Proceedings of the Royal Society of Edinburgh, **130A**, 591–602, *2000*

The critical Fujita number for a semilinear heat equation in exterior domains with homogeneous Neumann boundary values

H. A. Levine
Department of Mathematics, Iowa State University,
Ames, IA 50011, USA

Q. S. Zhang
Department of Mathematical Sciences, University of Memphis,
Memphis, TN 38152-6429, USA

(MS received 8 June 1998; accepted 1 March 1999)

Let D be a domain in $\mathbb{R}^n$ with bounded complement and let $n \neq 2$. For the initial-boundary value problem

$$\begin{aligned} \Delta u - \partial_t u + u^p &= 0 \quad \text{in } D \times (0,\infty), \\ \frac{\partial u(x,t)}{\partial n} &= 0, \quad (x,t) \in \partial D \times (0,\infty), \\ u(x,0) = u_0(x) &\geqslant 0 \quad \text{in } D, \end{aligned}$$

we prove that there are no non-trivial global (non-negative) solutions if $0 < n(p-1) \leqslant 2$ and there exist both global non-trivial and non-global solutions if $n(p-1) > 2$.

1. Introduction

We consider the questions of global existence and finite-time blow up for the following semilinear parabolic Neumann initial-boundary value problem:

$$\left.\begin{aligned} \Delta u - \partial_t u + u^p &= 0 \quad \text{in } D \times (0,\infty), \\ \frac{\partial u(x,t)}{\partial n} &= 0, \quad (x,t) \in \partial D \times (0,\infty), \\ u(x,0) = u_0(x) &\geqslant 0 \quad \text{in } D, \end{aligned}\right\} \tag{1.1}$$

where Δ is the Laplacian and D is a domain whose complement is a bounded Lipschitz domain in $\mathbb{R}^n$ for $n \neq 2$. $\partial/\partial n$ is the outward normal derivative with respect to x, which is well defined on ∂D almost everywhere.

The corresponding initial Dirichlet problem on exterior domains was first studied in [2]. There, the authors considered the following problem:

$$\left.\begin{aligned} \Delta u - \partial_t u + u^p &= 0 \quad \text{in } D \times (0,\infty), \\ u(x,t) &= 0, \quad (x,t) \in \partial D \times (0,\infty), \\ u(x,0) = u_0(x) &\geqslant 0 \quad \text{in } D, \end{aligned}\right\} \tag{1.2}$$

https://doi.org/10.1017/S0308210500000317 Published online by Cambridge University Press

where D^c is a bounded smooth domain in $\mathbb{R}^n$. They showed that (a) problem (1.2) possesses no global non-trivial non-negative solutions if $1 < p < 1 + 2/n$; and (b) problem (1.2) has global positive solutions if $p > 1 + 2/n$. More recently, in [10, 12], it was shown that if $n \geqslant 3$, then $1 + 2/n$ is in the blow-up case (a). These studies were motivated by the earlier work of Fujita, who proved the following result for the Cauchy problem:

$$\left.\begin{aligned} \Delta u - \partial_t u + u^p = 0 \quad &\text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) \geqslant 0 \quad &\text{in } \mathbb{R}^n, \end{aligned}\right\} \tag{1.3}$$

(a) If $1 < p < 1 + 2/n$ and u_0 is non-negative and non-trivial, problem (1.1) possesses no global non-negative solutions.

(b) If $p > 1 + 2/n$ and u_0 is smaller than a small Gaussian, then (1.1) has global positive solutions. It was later shown that $1 + 2/n$ belongs to case (a).

Since Fujita's work, several authors have considered similar questions for other problems. See [9] for a survey of the literature prior to 1990.

In view of the above results, it is a logical question to consider the initial Neumann problem (1.1). At first glance, it might seem easy to find a blow-up result for (1.1) when $p \leqslant 1 + 2/n$. One can argue that any non-negative solution of (1.1) dominates a solution of (1.2). However, in the existing literature concerning the blow-up properties of (1.2), some extra assumptions on the growth of solutions near infinity are always made. Therefore, we can not quote these results directly. In this paper we will take a direct approach without making any *a priori* assumptions on the solution. We should add that Pinsky [11] has proven blow-up results for Cauchy problems of the equation in (1.2) without any assumptions on the growth of solutions near infinity. However, we are not aware of any similar results concerning initial-boundary value problems.

The establishment of global existence is also subtle. In general, it is difficult to find a super solution of a nonlinear equation with Neumann boundary conditions because the value of the function on the boundary is unknown *a priori*. Existence results of (1.2) are not helpful because solutions of (1.1) dominate those of (1.2). It is also difficult to compare solutions of (1.1) to those of the Cauchy problem. Therefore, we take a different approach as described in remark 1.6 below.

Definition 1.1. Given $\tau \in (0, \infty]$, a continuous function $u = u(x, t)$ defined in $D \times (0, \tau)$ is called a solution of problem (1.1) if

$$\begin{aligned} \int_0^t \int_D u \Delta \psi \, \mathrm{d}y \, \mathrm{d}s - \int_0^t \int_{\partial D} u \frac{\partial \psi}{\partial n} \, \mathrm{d}S_y \, \mathrm{d}s + \int_0^t \int_D |u|^{p-1} u \psi \, \mathrm{d}y \, \mathrm{d}s \\ + \int_0^t \int_D u \partial_s \psi \, \mathrm{d}y \, \mathrm{d}s - \int_D [u(y, t) \psi(y, t) - u_0(y) \psi(y, 0)] \, \mathrm{d}y = 0 \end{aligned}$$

for $t \in (0, \tau)$ and all $\psi \in C^2(\mathbb{R}^n \times [0, \tau])$ such that ψ is compactly supported in $\mathbb{R}^n \times [0, \tau]$. If $\tau = \infty$, then u is called a global solution.

The above definition is fairly standard. It includes solutions which may change sign. When ∂D and u_0 are smooth, solutions thus defined are classical. It also has

https://doi.org/10.1017/S0308210500000317 Published online by Cambridge University Press

the advantage that no assumptions on the growth of solutions near infinity are required.

In this paper we establish the following result.

THEOREM 1.2.

(a) *Global existence. Suppose $n \neq 2$ and $p > 1 + 2/n$. Given $\delta > 0$, there exists a constant $b_0 > 0$ such that for each non-negative $u_0 \in C^2(D)$ satisfying $u_0(x) \leqslant b_0(1 + |x|)^{-n-\delta}$, for all $x \in D$, there exists a global non-negative solution of (1.1).*

(b) *Blow up. Suppose $n \geqslant 1$. If $1 < p \leqslant 1+2/n$, then the only global non-negative solution of (1.1) is zero.*

REMARK 1.3. It should be remarked that when the initial function is large in a certain sense, the solution need not be global, even in case (a) in the theorem. Indeed, it is well known [8] that if the initial potential energy is negative, the solution cannot be global.

REMARK 1.4. We will prove in remark 2.4 below that (1.1) has a local (in-time) solution for all $p > 1$ and all bounded non-negative u_0.

REMARK 1.5. We list a few open problems and questions.

(i) Does (1.1) have global solutions when $n = 2$ and $p > 1 + 2/n$?

(ii) Suppose the Neumann condition is replaced by the Robin condition

$$\frac{\partial u}{\partial n} + \alpha(x)u = 0 \quad \text{on } \partial D,$$

where $\alpha > 0$. The limiting cases $\alpha = 0$ and $alpha = +\infty$ are included here and in [10], respectively.

(iii) What is the situation for domains such as cones or other unbounded domains with unbounded complements?

(iv) There are also corresponding open problems for weakly coupled systems and other parabolic problems, as discussed, for example, in [9].

REMARK 1.6. Let us briefly discuss the method of proof. First, we use the contraction mapping principle to prove local and global existence. To do this, we construct a suitable function space for global solutions and show that the integral operator defined in (2.4) below will be a contraction if $p > 1 + 2/n$ and $\|u_0\|_{L^\infty(D)}$ is small enough. In this respect, the argument is similar to that of Fujita [6]. However, to establish global existence in the current case, we will need two new estimates for Green's function. One of them [5] is a global Gaussian upper bound for Green's functions with zero Neumann boundary conditions on exterior domains. The other is a convolution inequality for Green's function of the heat equation (see [13]).

To establish the blow-up result, we derive a contradiction by showing that if there were any global non-trivial solutions, the L^p norm of u on certain space-time cylinders must tend to zero when $p \leqslant 1+2/n$ and then by showing that this cannot happen on these cylinders (see [3, 4] for the elliptic case).

To conclude this section, we list some of the notation we will use in the sequel.

Let $G = G(x, t; y, s)$ denote Green's function for the heat equation on D with zero Neumann boundary conditions.

For any $a > 0$, we denote the standard Gaussian by

$$G_a(x, t; y, s) \equiv \frac{1}{[4\pi(t-s)]^{n/2}} \exp\left(-a\frac{|x-y|^2}{t-s}\right), \quad t > s. \tag{1.4}$$

Let u_0 be a non-negative function in $L^\infty(D)$. With $a > 0$, we write

$$h_a(x, t) = \int_D G_a(x, t; y, 0)u_0(y)\,\mathrm{d}y, \tag{1.5}$$

$$h(x, t) = \int_D G(x, t; y, 0)u_0(y)\,\mathrm{d}y. \tag{1.6}$$

2. Proof of theorem 1.2 (a)

When $n = 1$, the exterior domain consists of two half lines $[a, \infty)$ and $(-\infty, -a]$, say. The construction of a global solution is easy to carry out. Since $p > 3$, the full Cauchy problem has a global solution with initial data $\epsilon/(1 + |x|^{1+\delta})$, which decays like $t^{-1/(p-1)}$ uniformly in x (see [7, theorem 3.8]). Here, $\epsilon > 0$ is small and $\delta > 0$. Since the initial datum is symmetric in x, a standard uniqueness argument, using the decay property of such solutions, shows that any such solution is likewise symmetric in x and hence $u_x(0, t) = 0$. Our desired global solution is then defined as $u(x - a, t)$ for $x \geqslant a$ and $u(x + a, t)$ for $x \leqslant -a$.

When $n \geqslant 3$, the situation is more complicated. First, we present two elementary propositions, the proofs of which are quite easy (see, for example, [13]).

Proposition 2.1. *Given $a > 0$, let*

$$h_a(x, t) = \int_D G_a(x, t; y, 0)u_0(y)\,\mathrm{d}y, \tag{2.1}$$

where u_0 is a bounded non-negative function. The following two statements hold.

(a) *Given $p > 1$, there exists a constant $C(p)$ such that*

$$h_a^p(x, t) \leqslant C(p)\|u_0\|_{L^\infty}^{p-1} h_a(x, t), \tag{2.2}$$

for all $t > 0$.

(b) *If $\lim_{|x|\to\infty} u_0(x) = 0$, then $\lim_{|x|\to\infty} h_a(x, t) = 0$ uniformly with respect to $t > 0$.*

Proposition 2.2. *Suppose $0 \leqslant u_0(x) \leqslant A/(1 + |x|^{n+\delta})$ for some A, $\delta > 0$. Then*

$$h_a(x, t) \leqslant \frac{CA}{1 + |x|^n},$$

for all $t > 0$, $x \in \mathbb{R}^n$ and some $C = C_{n,\delta} > 0$.

https://doi.org/10.1017/S0308210500000317 Published online by Cambridge University Press

We recall from [5] that when $n \geqslant 3$, there are positive constants C and b such that

$$G(x,t;y,s) \leqslant \frac{C}{(t-s)^{n/2}} \exp\left(-b\frac{|x-y|^2}{t-s}\right) = CG_b(x,t;y,s), \tag{2.3}$$

for all $t > s$ and x, $y \in D$.

For any constants $a > 0$ and $M > 1$, we define the space

$$S = S(u_0) = \{u(x,t) \in C(D \times [0,\infty)) \mid 0 \leqslant u(x,t) \leqslant Mh_a(x,t)\},$$

where the function h_a is given by (1.5). To give S a metric space structure, we endow it with the sup norm in $D \times (0,\infty)$. We define the integral operator

$$Tu(x,t) = h(x,t) + \int_0^t \int_D G(x,t;y,s)u^p(y,s)\,\mathrm{d}y\,\mathrm{d}s. \tag{2.4}$$

for $u \in S$. Here, G is Green's function for the heat equation on D with zero Neumann boundary conditions. Solutions of (2.4) are sometimes called 'mild' solutions of (1.1).

Obviously, not every function in S satisfies $\partial u/\partial n = 0$ on the boundary $\partial D \times (0,\infty)$. Indeed, this is the case for some of the elements of of S in $T(S)$. However, we claim that every fixed point of $T(\cdot)$ is a solution of (1.1) in the sense of definition 1.1.

This can be shown as follows. Suppose u is a fixed point of T. Then $u \in S$ is uniformly bounded by the choice of S. By [5], G is bounded from above by a Gaussian. From this, a standard argument via integration by parts shows that u satisfies definition 1.1.

Hence it only remains to show that T has a fixed point in S.

We fix the number a, $0 < a < b$, where b is the constant in the Gaussian upper bound for G. This choice of a is critical for the proof of the theorem below. Since $a < b$, we have

$$G(x,t;y,s) \leqslant CG_b(x,t;y,s) \leqslant CG_a(x,t;y,s),$$
$$h(x,t) \leqslant Ch_b(x,t) \leqslant Ch_a(x,t).$$

To invoke the contraction mapping principle, we check the following conditions.

(i) S is non-empty, closed, bounded and convex.

(ii) $TS \subset S$.

(iii) T is a contraction.

(i) It is clear that S is convex. It is closed and bounded in the sup norm given above because

$$0 \leqslant u(x,t) \leqslant Mh_a(x,t) \leqslant CM\|u_0\|_{L^\infty},$$

since

$$\int_{D^c} G_a(x,t;y,0)\,\mathrm{d}y \leqslant C.$$

(ii) Next we show that $0 \leqslant Tu \leqslant Mh_a$ when $0 \leqslant u \leqslant Mh_a$.

https://doi.org/10.1017/S0308210500000317 Published online by Cambridge University Press

Since $p > 1 + 2/n$, we can write $p = p_1 + p_2$ such that $p_1 > 1$ and $p_2 > 2/n$. For any $u \in S$, $u \leqslant M h_a$. Since $\|u_0\|_{L^\infty} \leqslant b_0$, from proposition 2.1 (a), we obtain

$$u^{p_1}(y,s) \leqslant CM^{p_1}\|u_0\|_{L^\infty}^{p_1-1} h_a(y,s) \leqslant CM^{p_1} b_0^{p_1-1} h_a(y,s). \tag{2.5}$$

Similarly, from proposition 2.2,

$$u^{p_2}(y,s) \leqslant M^{p_2} h_a^{p_2}(y,s) \leqslant M^{p_2}\biggl[\frac{C}{1+|y|^n}\biggr]^{p_2} \leqslant (CM)^{p_2} V(y) \tag{2.6}$$

for all $s > 0$, where

$$V(y) = \frac{1}{1+|y|^{p_2 n}}.$$

Therefore,

$$u^{p_2}(y,s) \leqslant CM^{p_2} V(y).$$

Recalling the definition of h_a in (2.1) and using the previous inequalities, we have

$$u^p(y,s) = u^{p_1}(y,s)u^{p_2}(y,s) \leqslant CM^p b_0^{p_1-1} V(y) \int_D G_a(y,s;z,0)u_0(z)\,\mathrm{d}z. \tag{2.7}$$

Using the upper bound given in (2.7) and Fubini's theorem we obtain

$$\begin{aligned} Tu(x,t) &\leqslant h(x,t) \\ &\quad + CM^p b_0^{p_1-1}\int_D\int_0^t\int_D G(x,t;y,s)|V(y)|G_a(y,s;z,0)\,\mathrm{d}y\,\mathrm{d}s\,u_0(z)\,\mathrm{d}z. \end{aligned} \tag{2.8}$$

Using (2.3), we have

$$\begin{aligned} &\int_0^t\int_D G(x,t;y,s)|V(y)|G_a(y,s;z,0)\,\mathrm{d}y\,\mathrm{d}s \\ &\qquad \leqslant C\int_0^t\int_D G_b(x,t;y,s)|V(y)|G_a(y,s;z,0)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

From [13], we have that given $b > a > 0$ and any Borel measurable function $V = V(x,t)$, there exist positive constants c, $C_{a,b}$ such that

$$\int_0^t\int_{\mathbb{R}^n} G_b(x,t;y,s)|V(y,s)|G_a(y,s;z,0)\,\mathrm{d}y\,\mathrm{d}s \leqslant C_{a,b}N_{c,\infty}(V)G_a(x,t;z,0), \tag{2.9}$$

for all $t > 0$, x, $y \in \mathbb{R}^n$, where

$$\begin{aligned} N_{c,\infty}(V) \equiv \sup_{x,t}&\int_0^t\int_{\mathbb{R}^n}|V(y,s)|G_c(x,t;y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &+ \sup_{y,s}\int_s^\infty\int_{\mathbb{R}^n}|V(x,t)|G_c(x,t;y,s)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

As shown in [13], $N_{c,\infty}(V)$ is a finite number for $V(y) = 1/(1+|y|^{p_2 n})$ since $p_2 n > 2$ by our choice of p_2. Indeed, for $n \geqslant 3$,

$$\int_0^\infty G_c(x,t;y,0)\,\mathrm{d}t = \frac{C}{|x-y|^{n-2}}.$$

https://doi.org/10.1017/S0308210500000317 Published online by Cambridge University Press

As V is time independent, we have

$$N_{c,\infty}(V) \leqslant C \sup_x \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}(1+|y|^{p_2 n})}\,\mathrm{d}y < \infty,$$

since $p_2 n > 2$.

Combining (2.9) with (2.8), we obtain

$$Tu(x,t) \leqslant h(x,t) + CM^p b_0^{p_1-1} C_{a,b} N_{c,\infty}(V) \int_D G_a(x,t;z,0)u_0(z)\,\mathrm{d}z,$$

which yields

$$Tu(x,t) \leqslant (C + CM^p b_0^{p_1-1} C_{a,b} N_{c,\infty}(V))h_a(x,t). \tag{2.10}$$

Taking $M > 2C$ and b_0 suitably small, we find that

$$0 \leqslant Tu(x,t) \leqslant Mh_a(x,t). \tag{2.11}$$

Thus condition (ii) is satisfied.

(iii) Given u_1 and u_2 in S, we have, by (2.4),

$$(Tu_1 - Tu_2)(x,t) = \int_0^t \int_D G(x,t;y,s)[u_1^p(y,s) - u_2^p(y,s)]\,\mathrm{d}y\,\mathrm{d}s. \tag{2.12}$$

Now,

$$|u_1^p(y,s) - u_2^p(y,s)| \leqslant p\max\{u_1^{p-1}(y,s), u_2^{p-1}(y,s)\}|u_1(y,s) - u_2(y,s)|.$$

Using the assumption that $0 \leqslant u_1,\, u_2 \leqslant Mh_a$, $u_0(y) \leqslant b_0(1+|y|)^{-n-\delta}$, and applying proposition 2.2 with $A = b_0$, we have, for $i = 1, 2$,

$$0 \leqslant u_i \leqslant Mh_a(y,s) \leqslant \frac{Cb_0M}{1+|y|^n}.$$

Consequently,

$$|u_i|^{p-1} \leqslant \frac{CM^{p-1}b_0^{(p-1)}}{(1+|y|^n)^{p-1}}.$$

It follows that

$$|u_1^p(y,s) - u_2^p(y,s)| \leqslant \frac{Cb_0^{p-1}M^{p-1}}{(1+|y|^{p_2 n})^{(p-1)/p_2}}|u_1(y,s) - u_2(y,s)|.$$

Thus

$$|u_1^p(y,s) - u_2^p(y,s)| \leqslant Cb_0^{p-1}M^{p-1}[V(y)]^{(p-1)/p_2}|u_1(y,s) - u_2(y,s)|.$$

Substituting this last inequality in (2.12), we obtain

$$\begin{aligned}\|Tu_1 - Tu_2\|_{L^\infty} &\leqslant Cb_0^{p-1}M^{p-1}\|u_1 - u_2\|_{L^\infty} \int_0^t \int_D G(x,t;y,s)[V(y)]^{(p-1)/p_2}\,\mathrm{d}y\,\mathrm{d}s \\ &\leqslant Cb_0^{p-1}M^{p-1}\|u_1 - u_2\|_{L^\infty} N_{b,\infty}(V^{(p-1)/p_2}),\end{aligned}$$

https://doi.org/10.1017/S0308210500000317 Published online by Cambridge University Press

for some computable constant C. Since $p = p_1 + p_2$ and $p_1 > 1$, we have $p - 1 > p_2$. Noting that $V(x) \leqslant 1$, we have $N_{b,\infty}(V^{(p-1)/p_2}) \leqslant N_{b,\infty}(V)$, which is a finite constant. For b_0 sufficiently small, T will be a contraction. Since, as remarked above, a mild solution is also a weak solution, the fixed point is thus a global solution of (1.1). □

REMARK 2.3. The referee remarked that the above argument still holds if the L^∞ norm is replaced by the weighted norm $\|wu\|_{L^\infty}$, where $w = 1 + |x|^{n+\delta}$.

REMARK 2.4. We conclude the section by proving the claim of local existence made in remark 1.4. We use the same notation as before. By (2.3), we have

$$\int_D G(x,t,y,0)\,\mathrm{d}y \leqslant C.$$

Therefore, for some positive C_1 and C_2,

$$0 \leqslant Tu(x,t) \leqslant C_1\|u_0\|_{L^\infty} + C_1 t\|u\|^p_{L^\infty}, \tag{2.13}$$

$$|Tu_1(x,t) - Tu_2(x,t)| \leqslant C_2 t \max\{\|u_1\|^{p-1}_{L^\infty}, \|u_2\|^{p-1}_{L^\infty}\}\|u_1 - u_2\|_{L^\infty}. \tag{2.14}$$

Next, define, for a fixed $M > 2C_1\|u_0\|_{L^\infty}$,

$$S_M = \{u \in C(D \times [0,s]) \mid \|u\|\_L^\infty \leqslant M\}.$$

Selecting $s > 0$ so that $C_1 s M^{p-1} \leqslant \frac{1}{2}$ and $sC_2M^{p-1} < \frac{1}{2}$, then, from (2.13), for $u \in S_M$, it follows that

$$0 \leqslant Tu(x,t) \leqslant \tfrac{1}{2}M + C_1 s M^p \leqslant \tfrac{1}{2}M + \tfrac{1}{2}M = M$$

when $t \in [0,s]$. By (2.14), for u_1, $u_2 \in S_M$,

$$|Tu_1(x,t) - Tu_2(x,t)| < \tfrac{1}{2}\|u_1 - u_2\|_{L^\infty}.$$

Hence T is a contraction from S_M to S_M, a closed bounded set in $C(D \times [0,s])$. This establishes the local existence for all $p > 1$ and all L^∞ initial data.

3. Proof of theorem 1.2 (b)

In this section we establish the blow-up result for problem (1.1) when $p \leqslant 1 + 2/n$.

Let R be so large that $D^c \subset B_R(0) = \{x \mid |x| \leqslant R\}$. Define

$$Q_R = (B_{2R}(0) \cap D) \times [0, 2R^2].$$

Notice that these sets increase with R and $\cup_{R>0} Q_R = D \times [0,\infty)$.

We construct a cut-off function of the form

$$\psi_R = \phi_R(x)\eta_R(t),$$

where we define ϕ_R, η_R as follows. We define $\eta_R \in C^\infty[0,\infty)$ such that $\eta_R(t) = 1$ for $0 \leqslant t \leqslant R^2$, $\eta_R(t) = 0$ for $t \geqslant 2R^2$, $0 \leqslant \eta_R \leqslant 1$ and

$$-\frac{C}{R^2} \leqslant \eta_R'(t) \leqslant 0.$$

Likewise, we define $\phi_R \in C_0^\infty(\bar{D})$ such that $\phi_R(x) = 1$ when $x \in B_R(0) - D^c$, $\phi_R(x) = 0$ when $x \in B_{2R}(0)^c$ and $0 \leqslant \phi_R(x) \leqslant 1$ when $x \in B_{2R}(0) - B_R(0)$. We can also choose $\phi_R(x)$ to be radial in the ring $B_{2R}(0) - B_R(0)$ and to satisfy

$$\left|\frac{\partial \phi_R}{\partial r}\right| \leqslant \frac{C}{R}, \qquad \left|\frac{\partial^2 \phi_R}{\partial r^2}\right| \leqslant \frac{C}{R^2} \tag{3.1}$$

and

$$\frac{\partial \phi_R(x)}{\partial r} = 0 \tag{3.2}$$

when $|x| = R$ or $|x| = 2R$. The constants above can be chosen independent of R (greater than or equal to 1) by first constructing ϕ, η for $R = 1$ and then letting $\phi_R(x) = \phi(x/R)$ and $\eta_R(t) = \eta(t/R^2)$.

We shall argue by contradiction. Let u be a global non-trivial solution of (1.1). Set

$$I_R \equiv \int_{Q_R} u^p(x,t)\psi_R^{\,q}(x,t)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.3}$$

where $1/p + 1/q = 1$. Notice that since

$$I_R \geqslant \int_0^{R^2}\int_{B_R(0)-D^c} u^p(x,t)\,\mathrm{d}s\,\mathrm{d}t, \tag{3.4}$$

since $\psi_R \equiv 1$ on the region of integration, this inequality tells us that $\lim_{R\to\infty} I_R = 0$ cannot hold.

From definition 1.1, we have

$$\begin{aligned} I_R = {} & \int_{B_{2R}(0)-D^c} u(x,t)\psi_R^{\,q}(x,t)\big|_0^{2R^2}\,\mathrm{d}x \\ & - \int_{Q_R} u(x,t)\phi_R^q(x)q\eta_R^{q-1}(t)\eta_R'(t)\,\mathrm{d}x\,\mathrm{d}t \\ & - \int_0^{2R^2}\int_{\partial D} u(x,t)\frac{\partial \phi_R^q(x)}{\partial n}\eta_R^q(t)\,\mathrm{d}S_x\,\mathrm{d}t \\ & - \int_{Q_R} u(x,t)\Delta\phi_R^q(x)\eta_R^q(t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.5}$$

Therefore, from the definition of ϕ_R and η_R, together with the boundary condition on u, we obtain

$$\begin{aligned} I_R \leqslant {} & -\int_{B_{2R}(0)-D^c} u(x,0)\psi_R^{\,q}(x,0)\,\mathrm{d}x \\ & - \int_{Q_R} u(x,t)\phi_R^q(x)q\eta_R^{q-1}(t)\eta_R'(t)\,\mathrm{d}x\,\mathrm{d}t \\ & - \int_{Q_R} u(x,t)\Delta\phi_R^q(x)\eta_R^q(t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.6}$$

https://doi.org/10.1017/S0308210500000317 Published online by Cambridge University Press

Since $u(x,0) \geqslant 0$, $\Delta\phi_R^q = q\phi_R^{q-1}\Delta\phi_R + q(q-1)\phi_R^{q-2}|\nabla\phi_R|^2$, we have

$$I_R \leqslant -\int_0^{2R^2}\int_{B_{2R}(0)-D^c} u(x,t)\phi_R^q(x)q\eta_R^{q-1}(t)\eta_R'(t)\,\mathrm{d}x\,\mathrm{d}t \\ -\int_0^{2R^2}\int_{B_{2R}(0)-B_R(0)} u(x,t)q(\phi_R^{q-1}\Delta\phi_R)(x)\eta_R^q(t)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.7}$$

Since ϕ_R is radial on $B_{2R}(0)-B_R(0)$, we have $\Delta\phi_R = \phi_R'' + ((n-1)/r)\phi_R'$ there. Taking R sufficiently large, we obtain, for $|x| \geqslant R$,

$$|\Delta\phi_R| \leqslant \frac{C}{R^2}, \tag{3.8}$$

Using (3.8) in (3.7), we have

$$I_R \leqslant \frac{C}{R^2}\biggl\{\int_{R^2}^{2R^2}\int_{B_{2R}(0)-D^c} u(x,t)\phi_R^q\eta_R^{q-1}(t)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^{2R^2}\int_{B_{2R}(0)-B_R(0)} u(x,t)\phi_R^{q-1}\eta_R^q(t)\,\mathrm{d}x\,\mathrm{d}t\biggr\}.$$

Since ϕ_R, $\eta_R \leqslant 1$, by Hölder's inequality we have

$$I_R \leqslant \frac{C}{R^2}\biggl[\int_{R^2}^{2R^2}\int_{B_{2R}(0)-D^c} u^p\psi_R^{p(q-1)}(x,t)\,\mathrm{d}x\,\mathrm{d}t\biggr]^{1/p}\biggl[\int_0^{2R^2}\int_{B_{2R}(0)-D^c}\mathrm{d}x\,\mathrm{d}t\biggr]^{1/q} \\ + \frac{C}{R^2}\biggl[\int_0^{2R^2}\int_{B_{2R}(0)-B_R(0)} u^p\psi_R^{p(q-1)}(x,t)\,\mathrm{d}x\,\mathrm{d}t\biggr]^{1/p} \\ \times\biggl[\int_0^{2R^2}\int_{B_{2R}(0)-B_R(0)}\mathrm{d}x\,\mathrm{d}t\biggr]^{1/q},$$

which yields

$$I_R \leqslant CR^{(n+2)/q-2}\biggl[\int_{R^2}^{2R^2}\int_{B_{2R}(0)-D^c} u^p\psi_R^{p(q-1)}(x,t)\,\mathrm{d}x\,\mathrm{d}t\biggr]^{1/p} \\ + CR^{(n+2)/q-2}\biggl[\int_0^{2R^2}\int_{B_{2R}(0)-B_R(0)} u^p\psi_R^{p(q-1)}(x,t)\,\mathrm{d}x\,\mathrm{d}t\biggr]^{1/p}. \tag{3.9}$$

Therefore,

$$I_R \leqslant C_p R^{n+2-2q}. \tag{3.10}$$

When $1 < p < 1 + 2/n$, we have $2q > n+2$. Thus, as $R \to \infty$, $I_R \to 0$. This contradicts our earlier statement that $I_R \to 0$ is impossible.

When $p = 1 + 2/n$, then from (3.10) we see that $I_R \leqslant C_p$ when $R \geqslant 1$ on $[0,\infty)$. Therefore, by the monotone convergence theorem,

$$\int_0^\infty\int_D u^p(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C_p < \infty.$$

Therefore, given any $\epsilon > 0$, there is a compact region E in $D \times [0, \infty)$ such that

$$\int_{E^c} u^p(x,t)\,\mathrm{d}x\,\mathrm{d}t < \epsilon.$$

Taking R sufficiently large, we know that $(B_{2R}(0) - D^c) \times [R^2, 2R^2] \subset E^c$ and $(B_{2R}(0) - B_R(0)) \times [0, 2R^2] \subset E^c$.

Consequently,

$$\int_{R^2}^{2R^2} \int_{B_{2R}(0)-D^c} u^p(x,t)\,\mathrm{d}x\,\mathrm{d}t \to 0, \qquad \int_0^{2R^2} \int_{B_{2R}(0)-B_R(0)} u^p(x,t)\,\mathrm{d}x\,\mathrm{d}t \to 0$$

when $R \to \infty$. Since $n + 2 - 2q = 0$ in this case, from (3.9),

$$I_R \to 0$$

when $R \to \infty$. This again is a contradiction. Thus there is no non-trivial global solutions when $p \leqslant 1 + 2/n$. □

REMARK 3.1. In a recent paper [1], Andreucci and Tedeev obtained an interesting result in a different but related direction. They considered degenerate equations on domains with non-compact boundary.

Acknowledgments

We thank the referee for the careful review and helpful suggestions which greatly improved an earlier version of this paper. This research is supported in part by NSF grants for both authors.

References

1 D. Andreucci and A. Tedeev. A Fujita type result for a degenerate Neumann problem in domains with non-compact boundary. *J. Math. Analysis Appl.* **231** (1999), 543–567.

2 C. Bandle and H. A. Levine. On the existence and non-existence of global solutions of reaction-diffusion equations in sectorial domains. *Trans. Am. Math. Soc.* **655** (1989), 595–624.

3 H. Berestycki, I. D. Capuzzo and L. Nirenberg. Superlinear indefinite elliptic problems and nonlinear Liouville theorems. *Topological Methods Nonlinear Analysis* **4** (1995), 59–78.

4 I. Birindelli, I. D. Capuzzo and A. Cutri. Liouville theorems for semilinear equations on the Heisenberg group. *Ann. Inst. H. Poincaré Analyse Non Linéare* **14** (1997), 295–308.

5 Z. Q. Chen, R. J. Williams and Z. Zhao. A Sobolev inequality and Neumann heat kernel estimate for unbounded domains. *Math. Res. Lett.* **1** (1994), 177–184.

6 H. Fujita. On the blowup of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo* **13** (1966), 109–124.

7 T.-Y. Lee and W.-M. Ni. Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem. *Trans. Am. Math. Soc.* **333** (1992), 365–378.

8 H. A. Levine. Some non-existence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. *Arch. Ration. Mech. Analysis* **51** (1973), 371–386.

9 H. A. Levine. The role of critical exponents in blowup theorems. *SIAM Rev.* **32** (1990), 269–288.

10 K. Mochizuki and R. Suzuki. Critical exponent and critical blow-up for quasilinear parabolic equations. *Israel J. Math.* **98** (1997), 141–156.

https://doi.org/10.1017/S0308210500000317 Published online by Cambridge University Press

11 R. G. Pinsky. Existence and non-existence of global solutions for $u_t = \Delta u + a(x)u^p$ in $\mathbb{R}^d$. *J. Diff. Eqns* **133** (1997), 152–177.

12 R. Suzuki. Critical blow-up for quasilinear parabolic equations in exterior domains. *Tokyo J. Math.* **19** (1996), 397–409.

13 Q. S. Zhang. Global existence and local continuity of solutions for semilinear parabolic equations. *Commun. PDE* **22** (1997), 1529–1557.

(*Issued 26 May 2000*)