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A CLASS OF PARABOLIC EQUATIONS DRIVEN BY THE MEAN CURVATURE FLOW

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Abstract We study a class of parabolic equations which can be viewed as a generalized mean curvature flow acting on cylindrically symmetric surfaces with a Dirichlet condition on the boundary. We prove the existence of a unique solution by means of an approximation scheme. We also develop the theory of asymptotic stability for solutions of general parabolic problems.

Keywords: mean curvature flow; asymptotic behaviour; stability

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1. Introduction

The motion of surfaces by their mean curvature flow can be described as follows. Let $S(0)$ be a compact, convex and n-dimensional surface without boundary, which is smoothly embedded in $\mathbb{R}^{n+1}$, $n \geq 1$. Let $S(0)$ be locally represented by a diffeomorphism $u_0 : \Omega \to \mathbb{R}^{n+1}$, where $\Omega \subset \mathbb{R}^n$ and $u_0(\Omega) \subset S(0)$. For each $t > 0$ the problem of finding the maps $u(.,t) : \Omega \to \mathbb{R}^{n+1}$ satisfying the equation

$$\begin{cases} u_t(x,t) = -2H(x,t)\nu(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

has solutions $u(.,t)$ which represent locally manifolds $S(t)$. Here, $H(.,t)$ is the mean curvature of $S(t)$ and $\nu(.,t)$ is the unit outward normal vector to $S(t)$ at time t.

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If one assumes that the moving surface has axial symmetry with respect to one of the coordinate axis, say x_1, denoted hereafter by simply x, equation (1) transforms into

$$u_t = \frac{u_{xx}}{1+(u_x)^2} - \frac{n-1}{u}. \tag{2}$$

Local existence for (1) has been proved in [**7**, **10**, **14**]. In [**15**] it has been proved that the equation (1) possesses a smooth solution on a finite time interval $[0,T)$ and that $S(t)$ converges to a single point as $t \to T^-$, see also [**13**]. The equation of a graph parametrized surface moving according to its mean curvature flow with a non-homogeneous Dirichlet boundary condition has a classical solution that converges to a solution of the minimal surface equation as $t \to \infty$, see [**16**]. The curve shortening in the plane has been addressed in [**10**, **12**], as well as convexity properties of the evolving curve. The evolution of curves on a surface has been dealt with in [**2**–**4**]. The mean curvature flow on axially symmetric surfaces without boundary conditions has been studied in [**1**, **24**] and with Dirichlet boundary condition in [**5**, **8**]. Periodic solutions have been found in [**21**]. Formation of singularities of the mean curvature flow have been addressed in [**6**, **17**–**19**, **24**]. Equation (1) is important in material sciences, image processing and differential geometry, see [**11**].

In the present paper, we are interested in studying a more general equation with non-homogeneous Dirichlet boundary condition, namely

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} - \dfrac{N}{u^\alpha}, & (x,t) \in (-a,a) \times (0,T), \\ u(-a,t) = \beta_1, u(a,t) = \beta_2, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [-a,a], \end{cases} \tag{3}$$

where $0 < \alpha \leq 1$ and $N > 0$. The constants

$$0 < \beta_1 < \beta_2 \tag{4}$$

will be precisely determined in Lemma 1.1. The initial datum is non-decreasing and has regularity as

$$u_0 : [-a,a] \to (0,\infty) \text{ belongs to } H_{3+\eta_0} \quad \text{for some } 0 < \eta_0 \leq 1 \text{ and } u_0' \geq 0. \tag{5}$$

The space $H_{3+\eta_0}$ is defined in (18). We also assume the compatibility condition

$$\frac{u_0''(-a)}{1+(u_0'(-a))^2} - \frac{N}{u_0^\alpha(-a)} = 0 \quad \text{and} \quad \frac{u_0''(a)}{1+(u_0'(a))^2} - \frac{N}{u_0^\alpha(a)} = 0. \tag{6}$$

We prove the existence of a solution of (3) by perturbing the equation as follows

$$\begin{cases} u_t = (\Phi(u_x))_x - \dfrac{N}{u^\alpha}, & (x,t) \in (-a,a) \times (0,T), \\ u(-a,t) = \beta_1, u(a,t) = \beta_2, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [-a,a]. \end{cases} \tag{7}$$

We proceed to describe in detail what kind of function Φ we need in (7). Let $\ell > 0$ be a positive constant such that

$$\max_{x\in[-a,a]} u_0'(x) \le \ell$$

and let Φ_ℓ be the function,

$$\Phi_\ell(s) = \int_0^s \phi_\ell(x)\,\mathrm{d}x, \tag{8}$$

where

$$\phi_\ell(s) = \begin{cases} \dfrac{1}{1+s^2}, & |s| \le \ell, \\ p(|s|), & \ell < |s| \le 2\ell, \\ p(2\ell), & |s| > 2\ell, \end{cases} \tag{9}$$

and $p : \mathbb{R} \to \mathbb{R}$ is the polynomial defined by

$$\begin{aligned} p(s) = {} & \frac{1}{2}\frac{-3+\ell^2}{\ell^2(3\ell^4+\ell^6+1+3\ell^2)}s^4 - \frac{4}{3}\frac{3\ell^2-7}{(1+2\ell^2+\ell^4)\ell(1+\ell^2)}s^3 \\ & + \frac{12\ell^2-20}{(1+2\ell^2+\ell^4)(1+\ell^2)}s^2 - \frac{(16\ell^2-16)\ell}{(1+2\ell^2+\ell^4)(1+\ell^2)}s + \frac{1}{6}\frac{51\ell^4-11\ell^2+6}{(1+\ell^2)^3}. \end{aligned}$$

Let $g(s) = (1/(1+s^2))$. The polynomial p satisfies

$$p(\ell) = g(\ell), \qquad p'(\ell) = g'(\ell), \qquad p''(\ell) = g''(\ell), \qquad p'(2\ell) = 0 \quad \text{and} \quad p''(2\ell) = 0.$$

Since $\max_{x\in[-a,a]} u_0'(x) \le \ell$, it follows from the definition of the function Φ_ℓ that the compatibility condition (6) can be rewritten in the form

$$\Phi_\ell'(u_0'(-a))u_0''(-a) - \frac{N}{u_0^\alpha(-a)} = 0 \quad \text{and} \quad \Phi_\ell'(u_0'(a))u_0''(a) - \frac{N}{u_0^\alpha(a)} = 0. \tag{10}$$

Henceforth, Φ and Φ_ℓ denote the same function. If necessary, we write Φ_ℓ to make explicit the dependence on ℓ. We list a few properties of the function Φ. Note that

$$\Phi : \mathbb{R} \to \mathbb{R} \text{ belongs to } \mathcal{C}^3. \tag{11}$$

$$\text{There exists a constant } \gamma > 0 \text{ such that } \gamma \le \Phi'(s) \le 1 \text{ for every } s \in \mathbb{R}. \tag{12}$$

The best constant in (12) is $\gamma = p(2\ell) = (1/6)(3\ell^4 + 5\ell^2 + 6/(1+\ell^2)^3) > 0$. Moreover,

$$\Phi \text{ is an odd function;} \tag{13}$$

$$\Phi' \le 0 \text{ in } (0,\infty); \tag{14}$$

$$\Phi' \ge 0 \text{ in } (-\infty,0); \tag{15}$$

$$\text{and there is a constant } B > 0 \text{ such that } -B \le \Phi''(s) \le B \text{ for every } s \in \mathbb{R}. \tag{16}$$

The best constant in (16) is $B = |\Phi''(0)| = |-2| = 2$.

We remark that equations (2) and (3) are not exactly particular cases of (7), since one has $\arctan'(s) = 1/(1+s^2) \to 0$ as $|s| \to \infty$, and $\Phi'(s)$ does not tend to 0 as $|s| \to \infty$.

We obtain a solution of problem (3) by, roughly speaking, cutting off Φ. In this way we first show the general Theorem 1.2, which ensures existence of solution for (7). We show that the solution u of (7) has bounded derivative $|u_x| < \ell$ and, since $\Phi(s) = \arctan(s)$ for $|s| < \ell$, then u is also a solution of (3), and Theorem 1.7 is proved.

By a solution of (3) and (7) we mean a function $u : [-a, a] \times [0, T) \to \mathbb{R}$ belonging to

$$\mathcal{C}^{2,1}((-a, a) \times (0, T)) \cap \mathcal{C}^0([-a, a] \times [0, T))$$

for some $0 < T < \infty$ which satisfies the problem for every $(x, t) \in (-a, a) \times (0, T)$.

We denote by Γ the parabolic boundary

$$\Gamma = ((-a, a) \times \{0\}) \cup (\{-a, a\} \times [0, T)). \tag{17}$$

The norm of a point in $(-a, a) \times (0, T)$ is denoted by $|(x, t)| = \max\{|x|, |t|^{1/2}\}$. It is worth defining the following spaces when $0 < \eta \leq 1$ and $k \geq 1$:

$$\begin{aligned}&\mathcal{C}^{k,[k/2]}((-a, a) \times (0, T))\\ &\quad =\{u : (-a, a) \times (0, T) \to \mathbb{R} : \exists D_x^i D_t^j u \in \mathcal{C}^0((-a, a) \times (0, T)) \text{ for } i + 2j \leq k\}\end{aligned}$$

and

$$H_{k+\eta}((-a, a) \times (0, T)) = \{u \in \mathcal{C}^{k,[k/2]}((-a, a) \times (0, T)) : |u|_{k+\eta} < \infty\}, \tag{18}$$

where $[k/2]$ is the integer part of $k/2$,

$$|u|_{k+\eta} = \sum_{i+2j\leq k} \sup_{(-a,a)\times(0,T)} |D_x^i D_t^j u| + [u]_{k+\eta} + \langle u\rangle_{k+\eta}$$

with

$$[u]_{k+\eta} = \sum_{i+2j=k} \sup_{(x,t)\neq(y,s)} \frac{|D_x^i D_t^j u(x, t) - D_x^i D_t^j u(y, s)|}{|(x, t) - (y, s)|^\eta}$$

and

$$\langle u\rangle_{k+\eta} = \sum_{i+2j=k-1} \sup_{(x,t)\neq(y,s)} \frac{D_x^i D_t^j u(x, t) - D_x^i D_t^j u(y, s)}{|t - s|^{(1+\eta)/2}}.$$

The corresponding spaces of functions $u(x)$ defined on an interval $D = [-a, a]$ are the following

$$H_{2+\eta} = \{u : |u|^D_{2+\eta} < \infty\}, \tag{19}$$

where

$$|u|^D_{2+\eta} = |u|^D_{1+\eta} + |u_{xx}|^D_\eta$$

with

$$|u|^D_{1+\eta} = |u|^D_\eta + |u_x|^D_\eta$$

and

$$|u|^D_\eta = \sup_D |u(x)| + \sup_{x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\eta}.$$

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Analogously, we define

$$H_{3+\eta}(D) = \{u : |u|^D_{3+\eta} < \infty\}$$

where

$$|u|^D_{3+\eta} = |u|^D_{3+\eta} + |u_{xxx}|^D_\eta.$$

We state next that (7) has a stationary solution V satisfying

$$\begin{cases} (\Phi(V_x))_x - \dfrac{N}{V^\alpha} = 0, & x \in (-a, a), \\ V(x) > 0, & x \in [-a, a], \\ V_x(-a) = 0, \\ V(-a) = \beta_1, & V(a) = \beta_2. \end{cases} \tag{20}$$

Lemma 1.1. *Let $\beta_2 > 0$ be a positive constant and Φ satisfying (8), (9) and (12).*

(a) *If $\alpha = 1$, there exists $a_1 > 0$ such that for $0 < a < a_1$, there are exactly two constants β_1 and β_1' with $\beta_1 < \beta_2$, $\beta_1' < \beta_2$ and $\beta_1' < \beta_1$ such that (20) has exactly one solution W with boundary values $W(-a) = \beta_1'$ (instead of β_1) and $W(a) = \beta_2$, and exactly one solution V corresponding to β_1 and β_2, that is, $V(-a) = \beta_1$ and $V(a) = \beta_2$. Moreover, $W < V$ in $(-a, a)$, $W(-a) = \beta_1' < \beta_1 = V(a)$ and $W(a) = \beta_2 = V(a)$. In addition, $0 \leftarrow \beta_1'(a) < \beta_1(a) \rightarrow \beta_2$ as $a \rightarrow 0^+$.*
If $a = a_1$, there is exactly one $\beta_1 < \beta_2$ such that the problem (20) has exactly one solution V.
If $a > a_1$ there is no solution of (20).

(b) *If $0 < \alpha < 1$, there exists $a_2 > 0$ such that for $0 < a < a_2$, there is exactly one β_1 such that the problem (20) has exactly one solution V.*
In all items, β_1 and β_1' depend on a, β_2 and α. The functions V and W belong to $C^2[-a, a]$.

Throughout the paper we will work with the stationary solution V.
We suppose a condition relating u_0 and V,

$$u_0(x) \geq V(x) \quad \text{for every } x \in [-a, a]. \tag{21}$$

Theorem 1.2 deals with the existence of a solution for the general problem (7), which is obtained by proving the existence of a solution for an auxiliary problem, see Lemma 1.3 below. The regularity of the solution follows from Lemma 1.4.

Theorem 1.2. *Let $T > 0$, $\beta_2 > 0$. Assume (5), (6) and (21). Suppose that Φ satisfies (8)–(16) and that a, β_1 and V are as in Lemma 1.1. Then problem (7) has a unique positive solution.*

To prove Theorem 1.2, we first obtain a solution of the following auxiliary problem with the nonlinearity $f(u)$ instead of $-N/u^{\alpha}$ which is singular at $u=0$. Let

$$\begin{cases} u_t = \Phi'(u_x)u_{xx} + f(u), & (x,t) \in (-a,a)\times(0,\infty), \\ u(-a,t)=\beta_1, u(a,t)=\beta_2, & t>0, \\ u(x,0)=u_0(x), & x\in[-a,a]. \end{cases} \tag{22}$$

We define $\theta = \min_{x\in[-a,a]} V(x)$, $m=(\theta/4)$ and

$$f(z) = \begin{cases} -\dfrac{N}{z^{\alpha}}, & z \geq m, \\ \dfrac{N}{(2m-z)^{\alpha}} - \dfrac{2N}{m^{\alpha}}, & z<m. \end{cases} \tag{23}$$

Note that the function $f:\mathbb{R}\to\mathbb{R}$ is $\mathcal{C}^1$, $f(z)<0$ and $f'(z)\geq 0$, for every $z\in\mathbb{R}$.

Lemma 1.3. *Let $T>0$ and $\beta_2>0$ be positive constants. Suppose that Φ satisfies (8)–(16) and that a,β_1 and V are as in Lemma 1.1. Assume that the initial datum u_0 satisfies (5), (6) and (21). Then the problem (22) possesses a solution belonging to $\mathcal{C}^{2,1}((-a,a)\times(0,T))\cap\mathcal{C}^0([-a,a]\times[0,T))\cap H_{3+\eta}$ for some $0<\eta\leq 1$.*

To prove Lemma 1.3 we take advantage of the general theory of the parabolic equations and use a combination of the existence and the regularity (Theorem 14.24 and Lemma 14.11, respectively, both from the book [**20**, pp. 371, 382]). For the sake of completeness we state it below for future reference.

Lemma 1.4. *Let $F:(-a,a)\times(0,T)\times\mathbb{R}^3\to\mathbb{R}$ be a $\mathcal{C}^1$ function such that:*

(i) *there are constants k and c such that;*

$$zF(x,t,z,0,0)\leq k|z|^2+c \quad \text{for every } (x,t)\in(-a,a)\times(0,T),\ z\in\mathbb{R};$$

(ii) *there are constants $a_0>0$ and $a_1>0$ such that;*

$$a_0 \leq F_r(x,t,z,p,r)\leq a_1 \quad \text{for every } (x,t)\in(-a,a)\times(0,T),\ (z,p,r)\in\mathbb{R}^3;$$

(iii) *for every $K\geq 0$, there are constants $b_1=b_1(K)$, $b_2=b_2(K)$ and $0<\Theta\leq 1$ such that;*

$$|F(x,t,z,p,r)-F(y,s,w,q,r)|\leq\big[|(x,t)-(y,s)|+|z-w|+|p-q|\big]^{\Theta}\big[b_1+b_2|r|\big]$$

holds for every $(x,t),(y,s)$ in $(-a,a)\times(0,T)$ and $|z|+|w|+|p|+|q|\leq K$.

Then, for every $\phi\in H_{3+\eta}$, there is a solution $u\in\mathcal{C}^{2,1}((-a,a)\times(0,T))\cap\mathcal{C}^0([-a,a]\times[0,T))$ of

$$\begin{cases} -u_t + F(x,t,u,u_x,u_{xx}) = 0 & \text{in } (-a,a)\times(0,T), \\ u=\phi & \text{in } \Gamma. \end{cases} \tag{24}$$

Moreover, recall (6): if $-\phi_t+F(X,\phi,\phi_x,\phi_{xx})=0$ on $\{-a,a\}\times\{0\}$, then $u\in H_{3+\sigma}$ for some $0<\sigma\leq 1$ determined only by b_0 and b_1.

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The following two lemmas help us to estimate u_x from above and from below, where u is a solution of (3).

Lemma 1.5. *Suppose that Φ satisfies (8), (9), (11) and (12) and that a, β_1, β_2 and V are as in Lemma 1.1, u_0 is continuous and satisfies (21). If $u(x,t)$ is a solution of (7), then $u(x,t) \geq V(x)$ for every $(x,t) \in [-a,a] \times [0,T)$.*

Lemma 1.6. *Suppose that Φ satisfies (8)–(16) and that a, β_1, β_2 and V are as in Lemma 1.1, and u_0 satisfies (5) and (21). Let $u(x,t)$ be a solution of (7), then $u(x,t)_x \geq 0$ for every $(x,t) \in [-a,a] \times [0,T)$.*

We state the existence of a solution for problem (3).

Theorem 1.7. *Let $T > 0$ and $\beta_2 > 0$. Suppose that Φ satisfies (8)–(16) and that a, β_1 and V are as in Lemma 1.1. If u_0 satisfies (5), (6) and (21), then the problem (3) possesses a unique positive solution.*

In the sequel we state the stability of the stationary solutions of (7), in reality they are solutions of (20). Recall the values of γ and B from (12) and (16). Suppose that

$$\text{there exists } M > 0 \text{ and } \lambda > 0 \text{ such that } BM\lambda - \gamma\lambda^2 \leq -4 \tag{25}$$

and for these fixed M and λ we assume that

$$0 < a \leq \frac{1}{3\lambda} \ln\left(\frac{\beta_1(a)^{\alpha+1}}{4^{\alpha+1}\alpha N} + 1\right). \tag{26}$$

Note that such a choice of a is possible, since $\beta_2 > 0$ and by Lemma 1.1 there is $\beta_1(a) < \beta_2$ such that $\lim_{a\to 0^+} \beta_1(a) = \beta_2$.

The stationary solution V is a stable equilibrium, that is, u is asymptoticaly stable. The conclusion follows from the theory developed in § 4 for a general equation of the form $u_t = h(x,t,u,(\partial u/\partial x_i),(\partial^2 u/\partial x_i \partial x_j))$. The results are of independent interest and useful to deal with other parabolic problems beyond the scope of the present paper.

Theorem 1.8. *Let Φ satisfying (8)–(16) and a, β_1 and V as in Lemma 1.1, λ as in (25). Suppose (26) and additionally by (21) that $u_0(-a) = V(-a) = \beta_1$ and $u_0(a) = V(a) = \beta_2$. If u is the solution of (7), recall Theorem 1.2, then*

$$\lim_{t\to\infty} u(x,t) = V(x) \quad \textit{uniformly for } x \in [-a,a],$$

that is, V is a stable equilibrium of (7).

Corollary 1.9. *Let Φ satisfy (8)–(16), with a, β_1 and V as in Lemma 1.1, and λ as in (25). Suppose (26) and additionally by (21) that $u_0(-a) = V(-a) = \beta_1$ and $u_0(a) = V(a) = \beta_2$. If u is the solution of (3), recall Theorem 1.7, then*

$$\lim_{t\to\infty} u(x,t) = V(x) \quad \textit{uniformly for } x \in [-a,a],$$

that is, V is an stable equilibrium of (3).

Remark 1.10. Theorems 1.2 and 1.8, and Lemmas 1.3, 1.5 and 1.6 are true in a more general setting. For instance, assume Φ is a general function, not necessarily the one given by (8)–(9), and satisfies (11)–(16) with given γ and B. Compare with (6) and (10) – the hypotheses (5) and (21) are also needed, as well as the new compatibility conditions

$$\Phi'(u_0'(-a))u_0''(-a) - \frac{N}{u_0^\alpha(-a)} = 0 \quad \text{and} \quad \Phi'(u_0'(a))u_0''(a) - \frac{N}{u_0^\alpha(a)} = 0.$$

Note the following remarks and open problems.

If $0 < \beta_1' < \beta_2$ instead of (4) and if $u_0 \geq W$ in (21), recall Lemma 1.1, then both Theorems 1.2 and 1.7 are true, and the proofs remain the same. Theorem 1.8 is an open question in this situation. The stability assumption (26) is false with β_1 replaced by β_1', since β_1' tends to zero as $a \to 0$.

Assume that $0 < \beta_1' < \beta_2$ in place of (4) and that $W \leq u_0 \leq V$ in (21). Then W is a subsolution and V is a supersolution of (7), so Lemmas 3.1–3.3 apply, and thus there is a solution u of (7) with $W < u < V$ for $x \in (-a, a)$ and $t > 0$. The proofs of Lemmas 1.5 and 1.6 can be performed in the same way; thus there will be a solution of (3). In conclusion Theorems 1.2 and 1.7 hold. The stability stated in Theorem 1.8 is an open question.

Suppose that (4) and $W \leq u_0 \leq V$ in (21). Notice that W is a subsolution and V is a supersolution of (7). Lemmas 3.1–3.3 apply. Hence there is a solution u of (7) with $W < u < V$ for $x \in (-a, a)$ and $t > 0$. The asymptotic stability of u is an open problem. It is not clear now whether Lemmas 1.5 and 1.6 remain true, thus the existence of a solution of (3) as well as its stability are open questions. In synthesis, Theorem 1.2 holds, but Theorems 1.7 and 1.8 are open problems.

There is also the open question of the existence of solution u of problems (3) and (7) whenever u_0 is greater or equal to neither V nor W; for instance, without (21) we may have $W \leq u_0$, $V \leq u_0$ or no order relation between u_0, W and V. There also remains the open question of the stability of a possible solution u in such situations.

The outline of the paper is as follows.

In § 2 we prove Lemma 1.1 about the existence of stationary solution V.

Section 3 is devoted to the existence of solutions of the parabolic problems. We establish the existence of ordered sub and supersolutions (Lemmas 3.1–3.3). We prove Lemma 1.3 and in the sequel we prove Theorem 1.2, the existence of a solution to (7). To recover the solution of problem (3) we prove Lemmas 1.5 and 1.6, and we accomplish our aim with the proof of Theorem 1.7.

In § 4 we prove general results on stability of stationary solutions of nonlinear parabolic problems (Lemma 4.1, Theorem 4.2 and Corollary 4.3).

We apply the results of § 4 in § 5; there, we address the stability of the stationary solution of the problems (3) and (7), then prove Theorem 1.8 and Corollary 1.9. We conclude § 5 with a remark on an estimate from below for a solution of (3) or (7), see Proposition 5.1.

2. Existence of the stationary solution

In this section we only need the $\mathcal{C}^2$ regularity of the function Φ defined by (8) and (9).

Proof of Lemma 1.1.
(a) In this case $\alpha = 1$. First we show that

$$\begin{cases} (\Phi(V_x))_x - \dfrac{N}{V} = 0, \quad x \in (-a, a), \\ V_x(-a) = 0, \\ V(a) = \beta_2, \end{cases} \tag{27}$$

has a solution $V > 0$ in $[-a, a]$. Note that a solution of (27) is convex and $V_x(x) > 0$ for every $x \in (-a, a)$, then $V(-a) < V(a)$.

Define

$$\Psi(r) = \int_0^r \eta \Phi'(\eta)\, \mathrm{d}\eta. \tag{28}$$

Observe that the condition (12) implies that

$$\lim_{r\to\infty} \Psi(r) = \infty \tag{29}$$

and by (9) we conclude that

$$\int_0^{\beta_2} \frac{1}{\Psi^{-1}(\eta)}\, \mathrm{d}\eta < \infty. \tag{30}$$

Let $\beta_1 = V(-a)$. Then V is a solution of (27) if, and only if,

$$\Psi(V_x(x)) = N \ln\left(\frac{V(x)}{\beta_1}\right). \tag{31}$$

Equivalently,

$$\int_{\beta_1}^{V(x)} \frac{1}{\Psi^{-1}(N \ln(\eta/\beta_1))}\, \mathrm{d}\eta = x + a.$$

Then, the number of solutions of (27) is equal to the number of solutions β_1 of

$$\int_{\beta_1}^{\beta_2} \frac{1}{\Psi^{-1}(N \ln(\eta/\beta_1))}\, \mathrm{d}\eta = 2a. \tag{32}$$

Owing to the change of variable $y = N \ln(\eta/\beta_1)$ and (30), equation (32) becomes

$$\frac{\beta_1}{N} \int_0^{N \ln(\beta_2/\beta_1)} \frac{\mathrm{e}^{y/N}}{\Psi^{-1}(y)}\, \mathrm{d}y = 2a. \tag{33}$$

Setting $z = (\beta_2/\beta_1)$, define $H(z) = \int_0^{N \ln(z)} (\mathrm{e}^{y/N}/\Psi^{-1}(y))\, \mathrm{d}y$. Finding a solution $\beta_1 \in (0, \beta_2)$ of (33) is equivalent to find $z \in (1, \infty)$ satisfying

$$H(z) = \frac{2aN}{\beta_2} z. \tag{34}$$

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The function $H(z)$ is strictly increasing and strictly concave in z, since

$$H'(z) = \frac{N}{\Psi^{-1}(N\ln(z))} > 0, \tag{35}$$

and, by the fact that $\Psi' > 0$ implies $(\Psi^{-1})' > 0$, one obtains

$$H''(z) = \frac{-N[\Psi^{-1}(N\ln(z))]'}{[\Psi^{-1}(N\ln(z))]^2} = \frac{-N^2}{z}\frac{(\Psi^{-1})'(N\ln(z))}{[\Psi^{-1}(N\ln(z))]^2} < 0.$$

It follows from (28) that $\lim_{z\to\infty}\Psi^{-1}(N\ln(z)) = \infty$, consequently $\lim_{z\to\infty} H'(z) = 0$ and there exists a unique a_1 such that the line $y = (2a_1N/\beta_2)z$ is a tangent to the graph of H, since $H'' < 0 < H'$.

- For $a < a_1$, the equation $H(z) = (2aN/\beta_2)z$ has exactly two solutions, $z_1(a)$ and $z_2(a)$, with $z_1(a) < z_2(a)$. Therefore, $\beta_1 = (\beta_2/z_1(a))$ and $\beta_1' = (\beta_2/z_2(a))$, with $\beta_1' < \beta_1$, are the unique two solutions of (32). It is simple to verify that $z_1(a)$ is increasing for $a > 0$ while $z_2(a)$ is decreasing for $a > 0$. Moreover, $\lim_{a\to 0^+} z_1(a) = 1$ and $\lim_{a\to 0^+} z_2(a) = \infty$, in particular, $1 < z_1(a) < z_2(a)$. Since (31) and (34) are equivalent, there are two stationary solutions $V(x)$ and $W(x)$ such that $V(-a) = (\beta_2/z_1(a))$, $W(-a) = (\beta_2/z_2(a))$ and $\beta_1 = V(-a) > W(-a) = \beta_1'$.

  We claim that $V(-a) > W(-a)$ implies $V(x) > W(x)$ for every $x \in (-a, a)$. Assume by contradiction that $V(x) - W(x)$ has at least two zeros in $(-a, a]$, and note that $V(a) = W(a) = \beta_2$. Let $b \in (-a, a]$ be the second zero, that is, $V(b) = W(b)$. Then $V_x(b) \geq W_x(b)$. Therefore, by (28), ψ is monotone increasing and

  $$\Psi(V_x(b)) \geq \Psi(W_x(b)),$$

  hence

  $$\Psi(V_x(b)) - N\ln(V(b)) \geq \Psi(W_x(b)) - N\ln(W(b)).$$

  Since the left- and right-hand sides of the last inequality are constant functions as b varies in $[-a, a]$, again by (28), we obtain

  $$-N\ln(V(-a)) \geq -N\ln(W(-a)),$$

  since $N > 0$ we get

  $$V(-a) \leq W(-a)$$

  in contrast to $V(-a) > W(-a)$, a contradiction.

- For $a = a_1$, the equation $H(z) = (2a_1N/\beta_2)z$ has exactly one solution $z_0(a_1)$, and $\beta_1 = (\beta_2/z_0(a_1))$ is the unique solution of (32).

- For $a > a_1$, the equation $H(z) = (2aN/\beta_2)z$ has no solution.

(b) In this situation, $0 < \alpha < 1$. We are going to show that

$$\begin{cases} (\Phi(V_x))_x - \dfrac{N}{V^\alpha} = 0, & x \in (-a, a), \\ V_x(-a) = 0, \\ V(a) = \beta_2, \end{cases} \tag{36}$$

has a solution $V > 0$ in $[-a, a]$. Recall the situation (28) and properties (29) and (30).

As in the case $\alpha = 1$, a solution of (36) is convex and $V_x(x) > 0$ in $(-a, a)$, then $V(-a) < V(a)$. Let $\beta_1 = V(-a)$. Then V is a solution of (36) if, and only if,

$$\Psi(V_x(x)) = \frac{N}{1-\alpha}(V^{1-\alpha}(x) - \beta_1^{1-\alpha}),$$

which is equivalent to

$$\int_{\beta_1}^{V(x)} \frac{1}{\Psi^{-1}[(N/1-\alpha)(\eta^{1-\alpha} - \beta_1^{1-\alpha})]}\, \mathrm{d}\eta = x + a.$$

Then, the number of solutions of (36) is equal to the number of solutions β_1 of

$$\int_{\beta_1}^{\beta_2} \frac{1}{\Psi^{-1}[(N/1-\alpha)(\eta^{1-\alpha} - \beta_1^{1-\alpha})]}\, \mathrm{d}\eta = 2a. \tag{37}$$

Owing to the change of variable $y = (N/1-\alpha)(\eta^{1-\alpha} - \beta_1^{1-\alpha})$, (29) and (30), (37) becomes

$$\frac{1}{N}\int_0^{(N/(1-\alpha))(\beta_2^{1-\alpha} - \beta_1^{1-\alpha})} \frac{(((1-\alpha)y/N) + \beta_1^{1-\alpha})^{\alpha/(1-\alpha)}}{\Psi^{-1}(y)}\, \mathrm{d}y = 2a.$$

Setting $z = (N/(1-\alpha))(\beta_2^{1-\alpha} - \beta_1^{1-\alpha})$, we count the number of solutions $z \in (0, \infty)$ of

$$H(z) = 2a, \tag{38}$$

where we define $H(z) = \int_0^z ((((1-\alpha)y/N) + \beta_1^{1-\alpha})^{\alpha/(1-\alpha)}/N\Psi^{-1}(y))\, \mathrm{d}y$. Note that $H(z)$ is strictly increasing in z, since

$$H'(z) = \frac{(((1-\alpha)z/N) + \beta_1^{1-\alpha})^{\alpha/(1-\alpha)}}{N\Psi^{-1}(z)} > 0. \tag{39}$$

It follows from (28) and (12) that $(\gamma z^2/2) \leq \Psi(z) \leq (z^2/2)$, hence $\sqrt{2z} \leq \Psi^{-1}(z) \leq \sqrt{2/\gamma z}$ and $\lim_{z\to\infty} \Psi^{-1}(z) = \infty$, consequently, by (39), we have the following situations.

- If $(\alpha/1-\alpha) > (1/2)$, that is, $(1/3) < \alpha < 1$, we have

  $$H'(z) \geq \frac{(((1-\alpha)z/N) + \beta_1^{1-\alpha})^{\alpha/(1-\alpha)}}{N\sqrt{2/\gamma z}}$$

  and

  $$\lim_{z\to+\infty} H'(z) = +\infty.$$

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- If $(\alpha/1-\alpha) < (1/2)$, that is, $0 < \alpha < (1/3)$, we have

$$H'(z) \leq \frac{(((1-\alpha)z/N) + \beta_1^{1-\alpha})^{\alpha/(1-\alpha)}}{N\sqrt{2z}}$$

and

$$\lim_{z\to+\infty} H'(z) = 0.$$

- If $(\alpha/1-\alpha) = (1/2)$, that is, $\alpha = (1/3)$, we have

$$\frac{((2z/3N) + \beta_1^{2/3})^{1/2}}{N\sqrt{2/\gamma z}} \leq H'(z) \leq \frac{((2z/3N) + \beta_1^{2/3})^{1/2}}{N\sqrt{2z}},$$

in particular

$$H(z) \geq \frac{\gamma^{1/2}}{3^{1/2}N^{3/2}}z, \quad \forall z > 0.$$

Hence there exists a_2 such that for $a \leq a_2$, the equation $H(z) = 2a$ has a unique solution $z(a)$. Therefore, $\beta_1 = (\beta_2^{1-\alpha} - ((1-\alpha)/N)z(a))^{1/(1-\alpha)}$ is the unique solution of (37). □

3. Solution of the parabolic problems

We obtain the existence of ordered sub and supersolutions.

Lemma 3.1. *Let $T > 0$ and $\beta_2 > 0$. Suppose that Φ, a, β_1 and V are as in Lemma 1.1 and u_0 satisfies (21). Then $\overline{u}(x,t) = M = \max_{x\in[-a,a]} u_0(x)$ is a supersolution of (7) and $\underline{u}(x,t) = V(x)$ is a subsolution of (7) for every $(x,t) \in [-a,a]\times[0,T]$.*

Proof. We have

$$\overline{u}_t = 0 = \Phi'(\overline{u}_x)\overline{u}_{xx} \geq \Phi'(\overline{u}_x)\overline{u}_{xx} - \frac{N}{\overline{u}^\alpha},$$

$$\overline{u}(-a,t) = M \geq u_0(-a),$$

$$\overline{u}(a,t) = M \geq u_0(a)$$

and

$$\overline{u}(x,0) = M \geq u_0(x).$$

Therefore $\overline{u}$ is a supersolution.

By Lemma 1.1 we obtain

$$\underline{u}_t = 0 = \Phi'(\underline{u}_x)\underline{u}_{xx} - \frac{N}{\underline{u}^\alpha},$$

$$\underline{u}(-a,t) = V(-a) = \beta_1 = u_0(-a),$$

$$\underline{u}(a,t) = V(a) = \beta_2 = u_0(a)$$

and by (21)

$$\underline{u}(x,0) = V(x) \leq u_0(x). \qquad \square$$

Therefore $\underline{u}$ is a subsolution.

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The comparison between a subsolution and supersolution of (22) is stated next.

Lemma 3.2. *Let $T > 0$ and Φ satisfying (8), (9), (11) and (12). If $\underline{u}, \overline{u} \in \mathcal{C}^{2,1}((-a,a) \times (0,T)) \cap \mathcal{C}^0([-a,a] \times [0,T)) \cap H_{2+\eta}$, where $0 < \eta \leq 1$, are a subsolution and a supersolution of (22) with u_0 continuous, respectively, then $\underline{u} \leq \overline{u}$ for every $(x,t) \in [-a,a] \times [0,T]$.*

Proof. It follows from the hypotheses that

$$\begin{cases} \underline{u}_t \leq \Phi'(\underline{u}_x)\underline{u}_{xx} + f(\underline{u}), \quad (x,t) \in (-a,a) \times (0,\infty), \\ \underline{u}(-a,t) \leq u_0(-a), \, \underline{u}(a,t) \leq u_0(a), \quad t > 0, \\ \underline{u}(x,0) \leq u_0(x), \quad x \in [-a,a], \end{cases} \tag{40}$$

and

$$\begin{cases} \overline{u}_t \geq \Phi'(\overline{u}_x)\overline{u}_{xx} + f(\overline{u}), \quad (x,t) \in (-a,a) \times (0,\infty), \\ \overline{u}(-a,t) \geq u_0(-a), \, \overline{u}(a,t) \geq u_0(a), \quad t > 0, \\ \overline{u}(x,0) \geq u_0(x), \quad x \in [-a,a]. \end{cases} \tag{41}$$

Defining $w = \overline{u} - \underline{u}$ we obtain

$$\begin{cases} w_t \geq \Phi'(\overline{u}_x)\overline{u}_{xx} - \Phi'(\underline{u}_x)\underline{u}_{xx} + f(\overline{u}) - f(\underline{u}), \quad (x,t) \in (-a,a) \times (0,\infty), \\ w(-a,t) \geq 0, \, w(a,t) \geq 0, \quad t > 0, \\ w(x,0) \geq 0, \quad x \in [-a,a]. \end{cases}$$

By the first equation and the mean value theorem, one has

$$\begin{aligned} w_t &\geq \Phi'(\overline{u}_x)\overline{u}_{xx} - \Phi'(\underline{u}_x)\underline{u}_{xx} + f(\overline{u}) - f(\underline{u}) \\ &= \Phi'(\overline{u}_x)w_{xx} + \underline{u}_{xx}(\Phi'(\overline{u}_x) - \Phi'(\underline{u}_x)) + f(\overline{u}) - f(\underline{u}) \\ &= \Phi'(\overline{u}_x)w_{xx} + \underline{u}_{xx}\Phi''(\xi)w_x + f'(\widetilde{u})w, \end{aligned}$$

where ξ is between $\overline{u}_x$ and $\underline{u}_x$, and $\widetilde{u}$ is between $\overline{u}$ and $\underline{u}$. Therefore,

$$w_t - \Phi'(\overline{u}_x)w_{xx} - \underline{u}_{xx}\Phi''(\xi)w_x - f'(\widetilde{u})w \geq 0. \tag{42}$$

The maximum principle from the book [**22**, Lemma 2.1, p. 54] implies $w \geq 0$ and

$$\underline{u} \leq \overline{u} \quad \text{for every } (x,t) \in [-a,a] \times [0,T]. \qquad \square$$

Lemma 3.3. *Let $T > 0$, $\beta_2 > 0$ and Φ satisfy (8), (9), (11) and (12). Suppose that a, β_1 and V are as in Lemma 1.1 and $u_0(x)$ is continuous and satisfies (21), with $\overline{u}$ and $\underline{u}$ as in Lemma 3.1. Let $u \in \mathcal{C}^{2,1}((-a,a) \times (0,T)) \cap \mathcal{C}^0([-a,a] \times [0,T)) \cap H_{2+\eta}$, where $0 < \eta \leq 1$, be a solution of (22). Then*

$$M = \overline{u}(x,t) \geq u(x,t) \geq \underline{u}(x,t) = V(x) \quad \textit{for every } (x,t) \in [-a,a] \times [0,T).$$

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Proof. By Lemma 3.1, $\overline{u}(x,t) = M = \max_{x\in[-a,a]} u_0(x)$ is a supersolution of (22) and $\underline{u}(x,t) = V(x)$ is a subsolution of (22).

It follows from Lemma 3.2 that

$$\begin{aligned} \underline{u}(x,t) &\le u(x,t) \\ &\le \overline{u}(x,t) \quad \text{for every } (x,t) \in [-a,a]\times[0,T). \end{aligned} \tag{43}$$

□

Proof of Lemma 1.3. We rewrite below the problem (22) in an adequate form

$$\begin{cases} v_t = F(x,t,v,v_x,v_{xx}), \quad (x,t) \in (-a,a)\times(0,T), \\ v(-a,t) = \beta_1,\ v(a,t) = \beta_2, \quad t \in (0,T), \\ v(x,0) = u_0(x), \quad x \in [-a,a], \end{cases} \tag{44}$$

where

$$F(x,t,z,p,r) = \Phi'(p)r + f(z), \tag{45}$$

with Φ and f given by (8) and (23), respectively. We prove the existence of a solution for problem (44) by means of Lemma 1.4.

In fact, condition (i) of Lemma 1.4 is fulfilled by taking $k = 1/2$ and $c = \sup_{\mathbb{R}} |f|^2/2$, hence

$$zF(x,t,z,0,0) = zf(z) \le k|z|^2 + b_1, \quad \forall k > 0.$$

For condition (ii) we take $a_0 = \gamma$ and $a_1 = 1$.

The condition (iii) is satisfied if one takes $\Theta = 1$, $b_1 = \sup_{z\in\mathbb{R}} |f'(z)|$ and $b_2 = \sup_{s\in\mathbb{R}} |\Phi''(s)|$. Indeed,

$$|F(x,t,z,p,r) - F(y,s,w,q,r)| \le |(\Phi'(p) - \Phi'(q))r| + |f(z) - f(w)|.$$

By the mean value theorem applied to Φ' and f we obtain

$$|F(x,t,z,p,r) - F(y,s,w,q,r)| \le \sup_{\zeta\in\mathbb{R}} |\Phi''(\zeta)||p-q||r| + \sup_{z\in\mathbb{R}} |f'(z)||z-w|,$$

that is,

$$|F(x,t,z,p,r) - F(y,s,w,,q,r)| \le b_2|p-q||r| + b_1|z-w|$$

and then

$$|F(x,t,z,p,r) - F(y,s,w,q,r)| \le (|(x,t)-(y,s)| + |z-w| + |p-q|)(b_1 + b_2|r|).$$

Thus there is a solution of (44), which is equivalent to (22). The regularity follows from Lemma 1.4 and (6). The theorem is proved. □

Proof of Theorem 1.2. By Lemma 1.3, the problem (22) has a solution $u \in \mathcal{C}^{2,1}((-a,a)\times(0,T)) \cap \mathcal{C}^0([-a,a]\times[0,T)) \cap H_{3+\eta}$, for some $0 < \eta \le 1$.

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By Lemma 3.3, the solution obtained in Lemma 1.3 satisfies $m = (\theta/4) \leq V \leq u \leq M$. Therefore, the solution of (22) satisfies

$$m = \frac{\theta}{4} \leq \underline{u} \leq u \leq M. \tag{46}$$

It follows from the definition of f in (23) that $f(u) = -(N/u^\alpha)$. Therefore, u is a solution of problem (7). Uniqueness follows from Lemma 3.2. □

Proof of Lemma 1.5. Let $w(x,t) = u(x,t) - V(x)$. Since $V(x)$ is a stationary solution of (7), by the mean value theorem for ξ_1 between u_x and V_x, and ξ_2 between u and V (both ξ_1 and ξ_2 depend on x), we obtain

$$\begin{aligned} w_t &= \Phi'(u_x)w_{xx} + (\Phi'(u_x) - \Phi'(V_x))V_{xx} - N\left(\frac{1}{u^\alpha} - \frac{1}{V^\alpha}\right) \\ &= \Phi'(u_x)w_{xx} + \Phi''(\xi_1)V_{xx}w_x + \frac{N\alpha}{\xi_2^{\alpha+1}}w. \end{aligned}$$

Hence

$$\Phi'(u_x) - \Phi'(V_x) = \Phi''(\xi_1)(u_x - V_x)$$

and

$$\frac{1}{u^\alpha} - \frac{1}{V^\alpha} = \frac{\alpha}{\xi_2^{\alpha+1}}(u - V),$$

since $w(-a,t) = w(a,t) = 0$ and $w(0,x) = u_0(x) - V(x) \geq 0$. By the maximum principle from [**22**, Lemma 2.1, p. 54], we obtain $w \geq 0$ and

$$u(x,t) \geq V(x) \quad \text{for every } (x,t) \in (-a,a) \times [0,T). \qquad \square$$

Proof of Lemma 1.6. We distinguish two steps. We first prove a boundary estimate and second an interior estimate.

Step 1. u_x on $x = -a$ and $x = a$.

We define $L_s = (u_0(a) - u_0(-a))/2a$. Note that L_s is the smallest slope among all straight lines L passing through $(-a, u_0(-a))$ which have the property that $u_0(x) \leq L(x)$ for every $x \in [-a,a]$. This follows from the convexity of $u_0(x)$ assured by (5). Note that these lines L are not vertical because u_0 is $\mathcal{C}^3$ in the interval $[-a,a]$, see (5), thus the lateral derivatives of u_0 at $-a$ and a exist. Moreover, $L_s > 0$.

Define the operator

$$P(z) = -z_t + (\Phi(z_x))_x + f(z).$$

If u is a solution of (7), then $P(u) = 0 \geq P(L)$, because $f(L) \leq 0$ and $u \leq L$ on the parabolic boundary Γ, see (17). By the comparison principle of [**20**, Theorem 9.7, p. 222],

$u \le L$ in $[-a,a]\times[0,T)$. Then

$$u_x(-a,t) \le L_s \quad \text{for every } t\in[0,T). \tag{47}$$

Since $u(x,t)\ge V(x)$ for every $(x,t)\in[-a,a]\times[0,T)$, by Lemma 1.5, and $u(-a,t)=\beta_1=V(-a)$ we have

$$\frac{u(x,t)-u(-a,t)}{x+a} \ge \frac{V(x)-V(-a)}{x+a}.$$

Letting $x\to -a^+$, we obtain

$$0 = V_x(-a) \le u_x(-a,t) \quad \text{uniformly in } t. \tag{48}$$

By (47) and (48), we have

$$0 \le u_x(-a,t) \le L_s \quad \text{in } [0,T). \tag{49}$$

Let $v(x,t)=u(x,t)-u(a,t)$. Since v satisfies

$$v_t = (\Phi(v_x))_x + f(u),$$

with f as in (23), then

$$-v_t + (\Phi(v_x))_x = -f(u) \ge 0 \quad \text{for every } (x,t)\in[-a,a]\times[0,T).$$

Since

$$v(-a,t)=u(-a,t)-u(a,t)=\beta_1-\beta_2<0,$$

then $v(a,t)=0$ and $v(x,0)=u_0(x)-u_0(a)\le 0$. Thus, $v\le 0$ by the comparison principle [**20**, Theorem 9.7, p. 222]. Therefore

$$0 \le \frac{u(x,t)-u(a,t)}{x-a} \quad \text{for every } (x,t)\in[-a,a]\times[0,T).$$

Letting $x\to a^-$, we get

$$0 \le u_x(a,t) \quad \text{uniformly in } t. \tag{50}$$

Since $u(x,t)\ge V(x)$ and $u(a,t)=\beta_2=V(a)$, we obtain

$$\frac{u(x,t)-u(a,t)}{x-a} \le \frac{V(x)-V(a)}{x-a}.$$

Letting $x\to a^-$, we get

$$u_x(a,t) \le V_x(a) \quad \text{uniformly in } t. \tag{51}$$

By (51) and (52) we conclude that

$$0 \le u_x(a,t) \le V_x(a) \quad \text{for every } t\in[0,T). \tag{52}$$

Step 2. u_x in the interval $(-a,a)$.

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Let $h = u_x$. By the regularity of u provided by Lemma 1.3, we obtain

$$h_t - \Phi'(u_x)h_{xx} - \Phi''(u_x)u_{xx}h_x - \alpha u^{-(\alpha+1)}h = 0 \quad \text{for every } (x,t) \in (-a,a) \times [0,T).$$

Since $h(-a,t) \geq 0$ and $h(a,t) \geq 0$ for every $t \in [0,T)$ by Step 1, the fact that $h(x,0) = u_0'(x) \geq 0$, and using the maximum principle from [**22**, Lemma 2.1, p. 54], we obtain

$$u_x(x,t) \geq 0 \quad \text{for every } (x,t) \in (-a,a) \times [0,T). \qquad \square$$

Proof of Theorem 1.7. We first approximate problem (3) by a family of problems like (7) using Φ_ℓ defined by (8).

The hypotheses of Theorem 1.2 are satisfied here, hence there exists a solution u_ℓ of the problem

$$\begin{cases} z_t = (\Phi_\ell(z_x))_x - \dfrac{N}{z^\alpha}, & (x,t) \in (-a,a) \times (0,T), \\ z(-a,t) = \beta_1, z(a,t) = \beta_2, & t \in (0,T), \\ z(x,0) = u_0(x), & x \in [-a,a]. \end{cases} \tag{53}$$

Our aim is to show that there is $\ell_0 > 0$ such that the corresponding solution u_{ℓ_0} of (53) (and of (7)) satisfies

$$|(u_{\ell_0})_x(x,t)| < \ell_0 \quad \text{for every } (x,t) \in [-a,a] \times [0,T). \tag{54}$$

The function Φ_{ℓ_0} defined by (8) is such that

$$\Phi'_{\ell_0}((u_{\ell_0})_x) = \frac{1}{1 + ((u_{\ell_0})_x)^2}.$$

Therefore, we conclude that $u_{\ell_0} > 0$ is the solution of problem (3) and it belongs to $\mathcal{C}^{2,1}((-a,a) \times (0,T)) \cap \mathcal{C}^0([-a,a] \times [0,T)) \cap H_{3+\eta}$ for some $0 < \eta \leq 1$.

In this proof we denote $\Phi = \Phi_\ell$ and $u = u_\ell$. We estimate u_x on the interval $[-a,a]$.

Step 1. u_x on $-a$ and a.

By Lemma 1.6,

$$0 \leq u_x(-a,t) \leq L_s \quad \text{and} \quad 0 \leq u_x(a,t) \leq V_x(a). \tag{55}$$

We proceed to estimate u_x from below and from above for $x \in (-a,a)$.

Step 2. Estimate of u_x on $(-a,a)$ from below.

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The constant $U = \min_{(y,s)\in\Gamma} u_x(y,s)$ satisfies

$$U = \min_{(y,s)\in\Gamma} u_x(y,s) \geq \min_{x\in[-a,a]} u_0'(x)$$

by (55). Since u satisfies

$$u_t = \Phi'(u_x)u_{xx} + f(u) \quad \text{for every } (x,t) \in (-a,a) \times [0,T),$$

deriving with respect to x we obtain

$$u_{xt} = (\Phi'(u_x)u_{xx})_x + f'(u)u_x.$$

Defining $w = u_x$ we get

$$w_t = (\Phi'(w)w_x)_x + f'(u)w. \tag{56}$$

Define the operator

$$Qz = -z_t + (\Phi'(z)z_x)_x + f'(u)z.$$

Therefore,

$$Qw = 0 \leq Qv \quad \text{for every } (x,t) \in (-a,a) \times [0,T),$$

by virtue of

$$w(x,t) = u_x(x,t) \geq U \quad \text{for every } (x,t) \in \Gamma.$$

It follows from the comparison principle [**20**, Theorem 9.7, p. 222] that $w \geq v$ in $(-a,a) \times [0,T)$, that is,

$$u_x(x,t) \geq \min_{x\in[-a,a]} u_0'(x) \geq 0 \quad \text{for every } (x,t) \in (-a,a) \times [0,T). \tag{57}$$

Step 3. Estimate of u_x on $(-a,a)$ from above.

Define the constant $R = \max_{(y,s)\in\Gamma}\{u_x(y,s)\}$. The boundary and initial conditions imply

$$R \leq \max\{L_s, V_x(a), \max_{x\in[-a,a]} u_0'(x)\}.$$

Since u satisfies

$$u_t = \Phi'(u_x)u_{xx} + f(u),$$

deriving with respect to x gives

$$u_{xt} = (\Phi'(u_x)u_{xx})_x + f'(u)u_x.$$

Therefore, by (56) we obtain,

$$Qw = -w_t + (\Phi'(w)w_x)_x + f'(u)w = 0 \quad \text{for every } (x,t) \in (-a,a) \times [0,T),$$

and

$$w(x,t) = u_x(x,t) \leq R \leq \max\{L_s, V_x(a), \max_{x\in[-a,a]} u_0'(x)\} \quad \text{for every } (x,t) \in \Gamma.$$

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We can rewrite Qw in the form

$$Qw = -w_t + \Phi'(w)w_{xx} + \Phi''(w)w_x^2 + f'(u)w = 0.$$

Using the comparison principle [**20**, Theorem 9.5, p. 220] we obtain

$$\sup_{[-a,a]\times[0,T)} w(x,t) \leq \mathrm{e}^{(k+1)T} \sup_{(x,t)\in\Gamma} w^+(x,t)$$

where $k = \sup\{|f'(s)| : m \leq s \leq M\}$, see (46). Therefore

$$u_x(x,t) \leq \mathrm{e}^{(k+1)T} \max\{L_s, V_x(a), \max_{x\in[-a,a]} u_0'(x)\} \quad \text{for every } (x,t) \in [-a,a]\times[0,T). \tag{58}$$

Step 4. Conclusion of the proof.

By (57) and (58) we get

$$0 \leq u_x(x,t) \leq \mathrm{e}^{(k+1)T} \max\{L_s, V_x(a), \max_{x\in[-a,a]} u_0'(x)\} \quad \text{for every } (x,t) \in [-a,a]\times[0,T). \tag{59}$$

Therefore

$$\begin{aligned}(u_x(x,t))^2 \\ \leq \mathrm{e}^{2(k+1)T} \max\{L_s^2, (V_x(a))^2, \max_{x\in[-a,a]} (u_0'(x))^2\} \quad \text{for every } (x,t) \in [-a,a]\times[0,T).\end{aligned} \tag{60}$$

Since the right-hand side of (60) is finite constant and independent of $\ell > 0$, there exists $\ell_0 > 0$ large enough that

$$\mathrm{e}^{2(k+1)T} \max\{L_s^2, (V_x(a))^2, \max_{x\in[-a,a]} (u_0'(x))^2\} < \ell_0^2.$$

By (59) and (60) we conclude (54). The uniqueness of the solution follows from Lemma 3.2. □

4. An abstract result on asymptotic stability

This section is of independent interest. We generalize the results due to Reynolds [**23**, Theorem 1.9 and Corollary 1.10]. We prove the asymptotic stability for stationary solutions of a general parabolic problem. The improved results are Theorem 4.2 and Corollary 4.3; essentially, the conditions (66) and (67) with $\overline{C} > 0$ allow us to deal with singular equations like (3) and (7) (in [**23**] the requirement amounts to $\overline{C} < 0$).

We define the operator $\mathcal{L}$ by

$$\mathcal{L}u = h\left(x, t, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}\right) - u_t,$$

where

$$h\left(x, t, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}\right) = h\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_n^2}\right).$$

We also use the notation $h(x, t, z, p_1, \ldots, p_n, r_{11}, r_{1,2}, \ldots, r_{nn})$ or simply $h(x, t, z, p_i, r_{i,j})$, which is also equivalent to $h(x, t, z, p, r)$. The repeated indices i and j vary from 1 to n.

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First of all we show asymptotic stability for a stationary solution of the problem

$$\begin{cases} \mathcal{L}u = 0, & (x,t) \in \Omega \times (0,\infty), \\ u = \varphi(x,t), & (x,t) \in (\Omega \times \{0\}) \cup (\partial\Omega \times [0,\infty)), \end{cases} \tag{61}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $\varphi \in \mathcal{C}(\Omega \times \{0\}) \cup (\partial\Omega \times [0,\infty))$. More precisely, if v is a stationary solution, then one shows that the solution $u(x,t)$ of (61) tends to v as $t \to \infty$ uniformly for $x \in \Omega$.

We assume that the function $h : \Omega \times (0,\infty) \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$ is continuous. We also assume that the derivatives h_z, h_{p_i} and $h_{r_{ij}}$ are continuous in $\overline{\Omega \times (0,\infty)} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$.

Another assumption is the existence of a constant $K > 0$ such that

$$\sum_{i,j} h_{r_{i,j}}(x,t,z,p_i,r_{i,j})\xi_i\xi_j \geq K|\xi|^2 \tag{62}$$

for every $(x,t,z,p_i,r_{i,j}) \in \overline{\Omega \times (0,\infty)} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$ and every $\xi \in \mathbb{R}^n$, where $|\xi|$ denotes the Euclidean norm of ξ.

The following technical lemma is fundamental in the proof of the general stability Theorem 4.2.

Lemma 4.1. *Let λ, σ, χ, A be positive constants. Define $\phi(x) = \mathrm{e}^{\lambda R} - \mathrm{e}^{\lambda x_1}$, where R is a constant such that*

$$R > 3x_1 \quad \text{for every } x = (x_1, x_2, \ldots, x_n) \in \Omega.$$

Define

$$\psi(x,t) = \epsilon(\lambda)\frac{\phi(x)}{\delta} + \epsilon(\lambda)\frac{\phi(x)}{\delta_0} + \left[A\frac{\phi(x)}{\delta_0}\right]\mathrm{e}^{-\chi(t-\sigma)},$$

where

$$\delta = \inf_{(x,t)\in\Omega\times(0,\infty)} \mathrm{e}^{\lambda x_1}, \qquad \delta_0 = \inf_{(x,t)\in\Omega\times(0,\infty)} \phi(x), \qquad \delta_1 = \sup_{(x,t)\in\Omega\times(0,\infty)} \phi(x)$$

and

$$\epsilon(\lambda) = \frac{\epsilon}{\delta_1} \quad \text{with } 0 < \epsilon < 1.$$

Then, for every $A > 0$ and $\sigma > 0$ there exists a positive constant $M > 0$, independent of $\lambda > 0$ and $\chi > 0$, such that

$$|\psi_{x_1x_1}(x,t)| < M \quad \text{for every } (x,t) \in \overline{\Omega} \times [\sigma,\infty).$$

Proof. By definition of $\psi(x,t)$,

$$\psi_{x_1x_1}(x,t) = -\lambda^2\epsilon(\lambda)\frac{\mathrm{e}^{\lambda x_1}}{\delta} - \lambda^2\epsilon(\lambda)\frac{\mathrm{e}^{\lambda x_1}}{\delta_0} - \left[\lambda^2 A\frac{\mathrm{e}^{\lambda x_1}}{\delta_0}\right]\mathrm{e}^{-\chi(t-\sigma)}.$$

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A simple computation gives

$$\left|\lambda^2\epsilon(\lambda)\frac{\mathrm{e}^{\lambda x_1}}{\delta}\right| \le M_1\epsilon, \quad \left|\lambda^2\epsilon(\lambda)\frac{\mathrm{e}^{\lambda x_1}}{\delta_0}\right| \le M_2\epsilon,$$

$$\left|\lambda^2 A\frac{\mathrm{e}^{\lambda x_1}}{\delta_0}\right|\mathrm{e}^{-\chi(t-\sigma)} \le M_3 \quad \text{for every } t \ge \sigma,$$

where $M_i, i = 1, 2, 3$ are positive constants independent of λ. In this way, if $t \ge \sigma$, there exists a positive constant $M > 0$ independent of λ, such that

$$|\psi_{x_1x_1}(x,t)| < M \quad \text{for every } (x,t) \in \overline{\Omega} \times [\sigma, \infty).$$ □

The following assumptions are related to the general stability theorem that we state next.

Let $G = B(0, M)$ be a subset of $\mathbb{R}^{n^2}$, where M is the same as in Lemma 4.1. Suppose that there exists a constant B_G such that

$$|h_{p_i}| \le B_G \quad \text{for every } (x, t, z, p_i, r_{i,j}) \in \overline{\Omega \times (0,\infty)} \times \mathbb{R} \times \mathbb{R}^n \times G. \tag{63}$$

Recall (62), assume that there exists $\lambda > 0$ such that

$$B_G\lambda - K\lambda^2 \le -4. \tag{64}$$

For that fixed $\lambda > 0$ and R as in Lemma 4.1, let $\overline{C} > 0$ be a small enough constant that

$$\mathrm{e}^{\lambda(R-x_1)} \le \frac{1}{\overline{C}} + 1 \quad \text{for every } x_1 \text{ such that } x = (x_1, x_2, \ldots, x_n) \in \Omega. \tag{65}$$

Let $C : \Omega \times (0,\infty) \to \mathbb{R}$ be a measurable function essentially bounded from above. We assume that

$$\limsup_{t\to\infty} C(x,t) \le \overline{C} \quad \text{uniformly for } x \in \overline{\Omega} \tag{66}$$

and

$$h_z(x, t, z, p_i, r_{i,j}) \le C(x,t) \quad \text{for every } (x,t) \in \Omega \times (0,\infty). \tag{67}$$

We also assume that

$$\lim_{t\to\infty} h(x, t, 0, \ldots, 0) = 0 \quad \text{uniformly for } x \in \Omega \tag{68}$$

and

$$\lim_{t\to\infty} \varphi(x,t) = 0 \quad \text{uniformly for } x \in \partial\Omega. \tag{69}$$

Let $\bar{\sigma}$ be a sufficiently large constant that

$$C(x,t)(\mathrm{e}^{\lambda R} - \mathrm{e}^{\lambda x_1}) < 2\mathrm{e}^{\lambda x_1} \quad \text{for every } (x,t) \in \overline{\Omega} \times [\bar{\sigma}, \infty), \tag{70}$$

$$|h(x, t, 0, \ldots, 0)| < \epsilon(\lambda) \quad \text{for every } (x,t) \in \overline{\Omega} \times [\bar{\sigma}, \infty) \tag{71}$$

and

$$|\varphi(x,t)| < \epsilon(\lambda) \quad \text{for every } (x,t) \in \partial\Omega \times [\bar{\sigma}, \infty). \tag{72}$$

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Theorem 4.2. *Assume that (62)–(72) hold. If* $u \in \mathcal{C}^{2,1}(\Omega \times (0,\infty)) \cap \mathcal{C}(\overline{\Omega \times (0,\infty)})$ *is a solution of (61), then*

$$\lim_{t\to\infty} u(x,t) = 0 \quad \textit{uniformly for } x \in \overline{\Omega}.$$

Proof. Suppose that $\phi(x)$ and $\psi(x,t)$ are as in Lemma 4.1, with the constants

$$A = \sup_{\Omega} |u(x,\sigma)|, \quad \chi = \frac{2\delta}{\delta_1} \quad \text{and} \quad \sigma \geq \bar{\sigma}. \tag{73}$$

Note that $\psi(x,t) > 0$, $\psi_{x_1}(x,t) < 0$ and $\psi_{x_1x_1}(x,t) < 0$ for every $(x,t) \in \overline{\Omega} \times [\bar{\sigma},\infty)$.

By the definition of $\mathcal{L}$,

$$\mathcal{L}\psi(x,t) = h(x,t,\psi,\psi_{x_1},0,\ldots,0,\psi_{x_1x_1},0,\ldots,0) + \left[\frac{\chi A}{\delta_0}\right] \phi(x)\, \mathrm{e}^{-\chi(t-\sigma)}.$$

Writing h as $h(x,t,z,p_i,r_{i,j})$ and using the mean value theorem, we conclude that

$$\begin{aligned}\mathcal{L}\psi(x,t) = \; & h(x,t,0,\ldots,0) + \psi(x,t)h_z(\bar{v}) + \psi_{x_1}(x,t)h_{p_1}(\bar{v}) \\ & + \psi_{x_1x_1}(x,t)h_{r_{1,1}}(\bar{v}) + \left[\frac{\chi A}{\delta_0}\right]\phi(x)\mathrm{e}^{-\chi(t-\sigma)},\end{aligned}$$

where $\bar{v}$ is between $X = (x,t,0,\ldots,0)$ and $Y = (x,t,\psi,\psi_{x_1},0,\ldots,0,\psi_{x_1x_1},0,\ldots,0)$.

By virtue of the assumptions (62), (63) and (67), for every $(x,t) \in \overline{\Omega} \times [\bar{\sigma},\infty)$ we obtain

$$\begin{aligned}\mathcal{L}\psi(x,t) &\leq h(x,t,0,\ldots,0) + C(x,t)\psi(x,t) + B_G|\psi_{x_1}(x,t)| \\ &\quad + K\psi_{x_1x_1}(x,t) + \frac{\chi A}{\delta_0}\phi(x)\mathrm{e}^{-\chi(t-\sigma)} \leq h(x,t,0,\ldots,0) \\ &\quad + \frac{\epsilon(\lambda)}{\delta}(C(x,t)\phi(x) + B_G|\phi_{x_1}(x)| + K\phi_{x_1x_1}(x)) \\ &\quad + \frac{\epsilon(\lambda)}{\delta_0}(C(x,t)\phi(x) + B_G|\phi_{x_1}(x)| + K\phi_{x_1x_1}(x)) \\ &\quad + \frac{A}{\delta_0}\mathrm{e}^{-\chi(t-\sigma)}(C(x,t)\phi(x) + B_G|\phi_{x_1}(x)| + K\phi_{x_1x_1}(x)) \\ &\quad + \frac{\chi A}{\delta_0}\phi(x)\mathrm{e}^{-\chi(t-\sigma)}.\end{aligned} \tag{74}$$

By (64)–(66) we get

$$\begin{aligned} & C(x,t)\phi(x) + B_G|\phi_{x_1}(x)| + K\phi_{x_1x_1}(x) \\ &\quad = C(x,t)(\mathrm{e}^{\lambda R} - \mathrm{e}^{\lambda x_1}) + B_G\lambda\mathrm{e}^{\lambda x_1} - K\lambda^2\mathrm{e}^{\lambda x_1} \\ &\quad \leq -4\mathrm{e}^{\lambda x_1} + C(x,t)(\mathrm{e}^{\lambda R} - \mathrm{e}^{\lambda x_1}) \\ &\quad \leq -4\mathrm{e}^{\lambda x_1} + \overline{C}(\mathrm{e}^{\lambda R} - \mathrm{e}^{\lambda x_1}) \\ &\quad < -2\mathrm{e}^{\lambda x_1} \leq -2\delta.\end{aligned}$$

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In synthesis,

$$C(x,t)\phi(x) + B_G|\phi_{x_1}(x)| + K\phi_{x_1x_1}(x) < -2\delta \quad \text{for every } (x,t) \in \overline{\Omega} \times [\bar{\sigma}, \infty). \tag{75}$$

Substituting (75) into (74) we have

$$\begin{aligned} L\psi(x,t) &< \epsilon(\lambda) + \frac{\epsilon(\lambda)}{\delta}(-2\delta) + \frac{\epsilon(\lambda)}{\delta_0}(-2\delta) - \frac{2\delta A}{\delta_0}\mathrm{e}^{-\chi(t-\sigma)} + \frac{\chi A\delta_1}{\delta_0}\mathrm{e}^{-\chi(t-\sigma)} \\ &< -\epsilon(\lambda) - \frac{2\delta A}{\delta_0}\mathrm{e}^{-\chi(t-\sigma)} + \frac{\chi A\delta_1}{\delta_0}\mathrm{e}^{-\chi(t-\sigma)}. \end{aligned}$$

By (73),

$$L\psi(x,t) < -\epsilon(\lambda) \quad \text{for every } (x,t) \in \overline{\Omega} \times [\sigma, \infty). \tag{76}$$

And, by the definition of $\psi(x,t)$ (see Lemma 4.1),

$$\psi(x,\sigma) > A \quad \text{for every } x \in \Omega, \tag{77}$$

$$\psi(x,t) > \epsilon(\lambda) \quad \text{for every } (x,t) \in \partial\Omega \times [\sigma, \infty). \tag{78}$$

By the estimates (72), (76)–(78) and (73) and by the comparison theorem [**9**, Theorem 16, p. 52] in the cylindrical domain $\Omega \times (0, \infty)$, we obtain

$$u(x,t) < \psi(x,t) \quad \text{for every } (x,t) \in \overline{\Omega} \times [\sigma, \infty).$$

Repeating the same reasoning with $-\psi$ in place of ψ, we obtain

$$-\psi(x,t) < u(x,t) \quad \text{for every } (x,t) \in \overline{\Omega} \times [\sigma, \infty).$$

Therefore,

$$|u(x,t)| < \psi(x,t) \le M_4[\epsilon + \mathrm{e}^{-\chi(t-\sigma)}] \quad \text{for every } t \ge \sigma,$$

where M_4 is a positive constant independent of t. Letting $t \to \infty$ we conclude the proof of the theorem. □

Let h_0 and ω_0 be continuous functions and suppose that

$$h_0(x, u, u_{x_i}, u_{x_ix_j}) = 0 \quad \text{in } \Omega, \tag{79}$$

with

$$u = \omega_0 \quad \text{on } \partial\Omega, \tag{80}$$

has a unique solution $v \in \mathcal{C}^2(\Omega)$. We also suppose that $u \in \mathcal{C}^{2,1}(\Omega \times (0,\infty)) \cap \mathcal{C}(\overline{\Omega \times (0,\infty)})$ is a solution of (61).

Assume that

$$h(x, t, v, v_{x_i}, v_{x_ix_j}) \text{ is continuous on } \overline{\Omega \times (0,\infty)}, \tag{81}$$

and

$$\omega_0 \in \mathcal{C}(\overline{\Omega}). \tag{82}$$

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Corollary 4.3. *Assume (62)–(67), (81) and (82). Let u be a solution of (61). Suppose that $h(x,t,z,p,r) \to h_0(x,z,p,r)$ as $t \to \infty$ uniformly in x,z,p,r, where $x \in \Omega$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$, $r \in \mathbb{R}^{n^2}$ and $\varphi(x,t) \to \omega_0(x)$ uniformly for $x \in \partial\Omega$ as $t \to \infty$. Then*

$$\lim_{t\to\infty} u(x,t) = v(x) \quad \textit{uniformly for } x \in \overline{\Omega}.$$

Proof. The proof is analogous to [**23**, Corollary 1.10]. □

5. Asymptotic stability of stationary solutions of (3) and (7)

Proof of Theorem 1.8. Let $\Omega = (-a,a) \subset \mathbb{R}$ and let $M > 0$ as in Lemma 4.1. Recall (63) and take the interval $G = [-M, M]$.

We are going to use Theorem 4.2 and Corollary 4.3 with the function

$$h(x,t,z,p,r) = \Phi'(p)r + f(z), \tag{83}$$

in the set $(-a,a) \times (0,\infty) \times \mathbb{R} \times \mathbb{R} \times G$, where Φ and f are given by (8) and (23), respectively. Note that $h_p(x,t,z,p,r) = \Phi''(p)r$ is continuous. Then

$$\begin{aligned} &|h_p(x,t,z,p,r)| \\ &\quad = |\Phi''(p)||r| \le BM \quad \text{for every } (x,t,z,p,r) \in (-a,a) \times (0,\infty) \times \mathbb{R} \times \mathbb{R} \times G. \end{aligned} \tag{84}$$

Recalling (12) and (16), take $B_G = MB > 0$, $K = \gamma$ and $\lambda > 0$ large enough in (64) such that condition (25) holds. We conclude that

$$h_r(x,t,z,p,r)\xi\xi = \Phi'(p)\xi^2 \ge \gamma\xi^2. \tag{85}$$

Thus conditions (62)–(64) are satisfied.

The solution of (7) satisfies

$$\frac{\theta}{4} \le \underline{u} \le u \le M,$$

see the proof of Theorem 1.2. It follows from the definition of f in (23) that $f(u) = -(N/u^\alpha)$ and $f(V) = -(N/V^\alpha)$. Therefore,

$$\begin{aligned} h(x,t,u,u_x,u_{xx}) &= \Phi'(u_x)u_{xx} - \frac{N}{u^\alpha} = \Phi'(u_x)u_{xx} + f(u), \\ h_0(x,V,V_x,V_{xx}) &= \Phi'(V_x)V_{xx} - \frac{N}{V^\alpha} = \Phi'(V_x)V_{xx} + f(V). \end{aligned}$$

We can rewrite problem (7) as

$$\begin{cases} u_t = h(x,t,u,u_x,u_{xx}), \quad x \in (-a,a),\, t \in (0,\infty), \\ u(-a,t) = \beta_1, u(a,t) = \beta_2, \quad t \in (0,\infty), \\ u(x,0) = u_0(x), \quad x \in [-a,a], \end{cases} \tag{86}$$

where h is defined as in (83). From now on, we verify (62)–(67).

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We prove the stability condition for problem (7) by means of Corollary 4.3. Take

$$\varphi(x,t) = \frac{t}{1+t}V(x) + \frac{1}{1+t}u_0(x)$$

in (61) and $\omega_0(x) = V(x)$ in (82). Recall that $u_0(-a) = V(-a) = \beta_1$ and $u_0(a) = V(a) = \beta_2$, then

$$\varphi(x,0) = u_0(x),$$

$$\varphi(-a,t) = \frac{t}{1+t}V(-a) + \frac{1}{1+t}u_0(-a) = \beta_1$$

and

$$\varphi(a,t) = \frac{t}{1+t}V(a) + \frac{1}{1+t}u_0(a) = \beta_2.$$

Therefore, the solution u of (86) is such that $u = \varphi$ for each $(x,t) \in ((-a,a)\times\{0\}) \cup (\{-a,a\}\times(0,+\infty))$; we also have $\varphi(x,t) \to \omega_0(x) = V(x)$ as $t \to +\infty$ uniformly for $x \in \{-a,a\}$.

It remains to check (65)–(67). Condition (67) is verified, since $h_z(x,t,z,p,r) = f'(z)$ satisfies

$$f'(z) \leq \frac{\alpha N}{m^{\alpha+1}} = \frac{4^{\alpha+1}\alpha N}{\theta^{\alpha+1}} \quad \text{for every } z \in [-a,a]$$

(recall (23)), in (66) we take $C(x,t) = \overline{C} = (4^{\alpha+1}\alpha N/\theta^{\alpha+1})$.

Let V be the solution of the stationary problem, see Lemma 1.1. Thus

$$\begin{cases} (\Phi(V_x))_x + f(V) = 0, \quad x \in (-a,a), \\ V(-a) = \beta_1, V(a) = \beta_2, \end{cases} \tag{87}$$

where $f(V) = -(N/V^\alpha)$ and $V > 0$ in $[-a,a]$. Since $V_x \geq 0$ in $[-a,a]$, we have that $\theta = V(-a) = \beta_1(a)$. Therefore

$$f'(z) \leq \overline{C} = \frac{4^{\alpha+1}\alpha N}{\beta_1(a)^{\alpha+1}}.$$

We now verify (65). Since $\mathrm{e}^{\lambda(2a-x)} \leq \mathrm{e}^{\lambda 3a}$ for each $x \in [-a,a]$, (26) is equivalent to

$$\mathrm{e}^{\lambda 3a} \leq \frac{\beta_1(a)^{\alpha+1}}{4^{\alpha+1}\alpha N} + 1 = \frac{1}{\overline{C}} + 1.$$

Hence we conclude that

$$\mathrm{e}^{\lambda(2a-x)} \leq \frac{1}{\overline{C}} + 1.$$

We have verified (65)–(67).

Since $h(x,t,V,V_x,V_{xx})$ is continuous in $[-a,a]\times[0,\infty)$ and $h(x,t,z,p,r) = h_0(x,z,p,r) = \Phi'(p)r + f(z)$, by Theorem 4.2 and Corollary 4.3 we obtain

$$\lim_{t\to\infty} u(x,t) = V(x) \quad \text{uniformly in } x \in [-a,a]. \qquad \square$$

Then V is a stable equilibrium of (7).

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Proof of Corollary 1.9. In the proof of Theorem 1.7 we can take $\ell_0 > 0$ large enough that

$$|u_x(x,t)| < \ell_0 \quad \text{for every } (x,t) \in [-a,a] \times [0,T)$$

and

$$|V_x(x)| < \ell_0 \quad \text{for every } x \in [-a,a].$$

Therefore, we can rewrite

$$h(x,t,u,u_x,u_{xx}) = \frac{u_{xx}}{1+u_x^2} - \frac{N}{u^\alpha} = \Phi'_{\ell_0}(u_x)u_{xx} + f(u)$$

and

$$h_0(x,V,V_x,V_{xx}) = \frac{V_{xx}}{1+V_x^2} - \frac{N}{V^\alpha} = \Phi'_{\ell_0}(V_x)V_{xx} + f(V).$$

The conclusion follows from Theorem 1.8. □

We complete the paper with an additional remark on an estimate for the solution of (3) and of (7). Recall that

$$\frac{1}{\Phi'_\ell(s)} = s^2 + 1 \quad \text{if } |s| < \ell,$$

see (8) and (9). In the next proposition $0 < \alpha \leq 1$ is as in (3) and (7).

Proposition 5.1. *Suppose that Φ satisfies (8)–(16). Let u be the solution of (3) according to Theorem 1.7 with $0 < u_0 \leq 1$. For all constants σ and ξ such that $\sigma > 0$ and $0 < \xi < \alpha$, there exists $\gamma_0 > 0$ such that if*

$$0 < \sigma \leq \frac{1}{\gamma_0} \min\left\{\frac{\alpha-\xi}{2\xi}, 1\right\}$$

and

$$u_0' \geq \sigma(x+a)u_0^{-\xi} \quad \textit{for every } x \in [-a,a], \tag{88}$$

then

$$u(x,t) \geq C(x+a)^{2/(1+\xi)} \quad \textit{for every } x \in [-a,a], \tag{89}$$

where $C > 0$ is a constant depending only on ξ, σ, and γ_0.

Proof of Proposition 5.1. For $\ell > 0$, let Φ_ℓ be as in (8). We have $\Phi'_\ell(s) \geq \gamma = p(2\ell) > 0$, see the proof of Theorem 1.7. Note that $\gamma = \gamma_\ell = p(2\ell)$ goes to 0 when ℓ goes to $+\infty$. Hence, taking $\ell = \ell_0$ large enough as in Theorem 1.7:

$$0 < \sigma \leq \frac{1}{\gamma} \min\left\{\frac{\alpha-\xi}{2\xi}, 1\right\}. \tag{90}$$

We denote such γ by γ_0.

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Integrating from 0 to $s > 0$ we get

$$\Phi_\ell(s) \geq \gamma s$$

and by (88) we obtain

$$\Phi_\ell(u_0'(x)) \geq \gamma\sigma(x+a)u_0^{-\xi}. \tag{91}$$

Since $u_0 \leq 1$, it follows from Lemma 3.3 that

$$0 < u \leq M \leq 1. \tag{92}$$

Let

$$\epsilon = \gamma\sigma.$$

We set $J_\ell(x,t) = \Phi_\ell(u_x) - \epsilon(x+a)u^{-\xi}$. By (91) we have $J_\ell(x,0) \geq 0$ in $[-a,a]$ and, owing to the boundary conditions $u(-a,t) = \beta_1$ and $u(a,t) = \beta_2$, we have $J_\ell(-a,t) \geq 0$ and $J_{\ell x}(a,t) \geq 0$. Indeed, by Lemma 1.6, $u_x(-a,t) \geq 0$ and $u_x(a,t) \geq 0$, hence

$$J_\ell(-a,t) = \Phi_\ell(u_x(-a,t)) - \epsilon(-a+a)u(-a,t)^{-\xi} \geq \gamma u_x(-a,t) \geq 0.$$

Since by assumptions on ξ, α and $\epsilon = \gamma\sigma \leq 1$, we obtain

$$\begin{aligned} J_{\ell x}(x,t) &= \Phi_\ell'(u_x)u_{xx} - \epsilon\, u^{-\xi} + \epsilon\gamma(x+a)u^{-\xi-1}u_x \\ &\geq \Phi_\ell'(u_x)u_{xx} - u^{-\alpha} + \epsilon\gamma(x+a)u^{-\xi-1}u_x \\ &= u_t + \epsilon\gamma(x+a)u^{-\xi-1}u_x. \end{aligned}$$

Hence,

$$J_{\ell x}(a,t) \geq u_t(a,t) + \epsilon 2a\gamma u(a,t)^{-\xi-1}u_x(a,t) \geq 0,$$

because $u_t(a,t) = 0$.

To prove (89) it suffices to show that $J_\ell(x,t) \geq 0$ on $[-a,a] \times (0,T)$. In fact, if $J_\ell(x,t) \geq 0$ we have

$$\Phi_\ell(u_x) \geq \epsilon(x+a)u^{-\xi}.$$

Using the fact that $\Phi_{\ell_0}(s) \leq s$ for $s > 0$ and that $u_x \geq 0$ for every $(x,t) \in [-a,a] \times (0,T)$ we conclude that

$$u_x \geq \epsilon(x+a)u^{-\xi}.$$

Therefore,

$$\frac{1}{1+\xi}(u^{1+\xi}(x,t))_x \geq \epsilon(x+a).$$

Integrating from $-a$ to $x \leq a$ we obtain

$$u^{1+\xi}(x,t) \geq \epsilon(1+\xi)(x+a)^2 + \beta_1^{1+\xi} \geq \epsilon(1+\xi)(x+a)^2$$

and (89) follows.

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We will prove that $J_\ell(x,t) \geq 0$. A straightforward calculation gives

$$\frac{1}{\Phi'_\ell(u_x)} J_{\ell t} - J_{\ell xx} = (\alpha u^{-\alpha-1} - 2\epsilon\xi u^{-\xi-1})u_x - \epsilon\xi(x+a)\frac{1}{\Phi'_\ell(u_x)}u^{-\xi-\alpha-1} + \epsilon\xi(\xi+1)(x+a)u^{-\xi-2}u_x^2.$$

The first term on the right-hand side can be estimated from below by $\xi u^{-\alpha-1}u_x$, see (90) and (92).

To estimate the second term on the right-hand side, taking $\ell = \ell_0$ as in Theorem 1.7 and (90), note that by (54) we have the estimate $|u_x(x,t)|^2 < \ell_0^2$, and under assumption on Φ_ℓ we have

$$\frac{1}{\Phi'_{\ell_0}(u_x)} = u_x^2 + 1, \tag{93}$$

see (8) and (9). Therefore,

$$\frac{1}{\Phi'_{\ell_0}(u_x)} J_{\ell_0 t} - J_{\ell_0 xx} \geq \xi u^{-\alpha-1}u_x - \epsilon\xi(x+a)u^{-\xi-\alpha-1}u_x^2 - \epsilon\xi(x+a)u^{-\xi-\alpha-1} + \epsilon\xi(\xi+1)(x+a)u^{-\xi-2}u_x^2.$$

By virtue of the fact that

$$\Phi_{\ell_0}(s) \leq s \quad \text{for } s > 0,$$
$$u_x \geq 0 \quad \text{for every } (x,t) \in [-a,a] \times (0,T)$$

and the definition of J_{ℓ_0} we obtain

$$\begin{aligned}\frac{1}{\Phi'_{\ell_0}(u_x)} J_{\ell_0 t} - J_{\ell_0 xx} &\geq \xi u^{-\alpha-1}u_x - \epsilon\xi(x+a)u^{-\xi-\alpha-1}u_x^2 \\ &\quad - \epsilon\xi(x+a)u^{-\xi-\alpha-1} + \epsilon\xi(\xi+1)(x+a)u^{-\xi-2}u_x^2 \\ &\geq \xi u^{-\alpha-1}u_x + \epsilon\xi^2(x+a)u^{-\xi-2}u_x^2 + \xi u^{-\alpha-1}J_{\ell_0} - \xi u^{-\alpha-1}\Phi_{\ell_0}(u_x) \\ &\geq \xi u^{-\alpha-1}(u_x - \Phi_{\ell_0}(u_x)) + \xi u^{-\alpha-1}J_{\ell_0} \geq \xi u^{-\alpha-1}J_{\ell_0}.\end{aligned}$$

Using the maximum principle [**22**, Lemma 2.1, p. 54], we obtain from $u < 1$ and $\alpha < 1$ that

$$J_{\ell_0}(x,t) \geq 0 \quad \text{for every } (x,t) \in [-a,a] \times (0,T). \qquad \square$$

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