_s)\,\mathrm{d}s + M_t(g), \tag{5.6}$$

where

$$\begin{aligned}M_t(g) &:= \int_0^t g_3'(s, Y_s, X_s)\sqrt{Y_s}\,\mathrm{d}B_s\\ &\quad + \int_0^t\int_{\{|z|<1\}}(g(s, Y_{s-} + z\sqrt[\alpha]{Y_{s-}}, X_{s-}) - g(s, Y_{s-}, X_{s-}))\tilde{N}(\mathrm{d}s,\ \mathrm{d}z)\end{aligned}$$

$$+\int_0^t\int_{\{|z|\geq 1\}}(g(s, Y_{s-}+z\sqrt[\alpha]{Y_{s-}}, X_{s-})-g(s, Y_{s-}, X_{s-}))N(\mathrm{d}s,\ \mathrm{d}z)$$
$$-\int_0^t\int_{\{|z|\geq 1\}}(g(s, Y_s+z\sqrt[\alpha]{Y_s}, X_s)-g(s, Y_s, X_s))\hat{N}(\mathrm{d}s,\ \mathrm{d}z)$$

and $\mathcal{L}g$ is defined by

$$\begin{aligned}(\mathcal{L}g)(t, y, x) := {} & (a-by)g_2'(t, y, x)+(m-\theta x)g_3'(t, y, x)+\tfrac{1}{2}yg_{3,3}''(t, y, x)\\ & +\int_{\{|z|<1\}}(g(t, y+z\sqrt[\alpha]{y}, x)-g(t, y, x)-z\sqrt[\alpha]{y}g_2'(t, y, x))C_\alpha z^{-1-\alpha}\,\mathrm{d}z\\ & +\int_{\{|z|\geq 1\}}(g(t, y+z\sqrt[\alpha]{y}, x)-g(t, y, x))C_\alpha z^{-1-\alpha}\,\mathrm{d}z+\gamma\sqrt[\alpha]{y}g_2'(t, y, x)\end{aligned}$$

for $(t, y, x)\in\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\times\mathbb{R}$. By a change of variable $\tilde{z} := z\sqrt[\alpha]{y}$ and an easy computation, we see that $(\mathcal{L}g)(s, Y_s, X_s)=\mathrm{e}^{cs}(\mathcal{A}V)(Y_s, X_s)$, where $\mathcal{A}$ is given in (2.3). As a result, it follows from (5.6) that, for each $t\geq 0$,

$$g(t, Y_t, X_t)-g(0, Y_0, X_0)=\int_0^t\mathrm{e}^{cs}(\mathcal{A}V)(Y_s, X_s)\,\mathrm{d}s+\int_0^t g_1'(s, Y_s, X_s)\,\mathrm{d}s+M_t(g). \tag{5.7}$$

The rest of the proof is divided into three steps.

Step 1. We show that $(M_t(g))_{t\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, where $(\mathcal{F}_t)_{t\geq 0}$ is the same as in Section 2. To achieve this, we can use a similar argument as in [4]. Define

$$M_t^1(g) := \int_0^t g_3'(s, Y_s, X_s)\sqrt{Y_s}\,\mathrm{d}B_s,$$
$$\begin{aligned}M_t^2(g) := {} & \int_0^t\int_{\{|z|<1\}}(g(s, Y_{s-}+z\sqrt[\alpha]{Y_{s-}}, X_{s-})-g(s, Y_{s-}, X_{s-}))\tilde{N}(\mathrm{d}s,\ \mathrm{d}z),\\ & +\int_0^t\int_{\{|z|\geq 1\}}(g(s, Y_{s-}+z\sqrt[\alpha]{Y_{s-}}, X_{s-})-g(s, Y_{s-}, X_{s-}))N(\mathrm{d}s,\ \mathrm{d}z)\\ & -\int_0^t\int_{\{|z|\geq 1\}}(g(s, Y_s+z\sqrt[\alpha]{Y_s}, X_s)-g(s, Y_s, X_s))\hat{N}(\mathrm{d}s,\ \mathrm{d}z),\end{aligned}$$

where $t\geq 0$. By noting that $(t, y, x)\mapsto g_2'(t, y, x)$ is bounded for $(t, y, x)\in[0, T]\times\mathbb{R}_{\geq 0}\times\mathbb{R}$, where $T>0$ is constant, we can proceed in the same way as in [4, Theorem 2.1] to prove that $(M_t^1(g))_{t\geq 0}$ is a square-integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Note that $g(s, y+z, x)-g(s, y, x)=\beta\exp\{cs\}z$. Similarly to (5.2) and (5.4), we see that

$$\mathbf{1}_{\{|z|<1\}}(g(s, Y_{s-}+z\sqrt[\alpha]{Y_{s-}}, X_{s-})-g(s, Y_{s-}, X_{s-}))=\beta\mathrm{e}^{cs}\,\mathbf{1}_{\{|z|<1\}}\,z\sqrt[\alpha]{Y_{s-}}\in F_p^{2,\mathrm{loc}},$$
$$\mathbf{1}_{\{|z|\geq 1\}}(g(s, Y_{s-}+z\sqrt[\alpha]{Y_{s-}}, X_{s-})-g(s, Y_{s-}, X_{s-}))=\beta\mathrm{e}^{cs}\,\mathbf{1}_{\{|z|\geq 1\}}\,z\sqrt[\alpha]{Y_{s-}}\in F_p^1.$$

Following [14, pp. 62, 63], we see that

$$M_t^3(g) := \int_0^t\int_{\{|z|<1\}}(g(s, Y_{s-}+z\sqrt[\alpha]{Y_{s-}}, X_{s-})-g(s, Y_{s-}, X_{s-}))\tilde{N}(\mathrm{d}s,\ \mathrm{d}z),\qquad t\geq 0,$$

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is a local square-integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and

$$M_t^4(g) := \int_0^t \int_{\{|z|\geq 1\}} (g(s, Y_{s-} + z\sqrt[\alpha]{Y_{s-}}, X_{s-}) - g(s, Y_{s-}, X_{s-}))N(\mathrm{d}s,\, \mathrm{d}z) \\ - \int_0^t \int_{\{|z|\geq 1\}} (g(s, Y_s + z\sqrt[\alpha]{Y_s}, X_s) - g(s, Y_s, X_s))\hat{N}(\mathrm{d}s,\, \mathrm{d}z), \qquad t \geq 0,$$

is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Therefore, $(M_t^2(g))_{t\geq 0} = (M_t^3(g) + M_t^4(g))_{t\geq 0}$ is a local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. It remains to check that $(M_t^2(g))_{t\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Using the Lévy–Itô decomposition in (2.2), we obtain

$$\begin{aligned} M_t^2(g) &= \int_0^t \int_{\{|z|<1\}} \beta \mathrm{e}^{cs} z\sqrt[\alpha]{Y_{s-}}\tilde{N}(\mathrm{d}s,\, \mathrm{d}z) + \int_0^t \int_{\{|z|\geq 1\}} \beta \mathrm{e}^{cs} z\sqrt[\alpha]{Y_{s-}}N(\mathrm{d}s,\, \mathrm{d}z) \\ &\quad - \int_0^t \int_{\{|z|\geq 1\}} \beta \mathrm{e}^{cs} z\sqrt[\alpha]{Y_{s-}}\hat{N}(\mathrm{d}s,\, \mathrm{d}z) \\ &= \int_0^t \beta \mathrm{e}^{cs}\sqrt[\alpha]{Y_{s-}}\,\mathrm{d}L_s, \qquad t \geq 0. \end{aligned}$$

We can use [23, Remark 2.5] and Jensen's inequality to show that, for each $T > 0$, there exists some constant $c_2 > 0$ such that

$$\mathbb{E}_{(y,x)}\Big[\sup_{s\in[0,T]} |M_s^2(g)|\Big] \leq c_2\mathbb{E}_{(y,x)}\bigg[\bigg(\int_0^T Y_s\,\mathrm{d}s\bigg)^{1/\alpha}\bigg] \leq c_2\bigg(\int_0^T \mathbb{E}_{(y,x)}[Y_s]\,\mathrm{d}s\bigg)^{1/\alpha} < \infty,$$

where the last inequality follows from (5.3). Since $(M_t^2(g))_{t\geq 0}$ is a local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, there exists an increasing sequence of stopping times σ_n, $n \in \mathbb{N}$ with $\sigma_n \to \infty$ as n tends to ∞ almost surely such that $(M_{t\wedge\sigma_n}^2(g))_{t\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Then, by the dominated convergence theorem for conditional expectations, we obtain, for all $0 \leq s \leq t \leq T$,

$$\begin{aligned} \mathbb{E}[M_t^2(g) \mid \mathcal{F}_s] &= \mathbb{E}\Big[\lim_{n\to\infty} M_{t\wedge\sigma_n}^2(g) \,\Big|\, \mathcal{F}_s\Big] \\ &= \lim_{n\to\infty} \mathbb{E}[M_{t\wedge\sigma_n}^2(g) \mid \mathcal{F}_s] \\ &= \lim_{n\to\infty} M_{s\wedge\sigma_n}^2(g) \\ &= M_s^2(g), \end{aligned}$$

showing that $M_t^2(g)$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. As a result,

$$(M_t(g))_{t\geq 0} = (M_t^1(g) + M_t^2(g))_{t\geq 0}$$

is also a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. This completes the proof of step 1.

Step 2. We determine the constant $c > 0$ and find another constant $M > 0$ such that

$$(\mathcal{A}V)(y, x) \leq -cV(y, x) + M \quad \text{for all } (y, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}, \tag{5.8}$$

where $\mathcal{A}$ is given by (2.3). For the function V, we have $V \in C^2(\mathbb{R}_{\geq 0} \times \mathbb{R}, \mathbb{R})$,

$$\frac{\partial}{\partial y}V(y, x) = \beta, \qquad \frac{\partial}{\partial x}V(y, x) = \frac{\partial}{\partial x}h(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } |x| > 2, \\ h'(x) & \text{if } |x| \leq 2, \end{cases}$$

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and

$$\frac{\partial^2}{\partial x^2} V(y, x) = \frac{\partial^2}{\partial x^2} h(x) := \begin{cases} 0 & \text{if } |x| > 2, \\ h''(x) & \text{if } |x| \le 2, \end{cases}$$

where h' and h'' denote the first- and second-order derivatives of the function h, respectively. So

$$\begin{aligned} (\mathcal{A}V)(y, x) &= (a - by)\beta + (m - \theta x)\frac{\partial}{\partial x} h(x) + \frac{1}{2} y \frac{\partial^2}{\partial x^2} h(x) \\ &\quad + y \int_0^\infty (\beta(y+z) + h(x) - \beta y - h(x) - z\beta) C_\alpha z^{-1-\alpha} \, \mathrm{d}z \\ &= (a - by)\beta + (m - \theta x)\frac{\partial}{\partial x} h(x) + \frac{1}{2} y \frac{\partial^2}{\partial x^2} h(x). \end{aligned}$$

By choosing $\beta > 0$ large enough, we obtain, for all $(y, x) \in \mathbb{R}_{\ge 0} \times \mathbb{R}$,

$$\begin{aligned} (\mathcal{A}V)(y, x) &= a\beta - \frac{by\beta}{2} - \theta x \frac{\partial}{\partial x} h(x) + \bigg(-\frac{b\beta}{2} + \frac{1}{2} \frac{\partial^2}{\partial x^2} h(x) \bigg) y + m \frac{\partial}{\partial x} h(x) \\ &\le a\beta - \frac{by\beta}{2} - \theta (h(x) \, \mathbf{1}_{\{x > 2\}} + h(x) \, \mathbf{1}_{\{x < -2\}}) + 0 + c_3 \\ &\le a\beta - \frac{by\beta}{2} - \theta (h(x) \, \mathbf{1}_{\{|x| > 2\}} + h(x) \, \mathbf{1}_{\{|x| \le 2\}}) + c_4 \\ &= a\beta - \frac{by\beta}{2} - \theta h(x) + c_4 \\ &= -\frac{b\beta}{2} y - \theta h(x) + c_5, \end{aligned}$$

where we used the boundedness of $|h'|$, $|h''|$, and $|h| \, \mathbf{1}_{\{|x| \le 2\}}$ to obtain the first and second inequalities. Here c_3, c_4, and c_5 are some positive constants. Now we see that (5.8) holds with $c := \min(b/2, \theta)$ and $M := c_5$.

Step 3. We prove (5.1). Note that $(\mathcal{L}g)(s, Y_s, X_s) = \exp\{cs\}(\mathcal{A}V)(Y_s, X_s)$. By (5.7), (5.8), and the martingale property of $(M_t(g))_{t \ge 0}$, we obtain

$$\begin{aligned} \mathrm{e}^{ct} \mathbb{E}_{(y,x)}[V(Y_t, X_t)] - V(y, x) &= \mathbb{E}_{(y,x)}[g(t, Y_t, X_t)] - \mathbb{E}_{(y,x)}[g(0, Y_0, X_0)] \\ &= \mathbb{E}_{(y,x)}\bigg[\int_0^t (\mathrm{e}^{cs} (\mathcal{A}V)(Y_s, X_s) + c\mathrm{e}^{cs} V(Y_s, X_s)) \, \mathrm{d}s \bigg] \\ &\le \mathbb{E}_{(y,x)}\bigg[\int_0^t (\mathrm{e}^{cs} (-cV(Y_s, X_s) + M) + c\mathrm{e}^{cs} V(Y_s, X_s)) \, \mathrm{d}s \bigg] \\ &= \mathbb{E}_{(y,x)}\bigg[\int_0^t M\mathrm{e}^{cs} \, \mathrm{d}s \bigg] \\ &\le \frac{M}{c} \mathrm{e}^{ct} \end{aligned}$$

for all $(y, x) \in \mathbb{R}_{\ge 0} \times \mathbb{R}$ and $t \ge 0$, which implies (5.1). □

Remark 5.1. To see the existence of a function $h \in C^\infty(\mathbb{R}, \mathbb{R})$ that fulfills the conditions of Lemma 5.1, we can proceed in the following way: let $\rho \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $\rho(x) = 1$

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for $x \geq 2$, $\rho(x) = 0$ for $x \leq 1$, and $0 \leq \rho(x) \leq 1$ for $1 \leq x \leq 2$. Define $F\colon \mathbb{R} \to \mathbb{R}$ by $F(x) := \int_0^x \rho(r)\,\mathrm{d}r$, $x \in \mathbb{R}$. Then

$$F(x) = \begin{cases} 0, & x \leq 1, \\ \in [0,1], & 1 < x \leq 2, \\ x - 2 + \int_1^2 \rho(r)\,\mathrm{d}r, & x > 2. \end{cases}$$

We now define $h\colon \mathbb{R} \to \mathbb{R}$ by $h(x) := F(|x|) + 2 - F(2)$, $x \in \mathbb{R}$. Then h satisfies the conditions required in Lemma 5.1.

6. Exponential ergodicity of (Y, X)

In this section we prove our main result, namely, the exponential ergodicity of the affine two-factor model $(Y, X) = (Y_t, X_t)_{t\geq 0}$.

Let $\|\cdot\|_{\mathrm{TV}}$ denote the total variation norm for signed measures on $\mathbb{R}_{\geq 0} \times \mathbb{R}$, namely,

$$\|\mu\|_{\mathrm{TV}} := \sup\{|\mu(A)|\},$$

where μ is a signed measure on $\mathbb{R}_{\geq 0} \times \mathbb{R}$ and the above supremum is running for all Borel sets A in $\mathbb{R}_{\geq 0} \times \mathbb{R}$.

Let $\boldsymbol{P}^t(y, x, \cdot) := \mathbb{P}_{(y,x)}((Y_t, X_t) \in \cdot)$ denote the distribution of $(Y_t, X_t)_{t\geq 0}$ with the initial condition $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$.

By [4, Theorem 3.1] and the argument of [19, p. 80], there exists a unique invariant probability measure π for the two-dimensional process $(Y_t, X_t)_{t\geq 0}$. Roughly speaking, if, for each $(y, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, the convergence of the distribution $\boldsymbol{P}^t(y, x, \cdot)$ to π as $t \to \infty$ is exponentially fast with respect to the total variation norm then we say that the process $(Y_t, X_t)_{t\geq 0}$ is exponentially ergodic.

The main result of this paper is the following.

Theorem 6.1. *Consider the two-dimensional affine process $(Y, X) = (Y_t, X_t)_{t\geq 0}$ defined by (1.1) with parameters $\alpha \in (1, 2)$, $a > 0$, $b > 0$, $m \in \mathbb{R}$, and $\theta > 0$. Then $(Y_t, X_t)_{t\geq 0}$ is exponentially ergodic, i.e. there exist constants $\delta \in (0, \infty)$ and $B \in (0, \infty)$ such that*

$$\|\boldsymbol{P}^t(y, x, \cdot) - \pi\|_{\mathrm{TV}} \leq B(V(y, x) + 1)\mathrm{e}^{-\delta t}$$

for all $t \geq 0$ and $(y, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$.

Proof. We basically follow the proof of [16, Theorem 6.3]. The essential idea is to use the so-called Foster–Lyapunov criteria developed in [26] for the geometric ergodicity of Markov chains.

We first consider the skeleton chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\geq 0}}$, which is a Markov chain on the state space $\mathbb{R}_{\geq 0} \times \mathbb{R}$ with transition kernel $\boldsymbol{P}^n(y, x, \cdot)$. It is easy to see that the measure π is also an invariant probability measure for the chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\geq 0}}$.

Let the function V be the same as in Lemma 5.1, and the constant $\beta > 0$, again from Lemma 5.1, be sufficiently large. The Markov property, together with Lemma 5.1, implies that

$$\begin{aligned}\mathbb{E}[V(Y_{n+1}, X_{n+1}) \mid (Y_0, X_0), (Y_1, X_1), \ldots, (Y_n, X_n)] &= \int_{\mathbb{R}_{\geq 0}} \int_{\mathbb{R}} V(y, x)\boldsymbol{P}^1(Y_n, X_n, \mathrm{d}y\,\mathrm{d}x) \\ &\leq \mathrm{e}^{-c} V(Y_n, X_n) + \frac{M}{c},\end{aligned}$$

where c and M are the positive constants in Lemma 5.1. If we set $V_0 := V$ and $V_n := V(Y_n, X_n)$, $n \in \mathbb{N}$, then

$$\mathbb{E}[V_1 \mid Y_0, X_0] \le \mathrm{e}^{-c} V_0(Y_0, X_0) + \frac{M}{c} \tag{6.1}$$

and, for all $n \in \mathbb{N}$,

$$\mathbb{E}[V_{n+1} \mid (Y_0, X_0), (Y_1, X_1), \dots, (Y_n, X_n)] \le \mathrm{e}^{-c} V_n + \frac{M}{c}. \tag{6.2}$$

It follows from (6.1) and (6.2) that Condition (DD4) of [24, p. 564] holds. In order to apply [24, Theorem 6.3] for the chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\ge 0}}$, it remains to verify the following conditions:

(a) the Lebesgue measure λ on $\mathbb{R}_{\ge 0} \times \mathbb{R}$ is an irreducibility measure for the chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\ge 0}}$;

(b) the chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\ge 0}}$ is aperiodic (the definition of aperiodicity can be found in [27, p. 114]);

(c) all compact sets of the state space $\mathbb{R}_{\ge 0} \times \mathbb{R}$ are petite (see [25, p. 500] for a definition).

We now proceed to prove (a)–(c).

In order to prove (a), we will use the same argument as in [4, Theorem 4.1]. It is enough to check that, for each $(y_0, x_0) \in \mathbb{R}_{\ge 0} \times \mathbb{R}$, the measure $\boldsymbol{P}^1(y_0, x_0, \cdot)$ is absolutely continuous with respect to the Lebesgue measure with a density function $p_1(y, x \mid y_0, x_0)$ that is strictly positive for almost all $(y, x) \in \mathbb{R}_{\ge 0} \times \mathbb{R}$. Indeed, let A be a Borel set of $\mathbb{R}_{\ge 0} \times \mathbb{R}$ with $\lambda(A) > 0$. Then

$$\mathbb{P}_{(y_0,x_0)}(\tau_A < \infty) \ge \boldsymbol{P}^1(y_0, x_0, A) = \iint_A p_0(y, x \mid y_0, x_0)\, \mathrm{d}y\, \mathrm{d}x > 0$$

for all $(y_0, x_0) \in \mathbb{R}_{\ge 0} \times \mathbb{R}$, where the stopping time τ_A is defined by $\tau_A := \inf\{n \ge 0 \colon (Y_n, X_n) \in A\}$.

Next, we prove the existence of the density $p_1(y, x \mid y_0, x_0)$ with the required property. Recall that

$$Y_1 = \mathrm{e}^{-b}\bigg(y_0 + a\int_0^1 \mathrm{e}^{bs}\, \mathrm{d}s + \int_0^1 \mathrm{e}^{bs} \sqrt[\alpha]{Y_{s-}}\, \mathrm{d}L_s\bigg)$$

and

$$X_1 = \mathrm{e}^{-\theta}\bigg(x_0 + m\int_0^1 \mathrm{e}^{\theta s}\, \mathrm{d}s + \int_0^1 \mathrm{e}^{\theta s} \sqrt{Y_s}\, \mathrm{d}B_s\bigg),$$

provided that $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{\ge 0} \times \mathbb{R}$. For $(\bar{y}, \bar{x}) \in \mathbb{R}_{\ge 0} \times \mathbb{R}$, we have

$$\begin{aligned}\mathbb{P}_{(y_0,x_0)}(Y_1 < \bar{y}, X_1 < \bar{x}) &= \mathbb{E}_{(y_0,x_0)}[\mathbb{P}_{(y_0,x_0)}(Y_1 < \bar{y}, X_1 < \bar{x} \mid Y_1)] \\ &= \mathbb{E}_{(y_0,x_0)}[\mathbb{E}_{(y_0,x_0)}[\mathbf{1}_{\{Y_1<\bar{y}\}}\, \mathbf{1}_{\{X_1<\bar{x}\}} \mid Y_1]] \\ &= \mathbb{E}_{(y_0,x_0)}[\mathbf{1}_{\{Y_1<\bar{y}\}}\, \mathbb{E}_{(y_0,x_0)}[\mathbf{1}_{\{X_1<\bar{x}\}} \mid Y_1]]. \end{aligned} \tag{6.3}$$

Note that $(Y_t)_{t\ge 0}$ and the Brownian motion $(B_t)_{t\ge 0}$ are independent, since $(L_t)_{t\ge 0}$ and $(B_t)_{t\ge 0}$ are independent and $(Y_t)_{t\ge 0}$ is a strong solution. Therefore, the conditional distribution of X_1, given $(Y_t)_{t\in[0,1]}$, is a normal distribution with mean $x_0 \exp\{-\theta\} + m(1 - \exp\{-\theta\})/\theta$ and

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variance $\exp\{-2\theta\}\int_0^1 Y_s \exp\{2\theta s\}\,\mathrm{d}s$. Hence, we obtain, for $\bar{x}\in\mathbb{R}$,

$$\begin{aligned}\mathbb{E}_{(y_0,x_0)}&[\mathbf{1}_{\{X_1<\bar{x}\}} \mid Y_1]\\ &= \mathbb{E}_{(y_0,x_0)}[\mathbb{E}_{(y_0,x_0)}[\mathbf{1}_{\{X_1<\bar{x}\}} \mid (Y_t)_{0\le t\le 1}] \mid Y_1]\\ &= \mathbb{E}_{(y_0,x_0)}\bigg[\int_{-\infty}^{\bar{x}} \varrho\bigg(r-\mathrm{e}^{-\theta}x_0-\frac{m}{\theta}(1-\mathrm{e}^{-\theta});\ \mathrm{e}^{-2\theta}\int_0^1 \mathrm{e}^{2\theta s}Y_s\,\mathrm{d}s\bigg)\,\mathrm{d}r \;\bigg|\; Y_1\bigg], \qquad (6.4)\end{aligned}$$

where $\varrho(r;\sigma^2)=(\sigma\sqrt{2\pi})^{-1}\exp\{-r^2/(2\sigma^2)\}$, $r\in\mathbb{R}$, is the density of the normal distribution with variance $\sigma^2>0$. Note that the assumption $a>0$ ensures that $\mathbb{P}_{(y_0,x_0)}(\int_0^1 \mathrm{e}^{2\theta s}Y_s\,\mathrm{d}s>0)=1$. By [18, Theorem 6.3] and considering the conditional distribution of $\int_0^1 \mathrm{e}^{2\theta s}Y_s\,\mathrm{d}s$, given Y_1, we can find a probability kernel $K_{(y_0,x_0)}(\cdot,\cdot)$ from $\mathbb{R}_{\ge 0}$ to $\mathbb{R}_{\ge 0}$ such that

$$\mathbb{P}_{(y_0,x_0)}\bigg(\int_0^1 \mathrm{e}^{2\theta s}Y_s\,\mathrm{d}s\in\cdot \;\bigg|\; Y_1\bigg)=K_{(y_0,x_0)}(Y_1,\cdot)$$

and

$$K_{(y_0,x_0)}(z,\mathbb{R}_{>0})=1 \quad \text{for all } z\ge 0. \qquad (6.5)$$

So

$$\begin{aligned}\mathbb{E}_{(y_0,x_0)}&\bigg[\int_{-\infty}^{\bar{x}} \varrho\bigg(r-\mathrm{e}^{-\theta}x_0-\frac{m}{\theta}(1-\mathrm{e}^{-\theta});\ \mathrm{e}^{-2\theta}\int_0^1 \mathrm{e}^{2\theta s}Y_s\,\mathrm{d}s\bigg)\,\mathrm{d}r \;\bigg|\; Y_1\bigg]\\ &= \int_0^{\infty}\bigg(\int_{-\infty}^{\bar{x}} \varrho\bigg(r-\mathrm{e}^{-\theta}x_0-\frac{m}{\theta}(1-\mathrm{e}^{-\theta});\ \mathrm{e}^{-2\theta}w\bigg)\,\mathrm{d}r\bigg)K_{(y_0,x_0)}(Y_1,\,\mathrm{d}w)\\ &= \int_{-\infty}^{\bar{x}}\bigg(\int_0^{\infty} \varrho\bigg(r-\mathrm{e}^{-\theta}x_0-\frac{m}{\theta}(1-\mathrm{e}^{-\theta});\ \mathrm{e}^{-2\theta}w\bigg)K_{(y_0,x_0)}(Y_1,\,\mathrm{d}w)\bigg)\,\mathrm{d}r. \qquad (6.6)\end{aligned}$$

It follows from (6.3), (6.4), and (6.6) that, for all $(\bar{y},\bar{x})\in\mathbb{R}_{\ge 0}\times\mathbb{R}$,

$$\begin{aligned}\mathbb{P}_{(y_0,x_0)}&(Y_1<\bar{y},\,X_1<\bar{x})\\ &= \int_0^{\bar{y}}\int_{-\infty}^{\bar{x}}\bigg(\int_0^{\infty} \varrho\bigg(r-\mathrm{e}^{-\theta}x_0-\frac{m}{\theta}(1-\mathrm{e}^{-\theta});\ \mathrm{e}^{-2\theta}w\bigg)K_{(y_0,x_0)}(z,\,\mathrm{d}w)\bigg)f_{Y_1^{y_0}}(z)\,\mathrm{d}r\,\mathrm{d}z, \qquad (6.7)\end{aligned}$$

where $f_{Y_1^{y_0}}$ is given in (4.15). Define

$$p_1(y,x \mid y_0,x_0) := f_{Y_1^{y_0}}(y)\int_0^{\infty} \varrho\bigg(x-\mathrm{e}^{-\theta}x_0-\frac{m}{\theta}(1-\mathrm{e}^{-\theta});\ \mathrm{e}^{-2\theta}w\bigg)K_{(y_0,x_0)}(y,\,\mathrm{d}w).$$

Since $f_{Y_1^{y_0}}(y)>0$ for all $y>0$ and

$$0=\mathbb{P}_{(y_0,x_0)}\bigg(\int_0^1 \mathrm{e}^{2\theta s}Y_s\,\mathrm{d}s=0\bigg)=\int_0^{\infty}K_{(y_0,x_0)}(y,\{0\})f_{Y_1^{y_0}}(y)\,\mathrm{d}y,$$

it follows that $K_{(y_0,x_0)}(y,\{0\})=0$ for all $y\in\mathbb{R}_{\ge 0}\setminus N$, where N is some null set under the Lebesgue measure. By modifying the definition of the kernel $K_{(y_0,x_0)}(y,\cdot)$ for $y\in N$, we can make sure that $K_{(y_0,x_0)}(y,\{0\})=0$ for all $y\in\mathbb{R}_{\ge 0}$, or, equivalently, $K_{(y_0,x_0)}(y,\mathbb{R}_{>0})=1$ for all $y\in\mathbb{R}_{\ge 0}$. By (6.5) and the fact that $f_{Y_1^{y_0}}(y)$ is strictly positive for all $y>0$ (see

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Proposition 4.1), for each $(y_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, the density $p_1(y, x \mid y_0, x_0)$ is strictly positive for almost all $(y, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. Moreover, by (6.7), we have

$$\mathbb{P}_{(y_0,x_0)}(Y_1 < \bar{y}, X_1 < \bar{x}) = \int_0^{\bar{y}} \int_{-\infty}^{\bar{x}} p_1(y, x \mid y_0, x_0)\, \mathrm{d}y\, \mathrm{d}x \quad \text{for all } (\bar{y}, \bar{x}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}.$$

So $p_1(\cdot, \cdot \mid y_0, x_0)$ is the density function of (Y_t, X_t), given that $(Y_0, X_0) = (y_0, x_0)$.

To prove (b), i.e. the aperiodicity of the skeleton chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\geq 0}}$, we use a contradiction argument. Suppose that the period l of the chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\geq 0}}$ is greater than 1 (see [27, p. 114] for a definition of the period of a Markov chain). Then we can find disjoint Borel sets $A_1, A_2, \dots, A_l$ such that

$$\lambda(A_i) > 0, \quad i = 1, \dots, l, \qquad \bigcup_{i=1}^{l} A_i = \mathbb{R}_{\geq 0} \times \mathbb{R}, \tag{6.8}$$

$$\boldsymbol{P}^1(y_0, x_0, A_{i+1}) = 1 \quad \text{for all } (y_0, x_0) \in A_i,\ i = 1, \dots, l-1, \tag{6.9}$$

and $\boldsymbol{P}^1(y_0, x_0, A_1) = 1$ for all $(y_0, x_0) \in A_l$. By (6.9), we have

$$\iint_{(A_2)^c} p_1(y, x \mid y_0, x_0)\, \mathrm{d}y\, \mathrm{d}x = 0, \qquad (y_0, x_0) \in A_1,$$

and, further,

$$\iint_{A_1} p_1(y, x \mid y_0, x_0)\, \mathrm{d}y\, \mathrm{d}x = 0, \qquad (y_0, x_0) \in A_1.$$

However, since, for each $(y_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, the density $p_1(y, x \mid y_0, x_0)$ is strictly positive for almost all $(y, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, we must have $\lambda(A_1) = 0$, which contradicts (6.8). Therefore, the assumption that $l \geq 2$ does not hold. So we have $l = 1$.

In view of [24, Theorem 3.4(ii)], to prove (c) it is enough to check the Feller property of the skeleton chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\geq 0}}$. By [7, Theorem 2.7], the two-dimensional process $(Y_t, X_t)_{t\geq 0}$, as an affine process, possesses the Feller property. So the skeleton chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\geq 0}}$ also has the Feller property.

Now we can apply [24, Theorem 6.3] and, thus, find constants $\delta \in (0, \infty)$, $B \in (0, \infty)$ such that

$$\|\boldsymbol{P}^n(y, x, \cdot) - \pi\|_{\mathrm{TV}} \leq B(V(y, x) + 1)\mathrm{e}^{-\delta n} \quad \text{for all } n \in \mathbb{Z}_{\geq 0},\ (y, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}. \tag{6.10}$$

For the remainder of the proof, i.e. to extend inequality (6.10) to all $t \geq 0$, we can interpolate in the same way as in [26, p. 536]; we omit the details. □

Appendix A

Proof of Lemma 4.1. We will complete the proof in three steps.

Step 1. Consider $\rho \geq 2$ and $\vartheta \in [\pi/2 - \varepsilon, \pi/2 + \varepsilon]$, where $\varepsilon > 0$ is a small constant whose exact value will be determined later. We introduce a change of variables

$$z := \left(\left(\frac{1}{\alpha b} + (\rho \mathrm{e}^{\mathrm{i}\vartheta})^{1-\alpha}\right)\mathrm{e}^{b(\alpha-1)s} - \frac{1}{\alpha b}\right)^{1/(1-\alpha)}$$

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and define $\Gamma_0\colon [0,t] \to \mathbb{C}$ by

$$\Gamma_0(s) := \bigg(\bigg(\frac{1}{\alpha b} + (\rho e^{i\vartheta})^{1-\alpha}\bigg)e^{b(\alpha-1)s} - \frac{1}{\alpha b}\bigg)^{1/(1-\alpha)}, \qquad s \in [0,t].$$

Noting that

$$\begin{aligned}\frac{\partial}{\partial s}\Gamma_0(s) &= -b\bigg(\frac{1}{\alpha b} + (\rho e^{i\vartheta})^{1-\alpha}\bigg)e^{b(\alpha-1)s}\bigg(\bigg(\frac{1}{\alpha b} + (\rho e^{i\vartheta})^{1-\alpha}\bigg)e^{b(\alpha-1)s} - \frac{1}{\alpha b}\bigg)^{\alpha/(1-\alpha)}\\ &= -b\bigg[\bigg(\frac{1}{\alpha b} + (\rho e^{i\vartheta})^{1-\alpha}\bigg)e^{b(\alpha-1)s} - \frac{1}{\alpha b} + \frac{1}{\alpha b}\bigg]z^{\alpha}\\ &= -b\bigg(z^{1-\alpha} + \frac{1}{\alpha b}\bigg)z^{\alpha}\\ &= -b\bigg(z + \frac{z^{\alpha}}{\alpha b}\bigg),\end{aligned}$$

we obtain

$$\begin{aligned}\int_0^t v_s(\rho e^{i\vartheta})\,ds &= \int_0^t \bigg(\bigg(\frac{1}{\alpha b} + (\rho e^{i\vartheta})^{1-\alpha}\bigg)e^{b(\alpha-1)s} - \frac{1}{\alpha b}\bigg)^{1/(1-\alpha)}\,ds\\ &= -\frac{1}{b}\int_{\Gamma_0} z\bigg(z + \frac{z^{\alpha}}{\alpha b}\bigg)^{-1}\,dz\\ &= -\frac{1}{b}\int_{\Gamma_0}\bigg(1 + \frac{z^{\alpha-1}}{\alpha b}\bigg)^{-1}\,dz. \end{aligned}\tag{A.1}$$

Next, we derive a lower bound for $\mathrm{Re}(\int_0^t v_s(\rho e^{i\vartheta})\,ds)$.

Let Γ_0^* be the range of Γ_0. Since $\Gamma_0^* \subset \mathcal{O}$ and $z \mapsto (1 + z^{\alpha-1}/(\alpha b))^{-1}$ is analytic in $\mathcal{O}$, we have

$$\int_{\Gamma_0}\bigg(1 + \frac{z^{\alpha-1}}{\alpha b}\bigg)^{-1}\,dz = \int_{\rho e^{i\vartheta}}^{((1/\alpha b + (\rho\exp\{i\vartheta\})^{1-\alpha})\exp\{b(\alpha-1)t\} - 1/\alpha b)^{1/(1-\alpha)}}\bigg(1 + \frac{z^{\alpha-1}}{\alpha b}\bigg)^{-1}\,dz. \tag{A.2}$$

Here and in what follows, the notation

$$\int_{w_1}^{w_2}\bigg(1 + \frac{z^{\alpha-1}}{\alpha b}\bigg)^{-1}\,dz$$

means the integral $\int_{\Gamma_{[w_1,w_2]}}(1 + z^{\alpha-1}/(\alpha b))^{-1}\,dz$, where $\Gamma_{[w_1,w_2]}$ is the directed segment joining w_1 and w_2 and is defined by

$$\Gamma_{[w_1,w_2]}\colon [0,1] \to \mathbb{C} \quad \text{with } \Gamma_{[w_1,w_2]}(r) := (1-r)w_1 + rw_2,\ r \in [0,1].$$

By (A.1), (A.2), and the holomorphicity of $z \mapsto (1 + z^{\alpha-1}/(\alpha b))^{-1}$ on $\mathcal{O}$, we obtain

$$\begin{aligned}\int_0^t v_s(\rho e^{i\vartheta})\,ds &= \frac{1}{b}\int_{\exp\{i\vartheta\}}^{\rho\exp\{i\vartheta\}}\bigg(1 + \frac{z^{\alpha-1}}{\alpha b}\bigg)^{-1}\,dz\\ &\quad + \frac{1}{b}\int_{((1/\alpha b + (\rho\exp\{i\vartheta\})^{1-\alpha})\exp\{b(\alpha-1)t\} - 1/\alpha b)^{1/(1-\alpha)}}^{\exp\{i\vartheta\}}\bigg(1 + \frac{z^{\alpha-1}}{\alpha b}\bigg)^{-1}\,dz. \end{aligned}\tag{A.3}$$

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Since the second term on the right-hand side of (A.3) is continuous in $(t, \rho, \vartheta) \in [1/T, T] \times [2, \infty) \times [\pi/2 - \varepsilon, \pi/2 + \varepsilon]$ and converges to

$$\frac{1}{b} \int_{((\exp\{b(\alpha-1)t\}-1)/\alpha b)^{1/(1-\alpha)}}^{\exp\{\mathrm{i}\vartheta\}} \bigg(1 + \frac{z^{\alpha-1}}{\alpha b}\bigg)^{-1} \,\mathrm{d}z$$

(uniformly in $(t, \vartheta) \in [1/T, T] \times [\pi/2 - \varepsilon, \pi/2 + \varepsilon]$) as $\rho \to \infty$, and it must be bounded, i.e.

$$\bigg|\frac{1}{b} \int_{((\exp\{b(\alpha-1)t\}-1)/\alpha b)^{1/(1-\alpha)}}^{\exp\{\mathrm{i}\vartheta\}} \bigg(1 + \frac{z^{\alpha-1}}{\alpha b}\bigg)^{-1} \,\mathrm{d}z\bigg| \le c_3 \tag{A.4}$$

for all $t \in [1/T, T]$, $\vartheta \in [\pi/2 - \varepsilon, \pi/2 + \varepsilon]$, and $\rho \ge 2$, where $c_3 = c_3(\varepsilon, T) > 0$ is some constant.

Now define $\Gamma_\vartheta : [0, 1] \to \mathbb{C}$ by $\Gamma_\vartheta(r) := (1 - r)\exp\{\mathrm{i}\vartheta\} + r\rho\exp\{\mathrm{i}\vartheta\}$ for $r \in [0, 1]$, and let Γ_ϑ^* be the range of Γ_ϑ. We can calculate the real part of the first integral appearing on the right-hand side of (A.3) by

$$\begin{aligned}
\operatorname{Re}&\bigg(\int_{\exp\{\mathrm{i}\vartheta\}}^{\rho\exp\{\mathrm{i}\vartheta\}} \bigg(1 + \frac{z^{\alpha-1}}{\alpha b}\bigg)^{-1} \,\mathrm{d}z\bigg) \\
&= \operatorname{Re}\bigg(\int_0^1 \bigg(1 + \frac{(\Gamma_\vartheta(r))^{\alpha-1}}{\alpha b}\bigg)^{-1} \partial_r \Gamma_\vartheta(r) \,\mathrm{d}r\bigg) \\
&= \operatorname{Re}\bigg(\int_0^1 \frac{(\rho - 1)\mathrm{e}^{\mathrm{i}\vartheta}}{1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}} \,\mathrm{d}r\bigg) \\
&= \int_0^1 \bigg|\frac{(\rho - 1)\mathrm{e}^{\mathrm{i}\vartheta}}{1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}}\bigg| \cos\bigg(\arg\bigg(\frac{(\rho - 1)\mathrm{e}^{\mathrm{i}\vartheta}}{1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}}\bigg)\bigg) \,\mathrm{d}r.
\end{aligned} \tag{A.5}$$

For $r \in [0, 1]$, we have

$$\begin{aligned}
\arg(1 + (\Gamma_\vartheta(0))^{\alpha-1}(\alpha b)^{-1}) &\le \arg(1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}) \\
&\le \arg(1 + (\Gamma_\vartheta(1))^{\alpha-1}(\alpha b)^{-1}).
\end{aligned} \tag{A.6}$$

Define δ_ϑ by

$$\begin{aligned}
\delta_\vartheta &:= (\alpha - 1)\vartheta - \arg(1 + (\Gamma_\vartheta(0))^{\alpha-1}(\alpha b)^{-1}) \\
&= (\alpha - 1)\vartheta - \arg(1 + \mathrm{e}^{\mathrm{i}(\alpha-1)\vartheta}(\alpha b)^{-1}) \\
&\in (0, (\alpha - 1)\vartheta).
\end{aligned} \tag{A.7}$$

It is easy to see that

$$\arg(1 + (\Gamma_\vartheta(1))^{\alpha-1}(\alpha b)^{-1}) < (\alpha - 1)\vartheta. \tag{A.8}$$

For $r \in [0, 1]$, by (A.6)–(A.8), we have $\arg(1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}) \in [(\alpha-1)\vartheta-\delta_\vartheta, (\alpha-1)\vartheta)$. As a result,

$$\arg\bigg(\frac{(\rho - 1)\mathrm{e}^{\mathrm{i}\vartheta}}{1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}}\bigg) \in ((2 - \alpha)\vartheta, (2 - \alpha)\vartheta + \delta_\vartheta], \qquad r \in [0, 1]. \tag{A.9}$$

Note that $0 < \delta_{\pi/2} < (\alpha - 1)\pi/2$ by (A.7). Since δ_ϑ is continuous in ϑ, we see that

$$0 < \lim_{\vartheta \to \pi/2} \{(2 - \alpha)\vartheta + \delta_\vartheta\} = (2 - \alpha)\frac{\pi}{2} + \delta_{\pi/2} < \frac{\pi}{2}.$$

Set $c_4 := \pi/2 - ((2-\alpha)\pi/2 + \delta_{\pi/2}) \in (0, \pi/2)$. Now we choose $\varepsilon_0 > 0$ small enough such that, for all $\vartheta \in [\pi/2 - \varepsilon_0, \pi/2 + \varepsilon_0]$,

$$0 < (2-\alpha)\vartheta < (2-\alpha)\vartheta + \delta_\vartheta \le \frac{\pi}{2} - \frac{c_4}{2}. \tag{A.10}$$

It follows from (A.9) and (A.10) that, for all $\vartheta \in [\pi/2 - \varepsilon_0, \pi/2 + \varepsilon_0]$ and $r \in [0, 1]$,

$$\cos\biggl(\arg\biggl(\frac{(\rho-1)\mathrm{e}^{\mathrm{i}\vartheta}}{1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}}\biggr)\biggr) \ge \cos\biggl(\frac{\pi}{2} - \frac{c_4}{2}\biggr) =: c_5 > 0. \tag{A.11}$$

In view of (A.5) and (A.11), we obtain

$$\begin{aligned}
\mathrm{Re}\biggl(\int_{\exp\{\mathrm{i}\vartheta\}}^{\rho\exp\{\mathrm{i}\vartheta\}} \biggl(1 + \frac{z^{\alpha-1}}{\alpha b}\biggr)^{-1} \mathrm{d}z\biggr) &\ge \cos\biggl(\frac{\pi}{2} - \frac{c_4}{2}\biggr) \int_0^1 \biggl|\frac{(\rho-1)\mathrm{e}^{\mathrm{i}\vartheta}}{1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}}\biggr| \,\mathrm{d}r \\
&= c_5 \int_0^1 \frac{\rho - 1}{|1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}|} \,\mathrm{d}r \\
&\ge c_5 \int_0^1 \frac{\rho - 1}{1 + |(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}|} \,\mathrm{d}r \\
&= c_5 \int_0^1 \frac{\rho - 1}{1 + (1 - r + r\rho)^{\alpha-1}(\alpha b)^{-1}} \,\mathrm{d}r \\
&= c_5 \int_0^{\rho-1} \frac{1}{1 + (1 + r)^{\alpha-1}(\alpha b)^{-1}} \,\mathrm{d}r \\
&\ge \frac{c_5}{1 + (\alpha b)^{-1}} \int_0^{\rho-1} \frac{1}{(1 + r)^{\alpha-1}} \,\mathrm{d}r \\
&= c_5 \alpha b (1 + \alpha b)^{-1} (2-\alpha)^{-1} (\rho^{2-\alpha} - 1).
\end{aligned} \tag{A.12}$$

Combining (A.3), (A.4), and (A.12) yields

$$\mathrm{Re}\biggl(\int_0^t v_s(\rho \mathrm{e}^{\mathrm{i}\vartheta}) \,\mathrm{d}s\biggr) \ge c_6 \rho^{2-\alpha} - c_7, \tag{A.13}$$

for all $\rho \ge 2$, $\vartheta \in [\pi/2 - \varepsilon_0, \pi/2 + \varepsilon_0]$, and $t \in [1/T, T]$, where $c_6, c_7 > 0$ are constants that depend only on $a, b, \alpha, \varepsilon_0$, and T.

Step 2. The case with $\rho \ge 2$ and $\vartheta \in [-\pi/2 - \varepsilon_0, -\pi/2 + \varepsilon_0]$ can be similarly treated, and we thus obtain

$$\mathrm{Re}\biggl(\int_0^t v_s(\rho \mathrm{e}^{\mathrm{i}\vartheta}) \,\mathrm{d}s\biggr) \ge c_8 \rho^{2-\alpha} - c_9 \tag{A.14}$$

for all $\rho \ge 2$, $\vartheta \in [-\pi/2 - \varepsilon_0, -\pi/2 + \varepsilon_0]$ and $t \in [1/T, T]$, where $c_8, c_9 > 0$ are constants depending only on $a, b, \alpha, \varepsilon_0$, and T.

Step 3. Since $\int_0^t v_s(\rho \mathrm{e}^{\mathrm{i}\vartheta}) \,\mathrm{d}s$ is continuous in (t, ρ, ϑ), we can find a constant $c_{10} > 0$ such that

$$\mathrm{Re}\biggl(\int_0^t v_s(\rho \mathrm{e}^{\mathrm{i}\vartheta}) \,\mathrm{d}s\biggr) \ge -c_{10} \tag{A.15}$$

for all $0 \le \rho \le 2$, $\vartheta \in [-\pi/2 - \varepsilon_0, -\pi/2 + \varepsilon_0] \cup [\pi/2 - \varepsilon_0, \pi/2 + \varepsilon_0]$, and $t \in [1/T, T]$. The estimate (4.2) now follows from (A.13)–(A.15). □

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Proof of Lemma 4.2. Let $\rho > 0$ and $\vartheta \in [\pi/2 + \varepsilon_0, \pi]$. Our aim is to show that

$$\left| \int_0^t v_s(\rho e^{i\vartheta})\, ds \right| \le C_3 + C_4 \rho^{2-\alpha} \tag{A.16}$$

for some constants $C_3, C_4 > 0$ depending only on $a, b, \alpha, \varepsilon_0$, and t. Using the change of variables

$$z := \left(\frac{1}{\alpha b} + (\rho e^{i\vartheta})^{1-\alpha} \right) e^{b(\alpha-1)s} - \frac{1}{\alpha b},$$

we obtain

$$\int_0^t v_s(\rho e^{i\vartheta})\, ds = \int_0^t \left(\left(\frac{1}{\alpha b} + (\rho e^{i\vartheta})^{1-\alpha} \right) e^{b(\alpha-1)s} - \frac{1}{\alpha b} \right)^{1/(1-\alpha)} ds \tag{A.17}$$

$$= \frac{1}{b(\alpha-1)} \int_{(\rho \exp\{i\vartheta\})^{1-\alpha}}^{(1/\alpha b + (\rho \exp\{i\vartheta\})^{1-\alpha}) \exp\{b(\alpha-1)t\} - 1/\alpha b} z^{1/(1-\alpha)} \left(z + \frac{1}{\alpha b} \right)^{-1} dz. \tag{A.18}$$

Since $\vartheta \in [\pi/2 + \varepsilon_0, \pi]$, we have $(1-\alpha)\vartheta \in [(1-\alpha)\pi, (1-\alpha)(\pi/2 + \varepsilon_0)]$, which implies that

$$|\sin((1-\alpha)\vartheta)| \ge \min\left\{ \sin((\alpha-1)\pi), \sin\left((\alpha-1)\left(\frac{\pi}{2} + \varepsilon_0 \right) \right) \right\} =: c_1 > 0. \tag{A.19}$$

We first consider the $0 < \rho < 2$ case. Note that, for $\rho \in (0, 2)$ and $\vartheta \in [\pi/2 + \varepsilon_0, \pi]$,

$$\begin{aligned} \left| \left(\frac{1}{\alpha b} + (\rho e^{i\vartheta})^{1-\alpha} \right) e^{b(\alpha-1)s} - \frac{1}{\alpha b} \right| &\ge \left| \operatorname{Im}\left(\left(\frac{1}{\alpha b} + (\rho e^{i\vartheta})^{1-\alpha} \right) e^{b(\alpha-1)s} - \frac{1}{\alpha b} \right) \right| \\ &= \rho^{1-\alpha} e^{b(\alpha-1)s} \sin((\alpha-1)\vartheta) \\ &\ge 2^{1-\alpha} e^{b(\alpha-1)s} c_1. \end{aligned} \tag{A.20}$$

Then, by (A.17) and (A.20), it follows that, for $\rho \in (0, 2)$ and $\vartheta \in [\pi/2 + \varepsilon_0, \pi]$,

$$\begin{aligned} \left| \int_0^t v_s(\rho e^{i\vartheta})\, ds \right| &\le \int_0^t \left| \left(\frac{1}{\alpha b} (\rho e^{i\vartheta})^{1-\alpha} \right) e^{b(\alpha-1)s} - \frac{1}{\alpha b} \right|^{1/(1-\alpha)} ds \\ &\le \int_0^t c_1^{1/(1-\alpha)} e^{-bs}\, ds \\ &= c_1^{1/(1-\alpha)} \frac{1}{b} (1 - e^{-bt}). \end{aligned}$$

We see that estimate (A.16) holds for $0 < \rho < 2$ and $\vartheta \in [\pi/2 + \varepsilon_0, \pi]$.

We now consider $\rho \ge 2$. Note that $z \mapsto z^{1/(1-\alpha)}(z + 1/(\alpha b))^{-1}$ is holomorphic on $\mathcal{O}$. So we have

$$\begin{aligned} &\int_{(\rho \exp\{i\vartheta\})^{1-\alpha}}^{(1/\alpha b + (\rho \exp\{i\vartheta\})^{1-\alpha}) \exp\{b(\alpha-1)t\} - 1/\alpha b} z^{1/(1-\alpha)} \left(z + \frac{1}{\alpha b} \right)^{-1} dz \\ &\quad = \int_{(\rho \exp\{i\vartheta\})^{1-\alpha}}^{(\rho \exp\{i\vartheta\})^{1-\alpha} + 2} z^{1/(1-\alpha)} \left(z + \frac{1}{\alpha b} \right)^{-1} dz \\ &\qquad + \int_{(\rho \exp\{i\vartheta\})^{1-\alpha} + 2}^{(1/\alpha b + (\rho \exp\{i\vartheta\})^{1-\alpha}) \exp\{b(\alpha-1)t\} - 1/\alpha b} z^{1/(1-\alpha)} \left(z + \frac{1}{\alpha b} \right)^{-1} dz. \end{aligned} \tag{A.21}$$

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Since

$$\lim_{\rho\to\infty}\int_{(\rho\exp\{i\vartheta\})^{1-\alpha}+2}^{(1/\alpha b+(\rho\exp\{i\vartheta\})^{1-\alpha})\exp\{b(\alpha-1)t\}-1/\alpha b} z^{1/(1-\alpha)}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z$$
$$=\int_{2}^{(\exp\{b(\alpha-1)t\}-1)/\alpha b} z^{1/(1-\alpha)}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z,$$

where the convergence is uniform in $\vartheta\in[\pi/2+\varepsilon_0,\pi]$, we can find a constant $c_2>0$ such that

$$\left|\int_{(\rho\exp\{i\vartheta\})^{1-\alpha}+2}^{(1/\alpha b+(\rho\exp\{i\vartheta\})^{1-\alpha})\exp\{b(\alpha-1)t\}-1/\alpha b} z^{1/(1-\alpha)}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z\right|\le c_2 \tag{A.22}$$

for all $\rho\ge 2$ and $\vartheta\in[\pi/2+\varepsilon_0,\pi]$.

We now proceed to estimate the first term on the right-hand side of (A.21). Define $\Gamma_{\vartheta,\rho}(r):=(\rho\exp\{i\vartheta\})^{1-\alpha}+r$, $r\in[0,2]$. By (A.19), we have

$$|\rho^{1-\alpha}\mathrm{e}^{(1-\alpha)i\vartheta}+r|\ge\rho^{1-\alpha}|\sin((1-\alpha)\vartheta)|\ge c_1\rho^{1-\alpha}, \tag{A.23}$$

where $r\in[0,2]$ and $\vartheta\in[\pi/2+\varepsilon_0,\pi]$. If $r\in[2\rho^{1-\alpha},2]$ then

$$|\rho^{1-\alpha}\mathrm{e}^{(1-\alpha)i\vartheta}+r|\ge r-\rho^{1-\alpha}\ge\frac{r}{2}. \tag{A.24}$$

It follows from (A.23) and (A.24) that, for $\rho\ge 2$ and $\vartheta\in[\pi/2+\varepsilon_0,\pi]$,

$$\begin{aligned}
&\left|\int_{(\rho\exp\{i\vartheta\})^{1-\alpha}}^{(\rho\exp\{i\vartheta\})^{1-\alpha}+2} z^{1/(1-\alpha)}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z\right|\\
&\quad=\left|\int_0^2(\Gamma_{\vartheta,\rho}(r))^{1/(1-\alpha)}\left(\Gamma_{\vartheta,\rho}(r)+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}r\right|\\
&\quad\le c_3\int_0^2|\Gamma_{\vartheta,\rho}(r)|^{1/(1-\alpha)}\,\mathrm{d}r\\
&\quad\le c_3\int_0^{2\rho^{1-\alpha}}(c_1\rho^{1-\alpha})^{1/(1-\alpha)}\,\mathrm{d}r+c_3 2^{1/(\alpha-1)}\int_{2\rho^{1-\alpha}}^2 r^{1/(1-\alpha)}\,\mathrm{d}r\\
&\quad=2c_3c_1^{1/(1-\alpha)}\rho^{2-\alpha}+c_3 2^{1/(\alpha-1)}\frac{\alpha-1}{\alpha-2}r^{(2-\alpha)/(1-\alpha)}\bigg|_{r=2\rho^{1-\alpha}}^{2}\\
&\quad\le c_4\rho^{2-\alpha}+c_5,
\end{aligned} \tag{A.25}$$

where $c_3,c_4,c_5>0$ are some constants. Combining (A.18), (A.21), (A.22), and (A.25) yields (A.16) for $\rho\ge 2$. □

Acknowledgements

We are grateful to the anonymous referees for their valuable comments and suggestions, especially for pointing out one mistake in the first version of this manuscript. The many helpful suggestions made by the associate editor are also gratefully acknowledged. J. Kremer would like to thank the University of Wuppertal for the financial support through a doctoral funding program.

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