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REVIEWS

Financial Calculus. By M. BAXTER and A. RENNIE (Cambridge University Press, 1996) £24.95
Options, Futures and other Derivatives (Third Edition). By J. C. HULL (Prentice-Hall, 1997) £27.50
Stochastic Calculus applied to Finance. By D. LAMBERTON and B. LAPEYRE (Chapman & Hall, 1996) £30.00
The Mathematics of Financial Derivatives. By P. WILMOTT, S. HOWISON and J. DEWYNNE, (Cambridge University Press, 1995) £14.99

The books listed above are entries in the burgeoning market for literature on financial mathematics. For some time there has been a plethora of books on derivatives markets and trading strategies, at an advanced level as regards market practice, but non-mathematical. There has been little for the mathematical reader between the elementary book by Hull and the very advanced book by Duffie (1996). The other books reviewed here aim to fill that gap.

Together, they do succeed in putting across the mathematics needed for an understanding of the modern approach to financial mathematics. As they are modestly priced, our recommendation is simple; buy them all. Any one is worth reading, but each one gives only part of the picture, and makes different demands of the reader. Together they comprise a well-rounded tour of the subject from several points of view, and the resulting whole is definitely greater than the sum of its parts.

We will describe each book in turn, commenting on where it contributes best to the overall picture, and throws light on the difficulties in the others.

We begin with Hull, because it is the most elementary and because it has been chosen as the text for the new Advanced Certificate in Derivatives. Previous editions went under the title *Options, Futures and Other Derivative Securities*; since its first publication this text has been very widely used for introductory courses on modern financial mathematics.

Hull proceeds methodically through forwards and futures, including interest rate derivatives and swaps, on to options and the Black-Scholes model, including some of its variants, and then on to more advanced topics like term structure models and exotics.

The strength of Hull is its moderate pace, backed up by a large number of practical examples and exercises (unfortunately without solutions). The key idea of no-arbitrage pricing is carefully explained right at the start, using forwards. This is helpful, because the associated hedging strategy is to set up the appropriate portfolio and then to sit tight and do nothing; the forward price is a solution of the Black-Scholes equation for which a continuously rebalanced hedge portfolio is not needed. Such more delicate (and contentious) aspects are reserved for later chapters, where their practical aspects do get a proper airing.

Hull does not neglect the nitty-gritty business of how derivatives are actually traded, exercised, margined and paid for. At each stage there is an appropriate extract from the *Wall Street Journal*, with examples of real trades, which are very helpful in setting the practical scene. None of the other books does this to any great extent.

Also useful to the first-time reader is Hull's use of 'general reasoning' arguments. There comes a point at which general reasoning can get in the way of progress, but it does help the learner. For example, in Chapter 11, Hull introduces American options. The mathematics of American options is considerably harder than the basic Black-Scholes, but certain useful results can be deduced by general reasoning, such as the non-optimality of early exercise of an American call option on a non-dividend paying stock. Hull explains this carefully, where some of the other authors obtain like results as mathematical corollaries, without really drawing them to the reader's attention. For instance, Wilmott *et al.* say, after a lengthy discussion of free boundary problems in partial differential equations (PDEs); "Note that if $D_0=0$, $x_f=\infty$, $S_f(T)=\infty$ and there is no free boundary: without dividends we recover the well-known result that it is always optimal to hold an American call to expiry.'' Until the reader has acquired some intuition, Hull's approach is more helpful.

In Chapter 9 the Binomial stock price model, and options based upon it, are introduced. This is a crucial chapter, since it contains the ideas of hedging and risk-neutral valuation. In the Binomial model these are particularly simple and easily grasped. The construction of a hedged portfolio (or, equivalently, a portfolio which replicates a derivative payoff) involves no more than the solution of a pair of linear equations, and the no-arbitrage argument then leads to a unique derivative price. The risk-neutral method follows by noting that the price of any security depending on the stock price has the form of a discounted expected payoff under a different probability distribution. This deceptively simple observation has profound consequences.

Chapters 10 and 11 set the mathematical level of the book, about which we shall say more later, with the description of Brownian motion (or the Weiner process) and the Black-Scholes model. Chapter 10 is devoted to the lognormal model of stock prices, and constructs Brownian motion by assuming that the relative stock price at the end of every small time interval is Normal, then proceeding to the limit. It is an adequate procedure for the bare purpose of 'deriving' the Black-Scholes equation, but it imposes a barrier to a more complete grasp of the situation, and the later parts of the book do suffer the consequences.

The presentation of the lognormal model of stock prices leads to the question of how good a model it is. Hull is generally quite fair in considering how well any assumption underlying a model compares with reality; a whole chapter, in fact, is devoted to this. It is also pointed out that the Markov property of Brownian motion is consistent with ideas of efficient markets, that the Black-Scholes model is vulnerable to jumps in prices, and so on. Later, Chapter 14 brings in the practical issues of transaction costs and discrete-time rebalancing, again with helpful examples, and Chapter 20 covers credit risk. There are references throughout to the attitude that practitioners take to the known imperfections of the mathematical models that they use, and why and how they still use them. This is not a book that ignores the practicalities.

The Black-Scholes analysis in Chapter 11 proceeds by stating Itô's lemma, and then deriving the Black-Scholes partial differential equation (PDE). Helpful emphasis is laid upon the fact that the stock and the derivative are both driven by the same stochastic process, and that the Black-Scholes PDE is satisfied by any simple derivative (it is the boundary condition that determines which derivative is which). In view of the approach via the PDE, it is striking that Hull does not then obtain the famous Black-Scholes formula by solving it, but instead switches to a risk-neutral argument and integrates the appropriate density. While quite legitimate, this is less than illuminating, since the contrast between the PDE approach and the risk-neutral (or martingale) approach is one of the major features of finance theory, and a potential hurdle for the new reader.

In similar vein, the risk-neutral approach in the Black-Scholes model does not build directly on the concrete example of the Binomial models, by passage to the limit, but, instead, derives from the observation that the Black-Scholes PDE does not involve investors' risk preferences. In many ways the Binomial approach is more easily grasped, especially because it emphasises that risk-neutral pricing is formally equivalent to a change of probability distribution. It is not immediately obvious, in the PDE approach, that the risk-free rate enters the analysis in two quite different ways: as a parameter in the risk-neutral probability distribution; and as the discount rate in the risk-neutral world. Unless this point is grasped, the presentation of the more general risk-neutral argument in Chapter 13 will be less easily followed.

The Black-Scholes analysis is extended to variants of the basic stock in Chapter 12, again using a mixture of PDE and risk-neutral arguments. This chapter gives the adjustments required to the formula to allow, in particular, for dividends or for options on currencies, and these are important, because in later chapters they are used as templates, and the argument sometimes proceeds by analogy. The introduction of Black's model (for options on futures contracts) is important, as it is used later in Chapter 16.

The presentation of the general risk-neutral argument for continuous-time models, in Chapter 13, is one of the weaker parts of the book. The key point is that a derivative on a non-traded quantity cannot be hedged, even in theory, as a derivative on a stock price can be hedged, and so prices can depend on investors' preferences (or, more precisely, on the drift rate in the 'real' world). This is actually stated very clearly, but only in Section 21.2, and not in this chapter. Indeed, the opening paragraph says: "The distinction between underlying variables that are the prices of traded securities and those that are not is not an important one in the valuation of derivatives." This is true in the sense that the general method covers both cases, but it could mislead.

The 'market price of risk' is introduced by assuming that there are at least two securities depending on some underlying process, and then constructing a riskless portfolio out of them in the usual way. It represents the residual risk, which cannot be hedged away if the underlying is non-tradeable. It would have been helpful to show where the dividend-paying stock model of Chapter 12 fits into this framework. We might consider three cases. The simplest case is where the underlying is tradeable, such as a non-dividend paying stock. Then the hedging argument leads to the Black-Scholes price. The second case is where the underlying is non-tradeable, but is related to some tradeable quantity in a deterministic way. An example is a dividend-paying stock with a known dividend yield; the stock price is not tradeable, but the stock with dividends reinvested is tradeable. Then a hedge can be constructed using the related, tradeable, quantity, leading to a price which will depend on the relationship (in this case, the dividend yield), but not otherwise on investors' preferences. The third case is where there is no such tradeable related to the underlying; Hull's example is where the underlying is the temperature in New Orleans. Then the Black-Scholes argument leads to a price which does involve the 'real world' probability distribution, through the market price of risk. The reader might find Chapter 4 of Baxter & Rennie very helpful in understanding Chapter 13 of Hull, although in their examples they deal only with the first two cases above.

Chapter 14 is a practical exposition of hedging strategies, not omitting the realities of discrete time or transaction costs, and it includes portfolio insurance by the construction of synthetic options, and a description of the 'Greeks'.

Chapter 15 on numerical procedures is clear and very useful. It includes Monte Carlo methods and discretisation methods such as binomial and, unusually, trinomial trees, and finite difference methods for solving PDEs. It keeps to a technical level, which should allow the reader who is not an expert in numerical analysis to solve a wide range of derivatives problems, including American option pricing. It is a little disappointing, however, that the only method suggested of generating pseudo-random Normal variates is the steam-age one of summing 12 uniform variates.

The single chapter of the previous edition on interest rate models has now been split into two. Chapter 16 deals with Black's model, which was first encountered as a model for options on futures contracts. There is, however, a crucial difference between its use in that context and its use with interest rates. Applied to futures contracts, it develops the usual hedging argument, noting that the hedging can be done costlessly (under the simplifying assumptions of the model), because a futures contract can be entered into and closed out without cost. The resulting Black-Scholes price depends crucially on the existence of the hedge; but in the interest rate market, the same pricing formula is applied where the underlying can be almost anything, such as an interest rate, and where there might be no hedge. Further, there is not necessarily any justification for assuming that the underlying has a log-normal distribution; this assumption is made because the usual formula can then be applied. It is stated that Black's model is here an approximate one, suitable only for short terms, but it is not stated clearly enough, in our view, that it is not really a model at all in this context, but simply a convenient calculation device of a more or less arbitrary nature. Some arguments have been advanced which make a case for the approximation, but it is a pity that it precedes the discussion of more transparently built models. Critics of Black-Scholes, who often fail to grasp the nature of a mathematical model, would be on stronger ground if they turned their fire on this model.

Chapter 17 deals with term structure models of two types; equilibrium models and arbitrage-free models. The former include the Rendleman & Bartter, Vasicek and Cox, Ingersoll & Ross (CIR) models. These are all based on the short rate process, from which a term structure can be derived. Rendleman & Bartter model the short rate as a diffusion, which has the same defects as modelling bond prices as diffusions; Vasicek adds mean-reversion; and CIR adjusts the variance process to eliminate negative yields. These models are analytically tractable, and exact expressions can be obtained for certain derivative prices, which seems to explain their popularity. They are not necessarily successful, either in fitting the current term structure or in modelling the evolution of the term structure. Unlike earlier parts of the book, little or nothing is said about this aspect; there are no examples of actual prices or yield curves. The arbitrage-free models include the Heath, Jarrow & Morton (HJM) approach, which specifies the volatility process of the forward price, and the Ho & Lee and Hull & White models, which modify short-rate models to allow for the initial term structure. There is a useful discussion of tree-building procedures for the last two models, including some quite recent work.

The problem for the reader, trying to penetrate the jungle of term structure models, is that there are lots of them, there is no single successful model in the same way as Black-Scholes dominates stock options, and the mathematics needed is a level above what can be got away with for the basic Black-Scholes. In fact, this area of theory is still evolving and is hard to describe as a rounded whole. Chapter 17, therefore, appears a little disconnected, as a collection of possible models with only brief discussion of their advantages and disadvantages, and no grounding in market data. The reader will, perhaps, gain a clearer understanding from Chapter 5 of Baxter & Rennie.

Chapter 18 is on exotic options. Here again, a proper treatment would outrun the mathematics which underpins the book, so, although numerous pricing formulae are given they are not derived, and the whole treatment is briefer than the earlier chapters. More useful are the sections on numerical procedures for path-dependent derivatives, including lookback and barrier options, and on hedging and replication issues.

Chapter 19 is one of those which make Hull a good book. It details some of the discrepancies between the Black-Scholes model and reality, including empirical research, and gives brief descriptions of some alternatives including jump processes for stock prices, models with stochastic interest and models with stochastic volatility. Section 19.7 is entitled 'How Black-Scholes is Used in Practice', and Section 19.9 looks at empirical research. This chapter is the answer to those actuaries who believe, mistakenly, that financial economics neglects the realities of the market. It is a pity, however, that there is no mention of incomplete markets, which seems likely to be an area of much research activity in future.

Chapter 20, on credit risk, is mainly about a model of which Hull is co-author, and its application to bonds, options, swaps, forwards and convertible bonds. The model depends upon an assumption of independence between factors affecting asset values in a default-free world and factors affecting asset values in the less forgiving real world. Section 20.6 describes the Bank for International Settlements' requirements for risk-based capital, though with little by way of comment. Actuaries familiar with the solvency regime in United Kingdom life assurance will find some echoes here.

In summary, the best features of Hull are its methodical introduction to derivatives markets, its examples, its clear treatment of the no-arbitrage pricing approach and its clear explanation of the Black-Scholes model at an elementary level. However, the mathematical underpinning is flimsy. Brownian motion and its consequences are treated at an absolutely minimal level, which does not permit the underlying structure of financial mathematics to be revealed. While the first 12 chapters are likely to leave the reader with a fair understanding, some later chapters are likely to leave the reader seeking alternative treatments. Fortunately, some are to hand.

We mentioned above the difference between the PDE approach and the martingale approach to derivative pricing. The latter is where the real understanding comes from, while the former is an important computational tool, at least for stock models. Baxter & Rennie go straight for the martingale approach, and apply it consistently and methodically to a variety of problems. They do derive the Black-Scholes PDE at one point, only to say: "Notoriously, this PDE, coupled with the boundary condition that $V(s,T)$ must equal $f(s)$, gives another way of solving the pricing equation."

Baxter & Rennie is uncompromisingly mathematical. It has no examples of data and hardly anything on the realism or otherwise of the models discussed. It also omits some major topics; there is only a brief mention, or nothing at all, of American options, exotic derivatives, numerical methods, and market trading. This does not matter if the reader also has Hull to hand, but it would be a drawback of this book on its own. In fact, this concentration on the 'core' models of financial mathematics works well, because the authors do succeed in presenting the martingale approach very clearly, as a less focussed treatment might not have done.

Chapter 1 might have been written for actuaries. It takes a simple derivative — a forward contract — and considers two candidates for the fair price. The first is an expected value, based on the strong law of large numbers, exactly analagous to pricing an insurance contract. Of this, the authors say "... seductive though the strong law is, it is also completely useless." The second candidate is the cost of setting up a perfect hedge. This example repays close study. Although it is not revealed until later chapters, the forward price and the associated hedge are consequences of the Black-Scholes model. The major mathematical difference between a forward and an option is that forwards can be hedged statically and options only dynamically. This helps to put the much-discussed practical problems of dynamic hedging into perspective. By showing how the Black-Scholes paradigm must beat the expected value paradigm in the easy case, the authors, in fact, do much to kick the expected value paradigm into touch for more general derivatives too; the practical problems of approximating a hedge are quite separate.

It is one of the striking features of the theory, however, that prices can be calculated as expected values under an artificial probability measure, which can be confusing. Actuaries who are used to the idea of using an artificially low rate of interest for certain purposes should have no difficulty in grasping the concept of using an artificial probability measure purely as a computational tool. This paradigm is one of the main planks of the authors' approach, and it is explained with exemplary clarity.

Chapter 2 sets the scene with a careful treatment of discrete-time price processes, or Binomial trees. The pace is moderate and the hedging/pricing argument is gone through with worked examples. Key ideas from stochastic processes include process, measure, filtrations, conditional expectations, previsibility and martingales. Since much of Chapter 3 depends on assuming that the same concepts apply in continuous time, this is an important part of the book. Perhaps the most important part, however, is the emphasis placed on the separation of process and measure; the fact that a process observed in nature can be imagined as coming ready equipped with probabilities, which we might try to estimate from data, is not true of a process as a mathematical object. There is a whole family of different probabilities which we might associate with a given process. In the case of stock price processes, one such probability is that which we might be able to observe, and another is the risk-neutral (probability) measure. The latter is an artificial probability, not the one observed in nature; it just so happens that prices have the mathematical form of expected values under the risk-neutral probability.

Once the ideas are in place, Chapter 3 applies them to continuous time processes, and this is where the real machinery is revealed. Some advanced tools from stochastic process theory are introduced, of which the most important are the Cameron-Martin-Girsanov (CMG) Theorem, and the Martingale Representation Theorem. It is rare to find these results presented at a level that is stripped of the difficulties, which, no doubt, has contributed to the fearsome reputation of this branch of financial mathematics. The authors do this very well, which makes this chapter a valuable introduction to any less forgiving approach, such as those in Lamberton & Lapeyre or Duffie (1996). The pricing machine works as follows:

(a) First use the CMG Theorem to find a probability measure under which the discounted stock price is a martingale. This is the risk-neutral measure, also called the equivalent martingale measure.
(b) Then form the conditional expectation of the derivative payoff with respect to the martingale measure, given information at any time t. This converts the random variable representing the payoff into a process; moreover a process which is a martingale.
(c) Finally, the Martingale Representation Theorem says that the conditional expectation must be representable as the stochastic integral of some process with respect to the discounted stock price; the process is exactly what is needed to hedge the derivative.

The presentation of this 'rocket science' machinery is excellent and very clear, and the new reader who buys Hull or Lamberton & Lapeyre might be well advised to buy Baxter & Rennie as well; for Chapter 3 alone it would be worth it.

In Chapter 4, the authors apply the machinery methodically to more complicated cases, including dividend-paying stocks and currency bonds. This brings up the question of what is a tradeable security, and the very simple answer; one whose discounted price is a martingale under the equivalent martingale measure. This simple piece of information is quite helpful in understanding Chapter 13 of Hull.

Chapter 5 is on interest rate models, and here the benefit of the martingale machine is apparent. The authors go straight for the HJM approach, whose apparent complications fall into place as just a change of measure under the CMG Theorem. It is emphasised that practically all of the other well-known models, including one-factor models like Vasicek or CIR, are special cases of the HJM approach; this unifying framework makes for a more satisfying exposition than the slightly taxonomic approach of Chapter 17 of Hull.

In Chapter 6, the machinery is extended to more advanced models, including several stocks. There is a discussion of numeraires, and, in general terms, of complete markets. Much of this chapter is tying up loose ends, and indicating how some of the simple restrictions imposed in earlier chapters can be relaxed.

The Appendix contains a glossary of terms, both financial and mathematical. Since the subject bristles with jargon, of which most new readers will, at best, know only half, a glossary is certainly useful. Answers to the exercises are also given.

In summary, Baxter & Rennie set out with the one clear purpose of explaining the martingale machinery, and do this very well indeed. The price to be paid is that the book is almost devoid of computational machinery, apart from the early examples. We have already quoted the authors' slightly dismissive comment on the PDE approach. The martingale machine shows the existence of a hedging strategy, it does not say what that strategy is. The authors evidently trust that the reader who needs to know will buy another book, such as Wilmott *et al.*

Lamberton & Lapeyre, like Baxter & Rennie, is a mathematician's book, with little or no reference to market data or practice. It is written at a more advanced level than Baxter & Rennie, and a newcomer to the modern stochastic process theory will be glad to have the latter to hand, and one or two other books as well.

Chapters 1 and 2 are introductory, in that they deal with discrete-time, finite state models, in which setting the ideas of martingale pricing do not require stochastic calculus. However, the reader would be advised to have Williams (1991) to hand to help out with the theory, unless he or she is already familiar with the measure-theoretic treatment of stochastic processes. The authors make few compromises about the level of mathematics (the probability space $(\Omega, \mathscr{F}, \mathbf{P})$ appears on page 1). Chapter 1 deals, in complete generality, with price processes, discounted price processes, admissible and self-financing strategies and viable and complete markets. American options are introduced, and the Snell envelope. The Cox-Ross-Rubinstein binomial option pricing model is presented as an extended exercise.

The equivalence of the completeness of a market (i.e. a hedging strategy existing for any contingent claim) and the existence of a unique equivalent martingale measure is sometimes called the 'fundamental theorem of asset pricing'. Not only does it delineate exactly what can be done in a Black-Scholes world, it also indicates what cannot, and is therefore the jumping off point for several lines of research into alternative models. Any proof must use some sophisticated mathematics; martingales can hardly be avoided, for example; but the presentation in Lamberton & Lapeyre is clear and direct, and very helpful preparation for an analysis of securities markets in an economic setting, such as that given by Duffie (1996). It is worth mentioning, for the reader with little economic background, that the principal tool used to prove the result (called the convex sets separation theorem by the authors, but more often called the Separating Hyperplane Theorem) is one which appears very often in mathematical economics.

Chapter 2 treats American options as a discrete-time optimal stopping problem. The treatment is not limited to vanilla puts and calls. The Snell envelope is the smallest supermartingale dominating the payoff process, and the optimal exercise time is bounded from below by the first time at which the payoff exceeds the expected value one period later, and from above by the first time at which the hedge portfolio generates an excess expected return; these have natural expressions as stopping times, and the latter is, in fact, an application of the same Doob-Meyer decomposition which is the basis of the modern approach to survival analysis. It is helpful to note the similarity with the linear complementarity PDE problem in Chapter 7 of Wilmott *et al.*; the optimal stopping argument throws some light on the intuitive content of that approach.

Chapter 3 introduces the machinery of continuous-time stochastic processes. Neither Hull nor Baxter & Rennie mention convergence of random variables; convergence concepts are not needed for a bare statement of Itô's lemma, but they are crucial in defining the Itô integral. Lamberton & Lapeyre provide quite an abstract treatment, based on extension of the integral defined as a linear map in a dense subspace; the key point, which is not stated at all obviously, is that the Itô integral is not defined using the strongest (almost sure) form of convergence of random variables, but a weaker (mean square) form of convergence. The Itô integral is followed by a rapid tour of Itô's lemma and stochastic differential equations. A more accessible treatment of all of the material of this chapter can be found in Øksendal (1995) Chapters 1—5. It is possibly true that this body of theory forms part of the dividing line between a superficial and a comprehensive understanding of financial mathematics. The treatment in Chapter 3 of Lamberton & Lapeyre is not the friendliest we have seen, and the first-time reader might be well advised to become acquainted with this material somewhere else first.

Chapter 4, on the Black-Scholes model, covers the same ground as Baxter & Rennie, Chapters 3 and 4. The main difference lies in the level at which the martingale machinery has previously been introduced, rather than in its application to the model. The chapter starts with a brief introduction to the geometric Brownian motion model and self-financing strategies, then states the Girsanov (CMG) Theorem and the Martingale Representation Theorem. The intuitive content of the Girsanov Theorem is not discussed, although it should be clear enough that it corresponds to the change of measure in Chapter 1. American options are treated as well as European options, following the optimal stopping arguments of Chapter 2. Fully half of the chapter is devoted to exercises, which include non-trivial applications such as time-dependent parameters, the Garman-Kohlhagen model, asset exchanges, consumption strategies, compound options and Asian options.

Chapter 5 ties together the martingale method and the PDE approach; the role of the latter is that of computational technique. The link is the fact that the risk-neutral price (of a quite general derivative) is the unique solution of a certain PDE with given boundary conditions. The first part of the chapter shows, at quite an abstract level, how the connection arises. Associated with a diffusion is a differential operator called an infinitesimal generator; in essence this provides the compensator which turns any function of the diffusion into a martingale. By choosing as the function the discounted value of a solution of the PDE, the compensator vanishes and the discounted value alone is a martingale, and hence gives the risk-neutral price. This approach gives a more comprehensive view of the place of PDEs, not only in option pricing, but also in Markov processes. The middle part of the chapter discusses the finite difference method of solution of PDEs; the treatment is quite accessible, even though it includes a discussion of the convergence of solutions. The chapter ends with a discussion of partial differential inequalities and American options; this could usefully be read along with Wilmott *et al.* Chapter 7.

Chapter 6 covers term structure models. The treatment is based upon arbitrage-free methods, assuming that there is a measure under which discounted forward bond prices are martingales, the discounting being at the spot rate. The Vasicek and CIR models fit into this framework, each giving a different specification of the spot rate process, and each leading to analytic solutions for bond option prices. The HJM approach also fits into this general framework. In discussing term structure models, the reader needs a much clearer map, even than for the Black-Scholes model, since there are several competing models, each with advantages and disadvantages. The martingale approach provides such a map, in two ways: (i) it gives a more general framework in which other models appear as special cases; and (ii) it is really needed to understand the HJM approach, which overcomes some of the departures from reality of single-factor models. For that reason, this chapter along with Chapter 5 of Baxter & Rennie is useful.

In Chapter 7 a particular asset model with jumps is analysed, in an attempt to drop one of the stronger assumptions of the Black-Scholes model. This takes us into territory perhaps more familiar to actuaries, namely how to price risk which cannot be hedged away (in which case the stock price has jumps), and much of the chapter has a familiar feel to it, though expressed with much greater precision than would be usual in the actuarial literature. The jumps are generated by a compound Poisson process, which we hope is within the comfort zone of most actuaries! To bring these within the martingale framework their compensators are introduced (though not called such); these have a natural role in survival analysis also. Finally, optimisation is carried out over the mean squared loss, as in the Wise-Wilkie model, and a hedging strategy is obtained for the optimum. This material is somewhat closer to the research level than that of previous chapters, and the authors do caution the reader to consult the recent literature which treats some features of this model which are hard to justify.

Chapter 8 presents some basic computer algorithms for simulating financial models, including the approximation of the Normal distribution function and an Euler scheme for simulating sample paths of the solution of an SDE.

The Appendix gives a brief tutorial on measure-theoretic probability, which is useful in defining terms, but for a newcomer is no substitute for a book such as Williams (1991).

Wilmott *et al.* take a contrasting approach to Baxter & Rennie and Lamberton & Lapeyre. Their book is an instruction manual for the application of PDEs to financial problems, and, as such, deserves its place on the bookshelf. It pays little more than lip service to the stochastic process approach, zipping through Itô's Lemma without probability theory (not unlike Hull's treatment) and getting on to the heat equation with the least possible delay. It is obvious where the authors' interests lie! In this way it complements the other books reviewed here, but it would, perhaps, be unilluminating read in isolation.

Many chapters are supplemented with technical footnotes. Chapter 1 gives a lightning tour of derivatives contracts and markets, concentrating on options. The actuary will note with interest that Section 1.7 (Interest Rates and Present Value) covers, in slightly less than one page, the theory of continuously compounded deterministic interest, at variable rates; this is in marked contrast with the time which is often considered necessary for this topic in actuarial courses! In financial mathematics, it is usually taken for granted that 'interest rate' means what an actuary would call 'force of interest'.

Chapter 2 covers the lognormal model of stock prices and Itô's Lemma, all in the space of 13 pages. It is in the same spirit as Hull's treatment, and avoids the introduction of stochastic processes *per se*; it is therefore equally as useful for giving quick access to the Black-Scholes analysis and equally as limited for progressing to more advanced material. However, since it does lead straight to the Black-Scholes PDE, excursions into stochastic process theory would be a diversion from the authors' approach.

The Black-Scholes analysis is introduced in Chapter 3, including a quick look at option payoffs, hedging strategies and implied volatility. The ideas of risk and arbitrage are introduced heuristically. The focus is on options; the authors say "The key words in the definition of arbitrage are 'instantaneous' and 'risk-free' ...". This is perhaps a little unfortunate, as it might give the impression that all hedging strategies derived from arbitrage-free arguments depend on rebalancing from one instant to the next. This possibility is one of the idealised assumptions needed in the case of some, but not all, derivatives; the fact that it can, at best, be approximated in practice, and might sometimes break down, leads to some quite misplaced criticism of the entire analysis. We find it more illuminating to begin with arbitrage in the case of forwards (as in Hull or Baxter & Rennie) which (i) requires quite weak assumptions; (ii) leads to a static hedge; (iii) is a special case of the Black-Scholes analysis; and (iv) is in disagreement with prices based on expected values. However, it would be unfair to criticise the authors for this when their pedagogic intent is clearly along different lines (from that of this reviewer!).

Not surprisingly, matters relating to PDEs are discussed much more thoroughly than in the other books reviewed here, which is this book's strength. Thus, in Chapter 3 the boundary conditions for the Black-Scholes PDE are introduced with care, while Chapter 4 is devoted to an elementary study of the heat equation. This is very useful background for non-specialists, but hardly sufficient to give access to the more advanced material on PDEs which follows. Just as the reader of Lamberton & Lapeyre ought to have a basic text on probability to hand, so the reader of Wilmott *et al.* will soon find themselves in need of a basic text on PDEs. That is not a criticism; someone who wants to know what PDE theory is useful in financial applications has to find out where to look, and Wilmott *et al.* will tell them.

In Chapter 5 Wilmott *et al.* actually solve the Black-Scholes PDE using PDE methods. (It is at this point that Hull shifts to a risk-neutral viewpoint, and so avoids having to solve the PDE. It seems to be quite rare for a direct solution to be given in books on derivatives.) Since there is an apparent dichotomy between the stochastic process and PDE approaches, it is fair to ask, which will be more useful? Wilmott *et al.* point out that the computation of the expected values in the risk-neutral world (that is, under the equivalent martingale measure) requires the dynamics of the asset price process under that measure to be found, and this often amounts to the solution of the corresponding PDE. On the other hand, Baxter & Rennie and Lamberton & Lapeyre lead the reader sufficiently far down the stochastic process path to be quite convinced that a deeper understanding of pricing models lies in that direction.

Chapter 6 presents variations on the Black-Scholes model, covering dividend-paying stocks, forwards and futures and (deterministic) time-varying parameters. The treatment of forwards and futures is deliberately placed within the Black-Scholes framework, though it is noted in passing that a once-and-for-all hedge can be set up for forwards, and valuation is relatively independent of model assumptions. Currency options are not mentioned.

American options are introduced in Chapter 7, at which point the authors get into their stride. European options (or more generally, simple derivatives depending on the price of the underlying at one time only) have a complete, satisfying theory in terms of stochastic processes; how well that theory agrees with reality is, of course, an important question, but the theory itself is a triumph, in which PDE methods are somewhat overshadowed. In the case of American, path-dependent and exotic options the theory is both more difficult and less complete, and there is, perhaps, more scope for the direct application of PDE methods. This is what the authors set out to describe in the remainder of the book.

Section 7.4 gives a heuristic discussion of the price behaviour at the exercise boundary of an American put option. The fundamental difficulty is that the optimal time of exercise must be obtained as part of the solution; in PDE terms a free boundary problem. The 'obstacle problem' is introduced to provide analogies. (A string is stretched over an obstacle; the shape of the string depends on the region of contact with the obstacle, which is not specified a priori.) The principal method of setting up schemes for numerical solution is to reformulate the problem in linear complementarity form. (At this point it is useful to refer to Lamberton & Lapeyre, Chapter 2 and Section 5.3, where some connections with the stochastic process approach become much clearer.) Since analytical solution is generally impossible, this leads naturally into Part Two, on numerical methods.

Chapter 8 presents finite difference methods for fixed boundary problems in some detail, including pseudocode. Explicit and implicit schemes are given, including iterative solutions of the latter (because these can be extended to free boundary problems). These sections are clear and directly useful. Chapter 9 adapts the implicit scheme for American option prices.

Chapter 10 gives a particularly nice discussion of Binomial methods. First the Binomial tree is introduced as a discretised random walk. This can sometimes be slightly mysterious because, although less sophisticated than Brownian motion, certain assumptions have to be concocted, apparently out of thin air, so that the process tends to Brownian motion in the limit. The authors discuss two common (but different) assumptions, which helps to clear up any conflicts that might seem to arise between other texts. The discrete model is then applied simply to pricing problems, including American options. Again simulation is not mentioned.

Chapters 11 (Exotic and Path-dependent Options) and 12 (Barrier Options) are short and descriptive. Chapter 13 puts forward a general method, which has two parts: (i) extend the option payoff to include time integrals of functions of the stock price; in fact this induces a very simple modification of the Black-Scholes PDE; and (ii) solve the Black-Scholes PDE backwards from times at which any discontinuity arises, such as the calculation of running average prices at discrete times. Chapter 14 applies this to Asian options. Mostly it is only possible to set up the appropriate PDE for a numerical solution; only in the case of continuous geometric average options are there explicit solutions. Chapter 15 does the same for lookback options. This requires a result depending on the continuity of the stock price sample paths; oddly enough, it is only at this point that the authors draw the reader's attention to this assumption, although it is implicit in the version of Itô's Lemma used, and therefore in almost all of the book.

Chapter 16 describes Leland's model for option prices with discrete-time rebalancing and proportional transaction costs. This is particularly interesting, because a hedging strategy is chosen that mimics the Black-Scholes hedge, under these weakened assumptions, and it is then possible to find, approximately, what effect transaction costs have on the option price, and to strike a balance between frequent rebalancing (with reduced risk) and rising costs. Some graphical examples are given for portfolios of options, unfortunately without comment.

The last two chapters turn to bond pricing. After a brief survey of deterministic interest rates and the yield curve, Chapter 17 presents a diffusion model for the spot rate. The only tool which the authors employ is the derivation of PDEs by the hedging argument; in the case of bonds this leads to hedging using bonds of different durations and so to the market price of risk. The focus is on a class of one-factor models based on diffusion processes, which lead to a simple exponential form of bond prices. The Vasicek, CIR and Hull-White models are special cases, but, in its full generality, the model allows the current yield curve to be fitted. If the diffusion parameters are constant, bond prices can be found explicitly and derivatives such as swaps priced within the PDE framework. There is little development of this topic, and no attempt to delineate, in wider terms, the more complex world of term structure models.

Chapter 18 combines the stock and interest rate models to derive a PDE for a convertible bond, with hints for a solution in a few special cases.

A good selection of exercises is included, with hints for solution.

What overall conclusions can we draw from these four books? The most important point is that they are not substitutes for each other; each brings a different point of view, and we would not describe any of them as superfluous. Hull has many examples, which the others lack, and it is a good, moderately-paced entry point to this subject. Baxter & Rennie is an exceptionally clear introduction to the more advanced mathematics which Hull avoids, and Lamberton & Lapeyre takes this a step further. Wilmott *et al.* is a guide to the PDE methods which must often be used, even where the insight comes from the martingale approach. Taken together, these books give a rounded picture which none of them does on its own.

Topics not covered include the connections with other parts of economic theory (though this might not be too important to most actuaries), especially market equilibrium theory, in which martingales play the same striking role as in derivatives pricing. Material on these links can be found in Dothan (1990) and Duffie (1996).

If these books have a collective weakness, it is the scant reference to any data (the first half of Hull apart). This is certainly not because such empirical research does not exist; it is plentiful in the financial economics literature. But, if the reader walks into a good bookshop in search of a book on derivatives, he or she will find a dozen texts on market practice and empirical work before finding one which gives a proper mathematical treatment; these authors have performed valuable service by filling that gap, and it would be wrong to take them to task for not reproducing what is already available. We trust that the reader will keep that in mind while enjoying these excellent books.

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