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SOME SEPARABLE SPACES AND REMOTE POINTS

ALAN DOW

0. Introduction. A point $p \in \beta X \backslash X$ is called a remote point of X if $p \notin \text{cl}_{\beta X} A$ for each nowhere dense subset A of X. If X is a topological sum $\sum \{X_n : n \in \omega\}$ we call $\mathscr{F} \subset \mathscr{P}(X)$ *nice* if $\{n : F \cap X_n = \emptyset\}$ is finite for each $F \in \mathscr{F}$. We call $\mathscr{F}$ *remote* if for each nowhere dense subset A of X there is an $F \in \mathscr{F}$ with $F \cap A = \emptyset$ and *n-linked* if each intersection of at most n elements of $\mathscr{F}$ is non-empty.

For a space $X = \sum X_n$, remote points have been constructed in a variety of cases and under varying set-theoretic assumptions. Assuming CH, there are remote points if $|C^*(X)| = c$ (cf. [**5**]). Van Douwen, and independently Chae and Smith, constructed remote points if X has countable π-weight and van Mill did so if each X_n is a product of at most ω_1 spaces with countable π-weight. In [**3**], I extend van Mill's result to products of arbitrarily many factors. In [**2**], assuming MA, remote points are constructed if X is *ccc* and of weight at most c. In each of the above constructions, not only are remote points constructed, but so are nice remote filters. In [**6**], van Mill requires that he can construct nice remote filters on certain spaces to construct special points in $\beta\omega \backslash \omega$. It is unknown if every *ccc* (or separable) nonpseudocompact space has remote points. We present our examples for two major reasons. Firstly, in each of the above constructions which take place in ZFC, a remote filter $\mathscr{F}$ on $X = \sum X_n$ can be found which is not only nice but also n-linked on X_n. Secondly, in the constructions using special set-theoretic assumptions $\mathscr{F}$ can always be found to be nice. We give an example of a compact separable space K which does not have any remote 2-linked collections of closed sets but $\omega \times K$ has remote points. It is shown that it is consistent that there is a K so that $\omega \times K$ has no nice remote filters. Also K may be chosen so that it is unknown if $\omega \times K$ has remote points.

We hope that these examples are getting close to settling the question of there being a *ccc* space without remote points. The proof of the non-existence of nice remote filters is more difficult than the rest because it requires a new consistency result. We defer the proof of this result until the last section. Our notation and terminology is standard. We identify cardinals with initial ordinals and an ordinal is the set of its predecessors.

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For sets A, B ${}^A B$ is the set of functions from A to B. For a cardinal λ and a set A,

$$[A]^\lambda = \{B \subset A : |B| = \lambda\};$$

$[A]^{\leqq\lambda}$ and $[A]^{<\lambda}$ have the obvious meanings.

Let u be a filter on ω and $f, g \in {}^\omega\omega$, define $f <_u g$ if and only if $\{n : f(n) < g(n)\} \in u$. If u is the cofinite filter we shall often suppress the subscript u. For a filter u on ω, we shall let λ_u denote the least cardinal of a cofinal subset of $({}^\omega\omega, <_u)$. The cardinals

$$d = \lambda_{\text{cofinite}} \quad \text{and}$$

$$b = \min \{|B| : B \subset {}^\omega\omega \text{ is unbounded in } ({}^\omega\omega, <_{\text{cof.}})\}$$

are well known. We shall define the cardinal κ to be the smallest cardinal such that $\lambda_u < \kappa$ for all $u \in \omega^*$. It is well known that $\kappa > \omega_1$. We shall call $D \subset {}^\omega\omega$ a *u-scale* if D is cofinal in $({}^\omega\omega, <_u)$ and $(D, <_u)$ is of order type λ_u. Note that if $u \in \omega^*$, a u-scale always exists.

1. The examples. We construct many examples with the same construction. We shall need special subsets of ${}^\omega\omega$ for this purpose.

1.1 *Definition.* A subset $F \subset {}^\omega\omega$ is *admissable* if F contains the constant functions, F is a $\vee$-subsemilattice of ${}^\omega\omega$

$$(f \vee g(n) = \max (f(n), g(n)))$$

and countable subsets of F are bounded in $(F, <_{\text{cof}})$.

Let $S = \bigcup_{n\in\omega} {}^n\omega$, i.e., S is the set of finite sequences of integers. For $s \in S$, let $\operatorname{dom}(s)$ be the domain of s and $l(s) = |\operatorname{dom}(s)|$. For each $s \in S$ and $f \in {}^\omega\omega$ define

$$U(s,f) = \{t \in S : s \subset t \text{ and for } l(s) \leqq n < l(t), t(n) > f(n)\}.$$

Then for each admissable $F \subset {}^\omega\omega$,

$$B_F = \{U(s,f) : s \in S, f \in F\}$$

forms a clopen base for a topology on S. Let B_F' be the boolean algebra of subsets of S generated by B_F and let K_F be the Stone space of B_F'. We can think of S as being densely embedded in K_F and

$$\{\operatorname{cl}_{K_F} U(s,f) : s \in S, f \in F\}$$

forms a π-base.

If $F = {}^\omega\omega$ then the topology on S obtained from F is homeomorphic to the subspace of the box product of countably many copies of the converging sequence $\{1/n : n \in \omega\} \cup \{0\}$ consisting of those elements which are eventually 0. Notice that $U(s,f) \cap U(t,g) \neq \emptyset$ if and only if $s \subset t$, $t(n) > f(n)$ for $l(s) \leqq n < l(t)$ or $t \subset s$ and $s(n) > g(n)$ for $l(t) \leqq n < l(s)$.

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2. Remote 2-linked collections. As mentioned in the introduction all of the spaces for which there are ZFC constructions of remote points points can be constructed from n-linked remote collections. The space K_F, however, can be chosen so that it does not have a remote 2-linked collection.

2.1 THEOREM. *Let $F \subset {}^\omega\omega$ be admissable and unbounded in $({}^\omega\omega, <_{cof})$. There are no remote 2-linked collections of closed subsets of K_F.*

Proof. Suppose that $\mathscr{F}$ is such a collection on $K_F = K$. For each $f \in F$, let

$$C_f = \{U(s,f) : l(s) > 0\};$$

$\cup C_f$ is dense open in K and is proper as there is no finite dense subcollection. Therefore $K \setminus \cup C_f$ is nowhere dense so there is a compact $H_f \in \mathscr{F}$ with

$$H_f \cap K \setminus \cup C_f = \emptyset.$$

Hence we may choose a finite set $S_f \subset S$, such that

$$H_f \subset \cup \{U(s,f) : s \in S_f\}.$$

Let $n(f) = \max \{l(s) : s \in S_f\}$. Since a countable union of bounded subsets of $({}^\omega\omega, <)$ is bounded, there is an $n \in \omega$ and an unbounded set $G \subset F$ such that $n(g) = n$ for each $g \in G$. Therefore there is a $j > n$ such that $\{g(j) : g \in G\}$ is infinite. Choose $f \in F$ arbitrarily and let

$$C = \{U(s,f) : l(s) > j\}.$$

Notice that for $g \in G$, $s \in S_g$, $l(s) < j$. It is clear that $\cup C$ is dense in K since for each $U(s, h)$ there is a $t \supset s$ with $l(t) > j$ and $t \in U(s, h)$. Therefore, as above, we may choose $H \in \mathscr{F}$ and a finite $T \subset S$ so that

$$H \subset \cup \{U(t,f) : t \in T\} \subset \cup C.$$

However, by the finiteness of T, there is an $m \in \omega$ such that $t(j) < m$ for each $t \in T$. So choose $g \in G$ with $g(j) \geqq m$, then $H_g \cap H = \emptyset$. For if $s \in S_g$, $t \in T$ then $l(s) < l(t)$, so in order that $U(s, g) \cap U(t,f) \neq \emptyset$ it must be true that $t(j) > g(j)$. This contradicts that $\mathscr{F}$ is 2-linked.

3. Remote points. In [**2**], a length c induction was used to construct remote filters on *ccc* spaces with weight c. However it is necessary to assume that $\kappa = c^+$ to carry out such an induction. For the spaces $X_F = \omega \times K_F$ we are able to complete such an induction at stage $|F|$, thereby not requiring special set theoretic assumptions.

3.1 THEOREM. *If $|F| < \kappa$ and F is admissible then $X = \omega \times K_F$ has remote points.*

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Proof. Let $X_n = \{n\} \times K_F$ and $U(n, s, f) = \{n\} \times U(s, f)$ for $n \in \omega$, $s \in S, f \in F$. By the definition of κ, there is a $u \in \omega^*$ and a u-scale $D \subset {}^\omega\omega$ with $\lambda = \lambda_u \geqq |F|$. Let $\{f_\alpha : \alpha < \lambda\}$ be an indexing of F (with possible repetitions) and let $D = \{h_\alpha : \alpha < \lambda\}$ be a $<_u$-order preserving indexing. Also define

$$\Gamma = \{\sigma : \exists\, f \in F \text{ with } \sigma \subset \{U(n, s, f) : n \in \omega, s \in S\} \text{ and } \cup\, \sigma \text{ is dense in } X\}.$$

Let $\sigma \in \Gamma$; choose $\alpha < \lambda$ so that

$$\sigma \subset \{U(n, s, f_\alpha) : n \in \omega, s \in S\}.$$

Fix an ordering $\{s_k : k \in \omega\}$ of S and define, for $n \in \omega$,

$$g_0(n) = \min\{k : U(n, s_k, f_\alpha) \in \sigma\}$$

and choose $\alpha_0 \geqq \alpha$ so that $g_0 \leqq_u h_{\alpha_0}$. Now, to start an induction, for each $\beta \leqq \alpha_0$ define

$$g_\beta(n) = \min\{k : \text{for each } i \leqq h_{\alpha_0}(n) \text{ there is a } j \leqq k \text{ with } U(n, s_j, f_\alpha) \in \sigma \text{ and } U(n, s_j, f_\alpha) \cap U(n, s_i, f_\beta) \neq \emptyset\},$$

for $n \in \omega$. Now, choose $\alpha_1 \geqq \alpha_0 \in \lambda$ so that $g_\beta \leqq_u h_{\alpha_1}$ for each $\beta \leqq \alpha_0$.

Suppose, for $j < N$, we have chosen $\alpha_j \geqq \alpha_{j-1}$ satisfying $h_{\alpha_j}\, {}_u\!\geqq g_z$ for each sequence $z = (\beta_0, \ldots, \beta_{j-1}) \in {}^j(\alpha_{j-1} + 1)$ where $g_z(n)$ is the smallest integer such that for each of the finitely many functions

$$r \in {}^j(h_{\alpha_{j-1}}(n) + 1), \cap \{U(n, s_{r(i)}, f_{\beta_i}) : i < j\} \neq \emptyset$$

implies there is an $m < g_z(n)$ with

$$U(n, s_m, f_\alpha) \in \sigma \quad \text{and}$$
$$U(n, s_m, f_\alpha) \cap \cap \{U(n, s_{r(i)}, f_{\beta_i}) : i < j\} \neq \emptyset.$$

To find α_N, we define g_z for each $z \in {}^N(\alpha_{N-1} + 1)$ as above. Note that for each $n \in \omega$, $g_z(n)$ exists because there are only finitely many sets to meet and $\cup\, \sigma$ is dense in X. We simply choose $\alpha_N < \lambda$, $\alpha_{N-1} \leqq \alpha_N$ such that $g_z \leqq_u h_{\alpha_N}$ for all $z \in {}^N(\alpha_{N-1} + 1)$ which we may do since $\{h_\gamma : \gamma < \lambda\}$ is a u-scale. Define

$$H_\sigma = \bigcup_{n \in \omega} \cup \{U(n, s_k, f_\alpha) \in \sigma : k \leqq \max\{h_{\alpha_j}(n) : j \leqq n\}\}.$$

We shall refer to the above ordinals by $\alpha(\sigma), \alpha_i(\sigma), i \in \omega$ and the function g_z by $g_{z,\sigma}$.

We show that $\{H_\sigma : \sigma \in \Gamma\}$ is a filter base and is remote. Let $\Gamma_1 \subset \Gamma$ with $|\Gamma_1| = N$; recursively select, for $j < N$, $\sigma_j \in \Gamma_1$ so that $\alpha_j(\sigma_j)$ is a minimum for

$$\{\alpha_j(\sigma) : \sigma \in \Gamma_1 \backslash \{\sigma_0, \ldots, \sigma_{j-1}\}\}.$$

Let $\beta_i = \alpha(\sigma_i)$ for $i < N$. First note that for each $i < j < N$,

$$\beta_i \leqq \alpha_0(\sigma_i) \leqq \alpha_i(\sigma_i) \leqq \alpha_{j-1}(\sigma_j)$$

so, for $0 < j < N$,

$$z_j = (\beta_0, \ldots, \beta_{j-1}) \in {}^j(\alpha_{j-1}(\sigma_j) + 1) \quad \text{and}$$

$$g_{z_j,\sigma_j} \leqq_u h_{\alpha_j(\sigma_j)}.$$

Also for $i < j < N$,

$$h_{\alpha_i(\sigma_i)} \leqq_u h_{\sigma_{j-1}(\sigma_j)}.$$

It follows that we may choose $U \in u$ so that for $n \in U$ all of the following hold:

(i) $n > N$,

(ii) $g_{0,\sigma_0}(n) \leqq h_{\alpha_0(\sigma_0)}(n)$,

(iii) for $i < j < N$,

$$h_{\alpha_i(\sigma_i)}(n) \leqq h_{\alpha_{j-1}(\sigma_j)}(n) \quad \text{and}$$

(iv) for $i < j < N$, $g_{z_j,\sigma_j}(n) \leqq h_{\alpha_j(\sigma_j)}(n)$.

Now let $n \in U$ and choose $r(0) \leqq h_{\alpha_0(\sigma_0)}(n)$ such that

$$U(n, s_{r(0)}, h_{\beta_0}) \in \sigma_0.$$

From (iii) and the definition of $g_{z_1,\sigma_1}(n)$ there is an $r(1) \leqq g_{z,\sigma_1}(n)$ such that

$$U(n, s_{r(1)}, f_{\beta_1}) \in \sigma_1 \quad \text{and}$$

$$U(n, s_{r(1)}, f_{\beta_1}) \cap U(n, s_{r(0)}, f_{\beta_0}) \neq \emptyset.$$

By (iv), $r(1) \leqq h_{\alpha_1(\sigma_1)}(n)$. Suppose, for $i < j < N$, we have chosen $r(i) \leqq h_{\alpha_i(\sigma_i)}(n)$ such that

$$U(n, s_{r(i)}, f_{\beta_i}) \in \sigma_i \quad \text{and} \quad \bigcap_{i<j} U(n, s_{r(i)}, f_{\beta_i}) \neq \emptyset.$$

Again from (iii) and the definition of $g_{z_j,\sigma_j}(n)$ there is an $r(j) \leqq g_{z_j,\sigma_j}(n) \leqq h_{\alpha_j(\sigma_j)}(n)$ such that

$$\bigcap_{i \leqq j} U(n, s_{r(i)}, f_{\beta_i}) \neq \emptyset \quad \text{and} \quad U(n, s_{r(j)}, f_{\beta_j}) \in \sigma_j.$$

Therefore

$$\bigcap_{i<N} U(n, s_{r(i)}, f_{\beta_i}) \neq \emptyset.$$

Also, for $i < N$,

$$U(n, s_{r(i)}, f_{\beta_i}) = U(n, s_{r(i)}, f_{\alpha(\sigma_i)}) \subset H_{\sigma_i}$$

because $r(i) \leqq h_{\alpha_i(\sigma_i)}(n)$. Hence $\{H_\sigma : \sigma \in \Gamma\}$ is a filter base.

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Let $A \subset X$ be a nowhere dense set and σ' a countable collection of π-base members whose union is dense and misses A. Choose $\alpha < \lambda$ such that, for each $U(n, s, f) \in \sigma'$, $f \leqq f_\alpha$ which we may do since F is admissible. Let $U(n, s, f) \in \sigma'$ be arbitrary and choose $N \in \omega$ such that $f_\alpha(k) \geqq f(k)$ for $k \geqq N$. So for each $t \in U(n, s, f)$ with $l(t) \geqq N$, $U(n, t, f_\alpha) \subset U(n, s, f)$. Recalling the definition of Γ, we see that there is a $\sigma \in \Gamma$ with $\cup \sigma \subset \cup \sigma'$. Therefore $H_\sigma \cap A = \emptyset$ and $\{H_\sigma : \sigma \in \Gamma\}$ is remote. Each point $p \in \cap \{\mathrm{cl}_{\beta X} H_\sigma : \sigma \in \Gamma\}$ is a remote point of X.

3.2 COROLLARY. *There is a compact separable space K_F such that $\omega \times K_F$ has remote points but K_F has no remote 2-linked collections of closed sets.*

Proof. By the definition of b, there is a sequence $\{f_\alpha : \alpha < b\} \subset {}^\omega\omega$, well-ordered by $<_{\mathrm{cof}}$ which is unbounded in $({}^\omega\omega, <)$. Since b is regular and uncountable it is clear that $F = \{f_\alpha : \alpha < b\}$ is admissible by simply insisting that it contain the constants. Therefore, by 2.1, K_F has no remote 2-linked collections. For each $u \in \omega^*$, $\lambda_u \geqq b$ because a subset of ${}^\omega\omega$ which is bounded in $<_{\mathrm{cof}}$ is also bounded in $<_u$. Therefore $\kappa > b$ and by 3.1, X has remote points.

3.3 COROLLARY. *If $\kappa > d$ then $\omega \times K_F$ has remote points where $F = {}^\omega\omega$.*

Proof. If $D \subset {}^\omega\omega$ is dominating then $\{U(s, f) : s \in S, f \in D\}$ is a π-base for K_F. The proof of 3.1 may be carried out by replacing Γ with $\Gamma' = \{\sigma : \cup \sigma$ is dense in $\omega \times K$ and there is an $f \in D$ with $\sigma \subset \{U(n, s, f) : n \in \omega, s \in S\}$.

3.4 *Remark.* If $\kappa < d$, for instance when d is singular, it is not known if $\omega \times K_F$ has remote points. It seems very unlikely to the author that in this case $\omega \times K_F$ will have remote points.

4. Nice remote filters. As mentioned in the introduction we require an additional set theoretic assumption to show that $\omega \times K$ has no nice remote filters. We shall state this property below and defer the proof until Section 5. Let us assume that $F = {}^\omega\omega$ throughout this section, and let $X = \omega \times K_F$.

4.1 THEOREM. *If $b = d$ then X has nice remote filters.*

Proof. In the proof of 3.1 and 3.3, the remote filter $\mathscr{F}$ we constructed has the property that for each $H \in \mathscr{F}$,

$$\{n : H \cap X_n \neq \emptyset\} \in u.$$

Hence $\mathscr{F}$ may be constructed to be nice in case u is the cofinite filter. It is not difficult to see that this is the case if $b = d$.

Let "hockey stick" ($\diagdown\!\!\!\!\smile$) abbreviate the statement: there is a set $\{g_\alpha : \alpha < \omega_1\} \subset {}^\omega\omega$ and a sequence $\{S_\alpha : \alpha < \omega_1\}$ of countable subsets of

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ω_1 such that if $S \in [\omega_1]^{\omega_1}$ there is an $S_\alpha \subset S$ and an $n \in \omega$ with $\{g_\beta(n) : \beta \in S_\alpha\}$ infinite.

4.2 THEOREM. *Assume $\omega_2 < \kappa$ and $\diagdown\!\!\!\smile$. Then X has no nice remote filters.*

Proof. Let $G = \{g_\alpha : \alpha < \omega_1\}$ and $\{S_\beta : \beta < \omega_1\}$ exhibit $\diagdown\!\!\!\smile$. We may assume, without loss of generality, that each g_α is increasing. Let, for each $\alpha < \omega_1$,

$$\sigma_\alpha = \{U(n, s, g_\alpha) : s \in S, n \in \omega \text{ and } l(s) > g(n)\}.$$

Assume that $\mathscr{F}$ is a remote filter on X. We can choose, for $\alpha < \omega_1$ and $n \in \omega$, a finite set $\sigma_\alpha(n) \subset \sigma_\alpha$ such that

$$\cup\, \sigma_\alpha(n) \subset X_n \quad \text{and} \quad \bigcup_{n\in\omega} \cup\, \sigma_\alpha(n) = H_\alpha \in \mathscr{F}.$$

Define, for $\alpha < \omega_1$, $h_\alpha \in {}^\omega\omega$ as follows:

$$h_\alpha(n) = \max\{s(g_\alpha(n)) : U(n, s, g_\alpha) \in \sigma_\alpha(n)\}.$$

Now, for $\beta < \omega_1$, choose $S_\beta' \subset S_\beta$ so that for some $n = n(\beta) \in \omega$, $g_\delta(n) \neq g_\gamma(n)$ for $\delta \neq \gamma \in S_\beta'$. Notice that for $k > n$ and $m \in \omega$, $\{\alpha \in S_\beta' : g_\alpha(k) = m\}$ is finite because each g_α is increasing. Define, for $\beta \in \omega_1$ and $k \geqq n(\beta)$, $H_{\beta,k} \in {}^\omega\omega$ by

$$H_{\beta,k}(n) = \sum \{h_\alpha(k) : g_\alpha(k) = \min\{m : \exists\ \alpha \in S_\beta' \text{ such that } g_\alpha(k) = m \geqq n\}\}.$$

Since $d > \omega_1$ we can choose $f \in {}^\omega\omega$ so that for each $\beta < \omega_1$ and $k \geqq n(\beta)$, $\{n : f(n) > H_{\beta,k}(n)\}$ is infinite. We may also choose f to be increasing.

Let

$$\sigma_f = \{U(n, s, f) : s \in S, n \in \omega, l(s) > 0\}$$

and suppose that $\sigma_f(n)$ is a finite subset of σ_f with

$$\cup\, \sigma_f(n) \subset X_n \quad \text{and} \quad H = \bigcup_{n\in\omega} \cup\, \sigma_f(n) \in \mathscr{F}.$$

For sake of contradiction, suppose that $\mathscr{F}$ is nice. Hence for each $\alpha \in \omega_1$ there is an $n \in \omega$ such that

$$H \cap H_\alpha \cap X_k \neq \emptyset \quad \text{for } k > n.$$

It follows easily that there is an $n_1 \in \omega$ and an $A \in [\omega_1]^{\omega_1}$ such that

$$H \cap H_\alpha \cap X_k \neq \emptyset \quad \text{for } k > n_1 \quad \text{and} \quad \alpha \in A.$$

By $\diagdown\!\!\!\smile$, there is a $\beta < \omega_1$ such that $S_\beta \subset A$, hence $S_\beta' \subset A$. So we first choose $k > \max(n_1, n(\beta))$ and let

$$M = \max\{l(s) : s \in \sigma_f(k)\}.$$

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We shall show that there is an $\alpha \in S_\beta'$ such that

$$H \cap H_\alpha \cap X_k = \emptyset$$

which will be our contradiction. Since $\{n : f(n) > H_{\beta,k}(n)\}$ is infinite, there is an $n > M$ with $f(n) > H_{\beta,k}(n)$. By the definition of $H_{\beta,k}$ there is an $\alpha \in S_\beta'$ with

$$g_\alpha(k) = m > n \quad \text{and} \quad H_{\beta,k}(n) = H_{\beta,k}(m) \geqq h_\alpha(k)$$

since $g_\alpha(k) = m$. Since f is increasing,

$$f(m) \geqq f(n) > H_{\beta,k}(n) = H_{\beta,k}(m).$$

To show that $H \cap H_\alpha \cap X_k = \emptyset$, let

$$U(k, t, f) \in \sigma_f(k) \quad \text{and} \quad U(k, s, g_\alpha) \in \sigma_\alpha(k).$$

Since $l(t) \leqq M$ and $l(s) > g_\alpha(k) > M$, we have that

$$U(k, t, f) \cap U(k, s, g_\alpha) \neq \emptyset$$

implies

$$s(g_\alpha(k)) \geqq f(g_\alpha(k)).$$

However, since $g_\alpha(k) = m$, this is not the case and the proof is complete.

4.3 COROLLARY. *If $\diagdown\!\!\diagup$ and $\omega_2 \leqq d < \kappa$ then X has remote points but no nice remote filters.*

5. Consistency of $\diagdown\!\!\diagup + \kappa > d = \omega_2$. In this section we shall show that the model introduced by Shelah in [**7**] is a model of $\diagdown\!\!\diagup + \kappa > d = \omega_2$. We shall use the notation of [**4**] and the reader is referred to [**4**] for more details of forcing. We remind the reader of the following notions. For a stationary $S \subset \omega_2$, $\Diamond_S$ means there are $S_\alpha \subset \alpha$ for $\alpha \in S$ such that for any $A \subset \omega_2$, $\{\alpha \in S : A \cap \alpha = S_\alpha\}$ is stationary. Jensen introduced this principle and showed that it holds in $V = L$. Recall also that GCH holds in $V = L$.

We start with $M = L$. First add ω_3 subsets of ω_1 by forcing over

$$P_0 = \{f : f \text{ is a function from a countable } A \subset \omega_3 \times \omega_1 \text{ to } \omega_1\}$$

ordered by inclusion. (So if G_0 is L-generic over P_0 then in $M[G_0]$, $2^\omega = \omega_1$, $2^{\omega_1} = \omega_3 = 2^{\omega_2}$ and cardinalities are preserved (§ 5 of [**7**]).) We next collapse ω_1 by forcing over

$$P_1 = \{g : g \text{ is a function from a finite subset of } \omega \text{ to } \omega_1\}$$

(so P_1 collapses ω_1 and preserves cardinals not equal to ω_1, and preserves 2^λ for $\lambda \neq \omega$ [**7**]). We let G_0 be L-generic over P_0 and $M_0 = M[G_0]$. Next let G_1 be M_0-generic over P_1 and $M_1 = M_0[G_1]$. We show that M_1 is as required by a series of facts. Let $\leqq_c$ be the order on ${}^{\omega_1}\omega_1$ $f \leqq_c g$ if and only if $\{\beta \in \omega_1 : f(\beta) > g(\beta)\}$ is countable.

Fact 1. There is a $B \in L$, $B \subset {}^{\omega_1}\omega_1$ such that B is well ordered with respect to $\leqq_c$ and B is unbounded with respect to $\leqq_c{}^{M_0}$. This is essentially the same as Theorem 2.3 of VIII in [**4**] so we omit the proof. Let $B = \{b_\alpha : \alpha < \omega_2\}$ be an order preserving indexing.

Fact 2. Let $A \subset \omega_2$, $A \in M_0$, $|A| = \omega_2$; then for all $\gamma \in \omega_1$ there is a $\delta \in \omega_1$, $\delta > \gamma$ such that for each $\xi \in \omega_1$,

$$|\{\alpha \in A : b_\alpha(\delta) > \xi\}| = \omega_2.$$

Proof. Suppose that this is not the case. We shall show that $\{b_\alpha : \alpha \in A\}$ is bounded with respect to $\leqq_c$ which is a contradiction. Let $\gamma \in \omega_1$ be such that for each $\delta \in \omega_1$, $\delta > \gamma$ there is an $h(\delta) = \xi$ such that

$$|\{\alpha \in A : b_\alpha(\delta) > \xi\}| < \omega_2.$$

Let

$$A' = \{\alpha \in A : b_\alpha(\delta) > h(\delta) \text{ for some } \delta > \gamma\},$$

$|A'| \leqq \omega_1$. Let $h_1 \in B$ such that $h_1 \,{}_c\!> b_\alpha$ for all $\alpha \in A'$. Hence $b_\alpha <_c h + h_1$ for each $\alpha \in A$.

Let

$$S = \{\alpha \in \omega_2 : \alpha \text{ has cofinality } \omega\};$$

S is stationary in ω_2.

Fact 3. There is a sequence $\{S_{\alpha,\delta} : \alpha \in S, \delta \in \omega_1\} \in M_0$ of countable subsets of ω_2, such that if $A \in [\omega_2]^{\omega_2}$, $A \in M_0$ there is an $\alpha \in S$, $\delta \in \omega_1$ such that $S_{\alpha,\delta} \subset A$ and $\{b_\beta(\delta) : \beta \in S_{\alpha,\delta}\}$ is infinite.

Proof. By $\Diamond_S$ we can define $M_\alpha = (\alpha, \leqq_\alpha, R_\alpha)$ for $\alpha \in S$ such that for any partial order $\leqq^*$ on ω_2, and two-place relation R on ω_2, for a stationary set of α's, $\leqq_\alpha = \leqq^*|_\alpha$, $R_\alpha = R|_\alpha$. For each $\alpha \in S$, $\delta \in \omega_1$, choose recursively, if possible, $\beta_{\alpha,\delta}{}^i\ \gamma_{\alpha,\delta}{}^i$, $i \in \omega$ such that $\beta_{\alpha,\delta}{}^0 = 0$,

$$b_{\gamma^i_{\alpha,\delta}}(\delta) \notin \{b_{\gamma^j_{\alpha,\delta}}(\delta) : j < i\},$$

$R_\sigma(\beta_{\alpha,\delta}{}^i, \gamma_{\alpha,\delta}{}^i)$ and $\beta_{\alpha,\delta}{}^i$ $(i \in \omega)$ is increasing with respect to $\leqq_\alpha$. If we succeed, let

$$S_{\alpha,\delta} = \{\gamma_{\alpha,\delta}{}^i : i \in \omega\}.$$

If not, let $S_{\alpha,\delta}$ be any countable set with $\{b_\gamma(\delta) : \gamma \in S_{\alpha,\delta}\}$ infinite.

Suppose that $A \in M_0$ is an unbounded subset of ω_2. By Fact 2 and VII.3.6 of [**4**] there is a $p_0 \in P_0$ and a $\delta \in \omega_1$ such that

$$p_0 \Vdash (\forall\ \xi \in \omega_1\{\alpha \in A : b_\alpha(\delta) > \xi\} \text{ is unbounded in } \omega_2).$$

Now, working in L, choose $Q \subset P_0$ such that $p_0 \in Q$, $|Q| = \omega_2$, any chain of Q of countable length has an upper bound in Q (P_0 is countably complete) and for every $\alpha < \omega_2$, $q \in Q$ and $\xi \in \omega_1 \exists\ \alpha' \geqq \alpha\ q' \geqq q$ and $\xi' > \xi$ such that $b_{\alpha'}(\delta) = \xi'$ and $q' \mid\vdash (\alpha' \in A)$. Let

$$Q = \{q(\beta) : \beta \in \omega_2\},$$

$q(0) = p_0$ and define $\beta \leqq^* \gamma$ if and only if $q(\beta) \leqq q(\gamma)$. Define

$$R = \{(\beta, \gamma) : q(\beta) \mid\vdash (\gamma \in A)\}.$$

So, let

$$C = \{\alpha \in \omega_2 : \forall\ \beta \in \alpha\ \forall\ \xi \in \omega_1 \exists\ \gamma \in \alpha(R(\beta, \gamma) \text{ and } b_\gamma(\delta) > \xi)\}.$$

It is easy to check that C is closed in ω_2. To show that C is unbounded, let $\alpha \in \omega_2$. Recursively, for $n \in \omega$, choose $\alpha_{n+1} > \alpha_n$ so that

$$\forall\ \beta \in \alpha_n\ \forall\ \xi \in \omega_1 \exists\ \gamma \in \alpha_{n+1}\ R(\beta, \gamma) \text{ and } b_\gamma(\delta) > \xi.$$

Therefore $\alpha' = \sup \{\alpha_n : n \in \omega\} \in C$. Therefore we may choose some $\alpha \in S \cap C$ such that M_α is an elementary submodel of $(\omega_2, \leqq^*, R)$. So we succeed in defining $\beta_{\alpha,\gamma}{}^i$, $\gamma_{\alpha,\delta}{}^i$, $i \in \omega$ as required. Let $q \in Q$ with $q \geqq q(\beta_{\alpha,\delta}{}^i)$, $i \in \omega$, so

$$q \mid\vdash (\gamma_{\alpha,\delta}{}^i \in A) \text{ for } i \in \omega.$$

Now since $q \geqq p_0$ and $q \mid\vdash (S_{\alpha,\delta} \subset A)$ we are done.

Let $g \in M_1$ be a set isomorphism between ω and $\omega_1{}^L$. It is clear, then, that g induces an obvious set isomorphism between ${}^\omega\omega$ and ${}^{\omega_1^L}\omega_1{}^L$, i.e., for $f \in {}^\omega\omega$ define $\mathscr{H}(f) \in {}^{\omega_1^L}\omega_1{}^L$ by

$$\mathscr{H}(f)(g(n)) = g(f(n)).$$

In this way we have

$$\hat{B} = \{f \in {}^\omega\omega : \mathscr{H}(f) \in B\}.$$

Fact 4. $\diagdown\!\!\diagup$ holds in M_1.

Proof. Recall that in M_1, $\omega_1{}^{M_1} = \omega_2{}^L$. Let $A \subset \omega_1{}^{M_1}$ with $A \in M_1$ and A unbounded. Since, in L, $|P_1| < \omega_2{}^L$, there is an $A' \subset A$ with $A' \in M_0$, and A' is unbounded in $\omega_2{}^{M_0} = \omega_2{}^L$. Therefore there is an $\alpha \in S$ and $\delta \in \omega_1{}^L$ such that $S_{\alpha,\delta} \subset A'$ and $\{b_\gamma(\delta) : \gamma \in S_{\alpha,\delta}\}$ is infinite. Therefore

$$\{\hat{b}_\gamma(g^{\leftarrow}(\delta)) : \mathscr{H}(\hat{b}_\gamma) = b_\gamma \quad \text{and} \quad \gamma \in S_{\alpha,\delta}\}$$

is infinite showing that $\hat{B}$ is an instance of $\diagdown\!\!\diagup$.

Finally it remains to show that in M_1, $d = \omega_2 < \kappa$. To this end we first note that M_1 can also be obtained by $M[G_1] = M'$ and $M_1 = M'[G_0]$. Since $P_0 \in L$ we can use a Δ-system argument to show that P_0 has the $\omega_1{}^L$-*cc* property (this is why it preserves cardinals). Therefore in M', P_0 is *ccc*. This means that if $h \in {}^{\omega}\omega$, $h \in M_1$, there is an M'-countable set $I \subset \omega_3{}^L$ such that

$$f \in M'[G \cap \{f : I \times \omega_1{}^L \to \omega_1{}^L : f \text{ is } L\text{-countable}\}]$$

(VIII 2.2 in [**4**]).

Fact 5. $d = \omega_2$ holds in M_1.

Proof. Suppose that $D \subset {}^{\omega}\omega$, $D \in M_1$ and $(|D| < \omega_2)^{M_1}$. Let

$$D = \{d_\alpha : \alpha < \omega_1{}^{M_1}\}$$

be an ordering in M_1 of D. Since $\omega_1{}^{M_1} = \omega_2{}^L$, by the above argument, we can find an $\alpha \in \omega_3{}^L$ such that $D \in M'[G \cap \{f : f$ is a function from a countable subset of $\alpha \times \omega_1{}^L$ to $\omega_1{}^L\}] = M''$. It suffices, therefore, to show that if we extend M'' by forcing with $P' = \{f : f$ is a function from an L-countable subset of $\omega_1{}^L$ to $\omega_1{}^L\}$ then we introduce a function in ${}^{\omega}\omega$ not dominated by D. So, for each $\alpha \in \omega_1{}^{M_1}$ and $n \in \omega$, let

$$E_{\alpha,n} = \{f \in P' : \exists\, \beta \in \omega_1{}^L \text{ with } g^{\leftarrow}(\beta) > n \text{ and } g^{\leftarrow}(f(\beta)) > d_\alpha(g^{\leftarrow}(\beta))\}$$

(i.e., $f \in E_{\alpha,n}$ if $\mathscr{H}^{\leftarrow}(f)(m) > d_\alpha(m)$ for some $m > n$). To see that $E_{\alpha,n}$ is dense in P', let $f \in P'$ and find a $\beta \in \omega_1{}^L$ with $g^{\leftarrow}(\beta) > n$ and $\beta \notin \operatorname{dom} f$. Extend f at β to be any γ such that

$$g^{\leftarrow}(\gamma) > d_\alpha(g^{\leftarrow}(\beta)).$$

So, by forcing over P' we introduce an element of ${}^{\omega}\omega$ not dominated by D.

Fact 6. $\omega_2 < \kappa$ holds in M_1. It suffices to exhibit a filter u on ω such that there is a u-scale of order type ω_2. As above let $g \in M'$ be an isomorphism from ω to $\omega_1{}^L$. For $\alpha \in \omega_3{}^L$ let $P_\alpha = \{f \in L : f$ is a function from an L-countable $A \subset \alpha \times \omega_1{}^L$ to $\omega_1{}^L\}$. Observe that M_1 can be obtained by starting with M' and iterating $\omega_3{}^L$-times to obtain

$$M_\alpha = M'[G_0 \cap P_\alpha], \text{ for each } \alpha < \omega_3.$$

For each $\alpha \in \omega_3{}^L$ we introduce a function $f_\alpha \in M_{\alpha+1} \setminus M_\alpha$ where $f_\alpha \in {}^{\omega_1^L}\omega_1{}^L$ and with obvious abuse of notation $\{\alpha\} \times f_\alpha|_\gamma \in G_0$ for each $\gamma \in \omega_1{}^L$. For $\alpha \in \omega_3{}^L$, let

$$u_\alpha = \{\{g^{\leftarrow}(\delta) : g^{\leftarrow}(f_\alpha(\delta)) > g^{\leftarrow}(f(\delta)) : f \in {}^{\omega_1^L}\omega_1{}^L \cap M_\alpha\} \text{ and}$$

$$u = \bigcup \{u_\alpha : \alpha \in \omega_3{}^L\}.$$

Let us show that u is a filter on ω. Let $\alpha_0 \leqq \ldots \leqq \alpha_n$ and

$$h_i \in M_{\alpha_i} \cap {}^{\omega_1^L}\omega_1{}^L \quad \text{for } i < n.$$

Recursively define $U_i \in u_{\alpha_i}$ as

$$U_i = \{g^{\leftarrow}(\delta) : g^{\leftarrow}(f_{\alpha_i}(\delta)) > \sum_{j<i} g^{\leftarrow}(h_j(\delta))\}.$$

Now $E_i = \{f \in P_{\alpha_{i+1}} : \forall\, \gamma \in \omega_1{}^L \,\exists\, \delta \in \omega_1{}^L$ with $\delta > \gamma$, $g^{\leftarrow}(\delta) \in U_j$ for each $j < i$ and $g^{\leftarrow}(f((\alpha_i, \delta)) > \sum_{j<i} g^{\leftarrow}(h_j(\delta))\}$ is dense in $P_{\alpha_{i+1}}$ so long as U_j is infinite for $j < i < n$. Since the density of E_i guarantees that U_i is infinite, each U_i is infinite by induction. This completes the proof that u is a filter. It remains only to show that $\{\mathscr{H}^{\leftarrow}(f_\alpha) : \alpha \in \omega_3{}^L\}$ is cofinal in $({}^\omega\omega, <_u)$. This, however, follows from the fact that for $f \in {}^\omega\omega \cap M_1$ there is an $\alpha \in \omega_3{}^L$ with $f \in {}^\omega\omega \cap M_\alpha$ and

$$\{g^{\leftarrow}(\delta) : g^{\leftarrow}(f_\alpha(\delta)) > g^{\leftarrow}(H(f))(\delta)\} = \{n : H^{\leftarrow}(f_\alpha)(n) > f(n)\} \in u.$$

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Vrije Universiteit,
Amsterdam, The Netherlands