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Tournaments and Orders with the Pigeonhole Property

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Abstract. A binary structure S has the pigeonhole property ($\mathcal{P}$) if every finite partition of S induces a block isomorphic to S. We classify all countable tournaments with ($\mathcal{P}$); the class of orders with ($\mathcal{P}$) is completely classified.

1 Introduction

A nontrivial graph G has the pigeonhole property ($\mathcal{P}$) if for every finite partition of the vertex set of G the induced subgraph on at least one of the blocks is isomorphic to G. The intriguing thing about ($\mathcal{P}$) is that few countable graphs satisfy it: by Proposition 3.4 of [3] the only countable graphs with ($\mathcal{P}$) are (up to isomorphism) $K_{\aleph_0}$ (the complete graph on $\aleph_0$-many vertices), $\overline{K_{\aleph_0}}$ (the complement of $K_{\aleph_0}$), and R (the random graph). Cameron in [2] originally asked which other relational structures satisfy ($\mathcal{P}$). In [1], the authors gave an answer to Cameron's question for various kinds of relational structures. However, in [1] the classification of countable tournaments with ($\mathcal{P}$) was left open.

The immediate goal of the present article is to present a complete classification of the countable tournaments with ($\mathcal{P}$) (see Theorem 1 below for an explicit list). In stark contrast to the situation for graphs, we find there are uncountably many non-isomorphic countable tournaments with ($\mathcal{P}$). Along the way, we classify the orders and quasi-orders with ($\mathcal{P}$) in each infinite cardinality (see Theorems 1 and 2). We close with a discussion on the classification of the oriented graphs with ($\mathcal{P}$).

2 Preliminaries

2.1 Binary Structures and the Pigeonhole Principle

Definition 1 A *binary structure* S consists of a vertex set (called S as well) and an edge set $E^S \subseteq S^2$. The *order* of S is the cardinality of the vertex set, written $|S|$. If $|S| > 1$, we say S is *nontrivial*.

If S is clear from context, we sometimes drop S from E^S and simply write E.

Example 1 Directed graphs (digraphs) are binary structures with an irreflexive edge set. An oriented graph is a binary structure with an irreflexive and asymmetric edge set. Graphs

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are binary structures with an irreflexive, symmetric edge set. Orders (or partial orders) are binary structures with an irreflexive and transitive edge set; for orders we write $x < y$ for $(x, y) \in E$. Tournaments are oriented graphs so that for each pair of distinct vertices x, y either (x, y) or (y, x) is in E.

Definition 2

1. Let S be a binary structure with $A \subseteq S$. Then $S \restriction A$ is the binary structure with vertices A and edges $E \cap A^2$. $S \restriction A$ is the *induced substructure* of S on A.
2. Given two binary structures S, T, we say that S and T are *isomorphic* if there is a bijective map $f\colon S \to T$ so that $(x, y) \in E^S$ if and only if $\big(f(x), f(y)\big) \in E^T$. We write $S \cong T$.

We use the notation $S \uplus T$ for the disjoint union of sets S and T.

Definition 3 A binary structure S has the *pigeonhole property* $(\mathcal{P})$ if S is nontrivial and whenever $S = S_1 \uplus \cdots \uplus S_n$ then for some $1 \leq i \leq n$, $S \restriction S_i \cong S$.

Note that every binary structure with $(\mathcal{P})$ is infinite.

2.2 Directed Graphs and Duality

Definition 4 Let D be a digraph with edge set E. The *converse* D^* of D is the digraph with vertex set D and edge set $E^* = \{(y, x) : (x, y) \in E\}$.

We will make use of the following well-known fact about digraphs.

Principle of Directional Duality For each property of digraphs, there is a corresponding property obtained by replacing every concept by its converse.

2.3 Results from [1]

We will use a few of the results from [1].

Definition 5 Let S be a binary structure. Define the *graph* of S, denoted by $G(S)$, to be the graph with vertices S, and edges $\{(x, y) : x, y \in S$ so that $x \neq y$ and $(x, y) \in E$ or $(y, x) \in E\}$.

Lemma 1 *If S is a binary structure with $(\mathcal{P})$, then $G(S)$ satisfies $(\mathcal{P})$.*

Definition 6 A graph G is *existentially closed* (or *e.c.*) if it satisfies the condition ($\clubsuit$): for every $n, m \geq 1$, if $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are vertices of G with $\{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_m\} = \varnothing$, then there is a vertex $x \in G$ adjacent to the x_i and to none of the y_j.

An e.c. graph embeds each countable graph; the random graph R is the unique countable e.c. graph; see Section 2.10 of [2] for details.

Proposition 1 *A graph G that satisfies $(\mathcal{P})$ that is neither null nor complete is e.c.*

Definition 7 Let D be a digraph.

1. For $x, y \in D$, $\neg xEy$ if and only if $(x, y) \in D^2 - E$.
2. Let $x \in D$ be a vertex.
   (a) $N_{\varnothing}(x) = \{y \in D : \neg yEx \text{ and } \neg xEy \text{ and } y \neq x\}$.
   (b) $N_o(x) = \{y \in D : \neg yEx \text{ and } xEy\}$. (x, y) is an *out-edge.*
   (c) $N_i(x) = \{y \in D : yEx \text{ and } \neg xEy\}$. (x, y) is an *in-edge.*
   (d) $N_u(x) = \{y \in D : yEx \text{ and } xEy\}$. (x, y) is an *(undirected) edge.*

The following property is an essential part of our classification.

Definition 8 A tournament T *has property* (\$) if for $\square \in \{i, o\}$, and for some $x \in T$, $N_{\square}(x) \neq \varnothing$ then for all $y \in T$, $N_{\square}(y) \neq \varnothing$.

T^{∞} is the generic (or random) tournament and is defined to be the Fraïssé limit of the class of finite tournaments; see specifically Example 1 of Section 3.3 of [2].

Proposition 2 *A countable tournament T is isomorphic to T^{∞} if and only if T satisfies* $(\mathcal{P})$ *and* (\$).

We will assume the reader is familiar with the basic facts about linear orderings and well-orderings. Rosenstein [4] is a good reference for our purposes. The set of natural numbers is denoted ω.

3 The Classification of Tournaments with $(\mathcal{P})$

The following is our main theorem.

Theorem 1 *The countable tournaments with $(\mathcal{P})$ are T^{∞}, $\{\omega^{\alpha}, (\omega^{\alpha})^* : \alpha$ a non-zero countable ordinal$\}$. In particular, there are uncountably many countable tournaments with $(\mathcal{P})$.*

Remark 1 We note that ω^{α} stands for ordinal exponentiation, not cardinal exponentiation.

The proof of Theorem 1 will take the rest of Section 3. To begin the proof, fix D a countable tournament with $(\mathcal{P})$. We consider the following two cases.

1. D satisfies (\$): by Proposition 2, $D \cong T^{\infty}$.
2. D does not satisfy (\$): we first show that D must be a linear order (see Proposition 3). We then show in Theorem 2 that a linear ordering with $(\mathcal{P})$ must be one of $\{\omega^{\alpha}, (\omega^{\alpha})^* : \alpha$ a non-zero countable ordinal$\}$.

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3.1 The Classification of Tournaments with $(\mathcal{P})$

3.1.1 From Tournaments to Linear Orders

Definition 9 Let T be a tournament.

1. A vertex $a \in T$ is a *source* if aEb for all $b \in T - \{a\}$.
2. A vertex $a \in T$ is a *sink* if bEa for all $b \in T - \{a\}$.
3. A vertex $a \in T$ is *special* if it is a source or a sink.

The following lemma is easy but makes our classification possible.

Lemma 2

1. *A tournament has no more than two special points; if it has exactly two special points, there must be exactly one source and one sink.*
2. *A nontrivial tournament has* $(\$)$ *if and only if it has no special points.*

Proof (1) A tournament with more than two special points would have at least two sinks or two sources, which is impossible.

(2) If T has $(\$)$ and $a \in T$ was special, then say $N_i(a) = \varnothing$. But then there is some $b \in T$ so that aEb, so that $N_i(b) \neq \varnothing$, which is a contradiction.

Conversely, assume T does not satisfy $(\$)$. Then for some $a, b \in T$, and some $\square \in \{i, o\}$, $N_\square(a) \neq \varnothing$ and $N_\square(b) = \varnothing$. But then b is special. ■

Proposition 3 *Let T be a countable tournament satisfying $(\mathcal{P})$. If $T \ncong T^\infty$ then T is a linear order.*

Proof If T satisfies $(\$)$, then $T \cong T^\infty$ by Proposition 2.

Assume T does not satisfy $(\$)$. We show that T must be a linear order. By Lemma 2 there are two cases: T has one or two special points.

Case 1 T has one special point.

Without loss of generality, we assume that T has a source 0 (the case when T has a sink will follow by the principle of directional duality). We aim to show that T does not have the intransitive 3-cycle D_3 as an induced subtournament; if we succeed then T is a linear order.

Assume T has D_3 as an induced subtournament. We find a contradiction. Define $S = \{y \in T : yEz \text{ for all } z \in X, \text{ where } X \text{ is an induced subtournament of } T \text{ isomorphic to } D_3\}$.

Claim 1 $S \neq \varnothing$.

We show that $0 \in S$. If not then either there is a z in a 3-cycle so that $zE0$, which is impossible as 0 is a source, or 0 itself is in 3-cycle, which is impossible as D_3 has no source.

Claim 2 S is a linear order.

Otherwise, D_3 embeds in S; let X be an induced subtournament of S isomorphic to D_3. But then $X \subseteq T$, so that for each $x \in X$, xEx (by the definition of S), contradicting irreflexivity.

Let $A = S$, $B = T - S$. If $B = \varnothing$ then D is a linear order by Claim 2 and we have our contradiction. Assume now that $B \neq \varnothing$.

Claim 3 $T \cong T \upharpoonright A$.

If not, as T satisfies $(\mathcal{P})$, then $T \cong T \upharpoonright B$. If so, then B contains a source $0'$; that is, for all $y \in B - \{0'\}$, $0'Ey$. But $0' \notin S$ implies that there is $X \subseteq T$ isomorphic to D_3 so that $0' \in X$ or there is some $y \in X$ so that $yE0'$. By the proof of Claim 2, $X \subseteq B$. As before, as $0'$ is a source in B either case leads to a contradiction.

Claims 2 and 3 contradict our assumption that T has D_3 as an induced subtournament. Hence, in Case 1, T is a linear order with first element 0 and no greatest element. If T has a sink, a similar argument shows that T is a linear order with last element and no first element.

Case 2 T has two special points.

Proceed as in Case 1. T is then a linear order with a first and last element. ■

3.1.2 The Classification of Orders with $(\mathcal{P})$

We classify orders (even the uncountable ones) with $(\mathcal{P})$. We can consider orders as binary structures with a binary relation $\leq$ that is reflexive, anti-symmetric, and transitive; we call these *reflexive orders* to distinguish them from their irreflexive counterparts (see Example 1 above). However, reflexive orders are not true oriented graphs (recall that we forbid loops). Nevertheless, the following result holds for both "irreflexive" and reflexive orders; when the distinction is irrelevant, we refer to either kind of structure simply as an order. In the irreflexive case, $\leq$ means "$<$ or $=$".

The next theorem, in the countable case, will complete the proof of Theorem 1.

Theorem 2 *Let P be an order satisfying $(\mathcal{P})$. Then P is an infinite antichain or P is one of ω^α or $(\omega^\alpha)^*$, where α is a non-zero ordinal.*

Proof An infinite antichain satisfies $(\mathcal{P})$.

Assume P is not an antichain and $|P| = \delta \geq \aleph_0$. For an order P, $G(P)$ is the comparability graph of P. By Lemma 1 and Proposition 1 above, $G(P)$ is e.c. or K_δ; the first case is impossible, as every e.c. graph embeds the 5-cycle C_5. Hence, $G(P) = K_\delta$ so that P is a linear ordering.

Claim 1 P has endpoints.

Otherwise, let $a, b \in P$ with $a < b$. Define $A = \{y \in P : y \geq a\} - \{b\}$, $B = P - A$. But $P \upharpoonright A$ has a least point and $P \upharpoonright B$ has a greatest point, so that neither A nor B is isomorphic to P, violating $(\mathcal{P})$.

By Claim 1, P has either a least point and no greatest point, a greatest point and no least point, or both a least and greatest point.

Case 1 P has a least point 0 and no greatest point.

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We show P is a well-ordering. We use the characterization that P is well-ordered if it has no subordering isomorphic to ω^*. Assume P is not a well-ordering. Define $S = \{x \in P : x < y$ for all $y \in X \subseteq P$ with X isomorphic to $\omega^*\}$.

Claim 2 $S \neq \varnothing$.

We show $0 \in S$. If not, then $0 \geq y$ where y is some element of an infinite descending chain in P, which is a contradiction.

Claim 3 S is well-ordered.

The proof is similar to the proof of Claim 2 of Theorem 3. We show there is no subordering X of S isomorphic to ω^*. Otherwise, say X is a subset of P isomorphic to ω^*. Fix $x \in X$. Then $x < x$, which is a contradiction.

Let $A = S$, $B = P - S$. By Claims 2 and 3 we may assume B is nonempty.

Claim 4 $P \cong P \restriction A$.

If not, then $P \cong P \restriction B$ by $(\mathcal{P})$. If so, B contains a least element $0'$. As $0' \notin S$, there is some $y \in X \subseteq P$ with X isomorphic to ω^* so that $y \leq 0'$. By the proof of Claim 3, $X \subseteq B$. But then there is an infinite descending chain below $0'$ in B so we arrive at a contradiction.

The contradiction shows that P is well-ordered, and hence, isomorphic to an ordinal α.

We now employ Cantor's normal form theorem (see Theorem 3.46 of [4]): there are ordinals $\alpha_1 > \cdots > \alpha_k$ for $k \in \omega - \{0\}$, and $n_1, \ldots, n_k \in \omega - \{0\}$ so that

$$\alpha = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k.$$

Claim 5 $k = 1$.

Otherwise, $k \geq 2$. Let $A_i = \omega^{\alpha_i} n_i$, with $1 \leq i \leq k$. By $(\mathcal{P})$ there is some i so that $P \cong P \restriction A_i$.

Claim 6 $n_i = 1$.

Otherwise, $\alpha = \omega^{\alpha_1} n_i = \omega^{\alpha_1} + \cdots + \omega^{\alpha_1}$ (n_i times). Again by $(\mathcal{P})$ α is isomorphic to some ω^{α_1}.

It remains to show sufficiency; namely, we must show that ω^{α} satisfies $(\mathcal{P})$ for α a nonzero ordinal. We proceed by transfinite induction on $\alpha \geq 1$.

As ω satisfies $(\mathcal{P})$ the induction commences. Let $2 \leq \alpha = \beta + 1$ be a successor ordinal. Then $\omega^{\alpha} = \omega^{\beta}\omega$. Let $\omega^{\alpha} = S_1 \uplus \cdots \uplus S_n$ for $n \geq 2$. We label the ω copies of ω^{β} in ω^{α} as $\{\omega^{\beta}(i) : i \in \omega\}$. For $i \in \omega$, $j \in \{1, \ldots, n\}$ define $S_{ij} = \omega^{\beta}(i) \cap S_j$.

Then for $j \in \{1, \ldots, n\}$

$$S_j = \sum_{i \in \omega} S_{ij}.$$

By the inductive hypothesis ω^{β} satisfies $(\mathcal{P})$; hence, for each $i \in \omega$ there is a $j(i) \in \{1, \ldots, n\}$ so that $S_{ij(i)} \cong \omega^{\beta}$. By the pigeonhole principle for sets, there is some $j \in \{1, \ldots, n\}$ with infinitely many $S_{ij} \cong \omega^{\beta}$.

Recall that for $\beta \geq 1$, $\varepsilon + \omega^{\beta} = \omega^{\beta}$ for $\varepsilon < \omega^{\beta}$. By applying this fact and the fact that the set of blocks equal to ω^{β} is cofinal in $\{S_{ij} : i \in \omega\}$, we have that $S_j \cong \sum_{i \in \omega} \omega^{\beta} = \omega^{\alpha}$.

Now, assume α is a limit ordinal that satisfies $\alpha > \omega$. Then $\omega^\alpha = \sum_{\beta<\alpha} \omega^\beta$. The argument in this case is similar to the case when α is a successor ordinal and so is omitted.

Case 2 P has a greatest point and no least point.

In this case, we find that P is of the form $(\omega^\alpha)^*$. The argument for Case 2 follows from the argument of Case 1, and by directional duality.

Case 3 P has a least element 0 and greatest element ∞.

We find a contradiction. Define $A = S$ as in Case 1 and $B = P - A$. It is immediate that $0 \in A - B$ and $\infty \in B - A$. As in Case 1, A is well-ordered.

By $(\mathcal{P})$ one of $P \restriction A$, $P \restriction B$ is isomorphic to P. If $P \restriction A$ is isomorphic to P, then P is a well-ordering and hence, isomorphic to an ordinal. But then by Case 1, P is of the form ω^α for some non-zero ordinal α contradicting that P has a greatest point.

If $P \restriction B \cong P$, then B has a first-element $0'$; but as $0' \in P - S$, $0' \geq y$ for some y in an isomorphic copy of ω^*. This contradiction finishes the proof. ■

3.1.3 Quasi-Orders with $(\mathcal{P})$

The classification of orders with $(\mathcal{P})$ also supplies a classification of quasi-orders (or pre-orders) with $(\mathcal{P})$. A binary structure is a *quasi-order* if it has a reflexive, transitive edge set. We write $a \leq b$ for $(a, b) \in E$. If we define $a \sim b$ by $a \leq b$ and $b \leq a$, then $\sim$ is an equivalence relation; further, the quasi-ordering of S induces an order on the set of blocks $S/{\sim}$: $[a] \leq [b]$ if and only if $a \leq b$.

Definition 10 A class of binary structures $\mathcal{K}$ is *equipped with an equivalence relation R* if for each $S \in \mathcal{K}$ there is an equivalence relation $R^S \subseteq S^2$ satisfying the following two conditions.

(E1) For $S, T \in \mathcal{K}$ if $f\colon S \to T$ is an isomorphism, then $(x, y) \in R^S$ if and only if $\big(f(x), f(y)\big) \in R^T$.

(E2) For all $S, T \in \mathcal{K}$ with $S \leq T$, $R^S = R^T \cap S^2$.

Lemma 3 *Let S be a member of a class of binary structures equipped with an equivalence relation R. If S has $(\mathcal{P})$, then S has either a single infinite R-block or has only singleton R-blocks.*

Proof If S has a single finite block, then S is finite and so cannot satisfy $(\mathcal{P})$. Assume S has $(\mathcal{P})$, has more than one R-block, and has some block with at least two elements. We find a contradiction.

Case 1 S has n blocks, for $1 < n < \omega$.

Let S have blocks $\{S_i : 1 \leq i \leq n\}$. By $(\mathcal{P})$ some $S \restriction S_i \cong S$, which is a contradiction, as an isomorphism preserves the number of blocks by (E1). Hence, we may assume S has infinitely many blocks.

Case 2 Every block of S is finite.

Fix a block S_i with cardinality $m \geq 2$. Let $A = \{S_i : |S_i| = m\}$, $B = S - A$. If $B = \varnothing$, then each block of S has size m. If $B \neq \varnothing$, then since A is a union of R-blocks and by (E2),

$S \restriction B$ has no block of size m, so by $(\mathcal{P})$, $S \restriction A \cong S$. In either case, each block of S has size m. Now, let C consist of one element from each block of S, with $D = S - C$. Then by (E2) neither $S \restriction C$ nor $S \restriction D$ have blocks of order m, which is a contradiction.

Case 3 S has some blocks finite, some infinite.

Let A be the union of the finite blocks, $B = S - A$. Then neither $S \restriction A$ nor $S \restriction B$ is isomorphic to S, which is a contradiction.

Case 4 S has all blocks infinite.

Let S_i, S_j be distinct infinite blocks. Fix $a \in S_i$, $b \in S_j$. Let $A = \big(S - (S_i \cup \{b\})\big) \cup \{a\}$, $B = S - A$. Then both $S \restriction A$, $S \restriction B$ have singleton blocks by (E2), contradicting our hypothesis. ■

Corollary 1 *The quasi-orders with $(\mathcal{P})$ have either a single infinite $\sim$-block or are reflexive orders (quasi-orders with singleton $\sim$-blocks) with $(\mathcal{P})$.*

Proof If $\mathcal{K}$ is the class of quasi-orders, $\mathcal{K}$ is equipped with the equivalence relation $\sim$. Apply Lemma 3. ■

3.2 Towards a Classification of Oriented Graphs with $(\mathcal{P})$

By Proposition 3.4 of [3] and Lemma 1, if D is a countable oriented graph with $(\mathcal{P})$, $G(D)$ is isomorphic to one of $\overline{K_{\aleph_0}}$, $K_{\aleph_0}$, or R. If $G(D) \cong \overline{K_{\aleph_0}}$, then D is just the countable edgeless oriented graph on $\aleph_0$-many vertices. If $G(D) \cong K_{\aleph_0}$, then D is a tournament, for which we have a complete classification.

Assuming $G(D) \cong R$, then for each $x \in D$, both $N_u(x)$ and $N_\varnothing(x)$ are infinite in $G(D)$. But then $N_i(x) \cup N_o(x)$ and $N_\varnothing(x)$ are each infinite in D. If for each $x \in D$, $N_i(x)$, $N_o(x)$ are nonempty, then we can show that D is isomorphic to the generic oriented graph O (the Fraïssé limit of the finite oriented graphs).

Definition 11 Let D be an oriented graph. D is 1-*e.c.* if for each $x \in D$ and each $\square \in \{\varnothing, i, o\}$, $N_\square(x)$ is nonempty.

The following proposition follows from results in [1].

Proposition 4 *A countable oriented graph D with $(\mathcal{P})$ is* 1*-e.c. if and only if $D \cong O$.*

We do not have an answer to the following problem.

Problem Is there a countable oriented graph D that is not 1-e.c. with $G(D) \cong R$ so that D has $(\mathcal{P})$?

If so, then there is an orientation of the random graph R, distinct from the orientation giving O, with $(\mathcal{P})$.

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