Proceedings of the Royal Society of Edinburgh, **134A**, 249–258, *2004*

Non-zero solutions for a Schrödinger equation with indefinite linear and nonlinear terms

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(MS received 26 February 2003; accepted 13 January 2004)

We prove the existence of a non-trivial solution for the nonlinear elliptic problem $-\Delta u + V(x)u = a(x)g(u)$ in $\mathbb{R}^N$, where g is superlinear near zero and near infinity, $a(x)$ changes sign and $V \in C(\mathbb{R}^N)$ is positive at infinity. For g odd, we prove the existence of an infinite number of solutions.

1. Introduction

In this paper we consider a class of nonlinear Schrödinger equations of the form

$$-\Delta u + V(x)u = a(x)g(u), \quad x \in \mathbb{R}^N, \tag{P}$$

where $V(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $a(x) \in C(\mathbb{R}^N)$ with $N \geqslant 3$. We are interested in establishing existence and multiplicity results when the nonlinear term $g(s)$ has a superlinear behaviour at zero, a power-like growth at infinity and satisfies the sign condition $g(s)s \geqslant 0$, for all $s \in \mathbb{R}$, while the weight function $a(x)$ is a sign-changing function in $\mathbb{R}^N$ that is *negative* at infinity in the sense that $\limsup_{|x|\to\infty} a(x) < 0$. In fact, as noted in [2], when $a(x)$ is *positive* at infinity, then Pohožaev-type identities will yield non-existence results under rather mild assumptions.

In [8], the authors considered the case of a bounded domain $\Omega \subset \mathbb{R}^N$ and showed an existence result for the equation

$$-\Delta u - \lambda u = a(x)g(u)$$

in $H_0^1(\Omega)$, provided $g(s)$ is a superlinear nonlinearity as described above, $a(x)$ a sign-changing function in Ω and $\lambda_k < \lambda < \lambda_{k+1}$, where the λ_j denote the eigenvalues of $-\Delta$ on $H_0^1(\Omega)$. In this case, we note that the operator $L = -\Delta - \lambda$ is *indefinite*, with its spectrum $\sigma(L)$ consisting solely of isolated eigenvalues of finite multiplicity and $0 \notin \sigma(L)$.

Our first theorem will extend the existence result of [8] to the case where $\Omega = \mathbb{R}^N$. Our second result will show existence of infinitely many solutions when g is odd. We

https://doi.org/10.1017/S0308210500003206 Published online by Cambridge University Press

observe that, in our present situation of $\Omega = \mathbb{R}^N$, the verification of *compactness conditions* is a rather delicate problem since, in contrast with the bounded domain case, the spectrum of the Schrödinger operator

$$L = -\Delta + V(x) : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

may now contain an essential part. In fact, when $\liminf_{|x|\to\infty} V(x) = v_\infty \in \mathbb{R}$, it is known (see [1]) that the essential spectrum $\sigma_{\mathrm{e}}(L)$ is contained in the half-line $[v_\infty, +\infty)$ and the spectrum of L in $(-\infty, v_\infty)$ consists of isolated eigenvalues of finite multiplicity. Furthermore, $\sigma_{\mathrm{e}}(L) = [\hat{v}_\infty, +\infty)$ in case the limit $\lim_{|x|\to\infty} V(x) = \hat{v}_\infty \in \mathbb{R}$ exists.

Here we assume that the potential $V(x)$ is *positive* at infinity (i.e. $v_\infty > 0$), that the operator L is indefinite (i.e. $\sigma(L) \cap (-\infty, 0) \neq \emptyset$ and $\sigma(L) \cap (0, +\infty) \neq \emptyset$) and, similarly to [8], that $0 \notin \sigma(L)$. We should note that when L is non-negative (i.e. when $\sigma(L) \cap (-\infty, 0) = \emptyset$), there are a number of existence and multiplicity results for (P) under various assumptions on the nonlinearity $a(x)g(s)$ (see, for example, [7] for related results for second- and fourth-order equations in the case where $V(x) \equiv 1$).

As far as we know, the only existence results for such problems in $\mathbb{R}^N$ with *indefinite* linear and nonlinear parts are those in [3, 4]. Our present results greatly generalize those of [3] by exploiting the full strength of the spectral method used in [4]. The basic idea is to obtain solutions of (P) as limits of solutions (u_n) of the equation in (P) considered in the spaces $H_0^1(B_{R_n}(0))$ with $R_n \to \infty$. The essence of the method consists in establishing compactness through information on the Morse indices of the approximated solutions (u_n). We point out that, in contrast to [4], this limiting process must be handled with some care in order to avoid the essential part of $\sigma(L)$ (see lemmas 2.2 and 3.3 below).

In § 2, after listing our precise hypotheses, we state and prove an existence result for problem (P), namely theorem 2.1, dealing with our class of *indefinite* superlinear nonlinearities. In § 3, we consider odd nonlinearities (i.e. $g(-s) = -g(s)$ for every s) and prove theorem 3.1 on existence of infinitely many solutions for (P).

2. Existence of one solution

We consider the Schrödinger equation

$$-\Delta u + V(x)u = a(x)g(u), \quad x \in \mathbb{R}^N, \tag{P}$$

under the following hypotheses on $V(x)$.

(H1) $V \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ $(0 < \alpha < 1)$ with

$$v_\infty := \liminf_{|x|\to\infty} V(x) > 0 \quad \text{and} \quad 0 \notin \sigma(-\Delta + V).$$

(H2) $\displaystyle\int_{\mathbb{R}^N} (|\nabla\varphi|^2 + V(x)\varphi^2) < 0$ for some $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^N)$.

As for the function $a(x)$, similarly to [4], we assume the following.

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(H3) $a \in C^1(\mathbb{R}^N)$ is sign changing, has only non-degenerate zeros (i.e. $\nabla a(x) \neq 0$ for every x such that $a(x) = 0$) and

$$\limsup_{|x|\to\infty} a(x) < 0.$$

Finally, regarding the nonlinearity $g(s)$, we make the following assumptions.

(H4) $g \in C^1(\mathbb{R})$, $g(0) = 0 = g'(0)$, $g(s)s \geqslant 0$ for every $s \in \mathbb{R}$ and there exist positive constants C, δ, ℓ_∞ and $p \in (2, 2N/(N-2))$ such that

$$G(s) \leqslant Cg(s)s \quad \forall |s| \leqslant \delta \quad \text{and} \quad \lim_{|s|\to\infty} \frac{g'(s)}{|s|^{p-2}} = \ell_\infty.$$

As usual, we denote $G(s) := \int_0^s g(\xi)\,\mathrm{d}\xi$ and recall from the introduction that L is the linear operator $L = -\Delta + V(x) : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. We now state the main result of this section.

Theorem 2.1. *Assume (H1)–(H4). Then the nonlinear Schrödinger equation (P) has a non-zero solution* $u \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

We point out that we *neither* assume a $L^\infty(\mathbb{R}^N)$ bound on $a(x)$ *nor* a global superlinear behaviour on $g(s)$, such as $g(s)s \geqslant (2+\delta)G(s)$ for every $s \in \mathbb{R}$ and some $\delta > 0$. On the other hand, it follows from our assumptions that there exist R_0, $\delta_0 > 0$ such that

$$a^-(x) \geqslant \delta_0 \quad \text{and} \quad V(x) \geqslant \delta_0 \quad \forall |x| \geqslant R_0, \tag{2.1}$$

and there exists $C > 0$ such that

$$0 \leqslant G(s) \leqslant Cg(s)s \quad \forall s \in \mathbb{R}. \tag{2.2}$$

Here we use the notation $a^+(x) := \max\{a(x), 0\}$ and $a^-(x) := a^+(x) - a(x)$. Moreover, it follows from (H1), (H2) that there exists $k \geqslant 1$ such that

$$\sigma(L) \cap (-\infty, 0] \text{ consists of } k \text{ non-zero eigenvalues of finite multiplicity.} \tag{2.3}$$

The rest of the section is devoted to the proof of theorem 2.1. We first state some auxiliary results. For that, we let

$$\|u\|^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \quad \forall u \in H^1(\mathbb{R}^N),$$

and we denote by I the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \int_{\mathbb{R}^N} a(x)G(u)$$

whenever it is defined for functions in $H^1(\mathbb{R}^N)$. In particular, if the space $H_0^1(B_R(0))$ is viewed as a subspace of $H^1(\mathbb{R}^N)$ by extending the functions by zero outside $B_R(0)$, then the functional I will be defined on $H_0^1(B_R(0))$ for all $R > 0$ (in fact, $I \in C^2(H_0^1(B_R(0)), \mathbb{R})$) and $H_0^1(B_R(0))$ can be viewed as a subspace of $H_0^1(B_{R'}(0))$ if $R' > R$.

LEMMA 2.2. *There exist $\eta > 0$ and $R_1 > 0$ such that, for every $R > R_1$, we can write $H_0^1(B_R(0)) = X_R \oplus Y_R$, where X_R has dimension k and*

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \leqslant -\eta \|u\|^2 \quad \forall u \in X_R,$$
$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \geqslant \eta \|u\|^2 \qquad \forall u \in Y_R.$$

Proof.

STEP 1. For each $i \in \mathbb{N}$, denote by $\lambda_i(R)$ the ith eigenvalue of the linear operator $-\Delta + V(x)$ in $H_0^1(B_R(0))$. To prove the lemma it is sufficient to show that there exist $\varepsilon > 0$ and $R_1 > 0$ such that

$$\lambda_k(R) \leqslant -\varepsilon < 0 < \varepsilon \leqslant \lambda_{k+1}(R) \quad \forall R \geqslant R_1.$$

STEP 2. The proof that $\lambda_k(R) \leqslant -\varepsilon < 0$ for large R is similar to that in [4, lemma 2.1], and therefore we omit it.

STEP 3. Let V be spanned by the eigenfunctions associated to the negative eigenvalues of the linear operator $L = -\Delta + V(x)$ in $H^2(\mathbb{R}^N)$. Since $0 \notin \sigma(L)$, there exists $\rho > 0$ such that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \geqslant 2\rho \int_{\mathbb{R}^N} u^2 \quad \forall u \in V^{\perp}.$$

Suppose that, for some fixed R, we have $\lambda_{k+1}(R) \leqslant \rho$. Then there exists a subspace $X \subset H_0^1(B_R(0))$ with dimension $k+1$ such that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \leqslant \rho \int_{\mathbb{R}^N} u^2 \quad \forall u \in X.$$

Using a density argument, we may assume that $X \subset D(\mathbb{R}^N)$. In particular, we have $X \subset H^2(\mathbb{R}^N)$. Since V has dimension k and X has dimension $k+1$, there exists $u \in X \cap (V^{\perp})$, $u \neq 0$. This is impossible by the definition of ρ. We conclude that $\lambda_{k+1}(R) > \rho$ for every R, and this completes the proof of the lemma. □

It is well known that the solutions u of

$$-\Delta u + V(x)u = a(x)g(u), \quad u \in H_0^1(B_R(0)), \tag{P_R}$$

are precisely the critical points of the energy functional I over the Hilbert space $H_0^1(B_R(0))$. For every such critical point u, we denote by $m(u)$ its Morse index, that is, the supremum of the dimensions of the linear subspaces of $H_0^1(B_R(0))$ on which the quadratic form $D^2I(u)$ is negative definite.

LEMMA 2.3. *There exist $R_2 > 0$ and $r > 0$ such that every solution u of* (P_R) *with $R > R_2$ and Morse index $m(u) \leqslant k-1$ satisfies $\|u\| \geqslant r$.*

Proof. Assume by contradiction that, for some sequence $R_n \to \infty$, problem (P_{R_n}) admits solutions u_n such that $m(u_n) \leqslant k-1$ and $\|u_n\| \to 0$. According to lemma 2.2,

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there exists a subspace $X \subset H_0^1(B_{R_1}(0))$ with dimension k and some constant $\eta > 0$ such that

$$\int_{B_{R_1}(0)} (|\nabla\varphi|^2 + V(x)\varphi^2) \leqslant -\eta\|\varphi\|^2 \quad \forall\varphi \in X.$$

Since $\|u_n\| \to 0$ and $g'(0) = 0$, it follows from the compact imbedding of $H_0^1(B_{R_1}(0))$ into $L^p(B_{R_1}(0))$ that there exists $n_0 \in \mathbb{N}$ such that

$$D^2I(u_n)(\varphi,\varphi) = \int_{B_{R_1}(0)} (|\nabla\varphi|^2 + V(x)\varphi^2) - \int_{B_{R_1}(0)} a(x)g'(u_n)\varphi^2 \leqslant -\tfrac{1}{2}\eta\|\varphi\|^2,$$

for every $\varphi \in X$ and every $n \geqslant n_0$. This contradicts the assumption that $m(u_n) \leqslant k - 1$. ☐

Lemma 2.4. *For every large $R > 0$, problem* (P_R) *has a non-zero solution $u_R \in H_0^1(B_R(0))$. The family of solutions (u_R) is bounded in $L^\infty(\mathbb{R}^N)$ and there exist $c, R_3 > 0$ such that*

$$0 < c \leqslant \max\{\|u_R\|, I(u_R)\} \quad \forall R > R_3. \tag{2.4}$$

Proof. In view of lemma 2.2, for each large R, problem (P_R) admits a non-zero solution u_R obtained through a minimax argument, with $m(u_R) \leqslant k + 1$ (see [5, proposition 2]). Moreover, by using arguments similar to those of [4, proposition 3.1], we claim that (u_R) is bounded in $L^\infty(\mathbb{R}^N)$.

Indeed, since our regularity assumptions imply that $u_R \in C(\bar{B}_R) \cap C^2(B_R)$, let us assume by contradiction that

$$M_n := \|u_n\|_\infty = \max_{\Omega_n} u_n = u_n(x_n) \to +\infty$$

for some $x_n \in \Omega_n \equiv B_{R_n}(0)$ and some sequence $R_n \to \infty$ (the case where $\|u_n\|_\infty = \max_{\Omega_n}(-u_n)$ is handled similarly). Since $\Delta u_n(x_n) \leqslant 0$ and $V \in L^\infty(\mathbb{R}^N)$, the equation in (P_{R_n}) shows that

$$a^-(x_n)g(M_n) \leqslant CM_n + a^+(x_n)g(M_n).$$

From assumption (H3), we see that (x_n) is bounded. Therefore, up to a subsequence, we can assume that $x_n \to x_0 \in \mathbb{R}^N$ and $a(x_0) \geqslant 0$. At this point, the blow-up argument in [8, sect3] can be applied, leading to a contradiction. In fact, since $m(u_n) \leqslant k + 1$ and $\|u_n\|_\infty \to \infty$, it is shown in [8] that the sequence

$$v_n(x) = u_n(\lambda_n x + x_n)/M_n$$

(where either $\lambda_n = M_n^{(2-p)/2}$ or $\lambda_n = M_n^{(2-p)/3}$, depending on whether $a(x_0) > 0$ or $a(x_0) = 0$, respectively) converges uniformly to 0 on compact sets. This is impossible, as $v_n(0) = 1$ by definition. Therefore, the family of solutions (u_R) is bounded in $L^\infty(\mathbb{R}^N)$.

Now, when $m(u_R) \geqslant k$ along a sequence $R = R_n \to \infty$, then, by construction (cf. [4, eqn (3.4)]), we have

$$I(u_R) \geqslant \inf\{I(u) : u \in Y_R,\ \|u\| = r\} \tag{2.5}$$

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for some small $r > 0$, where Y_R is given by lemma 2.2. Since a^+ has compact support and also $g(0) = 0 = g'(0)$ and $\lim_{|s|\to\infty} G(s)/|s|^p < \infty$, it follows that, for every $u \in Y_R$,

$$I(u) \geqslant \eta \|u\|^2 - \int_{\mathbb{R}^N} a^+ G(u) \geqslant \eta_1 \|u\|^2 - \eta_2 \|u\|^p,$$

for some η_1, η_2 independent of R, so that the right-hand side of (2.5) is bounded away from zero, if r is sufficiently small. This proves (2.4) for the case $m(u_R) \geqslant k$. The case where $m(u_R) \leqslant k-1$ along a sequence $R \to \infty$ is ruled out by lemma 2.3. □

Our next lemma will also be used in § 3.

LEMMA 2.5. *For any sequence $R_n \to \infty$, let (u_n) be a sequence of solutions of* (P_{R_n}) *that is bounded in $L^\infty(\mathbb{R}^N)$. Then there exists a solution u of equation* (P) *such that $u \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, u has finite energy and*

$$u_n \to u \quad \text{in } H^1(\mathbb{R}^N) \quad \text{and} \quad I(u_n) \to I(u). \tag{2.6}$$

Proof. Using (u_n) as a test function in (P_{R_n}) yields

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2 + a^- g(u_n)u_n) = \int_{\mathbb{R}^N} a^+ g(u_n)u_n. \tag{2.7}$$

Since a^+ has compact support by (H3), $g(s)s \geqslant 0$ by (H4) and (u_n) is bounded in $L^\infty(\mathbb{R}^N)$ by lemma 2.4, the above equation shows that (u_n) is also bounded in $H^1(\mathbb{R}^N)$. Therefore, up to a subsequence and for some $u \in H^1(\mathbb{R}^N)$, we have that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Clearly, u is a solution of (P) and $u \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Moreover, it follows from (2.2), (2.7) and Fatou's lemma (recall that $g(s)s \geqslant 0$) that $I(u) \in \mathbb{R}$.

In order to prove (2.6), denote

$$|||u|||^2 := \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{|x|\geqslant R_0} V(x)u^2,$$

where R_0 was defined in (2.1), and let $\hat{\ell} := \liminf |||u_n|||^2$. By convexity, $|||u|||^2 \leqslant \hat{\ell}$ and, by passing to a subsequence, we may assume that $\hat{\ell} = \lim |||u_n|||^2$. From the equation in (P) and the fact that $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we have

$$\begin{aligned} |||u_n|||^2 + \int_{|x|\leqslant R_0} V(x)u_n^2 + \int a^- g(u_n)u_n & \\ &= \int a^+ g(u_n)u_n \\ &= \int a^+ g(u)u + \mathrm{o}(1) \\ &= |||u|||^2 + \int_{|x|\leqslant R_0} V(x)u^2 + \int a^- g(u)u + \mathrm{o}(1) \end{aligned}$$

as $n \to \infty$, and it follows that

$$\hat{\ell} + \int_{|x|\leqslant R_0} V(x)u^2 + \liminf \int a^- g(u_n)u_n \leqslant |||u|||^2 + \int_{|x|\leqslant R_0} V(x)u^2 + \int a^- g(u)u.$$

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In other words,

$$\hat{\ell} + \liminf \int a^- g(u_n)u_n \leqslant |||u|||^2 + \int a^- g(u)u.$$

Therefore, in view of Fatou's lemma, we obtain

$$\hat{\ell} + \int a^- g(u)u \leqslant |||u|||^2 + \int a^- g(u)u$$

and, since $|||u|||^2 \leqslant \hat{\ell}$, we conclude that

$$|||u|||^2 = \hat{\ell} \quad \text{and} \quad \int a^- g(u_n)u_n \to \int a^- g(u)u. \tag{2.8}$$

In particular, it follows that $u_n \to u$ in $H^1(\mathbb{R}^N)$. Finally, since

$$a^-(x)G(u_n(x)) \to a^-(x)G(u(x)) \quad \text{a.e.}$$

and (cf. (2.2))

$$\int a^- G(u_n) \leqslant C \int a^- g(u_n)u_n,$$

we conclude from (2.8) and the Lebesgue–Vitali convergence theorem that

$$\int a^- G(u_n) \to \int a^- G(u).$$

The proof of lemma 2.5 is complete. □

End of proof of theorem 2.1. Fix any sequence $R_n \to \infty$ and let (u_n) be given by lemma 2.4. According to lemma 2.5, we can pass to a subsequence so that $u_n \to u$ in $H^1(\mathbb{R}^N)$, where u is a solution of (P). It follows from (2.4) and (2.6) that $u \neq 0$, which completes the proof of theorem 2.1. □

3. The symmetric case

In this section we will consider again the Schrödinger equation

$$-\Delta u + V(x)u = a(x)g(u), \quad x \in \mathbb{R}^N, \tag{P}$$

in the case where $g(s)$ is an odd nonlinearity, under the assumptions (H1)–(H4) stated in § 2.

THEOREM 3.1. *Assume that assumptions (H1)–(H4) hold and, in addition, that g is odd. Then the nonlinear Schrödinger equation (P) has infinitely many solutions $u \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.*

We now prove theorem 3.1. Let I denote the energy functional associated with (P) acting on $H_0^1(B_R(0))$ $(R > 0)$ and fix any $d \in \mathbb{R}$. Thanks to lemma 2.5, theorem 3.1 follows once we show that I has a critical point $u_R \in H_0^1(B_R(0))$ such that $I(u_R) \geqslant d$ and (u_R) is bounded in $L^\infty(\mathbb{R}^N)$, uniformly in R. As mentioned in the proof of lemma 2.4, the latter property will be a consequence of an estimate from

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above on the Morse indices $m(u_R)$. Summarizing, we must prove that *for every* $d \in \mathbb{R}$ *there exist* $\ell \in \mathbb{N}$ *and* $u_R \in H_0^1(B_R(0))$, *for every large* R, *such that*

$$I(u_R) \geqslant d \quad \text{and} \quad m(u_R) \leqslant \ell. \tag{3.1}$$

This kind of argument was also used in [4]. The proof is based in the following well-known existence result for even functionals.

PROPOSITION 3.2. *Let* I *be a real even* C^2 *functional over an Hilbert space* H *satisfying the Palais–Smale condition. Suppose that* $H = X \oplus Y$, *with* $\dim X = \ell \in \mathbb{N}$, *and that*

$$\inf\{I(u) : \|u\| = r,\ u \in Y\} \geqslant d \tag{3.2}$$

for some $d \in \mathbb{R}$ *and* $r > 0$. *Denote* $Z = X \oplus \operatorname{span}(e)$, *for some* $e \in Y$, $e \neq 0$, *and suppose that*

$$\sup\{I(u) : \|u\| \geqslant M,\ u \in Z\} < d, \tag{3.3}$$

for some $M > 0$. *Then* I *has a critical point* u *at level* $I(u) \geqslant d$ *and having Morse index* $m(u) \leqslant \ell + 1$.

A proof of this result can be found, for example, in [9, theorem 5.2] and [10, theorem 3.6]; the Morse index estimate is essentially proved, for example, in [6]. In order to apply proposition 3.2, a crucial fact is provided by our next lemma. We recall that the constant R_0 was introduced in (2.1).

LEMMA 3.3. *Given* $\varepsilon > 0$, *there exist* $\ell \in \mathbb{N}$ *and a subspace* $X \subset H^1(\mathbb{R}^N)$ *of dimension* ℓ *such that*

$$\int_{B_{R_0}(0)} u^2 \leqslant \varepsilon \|u\|^2 := \varepsilon \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \quad \forall u \in X^{\perp}. \tag{3.4}$$

Moreover, $X \subset H_0^1(B_{\bar{R}}(0))$ *for some* $\bar{R} > 0$.

Proof. Fix any sequence $R_n \to \infty$. For each n, denote by V_n the finite-dimensional space spanned by the eigenfunctions

$$\varphi_1^1, \dots, \varphi_n^1, \varphi_1^2, \dots, \varphi_n^2, \dots, \varphi_1^n, \dots, \varphi_n^n,$$

where φ_j^i denotes the jth eigenfunction of $(-\Delta, H_0^1(B_{R_i}(0)))$. Assume, by contradiction, that, for every n, there exists $u_n \in V_n^{\perp}$ such that

$$\|u_n\| = 1 \quad \text{and} \quad \int_{B_{R_0}(0)} u_n^2 \geqslant \varepsilon.$$

Up to a subsequence, (u_n) converges weakly to some $u \in H^1(\mathbb{R}^N)$ such that

$$\int_{B_{R_0}(0)} u^2 \geqslant \varepsilon.$$

In particular, $u \neq 0$.

Fix any R_i, $i \in \mathbb{N}$, and any eigenfunction φ_j^i. By assumption,

$$0 = \int_{\mathbb{R}^N} \langle \nabla u_n, \nabla \varphi_j^i \rangle + u_n \varphi_j^i \quad \forall n \geqslant j.$$

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Taking limits as $n \to \infty$,

$$0 = \int_{\mathbb{R}^N} \langle \nabla u, \nabla \varphi_j^i \rangle + u\varphi_j^i.$$

Since j is arbitrary,

$$0 = \int_{\mathbb{R}^N} \langle \nabla u, \nabla \varphi \rangle + u\varphi \quad \forall \varphi \in H_0^1(B_{R_i}(0)).$$

Since i is arbitrary,

$$0 = \int_{\mathbb{R}^N} \langle \nabla u, \nabla \varphi \rangle + u\varphi \quad \forall \varphi \in D(\mathbb{R}^N).$$

Since $D(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, we conclude that $u = 0$, a contradiction. □

End of proof of theorem 3.1. Given $d > 0$, let $\varepsilon = \varepsilon(d)$ be a small positive number to be specified below (cf. (3.5)). Let ℓ and X be given by lemma 3.3 and consider the energy functional I acting on $H_0^1(B_R(0))$, $R > \bar{R}$. It follows easily from (2.1), (3.4) and (H4) that, for small $\varepsilon > 0$, there exist $A = A(\varepsilon)$ and $B = B(\varepsilon) > 0$ in such a way that

$$I(u) \geqslant A\|u\|^2 - B\|u\|^p \quad \forall u \in X^\perp \cap H_0^1(B_R(0)). \tag{3.5}$$

In fact, since $2 < p < 2N/(N-2)$ and $H^1(\mathbb{R}^N) \subset L^{2N/(N-2)}(\mathbb{R}^N)$, we have that

$$\left(\int_{B_{R_0}(0)} |u|^p \right)^{1/p} \leqslant o(1)\|u\| \quad \text{for every } u \in X^\perp,$$

where $o(1) \to 0$ as $\varepsilon \to 0$, as follows from lemma 3.3, and so (3.5) shows that we can choose $\varepsilon = \varepsilon(d)$ and $r = r(d)$ such that the inequality in (3.2) holds.

In our context, condition (3.3) is not expected to be satisfied. However, as done in [4, p. 11], one can consider the following truncations introduced in [8],

$$g_j(s) = \begin{cases} A_j|s|^{p_j-2}s + B_j & \text{for } s \geqslant a_j, \\ g(s) & \text{for } 0 \leqslant s \leqslant a_j, \\ -g_j(-s) & \text{for } s \leqslant 0, \end{cases}$$

where $a_j \to +\infty$, $p_j \in (2, p)$, $p_j \to p$, and the coefficients are chosen in such a way that g_j is C^1. Then the corresponding energy functional

$$I_j(u) = \frac{1}{2}\int (|\nabla u|^2 + V(x)u^2) - \int a^+(x)G(u) + \int a^-(x)G_j(u),$$

where $G_j(s) := \int_0^s g_j(\xi)\,\mathrm{d}\xi$, satisfies (3.2), (3.3) and the Palais–Smale condition. According to proposition 3.2, this yields a critical point $u_{j,R}$ of I_j such that

$$I_j(u_{j,R}) \geqslant d \quad \text{and} \quad m(u_{j,R}) \leqslant \ell + 1, \tag{3.6}$$

where $m(u_{j,R})$ denotes the Morse index of $u_{j,R}$ as a critical point of I_j. Since $m(u_{j,R})$ is bounded independently of j and R, it turns out that $u_{j,R}$ is actually a critical point of I for j sufficiently large. This establishes (3.1) (with $\ell + 1$ in place of ℓ). As mentioned earlier, this also completes the proof of theorem 3.1. □

Acknowledgments

M.R. was partly supported by FCT (Fundação para a Ciência e a Tecnologia), programa POCTI (Portugal/FEDER-EU).

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(*Issued 7 May 2004*)